RATIONAL CHOICE ON NONFINITE SETS BY MEANS OF EXPANSION-CONTRACTION AXIOMS*

Mª Carmen Sánchez**

WP-AD 95-07

* I would like to express my gratitude to J.E. Peris and B. Subiza for their helpful comments. Any remaining errors are my exclusive responsability. This research was supported by a grant from Instituto Valenciano de Investigaciones Económicas. Financial support from the Spanish DGICYT under project PB91-0756 is acknowledged.

** University of Alicante.

Editor: **Instituto Valenciano de Investigaciones Económicas, S.A.**
Primera Edición Marzo 1995.
ISBN: 84-482-0902-8
Depósito Legal: V-1204-1995
Impreso por Copisteria Sanchis, S.L.,
Quart, 121-bajo, 46008-Valencia.
Printed in Spain.

RATIONAL CHOICE ON NONFINITE SETS BY MEANS OF EXPANSION-CONTRACTION AXIOMS

M. Carmen Sánchez

A B S T R A C T

The rationalization of a choice function in terms of assumptions which involve expansion or contraction properties of the feasible set over nonfinite sets is analyzed. Schwartz's results [15], stated in the finite case, are extended to this more general framework. Moreover a characterization result when continuity conditions are imposed on the choice function as well as on the binary relation which rationalizes it is presented.

Key words: Rational Choice; Expansion Contraction Axioms; Nonfinite Sets.

0. INTRODUCTION

The problem of describing human behavior in terms of rationality conditions has been analyzed in many papers since Samuelson's work [14]. In his work, Samuelson studies those situations in which the demand function can be induced by a utility function as a way of describing the rationality consumer's behavior. Arrow [1] generalizes this idea and analyzes the more general problem of "rationality of a choice function". He studies under which conditions a choice function can be generated by a preference binary relation, in particular by a weak order. But asking for transitivity in the rationalization process and, in particular for transitivity of indifference, has been criticized by many authors since Luce's work [9]. So many papers have been dealing with the problem of describing choices by means of weaker binary relations (more reasonable from the social choice perspective) such as semiorders, interval orders, quasitransitive relations... (see Sen [16]; Fishburn [4]; Schwartz [15];...).

All of these works attempt to find characterization results of the different kinds of rationality of a choice function by means of assumptions of a different nature. Some of them are based on the idea of "revealed preference" introduced by Samuelson [14], which consists of judging the rationality by comparing the choices made in different situations (see Richter [13]; Kim and Richter [8]; Kim [7]; Bandyopadhyay and Sengupta [2]; ...).

Other characterizations have involved properties on the expansion or contraction of the feasible set, that is, assumptions which regulate the choice function behavior when the set of alternatives which is presented for choice changes (Sen [16]; Schwartz [15]; Suzumura [17];...). These, then, will be called "expansion-contraction axioms".

However, most of these characterizations are stated in contexts with finite sets of alternatives[(1)]. Sometimes, it is pointed out that the results remain true in the nonfinite case, and as such finite sets are only considered in order to simplify the work. But this is not always true and, in many cases, the characterization results do not cover the nonfinite case.

In this paper the question of the characterizations made in the finite case by means of expansion contraction axioms being valid in the nonfinite case is studied. In the cases in which these characterizations fail, alternative assumptions are proposed, and the corresponding characterization results are proved. In Peris, Sánchez and Subiza [12] the same problem is analyzed but by means of revealed preference axioms.

---

1 Campbell [3] works in the context of sets consisting of all the n-dimensional commodity bundles satisfying a budget constraint, hence only for *certain* infinite sets.

Richter [13], Kim and Richter [8] and Kim [7] present results in particular nonfinite contexts, but by means of revealed preference axioms.

The paper is organized as follows: in Section 1 the basic definitions and well known characterizations from the finite case are presented. In Section 2, problems which arise whenever a nonfinite set of alternatives is considered are analyzed and alternative assumptions in order to obtain the characterization results in this more general framework are introduced. Section 3 is devoted to analysing the rationalization of hemicontinuous choice functions and finally, Section 4 presents some final comments.

1. PRELIMINARIES

Let X be the set of alternatives (not necessarily finite). A choice function F is a functional relationship that assigns a subset of alternatives (*choice set*) for any set within a distinguished family of nonempty subsets of X. So hereafter it will be considered that the domain of the choice function, namely $\mathcal{D}(X)$, consists of a specific family of nonempty subsets of X which includes all finite sets. It is important to note that, although for any finite set it will be required to have associated nonempty choice sets, it will be possible for nonfinite sets to have empty choice sets (which will be interpreted as "refusal from choice"). It is easy to justify this fact because the problem which is being analyzed here is how to characterize the choice by means of maximizing a binary relation (which will at least be acyclic), and there are many examples where the set of maximal elements for acyclic binary relations defined on nonfinite sets is empty, while it is always nonempty on finite sets. Formally we give the following definition.

Definition 1.1.

A *choice function* is a correspondence $F: \mathcal{D}(X) \longrightarrow\!\!\!\!\rightarrow X$ such that

$$\forall A \in \mathcal{D}(X) \qquad\qquad F(A) \subseteq A.$$

Furthermore if A is a finite set, then $F(A) \neq \emptyset$.

