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An Endogeneous Scoring Rule*

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Abstract

This paper presents a way of ranking alternatives on the basis of their relative support. This is measured by the eigenvector of a positive matrix whose entries are the Condorcet dominations (off-diagonal) and the Borda counts (diagonal). A characterization is also presented.

Keywords: Scoring rules, domination, Condorcet, Borda.

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An Endogeneous scoring rule

1 Introduction

Consider the problem of distributing the budget of a University among different Departments, taking into account their performance on different aspects (publications, research attainments), or their needs (students under tuition, labs, etc...). The different departments are ordered (probably with ties, or in some cases only partially ordered) according to the different criteria. Normally, the budget is allocated by parts, or using some traditional multicriteria scheme. Think also of the problem of assigning seats in a constituency, out of the preferences of the members of a society. The electors order the candidates according to their preferences, an in some cases, the order is only partial, and maybe there are also some ties. In general, seats are allocated taking into account only the most preferred candidate for each elector. Another problem is that of assigning some measure of power to parties or lobbies according to their capabilities of influence, out of the citizens' perception of their importance.

Previous problems are of a very different nature, but they share some fundamental properties: (1) we have to allocate a good (a budget, seats, measure of power), (2) the allocation should take into account some orders, and (3) these orders may be partial or contain indifferences.

A traditional way of approaching this problem is by using scoring rules, by simply attaching some given weights to the alternatives, depending upon their position in the individual rankings. Scoring rules, indeed, provide a complete order of the alternatives, taking into account the information on individual preferences. In the design of representative bodies, in general, only the best preferred alternative of each individual is taking into account, namely, the amount of power given to the different candidates is obtained out of a particular scoring rule: plurality (1 for the best preferred, 0 for any other alternative). In some other circumstances, the Borda rule or some other scoring rules are widely used (points in beauty contests,...).

In scoring rules, the weights attached to the positions are exogeneously given, so that the resulting social ordering or power allocation will depend on those external weights (or on the particular scoring rule chosen). A previous collective agreement on which particular scoring rule to use is called then for, but getting such an agreement may have exactly as many difficulties as agree upon the right way of constructing collective preferences.

Morales (1798, 1805) (a Spanish thinker that lived during the times of Borda and Condorcet), claimed that the ranking of social alternatives should be related to the "*amount of opinion*" of the citizens in favor of each of them. Morales interpreted the "amount of opinion" as the sum of the number of times an alternative beats any other. In other words, he associated the amount of opinion in favor of an alternative with its Borda score.

The idea of computing the *amount of opinion* lies in the proposal we present in this paper, with three main differences: (1) we focus on the *relative support* of the alternatives (i.e., how important one alternative is with respect to another one, taking the full set of alternatives into account), (2) not all "beats" are equally important, i.e., beating an alternative with a large support is more important than another one with small support, and (3) the relative supports (scores) are endogeneously obtained, rather than with the use of any predetermined parameters.

Previous ideas are reminiscent of those of *centrality* in networks (see Freeman (1977), Wasserman & Faust (1994), Newmann (2003)), or of the *importance or relevance* of the journals in a certain discipline (Pinsky & Narin (1976), Liebowitz & Palmer (1984), Palacios-Huerta & Volij (2004)), or how to order teams out of a competition (Keener (1993), Slutzski & Volij (2006), Anderson et al (2009). These ideas are also considered in Chebotarev & Shamis (1998).

The *relative support* of alternatives is a vector with positive components, that provides two different outcomes. On the one hand, a *natural order* of the alternatives, according to the size of their support. On the other hand, a measure of the *worth, power* or *importance* of the candidates or the parties in a political election, and as thus, can be used in themselves (for instance, as a way of allocating seats in a constituency).

There are some features worth mentioning of our approach: First, the measurement of the relative support can be made by considering the relative frequency of an alternative being considered as ideal, in a certain random process, in which agents, chosen at random, decide, on the basis of their preferences, to choose among two alternatives. Second, previous procedure provides a Markov chain, whose stable distribution provides the measurement of the relative support. Third, when individual preferences are strict, complete and transitive, this stable distribution corresponds to the dominant eigenvector of a non-negative matrix whose entries are the Borda scores of the alternatives (in the diagonal), and whose off-diagonal elements (ij), are the number of agents strictly preferring one alternative (i) over another (j) (we call this matrix the Condorcet-Borda matrix associated to the particular problem considered). Fourth, this way of ordering the alternatives satisfies a number of interesting properties, namely, anonimity, neutrality, Pareto, monotonicity, participation, and continuity. Fifth, the procedure also works when individual preferences present indifferences, or even are neither transitive, nor complete.

