

# **THE NASH BARGAINING SOLUTION UNDER FIXED LABOR SUPPLY\***

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## **A B S T R A C T**

Since 1950 Nash solution has been applied to solve most bargaining problems in the field of Labor Economics. In the present paper we show that this solution might not be appropriate under uncertainty over the state of nature and a fixed labor supply due to the non-fulfilment of the concavity axiom.

*JEL* classification: J50

Keywords: Nash bargaining solution, uncertainty.

## **R E S U M E N**

Desde 1950 la solución propuesta por Nash ha sido aplicada mecánicamente en la mayor parte de los problemas de negociación que se plantean en Economía Laboral. En este trabajo demostramos que la solución de Nash no debe utilizarse en aquellas situaciones en las que existe un número de trabajadores fijo e incertidumbre sobre el estado de la naturaleza debido al incumplimiento del, frecuentemente ignorado, axioma de concavidad.

*JEL* clasificación: J50

Palabras Clave: Nash bargaining solution, uncertainty.

## I. INTRODUCTION

Following Nash seminal work in 1950, Nash solution has been mechanically applied to most problems that involve bargaining among economic agents. Although the theory of bargaining has been enriched with further developments, new insights and solutions have not been broadly adopted by other fields of economic theory or applied economics. This is particularly the case of Labor Economics where, up to our knowledge, no other solution has been seriously considered as a plausible alternative. In this paper, we intend to prove that Nash solution might not be appropriate under all sorts of circumstances. In particular, it should not be used when the firm bargains over wages with a fixed number of workers.

As it is well known in bargaining theory, Nash solution does not necessarily fulfil the axiom of concavity (Thomson, 1994), which implies that the parties might not have an incentive to reach an agreement and sign a contract as long as the bargaining set remains unknown. Thus, we might encounter situations where the calculation of Nash solution is possible but, having the concavity axiom not satisfied, this solution becomes meaningless because it would not be implemented.

The intuition supporting our story is straightforward. A firm and a union bargain over wages in a right-to-manage setting. Depending on the wage and the (formerly revealed) *state of nature*, the firm chooses the level of employment that maximizes its profits. Labor supply is rationed in the sense that there exists a maximum amount of labor that the firm might hire. Within this framework, a positive<sup>1</sup> demand shock would induce the firm to employ additional workers at the bargained wage. However, extra workers cannot be hired. Given that the wage rate has been set during the bargaining process, neither the employment level nor the wage can be increased. The union knows beforehand that a good state of nature will not report either a higher wage or a larger level of employment -and thus an increase in workers' utility- but it will only lead to an increase in the firm's rents. In this context, the union does not have an incentive to sign a collective bargaining agreement in an uncertain environment. However, if labor supply was perfectly elastic, the union could benefit from a good state of nature through an increase in the number of workers employed and it would thus not reject to sign an employment contract based on a previously bargained wage rate.

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<sup>1</sup> In the next section, we specify more clearly the magnitude of the demand shock.

In terms of bargaining theory, a fixed labor supply means that the Pareto-optimal frontier of the bargaining set becomes an isolated point that corresponds with the Nash solution. This causes the non-fulfilment of the concavity axiom. Nash solution exists but cannot be implemented because the parties are not willing to reach an agreement on wages while uncertainty remains unsolved.

The paper proceeds as follows. In the next section, we display an example to illustrate the main argument of the paper, devoting a subsection to the construction of the bargaining sets under alternative demand shocks. Finally, the last section summarizes the conclusion.

## II. A SIMPLE EXAMPLE

A firm signs state-contingent employment contracts with a group of risk-neutral<sup>2</sup> workers ( $n$ ). We assume that  $n \leq n_0$ , where  $n_0$  denotes the maximum amount of labor that the firm might hire. For simplicity, we normalize  $n_0=1$  and, therefore,  $n \leq 1$ . All workers are homogeneous and affiliated to a single union. The reader may think of this situation as a bilateral monopoly where the supply of labor is fixed for some reason (e.g., training new workers requires a long period of time, a closed-shop arrangement prevails in the industry, etcetera..).

### *The parties objective functions*

The union's utility function is,

$$U(n,w;\beta) = nw + (1-n)\beta$$

where  $w$  denotes the wage that employed workers earn, and  $\beta$  represents some sort of non-labor income that unemployed workers receive (e.g., unemployment insurance provided by the state).