Henceforth R defined on X will be considered a complete and reflexive binary preference relation. The strict preference relation P and the indifference relation I are defined in the usual way:

$$xPy \iff xRy \text{ and } not[yRx]; \qquad xIy \iff xRy \text{ and } yRx.$$

We assume that preference relations satisfy one of the following successively weaker rationality conditions: it is a *weak order* iff $\forall x,y,z \in X$ $[x\ R\ y,\ y\ R\ z]$ implies $x\ R\ z$; a *semiorder* iff $\forall x,y,z,t \in X$ $[x\ I\ y,\ y\ P\ z,\ z\ P\ t]$ implies $x\ P\ t$ and $[x\ P\ y,\ y\ I\ z,\ z\ P\ t]$ implies $x\ P\ t$; an *interval order* iff $\forall x,y,z,t \in X$ $[x\ P\ y,\ y\ I\ z,\ z\ P\ t]$ implies $x\ P\ t$; a *quasiorder* iff $\forall x,y,z \in X$ $[x\ P\ y,\ y\ P\ z]$ implies $x\ P\ z$; and finally an *acyclic relation* iff $\forall x_1,x_2,..,x_n \in X$ $[x_1\ P\ x_2, x_2\ P\ x_3,\dots,x_{n-1}\ P\ x_n]$ implies $x_1\ R\ x_n$.

From a binary relation R defined on X, the set of *maximal elements* for this relation on any subset $A \subset X$ can be defined as follows:

$$M(A,R) = \{\ a \in A:\ a\ R\ z \quad \forall z \in A\ \}.$$

The problem of *rationality* for a choice function consists of finding a binary relation R whose maximal elements define the choice set for any set of $\mathcal{D}(X)$. Formally we give the following definition.

Definition 1.2.

Let $F: \mathcal{D}(X) \longrightarrow\!\!\!\!\rightarrow X$ be a choice function, it is said to be *rational* if a reflexive complete binary relation R exists such that

$$F(A) = M(A,R) \qquad \forall A \in \mathcal{D}(X)$$

In this case F is said to be *rationalized* by R and the relation R is called a *rationalization* of F.

According to the properties that are required for the rationalization of F, it will be called *transitive-rational* (if the rationalization R is transitive), *pseudotransitive-rational* (if R is pseudotransitive), *semiorder-rational* (if R is a semiorder), *quasitransitive-rational* (if R is quasitransitive), and finally *rational* (in the case where R is acyclic).

From a choice function binary relations based on it can always be defined. One of the most important is the *revealed preference* which states that an alternative x is revealed preferred to another y if there exists a subset of X such that both of them are available and x is chosen (i.e. $x\ R\ y$, iff $\exists A \in \mathcal{D}(X)$ such that $x,y \in A$ and $x \in F(A)$). It is easy to prove that, if there exists a binary relation which rationalizes a choice function, then the revealed preference relation is also a possible rationalization of the choice function. So henceforth, R will denote the revealed preference and I and P the associated indifference and strict preference relations respectively.

Let us consider the following set which will be used throughout the work:

$$IF(A) = \{\ x \in A \mid \ \forall z \in F(A) \quad \exists B_z :\ x,z \in B_z \quad x \in F(B_z)\ \}$$

that is, the set of alternatives which are not necessarily chosen, but are revealed to be indifferent to every chosen alternative. Notice that if the choice function is rational, then IF(A) is the set of elements revealed as indifferent to all of the maximal ones,

$$IF(A) = \{\ x \in A \mid \ x\ I\ z \quad \forall z \in F(A)\ \}$$

So, in the case of considering rational choice functions it is verified that $M(A,R) \subseteq IF(A)$. Furthermore, if quasitransitive rationality on finite sets or transitive rationality are considered, it is verified that $M(A,R) = IF(A)$.

Schwartz [15] characterizes the different kinds of rationality by means of "expansion-contraction" axioms in contexts where the domain of the choice function is the family of all nonempty finite subsets of X. We will analyze whether these characterizations are valid in the general framework considered in this paper. First of all, assumptions used by Schwartz and his characterization results are presented.

(A1). For any $A,B \in \mathcal{D}(X)$, then $F(A) \cap F(B) = F(A \cup B) \cap A \cap B$.

(A2). For any $A,B \in \mathcal{D}(X)$ if $B \subset A-F(A)$, then $F(A-B) \subset F(A)$.

(A3). For any $A,B \in \mathcal{D}(X)$ if $B \subseteq A$, not$[F(A) \subseteq B]$, not$[F(B) \subseteq F(A)]$, then $F(A-B) \subset F(A)$.

(A4). For any $A,B \in \mathcal{D}(X)$ if $B \subset A$ and not$[F(A) \subset B]$, then either not$[F(B) \subset F(A)]$ or $F(A-F(B)) - B \subseteq F(A)$ implies $F(A-B) \subset F(A)$.

The results he obtains are summed up in the following theorem:

Theorem 1.1. *[Schwartz, 1976]*

Let $F:\mathcal{D}(X) \longrightarrow X$ be a choice function, where X is a finite set of alternatives and $\mathcal{D}(X)$ is the family of all nonempty finite subsets of X. Then this choice function is

1. rational iff it satisfies (A1).
2. quasitransitive rational iff it satisfies (A1) and (A2).

3. pseudotransitive rational iff it satisfies (A1) and (A3).

4. semiorder rational iff it satisfies (A1) and (A4).

Some of these characterizations are often presented in terms of other well known assumptions which are equivalent to those presented in this theorem such as Chernoff's, Expansion or Aizerman's assumptions (Moulin [10], [11]). These are also expansion-contraction axioms and are stated as follows,

(B1). *Chernoff:* For any $A,B \in \mathcal{D}(X)$, if $A \subset B$ and $F(B) \cap A \neq \emptyset$, then $F(B) \cap A \subset F(A)$.

