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Distributionally adjusted life expectancy as a life table function

Francisco J. Goerlich*

Abstract

This paper investigates alternative measures of life expectancy that take into account distributional considerations in the length of life of the generation in a period life table. Virtually all the literature studying inequality in the length of life distribution has used inequality tools from the income distribution analysis, focusing on life expectancy and inequality as separate, albeit related, issues. We propose an alternative, integrated framework that allows us to combine both dimensions in a single index, and provide an axiomatic derivation that delivers the particular indexes to be used given a set of axioms. We illustrate the proposed index using data from the Human Mortality Database.

Keywords: Life expectancy, Life table, Duration, Indexes, Length of life, Inequality.
JEL classification: J10, J11, J14.

Resumen

Este trabajo investiga medidas alternativas de esperanza de vida que tienen en cuenta aspectos distributivos en relación a la duración de la vida en la generación de una tabla de vida de periodo. La práctica totalidad de la literatura que estudio la desigualdad en el tiempo de vida utiliza el instrumental derivado de la medición de la desigualdad en la distribución de la renta, centrándose en la esperanza de vida y la desigualdad como aspectos separados, aunque relacionados. El trabajo propone un marco alternativo integrado que permite combinar ambas dimensiones en un único índice, y proporciona una derivación axiomática que conduce a una familia concreta de índices. Los índices propuestos se ilustran con una aplicación a partir de la Human Mortality Database.

Palabras clave: Esperanza de Vida, Tabla de Mortalidad, Índices de Duración, Duración de la Vida, Desigualdad.
Clasificación JEL: J10, J11, J14.

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1. Introduction.

Life expectancy at birth summarizes in a single number the mortality conditions of a given population, and it does so in a way that is independent of the age structure of the underlying population. Essentially this means that the indicator is comparable, in time and across societies, with populations having very different age structures. This feature has contributed to make life expectancy one of the most widely used indicators in international comparisons on development. Additionally, life expectancy at birth is one of the simplest summary measures of population health for a community (Murray et al. 2002) and as a consequence, of its degree of development (Sen 1998, 1999).

For all these reasons life expectancy has become one essential index in the complex and elusive concept of quality of life: without life there is no possibility to enjoy the consumption opportunities represented by per capita income, the other widely used development indicator in international comparisons. However, the Stiglitz, Sen, and Fitoussi (2009) report recently recognized the need to look beyond GDP to measure the progress of actual societies. This was in fact the goal of the United Nations Development Program (UNDP 2019) Human Development Index, together with many other proposals to include life expectancy as part of synthetic quality of life indexes (Osberg and Sharpe 2002).

In the same vein this paper attempts to go beyond life expectancy by trying to introduce distributional aspects into a single life expectancy index constructed from the standard biometric functions of a period life table, but that can be computed more generally for a population or a real generation.

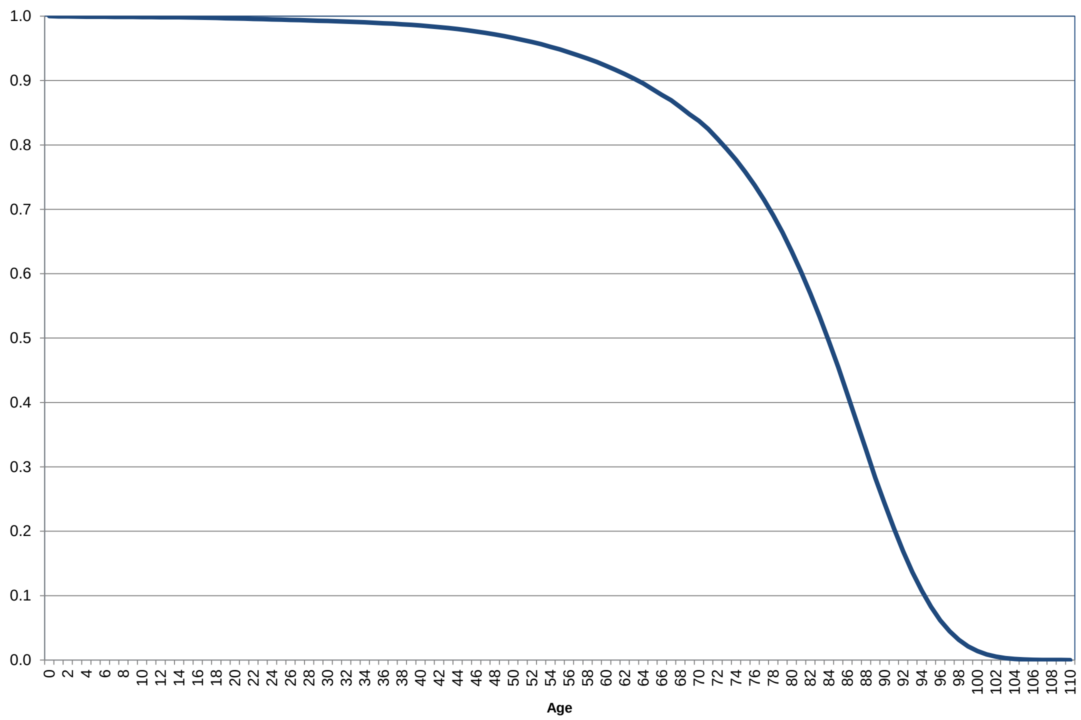
The structure of the paper is as follows. The next section tries to motivate the life expectancy index proposed by looking at standard biometric curves: the survival function and the distribution of age at death in the life table. A critical assessment of the literature on length of life inequality follows. The fourth section introduces the proposed measure and the fifth offers an illustrative application. A final brief section concludes.

2. Motivation.

Figure 1 represents the survival function of the Spanish life table population for 2009. The area under the curve represents life expectancy at birth, which was 81.64 years at that date, and the curve itself summarizes the mortality experience of the Spanish population in 2009 as projected onto a fictitious generation.[1]

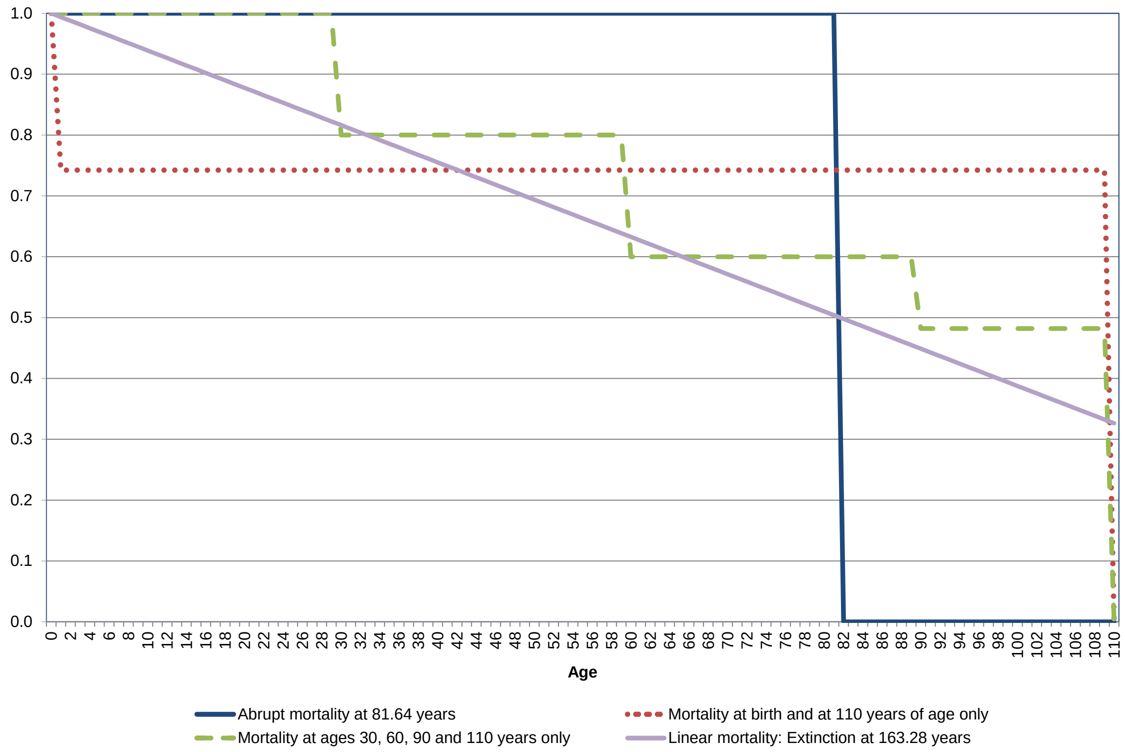
It should be obvious that many different shapes of the survival function are consistent with a given figure for life expectancy. Figure 2 depicts some of them. It is worth mentioning the implications of these curves for the mortality experience of the population they represent.

Figure 1: Survival function of total population. Spain. 2009
Life expectancy at birth 81.64 years



[1] Our set up is the standard period life tables since we end up by proposing a new biometric function – a new column – in these tables. It is well known that in period life tables age specific mortality rates for a given period (usually a year) are applied to a given number of newborns (usually 100,000), the fictitious life table generation, and they are followed until the generation dies out. This is of some importance because in making distributional adjustments to the measures of life expectancy we should take into account the special kind of data at hand.
This is to say, life expectancy is not obtained from survey data as it is for inequality or poverty measures. Similarly, life expectancy is a summary measure for a population, as is per capita income for a society; but we do not have individual 'life lines' for real people, although we are able to collect individual income data from surveys or administrative records for inequality analysis.
The particular structure of the data available will affect the way in which we can adapt the life expectancy index to incorporate distributional considerations.