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<sup>2</sup> To keep this example as simple as possible, we assume risk-neutrality. The reader should be aware, however, that the main result does not depend on this assumption. In fact, we have also solved a similar model with risk-averse workers and observed the non-fulfilment of the concavity axiom.

For our purposes,  $\beta$  is assumed exogenous to the bargaining process.<sup>3</sup> Obviously,  $w \geq \beta$ <sup>4</sup> (participation constraint); otherwise, workers would prefer to stay idle.

In this model, fixed costs are swept away in order to avoid unnecessary complexity. Production exhibits decreasing returns to scale. For simplicity, let the production function be

$$f(n) = n^{1/2}$$

Hence, the firm's objective function can be written as

$$\theta n^{1/2} - wn$$

where  $\theta$  stands for the *state of nature*. The related literature offers alternative interpretations for  $\theta$ . Basically,  $\theta$  is a random variable that might represent either a productivity shock or a demand shock. In terms of the labor market, changes in  $\theta$  lead to shifts of the labor demand function.

The time sequence is crucial. First, the firm and the union bargain over wages. Once the wage is decided upon, the state of nature ( $\theta$ ) is revealed. Therefore, the bargaining set is uncertain while bargaining takes place. Later both parties can observe  $\theta$ ; thus, there is not asymmetric information in this model. According to the wage and the state of nature, the firm unilaterally decides the level of employment.

### ***Employment determination***

As we consider a right-to-manage setting, the firm chooses the level of employment that maximizes its profits, which is

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<sup>3</sup>It would certainly be of much interest to explore the situation in which  $\beta$  is partially determined during the bargaining process. That is, a more realistic model would incorporate both unemployment insurance and the severance payment provided by the firm to unemployed workers.

<sup>4</sup>When  $w = \beta$ , a worker is indifferent between working at the firm or staying unemployed since effort does not cause disutility in this model.

$$n(w,\theta) = \frac{\theta^2}{4w^2} \quad \text{for } w \geq \frac{\theta}{2} \quad (1)$$

Since  $n \leq 1$ ,  $w \geq \theta/2$ .<sup>5</sup> Labor demand is thus a decreasing and convex function of the wage rate. Both the profit and the (indirect) utility function can then be written, respectively, as,

$$\Pi(w,\theta) = \frac{\theta^2}{4w} \quad (2)$$

$$U(w,\theta,\beta) = \frac{\theta^2}{4w^2}(w-\beta) + \beta \quad (3)$$

### ***The Nash solution***

Once we have specified the parties' objective functions, we proceed to write down the Nash maximand:

$$\max (\Pi - \Pi')^\alpha (U - U')^{1-\alpha} \quad (4)$$

$$s.t. \quad S(\theta) = [(\Pi, U)/\Pi \geq \Pi', U \geq U']$$

where  $\Pi'$  and  $U'$  refer to the fall-back incomes of each party, respectively, when an agreement over the wage rate is not reached. Since the model does not consider any type of fixed cost, failure to agree on wages implies that  $n=0$  and then  $\Pi'=0$ . Regarding  $U'$ , the only possible source of alternative income rests on  $\beta$ , thus  $U'=U(\beta)=\beta$ . For practical reasons, we refer to  $U-U'$  as  $V$ .

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<sup>5</sup> This constraint on the wage rate is crucial to drive further results.

### *Bargaining sets*

Let us now focus on the bargaining set,  $S(\theta)$ . In Nash static approach, all of the crucial information about the bargaining process might be fully described by a utility possibility set which represents the consequences in terms of utility of all possible outcomes, including agreement and disagreement. In order to depict the bargaining set associated to our problem, we adopt a methodology similar to Blair and Crawford (1984), that is, we represent the set  $V[\Pi]$ . Blair and Crawford (1984) explain the peculiarities of the utility possibility set referring to the firm-union bargaining. Within their framework, however, labor supply is perfectly elastic whereas in our model the constraint on employment ( $n \leq 1$ ) requires that we study in detail the construction of the bargaining set.