(B2). *Expansion:* For any $A,B \in \mathcal{D}(X)$, then $F(A) \cap F(B) \subset F(A \cup B)$.

(B3). *Aizerman:* For any $A,B \in \mathcal{D}(X)$ if $F(A) \subset B \subset A$, then $F(B) \subset F(A)$.

The relationship between these axioms and the ones presented above (if the domain of the choice function is the family of all nonempty finite subsets of X) is as follows: (B1) and (B2) are equivalent to (A1), whereas (B3) is equivalent to (A2).

2. RATIONALITY ON NONFINITE CONTEXTS

If we want to state the results from the previous section in the context of nonfinite sets, some problems arise. Throughout this section these problems are analyzed and some new assumptions are introduced to characterize the rationality in nonfinite contexts. The different kinds of rationality will be analyzed starting from the weakest one (acyclic rationality). The case of transitive rationality has not been mentioned because it is well known that this rationality is characterized by means of the Weak Axiom of Revealed Preference in nonfinite contexts.

The main problem that arises when we want to transfer the characterizations from the finite to the nonfinite case, is how to prove that M(A,R) is in general included in F(A) for any $A \in \mathcal{D}(X)$. Suzumura [17] imposes this condition directly as an assumption (*Generalized Condorcet Property*)[2] and this one together with Chernoff's Axiom characterizes the acyclic rationality. So the extension of Suzumura's characterization to the nonfinite case does not present any problem. This is not the case for Schwartz's characterization as the following example illustrates. In particular it proves that (A1) is not enough to ensure the acyclic rationality in nonfinite cases. So a stronger assumption will be required to obtain the characterization result.

---

2 *Generalized Condorcet Property:* $\forall A \subset \mathcal{P}(X)$, $M(A,R_b) \subset F(A)$ where R_b is the base relation ($x R_b y \Leftrightarrow x \in F(\{x,y\})$).

Hereafter a nonfinite set of alternatives will be assumed and the domain of the choice function is considered to be a specific family of nonempty subsets of X which includes all finite sets.

Example 2.1.

Let $X = [0,1]$ be the set of alternatives and $F\colon \mathcal{D}(X) \twoheadrightarrow X$ a choice function in which $\mathcal{D}(X)$ is considered to be a family closed under finite unions given by the sets $\{[0,a]: a\in[0,1)\}$ and all finite subsets of X.

$$F(X) = 1$$
$$F(A) = A \qquad \forall A\in\mathcal{D}(X),\ A\neq X$$

It is easy to prove that this choice function verifies (A1). However it is not rational because in the case of it being rational it has to verify $F(A) = M(A,R)$ $\forall A\in\mathcal{P}(X)$ where R is the revealed preference, and in this case it is clear that $0 \in M(X,R)$ but $0 \notin F(X)$.

The way to solve this problem is to consider the "natural" extension of (A1) to nonfinite contexts, which is as follows:

(A1'). For any arbitrary family $\{A_i\}_{i\in I} \subseteq \mathcal{D}(X)$, $I\neq\emptyset$, then

$$\bigcap_{i\in I} F(A_i) = F\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcap_{i\in I} A_i\right)$$

Theorem 2.1.

F is rational iff it satisfies (A1').

Proof.

Let F be a rational choice function, then $F(A) = M(A,R)$ for any $A \in \mathcal{D}(X)$. Therefore,

$$x \in \bigcap_{i\in I} F(A_i) \iff \begin{cases} x\ R\ y & \forall y\in A_i \quad \forall i\in I \\ x \in A_i & \forall i\in I \end{cases} \iff$$

$$\iff \begin{cases} x\ R\ y & \forall y\in \bigcup_{i\in I} A_i \\ x \in \bigcap_{i\in I} A_i & \end{cases} \iff x \in F\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcap_{i\in I} A_i\right)$$

Now let F be a choice function that satisfies (A1'). We have to prove that $F(A) = M(A,R^*)$ $\forall A\in\mathcal{D}(X)$ for some acyclic binary relation R^*. We are going to show that R_b, the *base relation* induced by the choice function ($x\ R_b\ y \iff x \in F(\{x,y\})$) is a rationalization of F.

If $x \in M(A,R_b)$ then $x\ R_b\ a$ $\forall a\in A$, that is $x \in F(\{x,a\})$ $\forall a\in A$. Moreover, considering that $\bigcup_{a\in A} \{x,a\} = A$ and $\bigcap_{a\in A} \{x,a\} = x$, by (A1') the following is obtained:

$$x \in \bigcap_{a\in A} F(\{x,a\}) = F(A) \cap \{x\}$$

In particular $x \in F(A)$, so $M(A,R_b) \subset F(A)$.

On the other hand, if $x \in F(A)$, assume that there exists an element $a^* \in A$ such that $a^*\ P_b\ x$, then $F(\{a^*,x\}) = a^*$ and by applying (A1'),

$$F\left(A \cup \{a^*,x\}\right) \cap A \cap \{a^*,x\} = F(\{a^*,x\}) \cap F(A)$$

and since $A \cup \{a^*,x\} = A$ and $x \in F(A) \cap \{a^*,x\}$, then $x \in F(\{a^*,x\})$ would be obtained, which is a contradiction.