Figure 2: Alternative survival functions with the same life expectancy at birth



- **Abrupt step (blue) line**: According to this line everybody survives until the age of 81.64 years. Everybody is born at the same instant and also dies at the same age, which coincides with the life expectancy for this population. Note that everybody has the same length of life.
- **Dotted (red) line**: According to this line 25.78% of the population dies at birth – they have no life– and the remaining 74.22% of the population survives until the age of 110. So mortality is concentrated at two points in the life of the generation, at the beginning and at the end; nobody dies between these two extremes.

These two survival functions represent extreme –and unrealistic– cases. In the first case everybody has the same life length and there is no inequality in the distribution of the length of life. Note that in this case the survival function is a perfect rectangular as Wilmoth and Horiuchi (1999) note. Hence rectangularization of a survival curve is directly associated with decreasing variability in the distribution of ages at death. As deaths become more concentrated in an increasingly narrow age range, the slope of the survival function becomes steeper in that range. And the curve itself ends up as a rectangle.

In the second case it is a lottery whether you die just at the start of life or at the end of life, dependent on health status and medical technology. There are only two possible outcomes and the inequality in the distribution of length of life is maximum: you either die at birth or survive for the maximum possible time.

In both cases, however, the population's life expectancy is identical, 81.64 years, although the two situations are extremely different and obviously society cannot value them in the same way.

- **Dashed (green) line**: According to this line every newborn survives until the age of 30 and then 20% of the population dies. The remaining 80% survives until the age of 60 and then another 20% of the initial population dies. The remaining 60% survive until the age of 90 and then a further 11.8% of the initial population dies. The survivors, 48.2% of the initial population, live to the age of 110 years and then they all die abruptly.
- **Continuously descending (purple) line**: According to this line the newborns decrease linearly by a constant amount. The generation lives until the incredible age of 163 years (twice the life expectancy of the actual survival curve as depicted in Figure 1), although the maximum age shown in Figure 2 is only 110 years.

All these survival functions represent the same life expectancy but have very different implications for the age at death distribution. Society can clearly value these functions in different ways. The purpose of this paper is to propose a way of incorporating this aspect into a more general measure of life expectancy, at any age, and at the same time to lay its foundations from an axiomatic point of view: a **distributionally adjusted life expectancy**.

It is worth stressing that although the economic literature has a clear preference for equality, which can be achieved for a given amount of total income or wealth resources just by a convenient set of transfers; there is no clear preference for equality in the length of life distribution in the demographic or public health literature. In this latter case there is no possibility for transfers, and life expectancy and measures of spread of the distribution are clearly related. Reducing childhood mortality will increase life expectancy but at the same time will reduce inequality. Reducing mortality at older ages will increase life expectancy but will also increase inequality in the length of life distribution. We return to these questions in future sections.

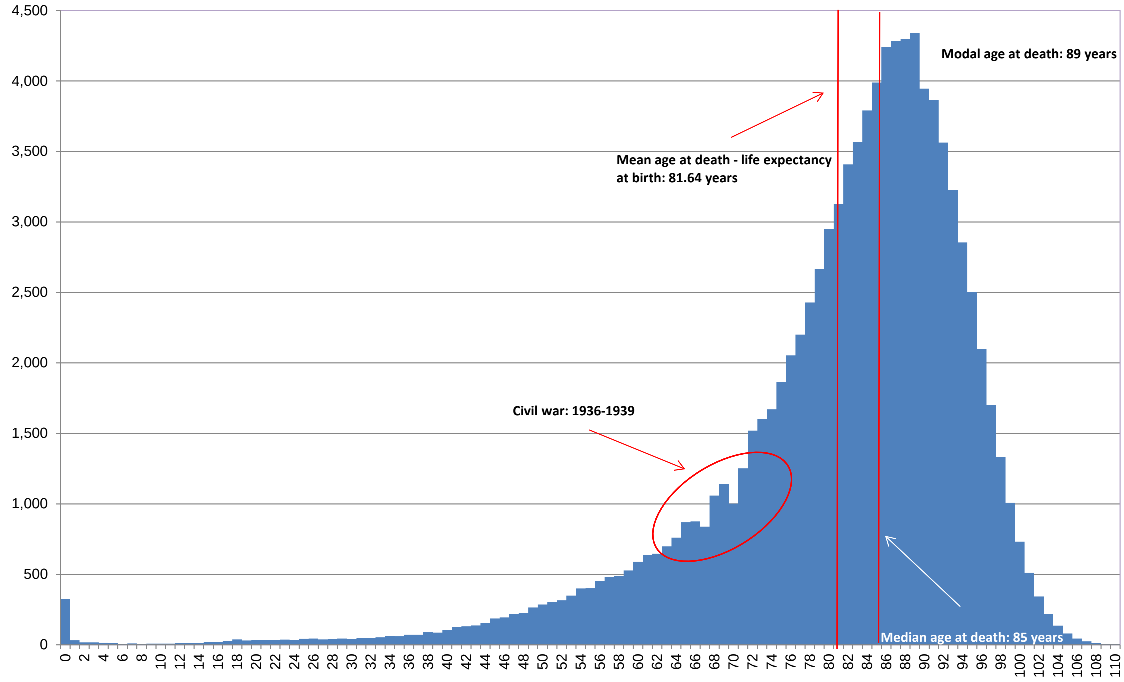
Nevertheless, given that life expectancy at birth –the area under the survival function in Figure 1– is in fact the average length of life of the fictitious generation in the life table, or equivalently the mean age at death of this generation, it is of interest to see what this distribution looks like. Figure 3 shows the age distribution of deaths corresponding to the survival function depicted in Figure 1.[2]

The general shape of this curve is well known. It has two modes, one at age 0, and the other at a much older age, 89 years in Figure 3. What is not apparent from the figure is that the relative importance of the two modes has switched over time. Mortality at birth was the dominant mode before the demographic transition. Reductions in mortality among younger groups increased life expectancy, decreased inequality in the length of life distribution, and amplified the importance of the modal age at older ages, which eventually became the dominant mode in mature societies.

Figure 3 makes it clear why our interest should reside not only in the mean of the distribution –life expectancy at birth– but also in other distributional patterns of the length of life distribution. Although we concentrate only on measures of location the mean is not the only interesting statistic. Some national statistical institutes published the median age at death –the age at which half of the generation has died and the other half is still alive– in their period life tables at the beginning of the 20th century (INE 1952, 1958). Also some authors have proposed the mode age at death as a more appropriate measure of longevity than life expectancy at birth (Canudas-Romo 2008).

[2] Dividing the ordinates in Figure 1 by the size of the generation, 100,000 people, we obtain the corresponding probability density function.

Figure 3: **Age distributions of death of total population. Spain. 2009**
Number of deaths in the life table generation



3. A critical assessment of the literature.

This paper does not directly take into account the question of measuring length of life inequality, but of course it will do so indirectly since its purpose is to develop a life expectancy index that incorporates distributional features.

A few papers in the demographic tradition deal with the issue of measuring inequality in length of life, essentially using the technical apparatus of the income distribution economic literature without too much theoretical discussion.[3] Many

[3] Given that inequality in length of life has extensively used the tools developed for the analysis of income distribution it seems of interest to compare the typical age distribution of deaths, as depicted in Figure 3, with the typical income distribution that can be found elsewhere.
First, we notice that the age distribution of deaths is always bimodal. Economists would say that it is a polarized distribution (Esteban and Ray 1994), so maybe polarization measures could also help in characterizing the length of life distribution. We do not pursue this point further in this paper.
Second, ignoring the mode at age zero, $x = 0$, the age distribution of deaths is a reflected mirror image of the typical income distribution. People dying in the early stages of their lives are the 'poor', but in this case they are very few. People dying at very old ages are the 'rich', and they account for a substantial part of the population. While the income distribution is right-skewed – so the mean is greater than the median, which in turns is greater than the mode – and has a very long right tail, the age distribution of deaths is left-skewed – so the mean is lower than the median, which in turns is lower than the mode as can be seen in Figure 3 – and has a long left tail bounded at $x = 0$.
These similarities can probably be exploited further in comparing income and length of life distributions, but again we do not pursue this further here.

others include Hanada (1983), Illsey and Le Grand (1987) –who compute inequality from distributions of deaths by age in real populations– Wilmoth and Horiuchi (1999), Anand and Nanthikesan (2000), Shkolnikov, Andreev, and Begun (2003), Edwards and Tuljapurkar (2005), Smits and Monden (2009), Edwards (2011), Seaman, Leyland, and Popham (2016), or Jordá and Niño-Zarazúa (2017).

In addition, a few papers have taken up the issue of incorporating inequality into the health dimension of the Human Development Index (HDI) –for example Hicks (1997), Foster, Lopez-Calva, and Szekely (2005), or Kovacevic (2010)– and given that life expectancy at birth is used to measure the health dimension in the HDI they are closest in spirit to this paper.

This literature, however, does not discuss many of the key assumptions used in measuring income or wealth inequality where the aversion to inequality has a more intuitive appeal. Some key assumptions commonly employed in this literature cannot be directly transposed to the health context. While reducing inequality in the income distribution is possible, without altering the mean, through a transfer of income from the rich to the poor – the so called Pigou (1932)–Dalton (1920) condition – a reduction of inequality in the length of life distribution cannot be achieved by this mechanism. We simply cannot reduce longevity of older people to increase the length of life of younger people. Although it seems sensible to assume that some aversion to inequality in the length of life does exist, this is not easily assessed.