The rest of the paper is organized as follows: In Section 2 we present the random process that provides a natural way of computing the relative support of the alternatives. Section 3 presents the formal model. Section 4 presents a characterization result. Finally, Section 5 provides some illustrations.

2 A natural way of obtaining the alternatives' support

Let $A = \{a, b, ..., z\}$ be a finite set of alternatives, with cardinal m, and $N = \{1, ..., n\}$ a finite set of agents, with n elements. Agents have strict preferences over the alternatives. Agents' preferences are complete, asymmetric and transitive, namely, they order all alternatives, and there are no indifferences. A preferences profile is a combination of preferences fulfilling previous requirements, one for each agent. When set A is fixed, a problem is specified by the set of agents and the profile of preferences of these agents over set A.

We want to find a procedure to socially order the alternatives, taking into account the preferences of the agents, so that such a procedure functions for any preferences profile.

Imagine that two alternatives i, j are selected at random at $t = 0$, and a randomly selected individual is asked which of the two alternatives she prefers. If i is preferred, it will be provisionally selected by society, if j is chosen, then it is provisionally selected by society; if the individual is indifferent, then either i or j are selected with equal probability; if the agent is unable to decide, no alternative is selected. This process is repeated infinitely many times. We may ask about the fraction of time that any alternative is selected as the ideal one in this process.[1]

For a preferences profile, let n_{ij} the number of individuals that strictly prefer i to j, e_{ij} be the number of individuals for which i and j are indifferent, and u_{ij} the number of individuals unable to compare i and j. We have that if n is the total number of individuals, then for any pair i, j, it holds that $n = n_{ij} + n_{ji} + e_{ij} + u_{ij}$.

[1] This procedure is similar to the "ping-pong" protocol (see Laslier (1997)) in tournaments.

Assume that, at time t the pair of alternatives selected were i, j. Then, the probability that alternative i is selected is $n_{ij}/n + e_{ij}/2n$; the probability that alternative j is selected is $n_{ji}/n+e_{ij}/2n$, and the probability that no alternative is selected is u_{ij}/n.

Now we can build a (row) stochastic matrix P, where the ji entry, p_{ji} is the probability that, conditional on i being the ideal alternative at the begining of a period, j becomes the ideal alternative for the next period. To compute p_{ji}, note that before any competing alternative appears at time t, the probability that alternative $j \neq i$ will become the new ideal alternative is $p_{ji} = n_{ji}/(n(m-1))$, and the probability that i remains as the ideal alternative for the next period is $p_{ii} = \sum_{j \neq i} n_{ij}/(n(m-1)) = B(i)/(n(m-1))$, where $B(i) = \sum_{j \neq i} n_{ij}$ is nothing but the Borda score of alternative i.

Hence, the entries of matrix P so defined are

$$p_{ji} = \tfrac{n_{ji}}{n(m-1)} \quad \text{for } i \neq j, \text{ and } \quad p_{ii} = \tfrac{B(i)}{n(m-1)}$$

That is, matrix P gives us a Markov chain. The fraction of time that alternative i is the ideal alternative is given by the positive eigenvector associated to the dominant eigenvalue (1) of matrix P. Now, if we compute that eigenvector w of P

$$Pw = w$$

we obtain that

$$Pw = \frac{1}{n(m-1)} \begin{pmatrix} B(1) & n_{12} & \cdots & n_{im} \\ n_{21} & B(2) & \cdots & n_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ n_{m1} & n_{m2} & \cdots & B(m) \end{pmatrix} w$$

and hence, the components of w fulfill the following relation:

$$w_i = \frac{\sum_{j \neq i} n_{ij} w_j}{\sum_{k \neq i} n_{ki}}$$

The interpretation of vector $w >> 0$, with $\sum_{i=1}^{m} w_i = 1$ as the frequency with which any of the alternatives is the ideal one in the process previously described, provides a natural measure of the *relative support* of the alternatives in our problem, and could be interpreted as a way of *naturally ranking* the alternatives.