Finally it is shown that R_b is acyclic. Consider $x_1, x_2, \ldots, x_t \in X$ such that $x_1 \ P_b \ x_2$, $x_2 \ P_b \ x_3$, $\ldots x_{t-1} \ P_b \ x_t$ and $A = \{x_1, x_2, \ldots, x_t\}$. Since $F(A) \neq \emptyset$, if $x_1 \notin F(A)$ then there exists $x_j \in F(A)$, $j=2,\ldots,t$. But $F(x_{j-1}, x_j) = x_{j-1}$ and by applying (A1')

$$F(x_{j-1}, x_j) \cap F(A) = F(A) \cap \{x_{j-1}, x_j\}$$

which implies that $x_j \in F(x_{j-1}, x_j)$, a contradiction. Therefore $x_1 \in F(A)$ and by (A1')

$$F(x_1, x_t) \cap F(A) = F(A) \cap \{x_1, x_t\}$$

which implies that $x_1 \in F(x_1, x_t)$ and hence that $x_1 \ R_b \ x_t$.

■

In Sen [16] a different characterization result appears for rational choice functions in finite contexts by means of other expansion-contraction axioms: Chernoff and Expansion (Moulin [10], [11]). However, the same problems as those in Schwartz's characterization arise if we try to transfer his result to nonfinite contexts. But if we modify the Expansion assumption used by Sen in the same way as has been done with Schwartz's one, then the characterization result can also be obtained in the nonfinite case. The modified assumption and the characterization result would be as follows:

(B2'). For any arbitrary family $\{A_i\}_{i\in I} \subseteq \mathcal{D}(X)$, $I\neq\emptyset$, then

$$\bigcap_{i\in I} F(A_i) \subset F\left(\bigcup_{i\in I} A_i \right).$$

Theorem 2.2.

F is rational iff it satisfies (B1) and (B2').

If quasitransitive rational choice functions are considered, the characterization obtained by Schwartz in the finite case does not remain true in the nonfinite case. The reason for this fact is that the application of either Schwartz's or Sen's result for rational choice functions is required and, as has been pointed out in the last theorem, neither of these are valid in the nonfinite case. However, substituting these characterizations by that of Theorem 2.1. is not enough to obtain the general characterization result as the following example shows.

Example 2.2.

Let X be the unit interval and consider the following binary relation defined on X.

$$x \; P \; y \quad \Leftrightarrow \quad x > y \qquad \forall x,y\in[0,1)$$

$$1 \; I \; x \qquad \forall x\in[0,1]$$

Let F be the choice function defined by means of the maximization of R on X in which $\mathcal{D}(X)=\mathcal{P}(X)$. It is obviously a quasitransitive rational choice function, but it does not verify (A2) (necessary in the finite case for the quasitransitive rationality). To show this consider X and B = (0.5,1), then B $\subset$ X-F(X) = [0,1) but F(X-B) = {0.5,1} is not included in F(X)

The assumption proposed to characterize quasitransitive rational choice functions in nonfinite sets is similar to the finite case but the set IF(A) is used instead of the choice set F(A).

(A2'). For any A,B $\in \mathcal{D}(X)$ if B $\subset$ A-IF(A), then F(A-B) $\subset$ F(A).

Note that if finite sets and quasitransitive relations are considered, it is verified that IF(A) coincides with F(A), so in this case the assumption is exactly the same as the one used in the finite case.

Theorem 2.3.

F is quasitransitive rational iff it satisfies (A1') and (A2').

Proof.

Let F be a quasitransitive rational choice function, in particular it is a rational one and by Theorem 2.1. (A1') is verified.

To show that (A2') is verified too, let us consider A,B$\in\mathcal{D}$(X) such that B $\subset$ A-IF(A) and prove that F(A-B) $\subseteq$ F(A).

If x$\in$F(A-B), since F(A-B) = M(A-B,R) then xRz $\forall$z$\in$A-B. So it is enough to prove that xRb $\forall$b$\in$B. But if b$\in$B, since B $\subset$ A-IF(A), there exists an element a*$\in$F(A) such that a*Pb, moreover a*$\in$A-B so xRa*. Therefore if there exists b$\in$B such that bPx, by the quasitransitivity, it is obtained that a*Px, which is a contradiction.

Conversely, if F is a choice function which satisfies (A1') and (A2'), then by Theorem 2.1. F is a rational choice function and it is enough to prove that the rationalization is quasitransitive.

Let a,b and c be elements in X such that a P b P c and consider A = {a,b,c}. Since F(A) ≠ ∅, it is clear that F(A) = {a} and considering B = {b} ⊂ A-IF(A), by (A2') it is obtained that F({a,c}) ⊂ F(A), which implies that F({a,c}) = {a}, that is a P c.

■

In the case of considering the alternative characterization by Schwartz for the quasitransitive case (by means of Chernoff's, Aizerman's and Expansion assumptions as we mentioned above), the problems which arise when the characterization is transferred to nonfinite contexts are exactly the same. The way to solving this consists of modifying the assumptions in a parallel way. In fact, the characterization result would be obtained by means of (B1), (B2') and the following modification of (B3):

(B3'). For any A,B ∈ $\mathcal{D}(X)$, if IF(A) ⊂ B ⊂ A, then F(B) ⊂ F(A).

Theorem 2.4.

F is quasitransitive-rational iff it satisfies (B1), (B2') and (B3').