The other important question that is neglected in this literature is whether the appropriate inequality measures for analyzing the length of life distribution should be scale invariant (relative) or translation invariant (absolute). Usually this is solved in practice by employing different kinds of indicators from both families (Wilmoth and Horiuchi 1999, Edwards 2011) but no real justification is offered. Scale invariant indicators are usually used for dealing with income variables since this guarantees that inequality is independent of the scale. Just as it is irrelevant whether we measure income in euros, dollars, or pounds, the same could be said with respect to inequality in the length of life distribution, which should be the same regardless of whether we measure length of life in years or months.

However, relative inequality measures have another important implication that may be not so appealing when we deal with health outcomes. Two length of life distributions with different means –life expectancy at birth– will be ranked as equal by scale invariant indicators if the relative distance between individuals in terms of the length of life is exactly the same. The following example, taken (and expanded) from Jordá and Niño-Zarazúa (2017), clearly shows this situation. Imagine we are interested in ranking two distributions with two individuals each according to their inequality only. In distribution A we have two individuals; one lived for only 5 years and the other for 50 years. Life expectancy at birth of this distribution is 27.5 years. In distribution B we have two individuals; one lived for only 6 years and the other for 60 years. Life expectancy at birth of this distribution is 33 years. Relative inequality indicators will rank both distributions as equally unequal, since the relative distance between the older and the younger is of the order of ten in both distributions. Notice that we have 20% longer life in distribution B than in distribution A, 11 years, and that distribution B can be obtained from distribution A simply by multiplying the length of life of each individual by the scale factor 1.2. By contrast, absolute inequality indicators will rank distribution B as more unequal than distribution A since the absolute difference in lifespan between individuals in distribution B is 54 years, but in distribution A it is just 45 years.

Now let us imagine we have distribution C with two individuals; one lived for 10.5 years and the other for 55.5 years. Life expectancy at birth of this distribution is 33 years, the same as in distribution B. Absolute inequality indicators will rank distribution A and distribution C as equally unequal since the absolute difference in lifespan between individuals in both distributions is the same: 45 years. By contrast, relative inequality indicators will rank distribution C as more equal than distribution A since the relative distance between the older and the younger is of the order of ten in distribution A but only of the order of 5.3 in distribution C. Notice that again we have 20% longer life in distribution C than in distribution A, 11 years, and that distribution C can be obtained from distribution A by simply adding half of this, 5.5 years of life, to each individual.

How the observed increments in life expectancy should be distributed among the generation in the life table to consider inequality constant when we compare the spread of the distribution independently of the mean is not a trivial question. Scale invariant or translation invariant inequality indexes provide different answers according to different perspectives. Whereas some demographers prefer absolute inequality measures (Wilmoth and Horiuchi 1999; Edwards 2011), others argue in favor of relative ones (Shkolnikov, Andreev, and Begun 2003; Smits and Monden 2009), and most of them present a plethora of different indexes in order to obtain robust results (Jordá and Niño-Zarazúa 2017).

Contrary to what happens in economics, the great tendencies – but not the finer details – seem to be robust to the choice of absolute vs. relative indicators of inequality (Smits and Monden 2009). Part of the reason is due to the strong negative association found between life expectancy at birth and length of life inequality, especially during the demographic transition.[4] It is widely known that the epidemiological transition goes hand in hand with a substantial mortality compression (Edwards and Tuljapurkar 2005). Although reducing mortality at any age will increase life expectancy, inequality reductions are only achieved if reductions in younger age mortality rates are greater than reductions in older age mortality rates. Figure 3 clearly shows that the reduction in younger age mortality rates compresses the age distribution of deaths while reducing older age mortality rates tends to expand the distribution. As a result, and given that modern mortality patterns are characterized by a high average age at death with much lower variability than in the past, there is evidence –especially in mature societies– that actual increases in life expectancy do not necessarily lead to lower inequality, which in addition may differ widely among societies at the same level of life expectancy.

The bottom line of this argument is that the choice between scale vs. translation invariant inequality indicators may be more important now than in the past, given that the relation between life expectancy at birth and length of life inequality is likely to be less strong in the future as mortality reductions shift to older ages.

[4] This is good news for demography because in economics there is no clear association, neither weak nor strong, between growth and inequality.

However, a more fundamental question about what type of inequality indicators, relative vs. absolute, are the most appropriate in this setting comes from the fact that the length of life distribution should be a bounded distribution, even if the actual bound is unclear. We know that for bounded distributions only absolute inequality indicators are consistent if we measure inequality in achievements, length of life in our case, and in shortfalls, life lost up to the bound in our case. This insight comes from the health economics literature (Lambert and Zheng 2011; Lasso de la Vega and Aristondo 2012), and also has implications for measuring the inequality of the poor (Aristondo, Goerlich and Lasso de la Vega 2015), which is also a bounded distribution.

Focusing only on length of life inequality may eventually lead us to unethical conclusions. Imagine that a policy objective is to reduce length of life inequality per se. In that case it could be argued that we should devote resources to prevent and treat diseases that can lead to premature deaths, while letting older people die, since that would reduce the inequality in the length of life. Equalizing the length of life distribution cannot be an objective on its own right. We should take into account that length of life inequality is just one aspect of health inequality, which in turn should not be confused with health inequity (Braveman and Gruskin 2003).

Instead of focusing on length of life inequality, a more productive line of work that to some extent escapes the discussion about relative vs. absolute inequality indexes would be to focus on developing life expectancy indicators that take into account the duration of lives at different ages so they are sensitive to distributional considerations. The idea parallels the measures of unemployment Sengupta (2009) and Shorrocks (2009a, 2009b) develop that take into account spell duration, with the difference that incidence is not an issue here given that everybody will eventually die. We develop this idea in the next section.

4. Life expectancy indexes in a life table population.

4.1. Life expectancy and inequality in length of life.

Consider the standard notation in a period life table in discrete time (Preston, Heuveline and Guillot 2001), where x denotes age, d_x the deaths occurred throughout the age interval $[x, x+1)$, and a_x the average number of years lived within the age interval $[x, x+1)$ for people dying at that age.[5] Hence the average length of life for those persons is $z_x = x + a_x$. As shown in the appendix, life expectancy at birth, e_0, can be written as

$$e_0 = \frac{\Sigma_{x\geq 0} d_x (x + a_x)}{\Sigma_{x\geq 0} d_x} = \Sigma_{x\geq 0} \omega_x z_x \tag{1}$$

where $\omega_x = \dfrac{d_x}{\Sigma_{x\geq 0} d_x}$, the proportion of deaths in the interval $[x, x+1)$, so $\Sigma_{x\geq 0}\omega_x = 1$. The same expression holds for life expectancy at age $x > 0$, simply by redefining the remaining length of life, conditional on having reached age x, and the corresponding weights, $e_x = \Sigma_{i\geq x}\omega_i z_i$; so without loss of generality we use (1) in the derivations that follow. The important point here is that our object of interest is a weighted distribution, $\{\omega_x; z_x\}_{x\geq 0}$, and the weights are more important than the survey sampling weights in the analysis of income inequality.

We follow an axiomatic approach similar to Shorrocks (2009b) in developing a life expectancy index adjusted for distributional considerations, taking into account that Shorrocks (2009b) deals with unemployment and its duration, which is a 'bad', whereas we deal with length of life, which is a 'good', and while not everybody is unemployed everybody will eventually die, so incidence is not an issue here but **intensity** and **inequality** are.

Let the vector $\mathbf{z} = (z_0, z_1, z_2, ..., z_x, ...)$ represent the length of life, which is naturally ordered, $z_x > z_{x+1}$ for $x \geq 0$, and collects the characteristic of interest: life length, $z_x = x + a_x$; and let $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2, ..., \omega_x, ...)$ represent the proportion of deaths in each interval $[x, x+1)$ for $x \geq 0$. As we have just observed, the mean of this distribution is just life expectancy at birth, $e_0 = \Sigma_{x\geq 0}\omega_x z_x$, which summarizes the life intensity of our distribution.

However, life expectancy is indifferent to how the total life time, $\Sigma_{x\geq 0} d_x z_x$, is distributed among the generation of interest. Let us consider that society has a

[5] See the appendix for the full notation and the derivations that follow.

preference, weak or strong, for equality in the distribution of length of life. In other words we favor rectangular survival functions, following Figure 2, or highly concentrated age distribution of deaths, following Figure 3.

A simple way, although not the only one, to incorporate distributional aspects into the measurement of life expectancy is to consider power means of order α,

$$U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) = \left[\Sigma_{x\geq 0}\omega_x z_x^\alpha\right]^{\frac{1}{\alpha}}; \qquad \alpha \leq 1 \tag{2}$$

For $\alpha = 1$ we are back to life expectancy at birth, $U_1(\boldsymbol{z};\boldsymbol{\omega}) = e_0$, but for $\alpha < 1$, $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ introduces a preference for equality as we shall see in the sequel.