2.1 Examples

Consider the following example. We have four alternatives, $A = \{a, b, c, d\}$, and there is a society made out of 21 individuals, with preferences as follows:

Num. Agents	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Call n_{ij} the number of individuals that prefer i to j. We then have

$n_{ab} = 8$; $n_{ac} = 8$; $n_{ad} = 8$; $n_{ba} = 13$; $n_{bc} = 10$; $n_{bd} = 21$; $n_{ca} = 13$; $n_{cb} = 11$; $n_{cd} = 14$; $n_{da} = 13$; $n_{db} = 0$; $n_{dc} = 7$

As it is clear, c is a Condorcet winner. Moreover, the Condorcet rule orders the alternatives as follows: $c \succ b \succ d \succ a$

On the other hand, the Borda rule gives to any alternative a score related to its position in the different preference profiles: 3 points whenever it ocuppies the first position; 2 points when in the second position, 1 point in the third position, and zero if it is the least preferred alternative. The Borda scores are as follows:

$B(a) = 24$; $B(b) = 44$; $B(c) = 38$; $B(d) = 20$, and thus, the Borda rule orders the alternatives in the following way: $b \succ c \succ a \succ d$

Consequently, the Markov matrix associated to this problem is

$$P = \frac{1}{21 \times 3} \begin{pmatrix} 24 & 8 & 8 & 8 \\ 13 & 44 & 10 & 21 \\ 13 & 11 & 38 & 14 \\ 13 & 0 & 7 & 20 \end{pmatrix}$$

and the dominant eigenvector w is given by $w^T = (0.17021; 0.40194; 0.32369; 0.10415)$, and if we interpret the relative frequencies of being the ideal alternative given above as a way of ranking the alternatives, we obtain $b \succ c \succ a \succ d$. In this example, the order given by the eigenvector is exactly the Borda order.

Take now a different example. Again we have four alternatives, and now 100 agents, with the following profiles:

Num. Agents	70	25	5
	a	b	c
	b	c	b
	c	d	a
	d	a	d

In this case, a is a Condorcet winner, and the Borda order is $b > a > c > d$. Now, the Markov matrix associated to this problem is the following:

$$P = \frac{1}{100 \times 3} \begin{pmatrix} 215 & 70 & 70 & 75 \\ 30 & 225 & 95 & 100 \\ 30 & 5 & 135 & 100 \\ 25 & 0 & 0 & 25 \end{pmatrix}$$

and the associated positive eigenvector is $w^T = (0.45294; 0.38685; 0.11903; 0.041176)$

If again, we interpret the frequencies given by the eigenvector components as the "relative support" of the alternatives, the order (of importance) of the alternatives according to this criterion is $a \succ b \succ c \succ d$, that is, in this case the proposed solution coincides with the Condorcet winner.

Previous examples indicate that this *natural way of ordering the alternatives* is not the Borda rule, neither is Condorcet consistent, but somehow we are using, simultaneously, the Condorcet and the Borda information together in order to naturally generate a certain order of the alternatives, clearly related with the *relative support* the alternatives enjoy from the population.

3 The Model

3.1 Case I: Linear orders

Given a problem, for any two alternatives, i, j, let us call n_{ij} the number of agents that strictly prefer i to j; and let n_{ji} stand for the number of agents that strictly prefer j to i. Whenever there are no indifferences, iIt is clear that, for any profile, it always happens that

$$n_{ij} + n_{ji} = n$$

Obviously, the comparison between n_{ij} and n_{ji} is the basis for the Condorcet comparison among the alternatives. Furthermore, for any alternative, it happens that the Borda score of that alternative can be computed as follows:

$$B(i) = \sum_{j \neq i} n_{ij}$$

We want to use all this information to endogeneously generate a social order in the set A, and moreover to get a vector of *weights or scores* that measure the importance of the alternatives.

Let us first condensate previous information about the different alternatives in a matrix, in the following way. Our matrix will be a square matrix, with m rows and colums, where m is the number of alternatives. Off-diagonal elements elements in the location ij, with $i \neq j$ are simply n_{ij}. Elements in the diagonal, i.e., elements in the location ii in the matrix are, for any alternative i, the Borda score of alternative i. We call the matrix constructed this way the *Condorcet-Borda matrix associated to problem p, CB(p).*

$$CB(p) = \begin{pmatrix} B(1) & n_{12} & \cdots & n_{im} \\ n_{21} & B(2) & \cdots & n_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ n_{m1} & n_{m2} & \cdots & B(m) \end{pmatrix}$$

Remark: Young (1994) introduces the *vote matrix* as the $m \times m$ matries whose ij entry is the number of voters than rank i over j. The vote matrix coincides with the *Condorcet-Borda* matrix in all off-diagonal elements, and it has zeroes in the diagonal. He uses this matrix to define the concept of *concordance* with the Condorcet criterion.