Suzumura [17] gives a characterization for the quasitransitive rationality in finite sets by adding the following assumption which ensures the quasitransitivity of the rationalization to the ones he uses to characterize the rational case (Generalized Condorcet and Chernoff), (B4). *Superset:* For any $A,B \in \mathcal{D}(X)$ if $A \subset B$, $F(B) \subset F(A)$, then $F(A) = F(B)$.

It is easy to show that this assumption is not enough to guarantee the quasitransitivity in the nonfinite case (Example 2.2), but it can be modified by using the set IF(A) and the characterization result can be stated as follows,

(B4') For any $A,B \in \mathcal{D}(X)$ if $A \subset B$, $IF(B) \subset F(A)$, then $F(A) = F(B)$.

Theorem 2.5.

F is quasitransitive-rational iff it satisfies Generalized Condorcet Property, (B1) and (B4').

If rationality by means of an interval order is considered, the problems which arise are of a different nature. On the one hand (A1') is required instead of (A1) in order to ensure that the choice function is rational in the nonfinite context. But in this case, we do not need to modify (A3) to obtain the pseudotransitive rationality, since the characterization in the nonfinite case is ensured by (A1') and (A3). However a different problem now arises, due to the way in which the result was proved, since this way of reasoning cannot be applied in nonfinite sets. In particular Schwartz uses the following Lemma which is not valid for nonfinite sets.

Lemma 2.1. *[Schwartz, 1976]*

If F is a quasitransitive rational choice function and $x \in A-F(A)$, then there exists an element $y \in F(A)$ such that $y P x$.

The characterization result can be proved by making use of the following result which extends Lemma 2.1. to nonfinite contexts (Peris, Sánchez and Subiza [12]).

Lemma 2.2. *[Peris, Sánchez and Subiza, 1994]*

A choice function F is quasitransitive rational iff for each $x \in A-F(A)$, there exists an element $y \in IF(A)$ such that $y P x$.

Theorem 2.6.

F is a pseudotransitive rational choice function iff it satisfies (A1') and (A3).

Proof.

Let F be a pseudotransitive rational choice function, in particular it is rational and by Theorem 2.1. (A1') is verified.

To show that (A3) is satisfied, consider $A,B \in \mathcal{D}(X)$ such that $B \subset A$, $not[F(A) \subset B]$ and $not[F(B) \subset F(A)]$, and prove that $F(A-B) \subset F(A)$. Since F is rational, $F(A) = M(A,R)$, so it is enough to prove that if $xRz \quad \forall z \in A-B$, then $xRb \quad \forall b \in B$. By contradiction, assume that there exists an element $b^* \in B$ such that b^*Px. If $b^* \in B-F(B)$, by Lemma 2.2. there exists an element $b' \in IF(B)$ such that $b'Px$, so without losing generality it can be considered that $b^* \in IF(B)$. Moreover, since $not[F(B) \subseteq F(A)]$ there exists an element $b_1 \in F(B)$ such that $b_1 \notin F(A)$ and by Lemma 2.2. there exists $a^* \in IF(A)$ such that a^*Pb_1.

Therefore $a^* \; P \; b_1 \; I \; b^* \; P \; x$ and by the pseudotransitivity $a^* \; P \; x$. But $a^* \in IF(A) \subset F(A)$ and since $a^* P b_1$ and $b_1 \in F(B)$ then $a^* \in A-B$, so xRa^* which is a contradiction.

Now, let F be a choice function which verifies (A1') and (A3). Since it verifies (A1'), by Theorem 2.1. we know that F is rational, that is $F(A) = M(A,R) \;\; \forall A \in \mathcal{D}(X)$. We have to prove that R is an interval order.

Consider $x,y,z,w \in X$ such that $x \; P \; y \; I \; z \; P \; w$. Since (A3) implies (A2'), the quasitransitivity is ensured and therefore $x \; R \; z$ and $x \; R \; w$. So considering $A = \{x,y,z,w\}$ and $B = \{y,z\}$ and applying (A3)

$B \subset A$, $F(B) = \{y,z\}$ is not included in $F(A)$ (since $y \notin F(A)$)

$F(A)$ is not included in B (since $x \in F(A)$)

hence $$F(A-B) = F(\{x,w\}) \subset F(A)$$

and since $w \notin F(A)$, it implies that $F(\{x,w\}) = x$, that is xPw.

■

Finally if the case of rationality by means of a semiorder is analyzed, the problems are exactly the same as with the pseudotransitive rationality. That is, except for (A1) which has to be changed for (A1'), the rest of the assumptions used in the finite case also characterize the nonfinite case, but the way of proving the result uses Lemma 2.2 instead of Lemma 2.1. The characterization result for this case would be as follows,

Theorem 2.7.

F is semiorder rational iff it satisfies (A1') and (A4).

3. CONTINUITY CONDITIONS

When the set of feasibles alternatives X is considered to be a topological space, it is usual to impose continuity conditions on the choice functions as well as to ask for continuity conditions on the binary relation which rationalizes it. Concretely, hemicontinuity on the choice function is required in order to ensure that small changes on the set of alternatives presented for choice produce small changes on the choice set.

Therefore we are going to analyze which conditions should verify the choice function to guarantee the continuity of its rationalization, and conversely, which conditions are necessary for rationalization to ensure the hemicontinuity of the choice function.

So, throughout this section we consider X to be a metric topological space and F a choice function defined on the family of nonempty compact subsets[3] of X, $\mathcal{C}(X)$, such that F(A) is closed for all A in $\mathcal{C}(X)$. Hausdorff topology over $\mathcal{C}(X)$ is considered.