Some properties of $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ are worth remembering (Steel 2004: chapter 8):

- For a given distribution $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is increasing in α, so $U_{\alpha^*}(\boldsymbol{z};\boldsymbol{\omega}) < U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ for $-\infty < \alpha^* < \alpha \leq 1$. Hence, $U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) < e_0$ for $\alpha < 1$.
- For $\alpha = 0$ we have the geometric mean (as a limit), $U_0(\boldsymbol{z};\boldsymbol{\omega}) = \Pi_{x\geq 0} z_x^{\omega_x}$.
- For $\alpha = -1$ we have the harmonic mean, $U_{-1}(\boldsymbol{z};\boldsymbol{\omega}) = \left[\Sigma_{x\geq 0}\dfrac{\omega_x}{z_x}\right]^{-1}$.
- As $\alpha \to -\infty$ we get the minimum value of $\boldsymbol{z}$, z_0, $U_{-\infty}(\boldsymbol{z};\boldsymbol{\omega}) = \min\{z_x\}_{x\geq 0} = z_0 = a_0$. Hence as α decreases, we increasingly focus on mortality of younger people.
- $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is homogeneous of degree one in $\boldsymbol{z}$, $U_\alpha(\lambda\boldsymbol{z};\boldsymbol{\omega}) = \lambda U_\alpha(\boldsymbol{z};\boldsymbol{\omega}); \quad \forall\lambda > 0$.
- $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is homogeneous of degree zero in $\boldsymbol{d} = (d_0, d_1, d_2, ..., d_x, ...)$, since this leaves unaltered the weights, $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2, ..., \omega_x, ...)$.
- $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is monotonically increasing in each of the elements of $\boldsymbol{z}$.
- Given the meaning we attach to $\boldsymbol{z}$, ages arranged in an increasing immutable order, $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is monotonically increasing in $\boldsymbol{\omega}$ in the following sense: $U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) < U_\alpha(\boldsymbol{z};\boldsymbol{\omega}')$ where $\boldsymbol{\omega}'$ is obtained from $\boldsymbol{\omega}$ in the following form: $\omega'_x > \omega_x$, $\omega'_{x-j} < \omega_{x-j}$ and $\omega'_{x-j} + \omega'_x = \omega_{x-j} + \omega_x$ for some $x > 0$ and some $0 < j \leq x$, while $\omega'_y = \omega_y, \quad \forall y \neq x$.

- $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is concave as a function of $\boldsymbol{z}$ for $\alpha \leq 1$, and strictly concave for $\alpha < 1$ (Magnus and Neudecker 1988, chapter 11, theorem 33).[6]

For $\alpha < 1$ the family measure $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ satisfies **six basic properties**:

(A1) Normalization: If there is no life $\omega_x = 0,\ \forall x$, so $\boldsymbol{\omega} = \boldsymbol{0}$, $U_\alpha(\boldsymbol{z};\boldsymbol{0}) = 0$.

(A2) Symmetry or anonymity: $U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) = U_\alpha(\boldsymbol{z}';\boldsymbol{\omega}')$ whenever $(\boldsymbol{z}';\boldsymbol{\omega}')$ is obtained from $(\boldsymbol{z};\boldsymbol{\omega})$ by a permutation, so $(\boldsymbol{z}';\boldsymbol{\omega}') = (\Pi\boldsymbol{z};\Pi\boldsymbol{\omega})$ for some permutation matrices, Π.

The symmetry or anonymity condition (A2) implies that what matters is the length of life vector and the associated deaths but not the characteristics of the individuals dying at a particular age, *x*. Personal features do not enter into the life expectancy index. This property is redundant if $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is calculated from a period life table, as in our case, since it is constructed from a fictitious generation, but we retain it just for the more general case in which $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is calculated from a real cohort or even a population, a possibility that it is not disregarded from our definition.[7]

(A3) Replication invariance: $U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) = U_\alpha(\boldsymbol{z}';\boldsymbol{\omega}')$ whenever $(\boldsymbol{z}';\boldsymbol{\omega}')$ is obtained from $(\boldsymbol{z};\boldsymbol{\omega})$ by a replication of the generation.

The replication invariance condition (A3) is a standard assumption when we have to compare societies of different sizes. As in the case of (A2) this property is redundant if $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is calculated from a period life table since in that case it is constructed from a fictitious generation with a given number of newborns, $l_0 = 100{,}000$, that will eventually die, $\Sigma_{x\geq 0} d_x = l_0$. Duplicating l_0 makes no difference to life expectancy by construction. As before, we retain (A3) to cover more general cases that are not excluded on a priori grounds.

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[6] This is in fact the reason for restricting the parameter space in the definition of $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ since power means are defined for any real value of α.

[7] Even if life expectancy at birth is constructed to allow comparison between societies with very different population structures, obtaining the average life expectancy of a population, the so called life potential index, is not meaningless from an economic point of view (Goerlich and Soler 2013). In addition, weighting life tables by actual population shares is common in global analysis of life span inequality (Edwards 2011; Jordá and Niño-Zarazúa 2017).

(A4) Monotonicity: $U_\alpha(\boldsymbol{z};\boldsymbol{\omega}) < U_\alpha(\boldsymbol{z}';\boldsymbol{\omega}')$ if some people live longer, but nobody lives less, so the total amount of life time increases, $\Sigma_{x\geq 0} d_x z_x < \Sigma_{x\geq 0} d'_x z'_x$. Hence, $\boldsymbol{\omega}'$ is obtained from $\boldsymbol{\omega}$ in the following form: $\omega'_x > \omega_x$, $\omega'_{x-j} < \omega_{x-j}$ and $\omega'_{x-j} + \omega'_x = \omega_{x-j} + \omega_x$ for some $x > 0$ and some $0 < j \leq x$, while $\omega'_y = \omega_y$, $\forall y \neq x$. $\boldsymbol{z}'$ can be equal to $\boldsymbol{z}$, or may have additional ages if extreme longevity is increased.

The monotonicity condition is more intuitively enunciated in terms of d_x. So given $\boldsymbol{d} = (d_0, d_1, d_2, ..., d_x, ...)$, we have a distribution $\boldsymbol{d}' = (d'_0, d'_1, d'_2, ..., d'_x, ...)$ in which some people live longer but nobody lives less. That is, $\boldsymbol{d}'$ is obtained from $\boldsymbol{d}$ in the following form: $d'_x > d_x$, $d'_{x-j} < d_{x-j}$ and $d'_{x-j} + d'_x = d_{x-j} + d_x$ for some $x > 0$ and some $0 < j \leq x$, while $d'_y = d_y$, $\forall y \neq x$.

This form of stating the monotonicity condition assumes $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is calculated from a life table, so if we have more deaths at age *x*, we should necessarily have fewer deaths at age $x - j$ since the size of the generation is fixed in advance, $\Sigma_{x\geq 0} d_x = l_0$. This restriction is not necessary if $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$ is calculated from a real population.

The monotonicity condition implies that the life expectancy index should increase as life is extended for at least one individual. Note that this will affect inequality, although we cannot anticipate in which direction. If the increment in the length of life takes place for an age below e_0 then inequality is reduced; if the increment in the length of life takes place for an age above e_0 then inequality is increased. If *x* is to one side of e_0 and $x - j$ to the other then we cannot say in advance. Contrary to what happens in the analysis of the income distribution, the level and the spread of the length of life distribution cannot be fully separated in a life table.

These four properties –normalization, symmetry, replication invariance, and monotonicity– are all satisfied by life expectancy, $U_1(\boldsymbol{z};\boldsymbol{\omega})$. The next one is not.

To motivate the next property let us consider a situation in which we are concerned with the distribution of some amount of total life time, $\Sigma_{x\geq 0} d_x z_x$, so we are back to Figures 1 and 2, and the question of interest is which survival curve do we prefer given that all of them have the same life expectancy at birth. If the proposed life expectancy index (2) makes sense, it should be the case that a rectangular survival function should be preferred over the rest. More generally, when comparing two survival functions with the same life expectancy, society should have a preference for the one closest to rectangularity. In other words society should prefer two individuals to each live for 50 years over one individual living 100 years and the other dying at the moment of birth; or society should prefer two individuals to live for 40 and 60 years over 20 and 80 years each. From the point of view of society as a whole, if a hypothetical social welfare function defined over our distribution of interest, $\{\omega_x; z_x\}_{x\geq 0}$, is increasing in life length and concave, this assumption can be justified (Sen 1973).[8]

(A5) Preference for Equality in the length of life: $U_\alpha(\mathbf{z};\omega) < U_\alpha(\mathbf{z}';\omega')$ if both distributions, $\{\omega_x; z_x\}_{x\geq 0}$ and $\{\omega'_x; z'_x\}_{x\geq 0}$, have the same total life time, $\Sigma_{x\geq 0} d_x z_x = \Sigma_{x\geq 0} d'_x z'_x$, but $\{\omega'_x; z'_x\}_{x\geq 0}$ is obtained from $\{\omega_x; z_x\}_{x\geq 0}$ by a life time equalization transformation.

In our case $\{\omega'_x; z'_x\}_{x\geq 0}$ is obtained from $\{\omega_x; z_x\}_{x\geq 0}$ by means of a life time equalization transformation if, given two ages *x* and *y*, the former is obtained from the latter in the following form: $\omega_x z_x > \omega'_x z'_x > \omega_y z_y$, where $\omega_x z_x + \omega_y z_y = \omega'_x z'_x + \omega'_y z'_y$ and $\omega_k z_k = \omega'_k z'_k, \ \forall k \neq x, y$.

The preference for equality in length of life is more intuitively enunciated in terms of d_x, given that what matters is the life time at different ages, $d_x z_x$. A life time equalization transformation implies $d_x z_x > d'_x z'_x > d_y z_y$, where $d_x z_x + d_y z_y = d'_x z'_x + d'_y z'_y$ and $d_k z_k = d'_k z'_k, \ \forall k \neq x, y$.

[8] The above argument is the essential justification for (2). It makes sense in income distribution since this is essentially Pigou–Dalton, and also in the case of unemployment duration (Sengupta 2009; Shorrocks 2009a, 2009b) where we can justify a preference for short spells or a preference for spell splits, given that it makes sense to assume a welfare loss from unemployment being an increasingly and strictly convex function of unemployment duration. But although this assumption is intuitive it is less appealing in terms of the length of life distribution, either from a demographic or an epidemiological perspective.