From the profit function given in (2),  $w = \theta^2/4\Pi$ . However, since  $w \geq \theta/2$ ,  $\Pi \leq \theta/2$ . Plugging  $w = \theta^2/4\Pi$  into equation (3), we can rewrite  $S(\theta)$  as

$$S(\theta) = \{(\Pi, V) / V = \Pi - \frac{4\beta}{\theta^2}\Pi^2 ; 0 \leq \Pi \leq \frac{\theta}{2}\}$$

When the constraint on  $\Pi$  is nonbinding, the function  $V(\Pi)$  intersects the x-axis at  $\Pi = \theta^2/4\beta$ . In this case,  $\theta \leq 2\beta$ . On the contrary, if  $\theta > 2\beta$  the bargaining set is truncated at  $\Pi = \theta/2$ .

We can now proceed and study the Nash solution to our bargaining problem. In order to do it, let us assume that  $\alpha = 1 - \alpha = 1/2$ , that is, both parties have equal power. The optimization problem (4) can be rewritten as

$$\begin{aligned} \max \quad & \Pi^{1/2} V^{1/2} \\ \text{s.t.} \quad & V = \Pi - \frac{4\beta}{\theta^2}\Pi^2 \\ & 0 \leq \Pi \leq \frac{\theta}{2} \end{aligned}$$

The Kuhn-Tucker conditions for the above problem (see Appendix A for further technical details) imply that Nash solution is given by

$$\frac{\theta}{2}, \frac{\theta}{2} - \beta \text{ if } \theta > 3\beta$$

$F(\theta) =$

$$\frac{\theta^2}{6\beta}, \frac{\theta^2}{18\beta} \text{ if } \theta \leq 3\beta$$

Both the Nash solution and the bargaining set have been depicted in Figure 1. If  $\theta \leq 3\beta$  Nash solution corresponds to point  $b$  and the bargaining set is the sum of areas A and B. Nash solution is Pareto optimal and tangent to the bargaining set. On the other hand, when  $\theta > 3\beta$  Nash solution is a point like  $a$ ; the bargaining set is truncated and corresponds to area A.

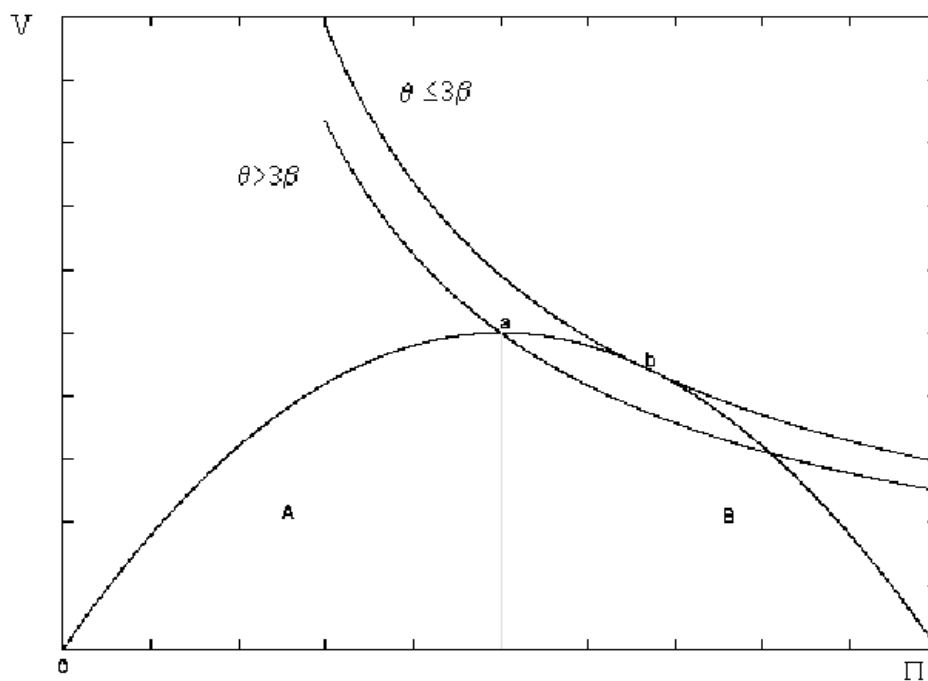


Figure 1



### *The concavity axiom*

In what follows we prove that the concavity axiom does not necessarily hold under uncertainty over the state of nature and a fixed labor supply. Thus, in our model an employment contract based on a previously bargained wage would not be signed as long as the state of nature remained unknown.

Let us first define concavity. For all  $\lambda \in [0,1]$ , Nash solution satisfies the axiom of concavity if

$$F(\lambda S_0 + (1-\lambda)S_1) \geq \lambda F(S_0) + (1-\lambda)F(S_1)$$

where  $F$  is a candidate solution. The fulfilment of this axiom guarantees that both agents benefit from an early agreement.