A correspondence $F:X \twoheadrightarrow X$ with nonempty closed images is *upper hemicontinuous (u.h.c.)* iff $\forall\{x_n\}\subset X$ which converges to x and $\{y_n\}$ which converges to y, where $y_n \in F(x_n)$, it is verified that $y \in F(x)$.

---

3 It is usual to consider that the domain of the choice function is given by the class of compact subsets on X whenever continuity conditions are imposed.

A preference relation R defined over X is *continuous* iff the sets $U(x) = \{y \in X \mid yPx\}$ and $L(x) = \{y \in X \mid xPy\}$ are open for all x in X. R is an *open relation* if and only if the graph of P $[G(P)=\{(x,y) \in X \times X \mid yPx\}]$ is an open set.

Theorem 3.1.

Let X be a metric topological space and $F:\mathcal{C}(X) \longrightarrow\!\!\!\!\rightarrow X$ a choice function. Then F satisfies (A1') and is u.h.c. iff F is rationalized by an acyclic open binary relation.

Proof.

Let F be an u.h.c. choice function which satisfies (A1'). By Theorem 2.1. F is rationalized by an acyclic binary relation R_b (base relation). We only need to prove that it is an open binary relation. Let $\{(x_n,y_n)\}$ be a sequence in $X \times X - G$ (where G denotes the graph of P_b) such that it converges to (x,y). On the one hand $\{x_n,y_n\}$ converges to $\{x,y\}$ in the Hausdorff topology and on the other hand since $x_n R_b y_n$ $\forall n \in \mathbb{N}$, $x_n \in F(\{x_n,y_n\})$ $\forall n \in \mathbb{N}$, so by the hemicontinuity of F we obtain that $x \in F(\{x,y\})$, that is $xR_b y$, so $(x,y) \in X \times X - G$.

Conversely if F is a choice function rationalized by an acyclic open binary relation, by Theorem 2.1 we know that it satisfies (A1'). Moreover we know that the base relation rationalizes F. To prove that F is u.h.c., let $\{S_n\}$ be a sequence of subsets of $\mathcal{C}(X)$ such that it converges to S in the Hausdorff topology and let $\{y_n\}$ be a sequence in $F(S_n)$ such that $\{y_n\}$ converges to y. If $y \notin F(S)=M(S,R_b)$, then there exists $z \in S$ such that $zP_b y$, that is $F(\{z,y\})=z$. Since $z \in S = \lim S_n$, there exists a sequence $\{z_n\}$ such that $z_n \in S_n$ $\forall n \in \mathbb{N}$ and $\{z_n\}$ converges to z. Since $y_n \in F(S_n)$ and $z_n \in S_n$, we obtain that $y_n R_b z_n$ $\forall n \in \mathbb{N}$ and by applying that P_b is open , $yR_b z$ which is a contradiction.

■

Similar results can be obtained for choice functions which are rationalized by means of quasitransitive relations, interval orders or semiorders by making use of the corresponding axioms instead of (A1'). Moreover, since in many contexts it is usual to impose continuity on the binary relation which rationalizes the choice function instead of requiring it to be open, we can state the following result which is an inmediate consequence of the previous theorem (since if R is an open binary relation then it is a continuous one).

Corollary 3.1.

Let X be a metric topological space and $F:\mathscr{C}(X) \longrightarrow X$ an u.h.c. choice function which satisfies (A1'). Then it is rationalized by a continuous acyclic binary relation.

4. FINAL COMMENTS

In finite contexts, it is usual to consider that the domain of the choice function is given by the family of all nonempty finite subsets of X, that is $\mathcal{D}(X) = \mathcal{P}(X)$. When more general domains are considered, sometimes finitely additive domains[4] are exploited (Hansson, [5]). However this domain fails in demand theory and many other disciplines. So, in order to simplify demand theory and following Arrow's suggestion (1959, p.122) many authors consider that the domain includes all finite sets. That is the kind of domain assumed by Sen [16] and Schwartz [15] (although they only restrict it to the family of all finite sets) and the one considered in this paper ($\mathcal{D}(X)$ is a specified family of nonempty subsets which includes all finite subsets. Other authors, (Jamison and Lau [6] and Fishburn [4]) take it as the family of all nonempty subsets of X (X finite or nonfinite).

In their work, Jamison and Lau [6] and Fishburn [4] obtain several results in nonfinite contexts. They assume rationalized choice functions and then analyze which assumptions determine the kind of rationalizing binary relation. Consequently their's are not rationality characterization results. However, we could consider the assumptions they obtain together with some of the ones presented in this paper to ensure the rationality of the choice function, and in this way obtain an alternative characterization result for rational choice functions in nonfinite contexts. In any case the number of axioms needed to obtain the characterization result will be greater than any of the ones presented in this paper.

---

4 A domain $\mathcal{D}(X)$ is *finitely additive* iff $\forall A,B\in\mathcal{D}(X)$, $A\cup B\in\mathcal{D}(X)$.

R E F E R E N C E S

1. K. J. Arrow, Rational choice functions and orderings, *Economica* **26** (1959), 121-127.

2. T. Bandyopadhyay and K. Sengupta, Revealed Preference Axioms for Rational Choice, *The Economic Journal* **101** (1991), 202-213.