Property (A5) is the analogue to the Pigou–Dalton principle of transfers in the analysis of income inequality, but as mentioned above it has no clear justification as such in this context because life years cannot be transferred. One way to justify its inclusion here is to have a preference for younger lives, so from society's point of view one additional year of life of a young person is valued more than the same additional year for an older person. This property is not satisfied for life expectancy but it is for $U_\alpha(\mathbf{z};\boldsymbol{\omega})$ when $\alpha<1$. The parameter α in $U_\alpha(\mathbf{z};\boldsymbol{\omega})$ governs the degree of preference for equality since the lower the value, the higher the preference for equality, but also the higher the preference for younger lives since as $\alpha\to-\infty$ we increasingly pay more attention to the life of the youngest people as we will see later on.

(A6) Homogeneity in the length of life. $U_\alpha(\lambda\mathbf{z};\boldsymbol{\omega})=\lambda U_\alpha(\mathbf{z};\boldsymbol{\omega});\ \forall\lambda>0$.

So if everybody's length of life doubles, the life expectancy index doubles.

Two characteristics of the family of life expectancy index $U_\alpha(\mathbf{z};\boldsymbol{\omega})$ make it attractive from a practical point of view. First, it can be shown to be a multiple of life expectancy at birth. Second, it can be multiplicative decomposed into two terms taking into account the two key aspects of the length of life distribution: average life and inequality. Hence we can measure the contribution of any of these aspects to the evolution of the life expectancy index.

Lemma 1: $U_\alpha(\mathbf{z};\boldsymbol{\omega})$ can by written as

$$U_\alpha(\mathbf{z};\boldsymbol{\omega})=e_0(1-A_\alpha) \qquad (3)$$

where $A_\alpha=1-\left[\sum_{x\geq 0}\omega_x\left(\frac{z_x}{e_0}\right)^\alpha\right]^{\frac{1}{\alpha}}$.

Proof:

$$A_\alpha=1-\left[\sum_{x\geq 0}\omega_x\left(\frac{z_x}{e_0}\right)^\alpha\right]^{\frac{1}{\alpha}} \quad\Rightarrow\quad \left[\sum_{x\geq 0}\omega_x\left(\frac{z_x}{e_0}\right)^\alpha\right]^{\frac{1}{\alpha}}=1-A_\alpha$$

Hence, multiplying both sides by e_0,

$$e_0(1-A_\alpha)=\left[\Sigma_{x\geq 0}\omega_x z_x^\alpha\right]^{\frac{1}{\alpha}}=U_\alpha(\mathbf{z};\boldsymbol{\omega})$$

■

$A_\alpha = 1 - \left[\sum_{x \geq 0} \omega_x \left(\frac{z_x}{e_0} \right)^\alpha \right]^{\frac{1}{\alpha}}$ is the relative Atkinson (1970) inequality index for $\alpha < 1$, so this parameter has a clear interpretation as the aversion to inequality. As $\alpha \rightarrow -\infty$, $A_\alpha \rightarrow 1 - \frac{z_0}{e_0}$, and A_α takes its maximum value.

This formulation makes clear that $U_\alpha(\mathbf{z};\boldsymbol{\omega})$ belongs to the class of 'equal equivalent length of life' indexes that could be derived from a social welfare function defined over our distribution of interest, $\{\omega_x ; z_x\}_{x \geq 0}$, which is symmetric, increasing in life length, and concave. So $U_\alpha(\mathbf{z};\boldsymbol{\omega}) = e_0(1 - A_\alpha)$ is well founded in welfare economics.

For a particular application it remains to choose the value of α. This relates to the question of our aversion to inequality in the length of life; a value not much lower than 1 will probably suffice. In economic applications a common value is $\alpha = 0$, so $A_0 = 1 - \frac{\Pi_{x \geq 0} z_x^{\omega_x}}{e_0}$, and $U_0(\mathbf{z};\boldsymbol{\omega}) = e_0(1 - A_0) = \Pi_{x \geq 0} z_x^{\omega_x}$, and the life index is the geometric mean of the distribution of the length of life distribution. Hence, using the geometric mean instead of the arithmetic mean as a life index incorporates a preference for equality in length of life.

4.2. Life length profiles and life length dominance.

In inequality analysis a common graphical device is the Lorenz (1905) curve, which plots cumulative income shares against population shares after income has been ordered in a non-decreasing fashion. Ordering distributions by non-intersecting Lorenz curves corresponds to unanimous inequality orderings according to a wide class of relative inequality measures that satisfy certain properties –essentially the Pigou–Dalton principle of transfers–.[9] This result was extended by Shorrocks (1983) when comparing income distributions with different means so levels matter, using the generalized Lorenz curve, which plots cumulative income means against population

[9] The Lorenz curve also appears in some studies of length of life inequality (Anand and Nanthikesan 2000; Shkolnikov, Andreev, and Begun 2003).

shares after income has been ordered in a non-decreasing fashion.[10] Ordering distributions by non-intersecting generalized Lorenz curves corresponds to unanimous welfare orderings according to a wide class of social welfare functions that satisfy certain properties –essentially symmetry, monotonicity, and concavity–.

Jenkins and Lambert (1997) establish similar results for poverty orderings. They name the corresponding curve the 'TIP' curve: the "Three I´s of Poverty", namely the incidence, intensity, and inequality dimensions of poverty, whereas for unemployment duration sensitive indexes Shorrocks (2009b) named it the duration profile. Since incidence is not an issue here we are back to a generalized Lorenz curve analogy.

For any $\{\omega_x; z_x\}_{x\geq 0}$, the length of life profile, $D(\boldsymbol{z},\boldsymbol{\omega}; p)$, is computed in a simple way: just plot cumulative means, $\Sigma_{x=0}^{p}\omega_x z_x$,[11] against cumulative death shares, $\Sigma_{x=0}^{p}\omega_x$, from the life table. As observed above, $\boldsymbol{z}$ is naturally ordered since it represents ages at death. Figure 4 depicts some typical length of life profiles.

The graph of the life of length profile provides a highly convenient way of summarizing information on life length in our distribution, $\{\omega_x; z_x\}_{x\geq 0}$. As Figure 4 shows, the curve starts at the origin and is continuous, non-decreasing, and convex. Its maximum corresponds to life expectancy at birth, as life expectancy increases the life length profiles shift upwards, and the curvature represents the inequality: the more convex the function the higher the inequality.

A distribution $\{\omega'_x; z'_x\}_{x\geq 0}$ life length dominates $\{\omega_x; z_x\}_{x\geq 0}$, in a weak form, whenever the curve of the former lies on or above that of the latter,

$$D(\boldsymbol{z}',\boldsymbol{\omega}'; p) \geq D(\boldsymbol{z},\boldsymbol{\omega}; p) \quad \forall p \in [0,1] \tag{4}$$

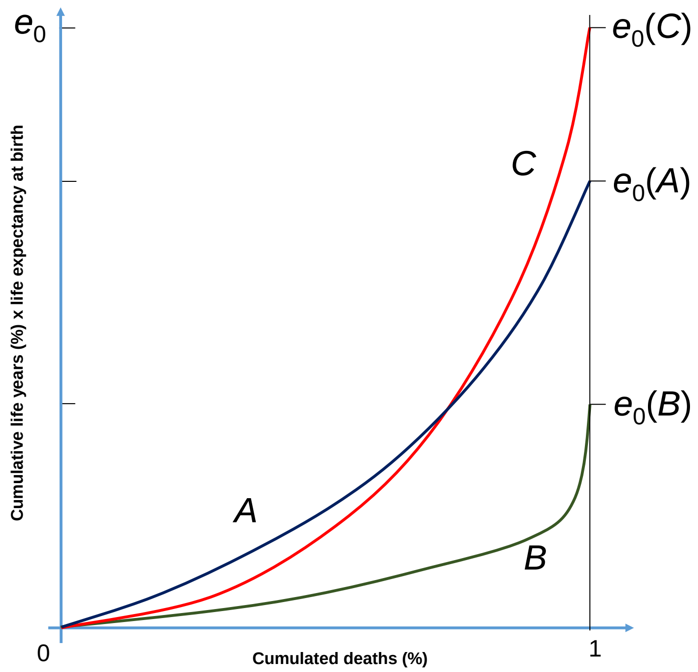
In terms of Figure 4 distributions *A* and *C* life length dominate distribution *B*, although neither *A* or *C* dominates over the other. Shorrocks (1983) establishes a

[10] The generalized Lorenz curve seems to be absent in length of life inequality studies.

[11] A Lorenz curve will just plot cumulative life years, $\Sigma_{x=0}^{p} d_x z_x / \Sigma_{x\geq 0}\, d_x z_x$, against cumulative death shares, $\Sigma_{x=0}^{p}\omega_x$, in the life table, but since we are not interested in inequality per se we focus on the generalized Lorenz curve.
Note in passing that the ordinates of the generalized Lorenz curve are just the ordinates of the Lorenz curve multiplied by the mean of the distribution, e_0.

fundamental result in ordering distributions by means of life expectancy indexes of the type (2), $U_{\alpha}(\mathbf{z};\boldsymbol{\omega})$. If $\{\omega'_x; z'_x\}_{x\geq 0}$ life length dominates $\{\omega_x; z_x\}_{x\geq 0}$, then $U_{\alpha}(\mathbf{z};\boldsymbol{\omega}) < U_{\alpha}(\mathbf{z}';\boldsymbol{\omega}')$ for $\alpha < 1$. Hence the family of life expectancy indexes $U_{\alpha}(\mathbf{z};\boldsymbol{\omega})$ for $\alpha < 1$ is fully consistent with partial orderings that come from length profiles, but the direct examination of life length profiles gives us a full picture of the evolution of life in its two dimensions: mean level – intensity – and inequality. So in a practical application the best course of action is to examine the life length profiles directly.