For simplicity, let  $\theta$  be a Bernoulli random variable with parameter  $\lambda$ , such that  $P(\theta_0) = \lambda$ , and  $P(\theta_1) = 1-\lambda$ , where  $\theta_0 \leq 3\beta$  and  $\theta_1 > 3\beta$ . The expected value of  $\theta$  is  $\lambda\theta_0 + (1-\lambda)\theta_1$ , which might obviously be higher or lower than  $3\beta$ . Let us first consider that  $E(\theta) > 3\beta$ . Under these circumstances,

$$F(S_0) = (\Pi_0, V_0) = \left( \frac{\theta_0^2}{6\beta}, \frac{\theta_0^2}{18\beta} \right)$$

$$F(S_1) = (\Pi_1, V_1) = \left( \frac{\theta_1}{2}, \frac{\theta_1}{2} - \beta \right)$$

$$F(\lambda S_0 + (1-\lambda)S_1) = (\Pi, V) = \left( \frac{E(\theta)}{2}, \frac{E(\theta)}{2} - \beta \right)$$

For the concavity axiom to be fulfilled,

$$\left( \frac{E(\theta)}{2}, \frac{E(\theta)}{2} - \beta \right) \geq \lambda \left( \frac{\theta_0^2}{6\beta}, \frac{\theta_0^2}{18\beta} \right) + (1-\lambda) \left( \frac{\theta_1}{2}, \frac{\theta_1}{2} - \beta \right) \quad (5)$$

The first component of each vector represents the firm profits under alternative realizations of  $\theta$ ; thus, if the inequality holds for the first component, the firm is willing to sign the contract. On the other hand, the second component indicates the union's attitude towards reaching an agreement on the wage rate beforehand. Solving (5), we find that the firm signs the contract only if  $\theta_0 \leq 3\beta$ , and the union signs when  $\theta_0 \geq 3\beta$ . Therefore, an agreement is reached on wages if and only if  $\theta_0 = 3\beta$ .<sup>6</sup>

We must now proceed and analyze whether the concavity axiom holds for  $E(\theta) < 3\beta$ . In this case,

$$F(S_0) = (\Pi_0, V_0) = \left( \frac{\theta_0^2}{6\beta}, \frac{\theta_0^2}{18\beta} \right)$$

$$F(S_1) = (\Pi_1, V_1) = \left( \frac{\theta_1}{2}, \frac{\theta_1}{2} - \beta \right)$$

$$F(\lambda S_0 + (1-\lambda)S_1) = (\Pi, V) = \left( \frac{[E(\theta)]^2}{6\beta}, \frac{[E(\theta)]^2}{18\beta} \right)$$

Thus, the fulfilment of concavity is guaranteed if

$$\left( \frac{[E(\theta)]^2}{6\beta}, \frac{[E(\theta)]^2}{18\beta} \right) \geq \lambda \left( \frac{\theta_0^2}{6\beta}, \frac{\theta_0^2}{18\beta} \right) + (1-\lambda) \left( \frac{\theta_1}{2}, \frac{\theta_1}{2} - \beta \right) \quad (6)$$

We can prove that (6) is inconsistent with  $\theta_1 > 3\beta$ .<sup>7</sup> Therefore, we have proved that the firm and the union would reach an agreement if and only if they knew beforehand that  $\theta_0 = 3\beta$ , which is incoherent with the assumption of an uncertain environment.

Let us now devote some time to provide an intuitive interpretation to our results. In particular, we must explain why the contract is signed only when  $\theta_0 = 3\beta$ . The fact that  $\theta_0$  must be

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<sup>6</sup> Notice that the value that  $\theta^1$  takes on does not matter at all.

<sup>7</sup> In order to avoid additional technical details in the text, we have moved the proof to Appendix B.

equal to  $3\beta$  depends on the assumption that both parties have the same bargaining power (i.e.,  $\alpha=1/2$ ). To prove this statement, let us go back to the optimization of the Nash maximand. Optimization of (4) leads to the following first-order condition:

$$\frac{\partial \Pi / \partial w}{\partial V / \partial w} = -\frac{(1-\alpha) \Pi - \Pi'}{\alpha U - U'}$$

which yields this solution for the optimal wage,

$$w(\alpha, \beta) = (2-\alpha)\beta$$

that is, the wage is a function of the parties' relative bargaining power and the alternative income. Notice that  $w$  does not depend on  $\theta$ . Assuming equal bargaining power for both parties (i.e.,  $\alpha=1/2$ ),  $w=(3/2)\beta$ . But  $w \geq \theta/2$  implies that  $\theta \leq 2w$ . Thus, for  $w=(3/2)\beta$ ,  $\theta \leq 3\beta$ .