3. D.E. Campbell, Revealed Preference and Demand Correspondences, *Journal of Economic Theory* **38** (1986), 364-370.

4. P.C. Fishburn, Semiorders and choice functions, *Econometrica* **43** (1975), 975-976.

5. B. Hansson, Choice Structure and Preference Relations, *Synthese* **18**, (1968), 433-458.

6. D.T. Jamison and L.J. Lau, Semiorders and the theory of choice, *Econometrica* **41** (1973), 901-912.

7. T. Kim, Intransitive indifference and revealed preference, *Econometrica* **55** (1987), 163-167.

8. T. Kim and M.K. Richter, Nontransitive-Nontotal Consumer Theory, *Journal of Economic Theory* **38**, (1986) 324-363.

9. R. D. Luce, Semiorder and a theory of utility discriminations, *Econometrica* **24** (1956), 178-191.

10. H. Moulin, Choice Functions over a finite set: A summary, *Social Choice and Welfare* **2,** (1985), 147-160.

11. H. Moulin, "Axioms of cooperative decision making". Cambridge University Press, New York, 1988.

12. J.E. Peris, M.C. Sánchez and B. Subiza, Revealed Preference Axioms for Rational Choice on nonfinite sets, *WP-A Discusión*, U.Alicante (1994).

13. M.K. Richter, Revealed Preference Theory, *Econometrica* **34**, 3 (1966), 635-645.

14. P. A. Samuelson, A note on the pure theory of consumer's behaviour, *Economica* **5** (1938), 61-71, 353-354.

15. T. Schwartz, Choice Functions, "Rationality" Conditions, and Variations on the Weak Axiom of Revealed Preference, *Journal of Economic Theory* **13,** (1976), 414-427.

16. A. K. Sen, Choice functions and revealed preference, *Review of Economic Studies* **38** (1971), 307-317.

17. K. Suzumura, "Rational choice, collective decisions and social welfare". Cambridge University Press, New York, 1983.

PUBLISHED ISSUES

WP-AD 90-01 "Vector Mappings with Diagonal Images"
C. Herrero, A.Villar. December 1990.

WP-AD 90-02 "Langrangean Conditions for General Optimization Problems with Applications to Consumer Problems"
J.M. Gutierrez, C. Herrero. December 1990.

WP-AD 90-03 "Doubly Implementing the Ratio Correspondence with a 'Natural' Mechanism"
L.C. Corchón, S. Wilkie. December 1990.

WP-AD 90-04 "Monopoly Experimentation"
L. Samuelson, L.S. Mirman, A. Urbano. December 1990.

WP-AD 90-05 "Monopolistic Competition: Equilibrium and Optimality"
L.C. Corchón. December 1990.

WP-AD 91-01 "A Characterization of Acyclic Preferences on Countable Sets"
C. Herrero, B. Subiza. May 1991.

WP-AD 91-02 "First-Best, Second-Best and Principal-Agent Problems"
J. Lopez-Cuñat, J.A. Silva. May 1991.

WP-AD 91-03 "Market Equilibrium with Nonconvex Technologies"
A. Villar. May 1991.

WP-AD 91-04 "A Note on Tax Evasion"
L.C. Corchón. June 1991.

WP-AD 91-05 "Oligopolistic Competition Among Groups"
L.C. Corchón. June 1991.

WP-AD 91-06 "Mixed Pricing in Oligopoly with Consumer Switching Costs"
A.J. Padilla. June 1991.

WP-AD 91-07 "Duopoly Experimentation: Cournot and Bertrand Competition"
M.D. Alepuz, A. Urbano. December 1991.

WP-AD 91-08 "Competition and Culture in the Evolution of Economic Behavior: A Simple Example"
F. Vega-Redondo. December 1991.

WP-AD 91-09 "Fixed Price and Quality Signals"
L.C. Corchón. December 1991.

WP-AD 91-10 "Technological Change and Market Structure: An Evolutionary Approach"
F. Vega-Redondo. December 1991.

WP-AD 91-11 "A 'Classical' General Equilibrium Model"
A. Villar. December 1991.

WP-AD 91-12 "Robust Implementation under Alternative Information Structures"
L.C. Corchón, I. Ortuño. December 1991.

WP-AD 92-01 "Inspections in Models of Adverse Selection"
I. Ortuño. May 1992.

WP-AD 92-02 "A Note on the Equal-Loss Principle for Bargaining Problems"
C. Herrero, M.C. Marco. May 1992.

WP-AD 92-03 "Numerical Representation of Partial Orderings"
C. Herrero, B. Subiza. July 1992.

WP-AD 92-04 "Differentiability of the Value Function in Stochastic Models"
A.M. Gallego. July 1992.

WP-AD 92-05 "Individually Rational Equal Loss Principle for Bargaining Problems"
C. Herrero, M.C. Marco. November 1992.

WP-AD 92-06 "On the Non-Cooperative Foundations of Cooperative Bargaining"
L.C. Corchón, K. Ritzberger. November 1992.

WP-AD 92-07 "Maximal Elements of Non Necessarily Acyclic Binary Relations"
J.E. Peris, B. Subiza. December 1992.

WP-AD 92-08 "Non-Bayesian Learning Under Imprecise Perceptions"
F. Vega-Redondo. December 1992.

WP-AD 92-09 "Distribution of Income and Aggregation of Demand"
F. Marhuenda. December 1992.

WP-AD 92-10 "Multilevel Evolution in Games"
J. Canals, F. Vega-Redondo. December 1992.

WP-AD 93-01 "Introspection and Equilibrium Selection in 2x2 Matrix Games"
G. Olcina, A. Urbano. May 1993.

WP-AD 93-02 "Credible Implementation"
B. Chakravorti, L. Corchón, S. Wilkie. May 1993.