Figure 4: Life length profiles



This result, the relation between life length dominance and partial orderings employing $U_{\alpha}(\mathbf{z};\boldsymbol{\omega})$ indexes, holds true for a wider class of life expectancy indexes. In fact it is true for all life expectancy indexes satisfying properties (A2)–(A5) above.

Two additional alternatives to the $U_{\alpha}(\mathbf{z};\boldsymbol{\omega})$ family, (2), are worth mentioning.

1. Consider the **alternative family of life expectancy indexes**

$$U_{\beta}(\mathbf{z};\boldsymbol{\omega}) = \Sigma_{x\geq 0}\omega_x z_x^{\beta}; \qquad \beta \leq 1 \tag{5}$$

For $\beta = 1$ we are back to life expectancy at birth, $U_1(\boldsymbol{z};\boldsymbol{\omega}) = e_0$, but for $\beta < 1$, $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ introduces a preference for equality. $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ satisfies properties (A1)–(A5), but not (A6), homogeneity in length of life, since $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ is homogeneous of degree β, $U_\beta(\lambda\boldsymbol{z};\boldsymbol{\omega}) = \lambda^\beta U_\beta(\boldsymbol{z};\boldsymbol{\omega});\ \forall\lambda > 0$. As a consequence $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ is not proportional to life expectancy at birth. However, since (5) is just a mean this measure is decomposable by population sub-groups.[12]

Lemma 2: $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ can by written as

$$U_\beta(\boldsymbol{z};\boldsymbol{\omega}) = e_0^\beta(1 - E_\beta) \tag{6}$$

where $E_\beta = 1 - \sum_{x\geq 0} \omega_x \left(\frac{z_x}{e_0}\right)^\beta$.

Proof:

$$E_\beta = 1 - \sum_{x\geq 0} \omega_x \left(\frac{z_x}{e_0}\right)^\beta \quad \Rightarrow \quad \sum_{x\geq 0} \omega_x \left(\frac{z_x}{e_0}\right)^\beta = 1 - E_\beta$$

Hence, multiplying both sides by e_0^β,

$$e_0^\beta(1 - E_\beta) = \Sigma_{x\geq 0}\omega_x z_x^\beta = U_\beta(\boldsymbol{z};\boldsymbol{\omega})$$

■

$E_\beta = 1 - \sum_{x\geq 0} \omega_x \left(\frac{z_x}{e_0}\right)^\beta$ is the counterpart of a generalized entropy index with coefficient β (Shorrocks 1980, 1984). As $\beta \to -\infty$, $E_\beta \to 1$, and E_β takes its maximum value. Because $U_\beta(\boldsymbol{z};\boldsymbol{\omega})$ cannot be written as proportional to e_0 this family has less appeal than $U_\alpha(\boldsymbol{z};\boldsymbol{\omega})$.

[12] A property not especially attractive in the context of the standard life table model since for example the life expectancy at birth of the total population cannot be written in a simple form as a weighted average of the life expectancy at birth of males and females because they are computed independently from artificial generations.

2. **Generalized Lorenz evaluation**.

Another way of obtaining an expression similar to (3) is to take twice the area under the life length profile, which yields

$$U_G(\mathbf{z};\boldsymbol{\omega}) = e_0(1-G) \tag{7}$$

where G is the Gini index for the distribution $\{\omega_x; z_x\}_{x\geq 0}$

$$G(\mathbf{z};\boldsymbol{\omega}) = \frac{1}{2e_0}\Sigma_{x\geq 0}\Sigma_{y\geq 0}\omega_x\omega_y\left|z_x - z_y\right| \tag{8}$$

which can also be obtained by any other formula proposed in the literature (Hanada 1983, Shkolnikov, Andreev, and Begun 2003).

$U_G(\mathbf{z};\boldsymbol{\omega})$ satisfies properties (A1)–(A6) since G is a relative inequality measure and hence homogeneous of degree zero, so $U_G(\lambda\mathbf{z};\boldsymbol{\omega}) = \lambda U_G(\mathbf{z};\boldsymbol{\omega});\ \forall\lambda > 0$. However, $U_G(\mathbf{z};\boldsymbol{\omega})$ cannot be written as some form of a generalized mean, such as (2) or (5), nor does it depend on a coefficient picking up the aversion to inequality. Both of these characteristics make it less attractive than $U_\alpha(\mathbf{z};\boldsymbol{\omega})$.

5. An illustrative example.

We now illustrate the historical evolution of our generalized life expectancy index, $U_\alpha(\mathbf{z};\boldsymbol{\omega})$, for a sample of countries using data from the Human Mortality Database (Shkolnikov, Barbieri, and Wilmoth 2019). The full set of complete, single year, period life tables were used. These cover 40 countries and 49 populations because for some countries we have more than one reference population; for example for Germany we have three sets of life tables: East Germany, West Germany and the aggregate. Temporal span varies according to the country and runs from 1751 for Sweden – the only country with data from the 18th century – to 2017 for most countries. Of course most of the data are available from the mid-20th century onwards. In total we have 13,920 complete life tables, covering the total population and both sexes, so we have 4,640 life tables for each sex.

For each complete period life table in the Human Mortality Database two additional columns were added for $\alpha = 0$[13], a moderate degree of preference for equality in the length of life: (*i*) the generalized life expectancy index and (*ii*) and the corresponding inequality index. These are computed for any age, x, so the two additional columns are complete from birth to 110 years old.[14]

Table 1 shows life expectancy, distributionally adjusted life expectancy, and the corresponding inequality index for two ages –at birth and at 65 years old– for selected years and countries as an illustration. By definition, distributionally adjusted life expectancy is lower than life expectancy. At birth Table 1 shows that the difference was important at the beginning of the 20th century, between 20 and 30 years, given that inequality in the length of life was substantial at that time. Early mortality was high, which tends to increase inequality. Over time life expectancy increases, but simultaneously inequality falls. The natural consequence is that distributionally adjusted life expectancy has increased historically much more than life expectancy at birth. In fact, at the beginning of the 21st century the difference is just between 2 and 4 years. We can see that inequality has decreased by a factor of ten or even more. The Spanish case is remarkable. While life expectancy at birth increased by more than 30 years between 1930 and 2016, distributionally adjusted life expectancy at birth increased almost twice as much to 60 years. The reason is the huge reduction in inequality in length of life in those years for Spain as Shkolnikov, Andreev, and Begun (2003) already noted.

Because by lemma 1 the generalized life expectancy index, $U_\alpha(z;\omega)$, can be written as the product of life expectancy, e_0, and the corresponding equality index, $(1-A_\alpha)$, relative changes in the distributionally adjusted life expectancy can be broken down using logarithmic approximation to growth rates into the change due to life expectancy and the change due to the reduction in inequality. Using this decomposition and focusing on the extreme years, 1930 and 2016, Table 1 shows that the contribution of the reduction in inequality in length of life to the increase in the

[13] A value $\alpha = 0.5$ produced similar qualitative results.

[14] An *R* script, available from the author upon request, was developed to perform this calculation. These extended, distributionally adjusted, period life tables are available from the author upon request.

distributionally adjusted life expectancy at birth exceeds 50% in all cases, reaching around 60% in Canada, Denmark and Spain. Historically, then, not only have we observed an important increment in life expectancy at birth in all countries, but also a huge reduction in inequality in the length of life.

This is good news for the length of life distribution because increments in the mean are associated overall with a reduction in dispersion, that is, with a fall in inequality. This is clearly a general pattern, so increments in life expectancy seem to have widespread benefits. However, in the income distribution literature the growth in per capita income is not always associated with a reduction in inequality, so growth does not benefit everybody in all cases.

At age 65 we have a similar but less marked pattern. Life expectancy increases and inequality falls, but the reduction in inequality is much less pronounced: on average it is reduced by a factor of two. It seems interesting to note that while inequality in the length of life was lower at the age of 65 than at birth for all countries in 1930, just the opposite is true in 2016. Today inequality at older ages is much higher than at birth. This is because the reduction in infant mortality is the main reason for the decrease in inequality in the length of life, and at the same time for the increase in life expectancy at birth.

Table 1: Life expectancy, distributionally adjusted life expectancy, and inequality (α = 0) for selected years and countries. Both sexes combined.