However, at this wage rate,  $w=(3/2)\beta$ , the union clearly prefers  $\theta=3\beta$  than  $\theta < 3\beta$ . The intuition is straightforward. The union has its preferences defined over both employment and wages. In fact, it is easy to prove that the union's indifference curves over wages and employment are negatively sloped and convex (i.e.,  $dw/dn < 0$  and  $d^2w/dn^2 \geq 0$ ), and union's utility, given  $w=(3/2)\beta$ , is highest when  $\theta=3\beta$ . Therefore, the union would sign an employment contract based on  $w=(3/2)\beta$  only if it knew that  $\theta=3\beta$ , which is not feasible under uncertainty.

Figure 3 depicts the labor market. We have drawn three alternative labor supply curves:  $w=\beta$  (the union does not have any power and, therefore,  $\beta$  is the lowest wage that could be set during the bargaining process),  $w=2\beta$  (the union has *all the* power and the resulting wage is highest), and an intermediate situation where  $\alpha=1/2$  (i.e., both parties have equal bargaining power) and  $w=(3/2)\beta$ . With respect to labor demand, we face multiple possibilities depending on the value that  $\theta$  takes on. In Figure 3, we explore the equilibria for two different values of  $\theta$ :  $\theta=3\beta$  and  $\theta=4\beta$ .

We can see in Figure 3 that for  $\alpha=1/2$  and  $w=(3/2)\beta$  the labor market is in equilibrium if  $\theta \leq 3\beta$ . Otherwise, for higher values of  $\theta$ , there would be an excess demand of labor. For instance, let  $\theta$  be equal to  $4\beta$ . In such a situation, the labor market would not reach an equilibrium. At the

ongoing wage  $w=(3/2)\beta$ , there is clearly an excess demand of labor which cannot be corrected through an increase in the wage rate.