WP-AD 93-03 "A Characterization of the Extended Claim-Egalitarian Solution"
M.C. Marco. May 1993.

WP-AD 93-04 "Industrial Dynamics, Path-Dependence and Technological Change"
F. Vega-Redondo. July 1993.

WP-AD 93-05 "Shaping Long-Run Expectations in Problems of Coordination"
F. Vega-Redondo. July 1993.

WP-AD 93-06 "On the Generic Impossibility of Truthful Behavior: A Simple Approach"
C. Beviá, L.C. Corchón. July 1993.

WP-AD 93-07 "Cournot Oligopoly with 'Almost' Identical Convex Costs"
N.S. Kukushkin. July 1993.

WP-AD 93-08 "Comparative Statics for Market Games: The Strong Concavity Case"
L.C. Corchón. July 1993.

WP-AD 93-09 "Numerical Representation of Acyclic Preferences"
B. Subiza. October 1993.

WP-AD 93-10 "Dual Approaches to Utility"
M. Browning. October 1993.

WP-AD 93-11 "On the Evolution of Cooperation in General Games of Common Interest"
F. Vega-Redondo. December 1993.

WP-AD 93-12 "Divisionalization in Markets with Heterogeneous Goods"
M. González-Maestre. December 1993.

WP-AD 93-13 "Endogenous Reference Points and the Adjusted Proportional Solution for Bargaining Problems with Claims"
C. Herrero. December 1993.

WP-AD 94-01 "Equal Split Guarantee Solution in Economies with Indivisible Goods Consistency and Population Monotonicity"
C. Beviá. March 1994.

WP-AD 94-02 "Expectations, Drift and Volatility in Evolutionary Games"
F. Vega-Redondo. March 1994.

WP-AD 94-03 "Expectations, Institutions and Growth"
F. Vega-Redondo. March 1994.

WP-AD 94-04 "A Demand Function for Pseudotransitive Preferences"
J.E. Peris, B. Subiza. March 1994.

WP-AD 94-05 "Fair Allocation in a General Model with Indivisible Goods"
C. Beviá. May 1994.

WP-AD 94-06 "Honesty Versus Progressiveness in Income Tax Enforcement Problems"
F. Marhuenda, I. Ortuño-Ortín. May 1994.

WP-AD 94-07 "Existence and Efficiency of Equilibrium in Economies with Increasing Returns to Scale: An Exposition"
A. Villar. May 1994.

WP-AD 94-08 "Stability of Mixed Equilibria in Interactions Between Two Populations"
A. Vasin. May 1994.

WP-AD 94-09 "Imperfectly Competitive Markets, Trade Unions and Inflation: Do Imperfectly Competitive Markets Transmit More Inflation Than Perfectly Competitive Ones? A Theoretical Appraisal"
L. Corchón. June 1994.

WP-AD 94-10 "On the Competitive Effects of Divisionalization"
L. Corchón, M. González-Maestre. June 1994.

WP-AD 94-11 "Efficient Solutions for Bargaining Problems with Claims"
M.C. Marco-Gil. June 1994.

WP-AD 94-12 "Existence and Optimality of Social Equilibrium with Many Convex and Nonconvex Firms"
A. Villar. July 1994.

WP-AD 94-13 "Revealed Preference Axioms for Rational Choice on Nonfinite Sets"
J.E. Peris, M.C. Sánchez, B. Subiza. July 1994.

WP-AD 94-14 "Market Learning and Price-Dispersion"
M.D. Alepuz, A. Urbano. July 1994.

WP-AD 94-15 "Bargaining with Reference Points - Bargaining with Claims: Egalitarian Solutions Reexamined"
C. Herrero. September 1994.

WP-AD 94-16 "The Importance of Fixed Costs in the Design of Trade Policies: An Exercise in the Theory of Second Best"
L. Corchón, M. González-Maestre. September 1994.

WP-AD 94-17 "Computers, Productivity and Market Structure"
L. Corchón, S. Wilkie. October 1994.

WP-AD 94-18 "Fiscal Policy Restrictions in a Monetary System: The Case of Spain"
M.I. Escobedo, I. Mauleón. December 1994.

WP-AD 94-19 "Pareto Optimal Improvements for Sunspots: The Golden Rule as a Target for Stabilization"
S.K. Chattopadhyay. December 1994.

WP-AD 95-01 "Cost Monotonic Mechanisms"
M. Ginés, F. Marhuenda. March 1995.

WP-AD 95-02 "Implementation of the Walrasian Correspondence by Market Games"
L. Corchón, S. Wilkie. March 1995.

WP-AD 95-03 "Terms-of-Trade and the Current Account: A Two-Country/Two-Sector Growth Model"
M.D. Guilló. March 1995.

WP-AD 95-04 "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets"
J. Alcalde. March 1995.

WP-AD 95-05 "Implementation of Stable Solutions to Marriage Problems"
J. Alcalde. March 1995.

WP-AD 95-06 "Capabilities and Utilities"
C. Herrero. March 1995.

WP-AD 95-07 "Rational Choice on Nonfinite Sets by Means of Expansion-Contraction Axioms"
M.C. Sánchez. March 1995.

WP-AD 95-08 "Veto in Fixed Agenda Social Choice Correspondences"
M.C. Sánchez, J.E. Peris. March 1995.

WP-AD 95-09 "Temporary Equilibrium Dynamics with Bayesian Learning"
S. Chatterji. March 1995.