	At birth (x = 0)	Australia	Canada	Switzerland	Denmark	Spain	France	United Kingdom	Iceland	Netherlands	Sweden
	Life expectancy	64.90	58.96	61.40	62.22	49.25	56.80	60.77	60.18	64.66	63.17
1930	Dist. adjusted life expectancy	47.29	34.08	43.99	40.31	21.41	34.88	40.41	42.50	46.20	44.51
	Inequality (A_0)	0.2714	0.4219	0.2836	0.3521	0.5653	0.3860	0.3350	0.2939	0.2855	0.2954
	Life expectancy	70.87	70.98	71.41	72.20	69.23	70.37	71.02	74.06	73.35	73.04
1960	Dist. adjusted life expectancy	60.96	58.89	61.13	62.01	52.85	58.51	60.62	66.20	64.42	64.59
	Inequality (A_0)	0.1398	0.1703	0.1440	0.1411	0.2366	0.1685	0.1465	0.1061	0.1218	0.1157
	Life expectancy	77.04	77.43	77.38	74.87	77.00	76.84	75.74	77.91	77.01	77.59
1990	Distributionally life expectancy	71.84	72.63	72.50	69.92	71.72	71.62	70.71	73.55	72.28	73.40
	Inequality (A_0)	0.0676	0.0620	0.0631	0.0660	0.0685	0.0679	0.0663	0.0560	0.0614	0.0541
	Life expectancy	82.89	82.16	83.46	80.88	83.07	82.39	81.04	82.07	81.55	82.34
2016	Dist. adjusted life expectancy	80.05	78.52	80.67	78.12	80.57	79.21	77.85	80.30	78.74	79.92
	Inequality (A_0)	0.0342	0.0443	0.0335	0.0342	0.0300	0.0386	0.0393	0.0216	0.0344	0.0295
	At 65 years old (x = 65)	**Australia**	**Canada**	**Switzerland**	**Denmark**	**Spain**	**France**	**United Kingdom**	**Iceland**	**Netherlands**	**Sweden**
	Life expectancy	13.76	13.37	11.90	13.08	11.71	12.28	12.57	14.48	13.32	13.70
1930	Dist. adjusted life expectancy	10.63	10.26	8.99	10.18	8.67	9.33	9.59	11.28	10.37	10.75
	Inequality (A_0)	0.2280	0.2325	0.2439	0.2217	0.2594	0.2399	0.2371	0.2215	0.2217	0.2155
	Life expectancy	14.19	14.82	14.11	14.39	14.20	14.36	13.91	16.09	14.97	14.54
1960	Dist. adjusted life expectancy	11.02	11.65	11.26	11.58	11.25	11.43	10.89	13.33	12.13	11.75
	Inequality (A_0)	0.2230	0.2136	0.2019	0.1951	0.2072	0.2040	0.2166	0.1713	0.1893	0.1919
	Life expectancy	17.28	17.75	17.54	16.03	17.52	17.97	16.12	17.93	16.86	17.27
1990	Dist. adjusted life expectancy	14.10	14.49	14.60	12.76	14.57	14.93	12.84	14.60	13.78	14.33
	Inequality (A_0)	0.1840	0.1837	0.1675	0.2040	0.1683	0.1693	0.2036	0.1855	0.1830	0.1701
	Life expectancy	21.23	21.05	21.26	19.48	21.16	21.41	19.85	19.83	19.76	20.30
2016	Dist. adjusted life expectancy	18.54	17.99	18.56	16.48	18.36	18.42	16.94	17.17	16.96	17.57
	Inequality (A_0)	0.1268	0.1456	0.1270	0.1542	0.1324	0.1395	0.1466	0.1342	0.1420	0.1345

Source: Own calculations from the Human Mortality Database.

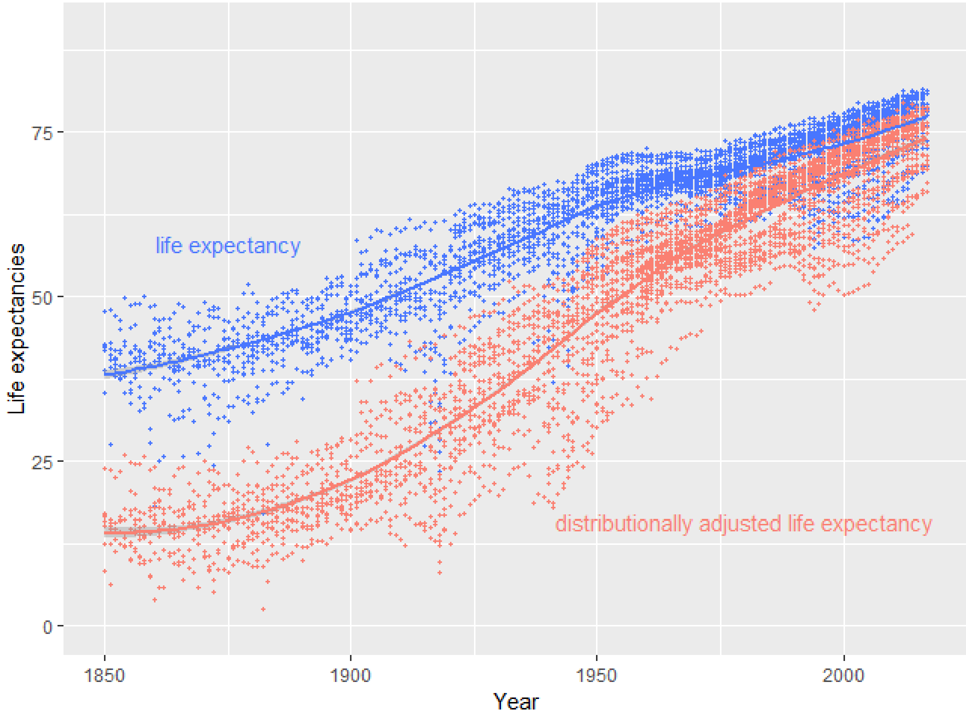
By looking at the relative contribution of both life expectancy and inequality to the relative changes in the distributionally adjusted life expectancy at age 65 for the period 1930–2016, we see now that the relative contribution of the inequality component is much lower at around 20%, with a maximum of 25% for Iceland. Hence at older ages the increment in the distributionally adjusted life expectancy comes mainly from increments in life expectancy, but the contribution of the reduction in inequality is not negligible. At age 65 we still find a strong negative relation between life expectancy and inequality.

At age 85 (results not shown) the pattern is still less marked but follows the same tendency as at age 65. Even now we still find a positive contribution of the inequality component to the increment in the distributionally adjusted life expectancy: around 15% for the period 1930–2016. Inequality still shows a decreasing temporal tendency at this age although its level is much higher than at younger ages.

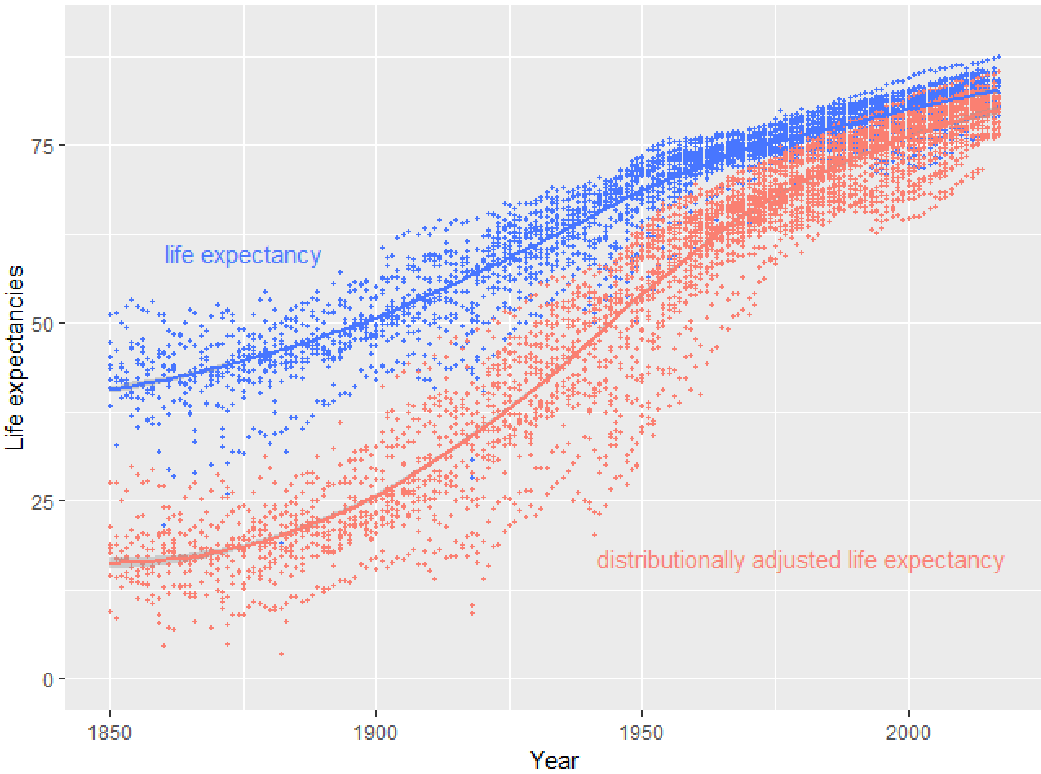
The details illustrated in Table 1 for selected years and countries are in fact quite general. Figure 5 shows this for all life tables from 1850 onwards and both sexes. Here we represent life expectancy and distributionally adjusted life expectancy at birth for men and women separately. In both cases we find the same convergence trend of the distributionally life expectancy approaching life expectancy over time. The reason behind this convergence is the strong and well known negative relation between the increases in life expectancy and the reduction in life length inequality (Wilmoth and Horiuchi 1999; Shkolnikov, Andreev, and Begun 2003; Smits and Monden 2009; Edwards 2011). This reduction in inequality was particularly strong during the mid-half of the 20th century and has slowed down considerably in recent years, essentially because infant mortality is so low in most developed countries that it is now difficult to reduce inequality in length of life any further.