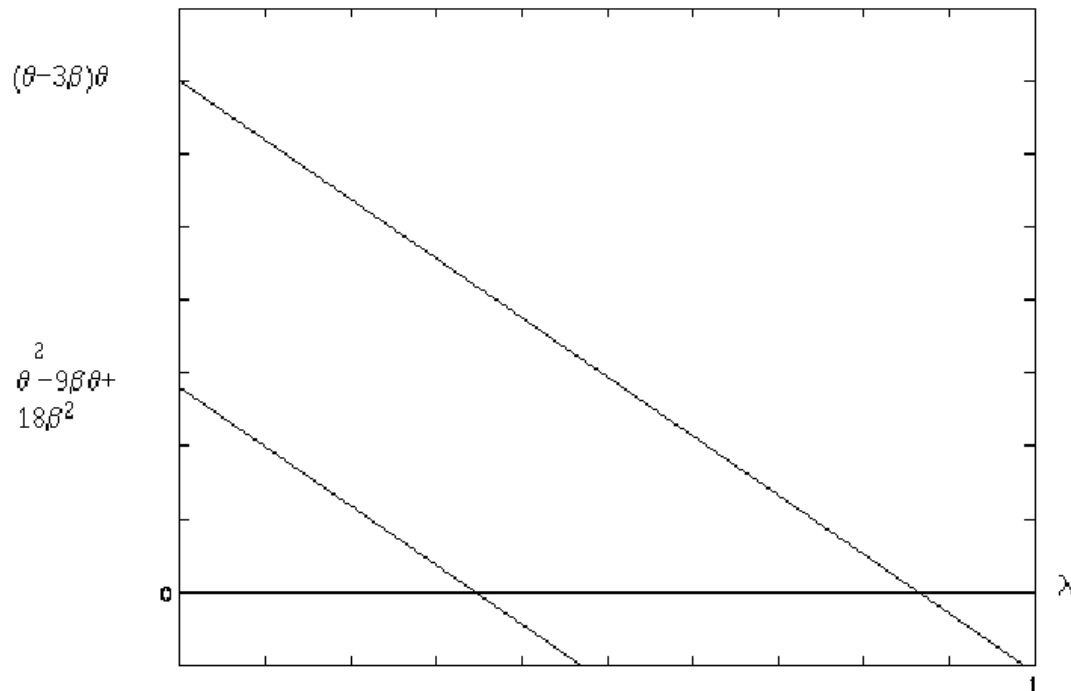


Figure 2

Regarding the equilibria in the labor market, problems arise when  $\theta$  takes on a value larger than  $3\beta$ . With respect to the bargaining sets, problems also appear when  $\theta$  takes on a high value with respect to  $\beta$ . We basically face two types of situations. When  $\theta \leq 3\beta$  we can draw bargaining sets *à la Blair and Crawford*. On the contrary, if  $\theta > 3\beta$  the bargaining set is *truncated*. In summary, in order to obtain *nicely shaped* utility possibility sets, labor supply must be perfectly elastic; otherwise, when the state of nature is good, the Pareto frontier,  $P(S)$ , *shrinks* and, beyond some level of  $\theta$ , might become a single point which is the Nash solution. This leads to the non-fulfilment of the concavity axiom.<sup>8</sup>

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<sup>8</sup> As Thomson (1994) indicates, neither the Nash nor the Kalai-Smorodinsky solutions fulfil the concavity axiom. Imposing the Kalai-Smorodinsky solution in our model would have led us to an identical result in terms of the non-fulfilment of the concavity axiom.

### III. CONCLUDING REMARKS

It was our main purpose to prove through a simple example that Nash bargaining solution might not be the appropriate tool to solve certain theoretical models. In particular, we have showed that Nash procedure fails in its implementation when the firm bargains over wages with a fixed number of workers. In this case, due to the non-fulfilment of the concavity axiom, agents do not sign a contract as long as the bargaining set remains unknown. Labor economists should thus be more careful when applying Nash's method and keep in mind the *axiomatic* nature intrinsic to such approach. This is the *bottom line* of this paper.

## APPENDIX A

The Kuhn-Tucker conditions for the optimization problem:

$$\begin{aligned} \max \quad & \Pi^{1/2}V^{1/2} \\ \text{s.t.} \quad & V = \Pi - \frac{4\beta}{\theta^2}\Pi^2 \\ & 0 \leq \Pi \leq \frac{\theta}{2} \end{aligned}$$

are the following:

$$\frac{\partial L}{\partial \Pi} \leq 0 \Rightarrow 2\Pi - \frac{12\beta}{\theta^2}\Pi^2 - \lambda \leq 0 \quad (7)$$

$$\frac{\partial L}{\partial \lambda} \geq 0 \Rightarrow \frac{\theta}{2} - \Pi \geq 0 \quad (8)$$

$$\frac{\partial L}{\partial \Pi} = 0 \quad (9)$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad (10)$$

$$\Pi \geq 0 \quad (11)$$

$$\lambda \geq 0 \quad (12)$$

If  $\lambda > 0$  (binding constraint), (10) and (8) imply that  $\Pi = \theta/2$  and therefore  $V = \theta/2 - \beta$ . On the other hand, if  $\lambda = 0$  (nonbinding constraint), (9) and (7) imply that

$$2\Pi^2 - \frac{12\beta}{\theta^2}\Pi^3 = 0$$

which yields  $\Pi = \theta^2/6\beta$  and  $V = \theta^2/18\beta$ .