Figure 5: Life expectancy and distributionally adjusted life expectancy at birth by sex

a) Men



b) Women

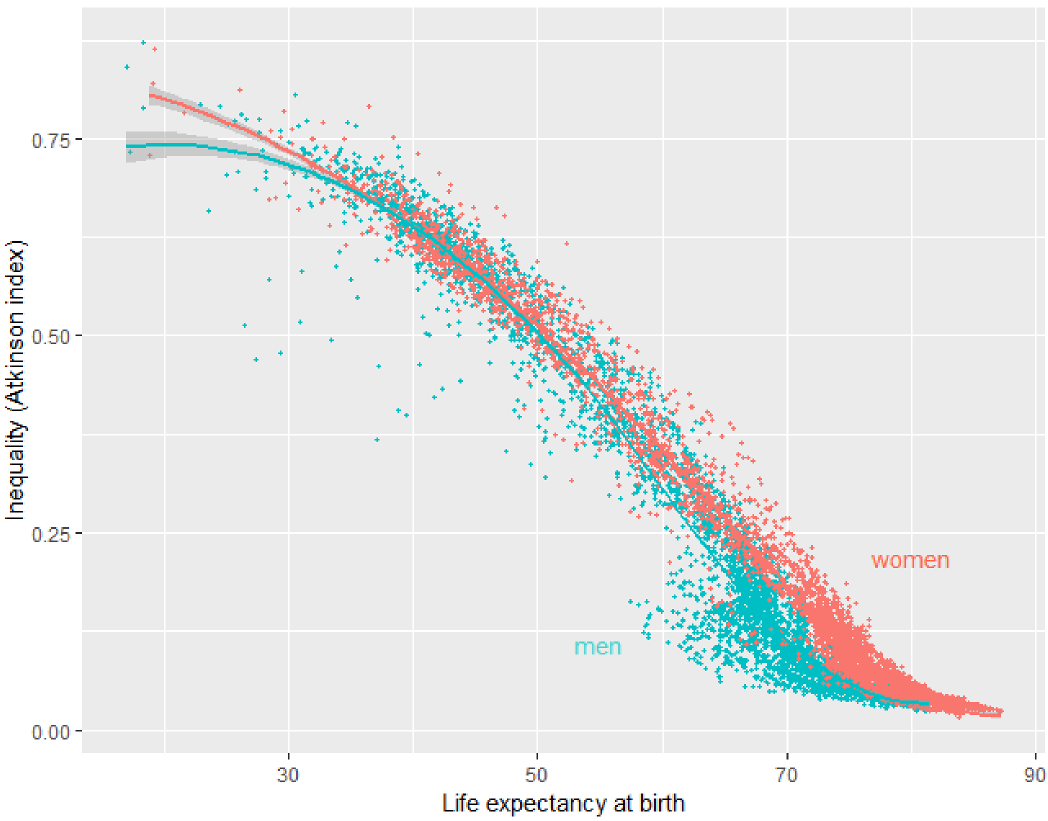


Source: Own calculations from the Human Mortality Database.

This negative correlation between life expectancy and inequality in length of life can alternatively be seen by plotting the inequality measure against life expectancy at birth as in Figure 6. This is the representation preferred in most of the literature (Wilmoth and Horiuchi 1999; Shkolnikov, Andreev, and Begun 2003; Smits and Monden 2009; Edwards 2011), essentially by using different inequality measures. As Wilmoth and Horiuchi (1999) show that the choice of the inequality measure for computing length of life inequality at country level is not a critical one, we can just focus on the Atkinson index that forms part of our distributionally adjusted life expectancy index.

As is apparent from Figure 6, the negative relationship between inequality in the length of life distribution and life expectancy is not a linear one since it clearly slows down for values of life expectancy at birth above 60. Note that for values of life expectancy at birth above 70 inequality is so low that it is extremely difficult to reduce it even more since additional increases in life expectancy come from reductions in mortality at older ages, which in addition tend to push inequality upwards.

Figure 6: Life expectancy at birth versus inequality in length of life



Source: Own calculations from the Human Mortality Database.

6. Final comments.

This paper has theoretically derived a distributionally adjusted life expectancy index that summarizes in a single number the life expectancy and the inequality in length of life distribution at any age. This distributionally adjusted life expectancy index can be added as an additional column in a period life table to inform about inequality in the length of life distribution. The essential axiom that our new index should verify to make sense is a preference for equality in length of life; if this preference cannot be socially justified our index serves no purpose and life expectancy is a sufficient statistic.

On the other hand if there is a preference for equality in the length of life distribution, then our set of axioms delivers a particular form of a distributionally adjusted life expectancy that is proportional to life expectancy, where the proportionality constant is one minus the Atkinson (1970) inequality family, a well-known family of inequality indexes in the income distribution literature. Other alternatives are available and are reviewed in the paper but the proposed measure is axiomatically justified and is the most appealing one.

Because the proposed index can be written as a product of life expectancy and an equality index, changes in distributionally adjusted life expectancy can be broken down into the relative contributions of changes in life expectancy and changes in equality in the distribution of length of life.

An illustration of our measure using data from the Human Mortality Database shows an alternative way of looking at the strong and well known negative relation between the increases in life expectancy and the reduction in life length inequality. Both the distributionally adjusted life expectancy index and the corresponding inequality index can be routinely added to period life tables as additional biometric functions for any age, thereby enriching the information on the length of life distribution.

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Appendix

Notation

We use the standard notation for period life tables in discrete time (Preston, Heuveline and Guillot 2001) as implemented for example in the Human Mortality Database methods protocol (Wilmoth et al. 2017).

Let x denote ages, q_x the probability of dying throughout the age interval $[x, x+1)$, its complementary, $p_x = 1 - q_x$, the probability of surviving, and $l_0 = 100{,}000$ the number of the newborns in the life table population.[15] Then, the number of survivors, l_x, (of the initial $100{,}000$) at age x is

$$l_x = l_{x-1}.p_{x-1} = l_0.\prod_{i=0}^{x-1} p_i \tag{A.1}$$

The distribution of deaths by age in the life table population is

$$d_x = l_x.q_x = l_x.(1-p_x) = l_x - l_x p_x = l_x - l_{x+1} \tag{A.2}$$

until its extinction.

The person-years, L_x, lived by the life table population in the age interval $[x, x+1)$ are

$$L_x = l_{x+1} + a_x.d_x = l_x - (1-a_x).d_x \tag{A.3}$$

where a_x represents the average number of years lived within the age interval $[x, x+1)$ for people dying at that age.

The person-years remaining for individuals of age x equal

$$T_x = \sum_{i \geq x} L_i \tag{A.4}$$

Remaining life expectancy at age x is

[15] We consider standard period life tables so the life table population corresponds to a fictitious generation and the population is closed.

$$e_x = \frac{T_x}{l_x} \tag{A.5}$$

For $x = 0$ we obtain life expectancy at birth, $e_0 = \frac{T_0}{l_0}$.

Life expectancy: Mean of the distribution in the length of life

Lemma A1: Life expectancy at birth, $x = 0$, viewed as the distribution of all life years of the life table population among the new born, $e_0 = \frac{T_0}{l_0}$, is identical to the mean age at death; that is, the mean of the length of life distribution of the life table population, $e_0 = \frac{\sum_{x \geq 0} d_x (x + a_x)}{\sum_{x \geq 0} d_x}$, since the length of life for an individual dying in the interval $[x, x+1)$ is $x + a_x$ and we have d_x people in the life table population dying in that interval.

Proof: First, because every newborn eventually dies, $\Sigma_{x \geq 0} d_x = l_0$.

Second, from the definition of T_x at $x = 0$

$$T_0 = \sum_{x \geq 0} L_x = \sum_{x \geq 0} (l_{x+1} + a_x . d_x)$$

So it remains to show that $\sum_{x \geq 0} l_{x+1} = \sum_{x \geq 0} x.d_x$. The RHS of this expression is

$$\begin{aligned}
\sum_{x \geq 0} x.d_x &= d_1 + 2.d_2 + 3.d_3 + 4.d_x + \ldots \\
&= \sum_{x \geq 1} d_x + \sum_{x \geq 2} d_x + \sum_{x \geq 3} d_x + \sum_{x \geq 4} d_x + \ldots \\
&= l_1 + l_2 + l_3 + l_4 + \ldots \\
&= \sum_{x \geq 0} l_{x+1}
\end{aligned}$$

since the life table population extinguishes, so $\sum_{i \geq x} d_i = l_x, \quad \forall x$.

■

Thus the distribution of interest is the distribution of the length of life, $z_x = x + a_x$, weighted by the number of people dying at that age, d_x.

To generalize the above lemma to any age, *x*, note that conditional on having reached age *x* the origin to measure the remaining length of life is not 0 but *x*, so the remaining length of life from that point onwards is $(i-x)+a_x$ for $i \geq x > 0$.

Lemma A2: Life expectancy at any age, *x*, viewed as the distribution of all the future remaining life years of the life table population among the current survivors, $e_x = \frac{T_x}{l_x}$, is identical to the mean age at death; that is, the mean of the length of life distribution for these survivors, $e_x = \frac{\sum_{i \geq x} d_i \left((i-x)+a_i\right)}{\sum_{i \geq x} d_i}$.

Proof: First, because survivors will eventually die, $\Sigma_{i \geq x} d_i = l_x$.

Second, from the definition of T_x

$$T_x = \sum_{i \geq x} L_i = \sum_{i \geq x} (l_{i+1} + a_i.d_i)$$

So it remains to show that $\sum_{i \geq x} l_{i+1} = \sum_{i \geq x} (i-x).d_i$. The RHS of this expression is

$$\sum_{i \geq x} (i-x).d_i = \sum_{i \geq x} i.d_i - x.\sum_{i \geq x} d_i = \sum_{i \geq x} i.d_i - x.l_x$$

and the first term of the second equality is

$$\begin{aligned}
\sum_{i \geq x} i.d_i &= x.d_x + (x+1).d_{x+1} + (x+2).d_{x+2} + (x+3).d_{x+3} + \ldots \\
&= x.\sum_{i \geq x} d_i + d_{x+1} + 2.d_{x+2} + 3.d_{x+3} + \ldots \\
&= x.l_x + d_{x+1} + 2.d_{x+2} + 3.d_{x+3} + \ldots
\end{aligned}$$

so

$$\begin{aligned}\sum_{i\geq x}(i-x).d_i &= d_{x+1} + 2.d_{x+2} + 3.d_{x+3} + 4.d_{x+4} + \ldots \\ &= \sum_{i\geq x+1} d_i + \sum_{i\geq x+2} d_i + \sum_{i\geq x+3} d_i + \sum_{i\geq x+4} d_i + \ldots \\ &= l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots \\ &= \sum_{i\geq x} l_{i+1}\end{aligned}$$

■

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