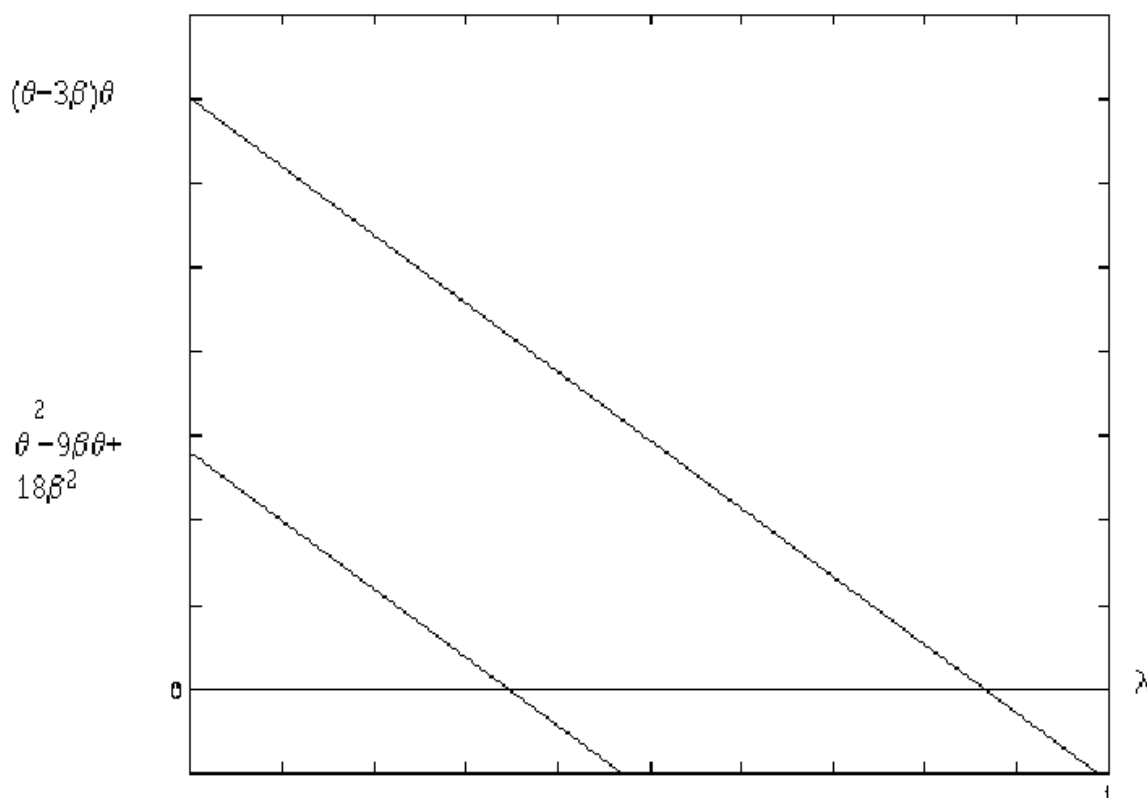
## APPENDIX B

The first and the second inequalities in expression (6) hold respectively if

$$-(\theta_1 - \theta_0)^2 \lambda + (\theta_1 - 3\beta)\theta_1 \geq 0$$

$$-(\theta_1 - \theta_0)^2 \lambda + \theta_1^2 - 9\beta\theta_1 + 18\beta^2 \geq 0$$

Hence, in terms of the probability associated to the *worse* state,  $\lambda$ , the union is more reluctant to reach an agreement than the firm, since it is more likely that the first inequality holds than the second inequality. Figure 2 illustrates this result. We know that  $(\theta_1 - 3\beta)\theta_1 > 0$ , but  $(\theta_1 - 9\beta)\theta_1 + 18\beta^2$  can be either positive or negative; what is certain is that it is lower than  $(\theta_1 - 3\beta)\theta_1$ . If  $(\theta_1 - 9\beta)\theta_1 + 18\beta^2 < 0$ , the union does not accept an agreement on the wage rate.



Thus, for the fulfilment of concavity we need that

$$\lambda \leq \frac{\theta_1^2 - 9\beta\theta_1 + 18\beta^2}{(\theta_1 - \theta_0)^2} \leq 1 \quad (13)$$

Assume that the contract is signed and (6) holds. Let us now prove that this is inconsistent with our previous assumption,  $E(\theta) < 3\beta$ . Substituting  $E(\theta)$  by  $\lambda\theta_0 + (1-\lambda)\theta_1$ ,  $E(\theta) < 3\beta$  implies that

$$\lambda > \frac{\theta_1 - 3\beta}{\theta_1 - \theta_0} \quad (14)$$

Therefore, (13) and (14) imply

$$\frac{\theta_1 - 3\beta}{\theta_1 - \theta_0} < \frac{\theta_1^2 - 9\beta\theta_1 + 18\beta^2}{(\theta_1 - \theta_0)^2}$$

which means that

$$\theta_0 > \frac{(6\theta_1 - 18\beta)\beta}{\theta_1 - 3\beta}$$

Since  $\theta_0 < 3\beta$ ,

$$\frac{(6\theta_1 - 18\beta)\beta}{\theta_1 - 3\beta} < 3\beta$$

which implies that  $\theta_1 < 3\beta$ . This contradicts our initial assumption:  $\theta_1 > 3\beta$ .



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