Notice that

$$P = \frac{1}{n(m-1)} CB(p)$$

and then, P and $CB(p)$ have the same dominant eigenvectors, that is, the dominant eigenvector of matrix $CB(p)$ also provides the natural ranking presented in previous section.

Matrix $CB(p)$ has a number of interesting properties:

- It is a non-negative matrix
- All its colums add up to $n(m-1)$
- $CB(p)$ has a positive eigenvalue, precisely $n(m-1)$, and all other eigenvalues are (in modulus) smaller than this one
- Associated to the dominant eigenvalue there is a positive eigenvector $v(p) > 0$, unique up to normalization
- Furthermore, the components of this eigenvector fulfill the following relationships, for all $i \in A$
$$v_i = \frac{\sum_{j \neq i} n_{ij} v_j}{\sum_{k \neq i} n_{ki}}$$
- Due to previous property, it is natural to interpret the components of vector v both as a measure of the *relative support* of the alternatives in problem p, as well as a natural way of ranking the alternatives.

Consequently, we may think of $v(p)$ as a way of constructing an "endogeneous score and thus a social order" of the alternatives in A, made out of a certain combination of the number of times the different alternatives dominates each other. This way, and exploiting a Condorcet-Borda type information, we end up with an endogeneous procedure that allow us to both evaluate the relative importance of the alternatives, and completely order them.

It is not a scoring rule. Suppose that there is a series of scores that provide the same order as our procedure. Take the case of three alternatives, a, b, c. In this case, we can rely on two scores, $0 \leq \alpha \leq \beta$, with $\beta > 0$.

Take the following profile p,

N. voters	30	3	27	10	10	1
	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

, $CB(p) =$

$$\begin{pmatrix} 103 & 43 & 60 \\ 38 & 105 & 67 \\ 21 & 14 & 35 \end{pmatrix}$$

. The dominant eigenvector is $v^T = (0.44177; 0.437; 0.12123)$, and the eigenorder is $a > b > c$

Let us compute the scores: $s(a) = 33\beta + 37\alpha$; $s(b) = 37\beta + 31\alpha$. Thus, $s(a) > s(b) \Longleftrightarrow 33\beta + 37\alpha > 37\beta + 31\alpha \Longleftrightarrow 6\alpha > 4\beta \Leftrightarrow \beta < \frac{3}{2}\alpha$

Take now the following profile q,

N. voters	6	5	6	2	6	3
	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

, $CB(q) = \begin{pmatrix} 29 & 15 & 14 \\ 13 & 30 & 17 \\ 14 & 11 & 25 \end{pmatrix}$. The dominant eigenvector is $w^T = (0.33484; 0.3642; 0.30096)$, and the eigenorder is $b > a > c$. Compute the scores: $s(a) = 9\beta + 11\alpha$; $s(b) = 11\beta + 8\alpha$. In this case, $s(b) > s(a) \Leftrightarrow 11\beta + 8\alpha > 9\beta + 11\alpha \Leftrightarrow 2\beta > 3\alpha \Leftrightarrow \beta > \frac{3}{2}\alpha$, exactly the opposite as before. So, there is no way of getting a series of scores agreeing with the eigenorders. ■

3.1.1 Some technical issues

In order to guarantee the strict positiveness of vector v, we need additional assumptions on the preferences profile p. Suppose there is an alternative such that it is unanimously ranked as the last one for all individuals, then this alternative can be considered as a *dummy*, namely, its vector component is going to be zero, and the score of the remaining alternatives does not change whether we consider this dummy alternative or we dispense with it. This is a special case in which the fact that an alternative has a zero component does not affect the relative ranking of the remaining alternatives.

We can guarantee the strict positiveness of vector v when matrix $CB(p)$ is irreducible. This could be granted whenever no alternative is unanimously dominated by another alternative. In such a case, matrix $CB(p) >> 0$.

3.2 The general case

Let $A = \{a, b, ..., z\}$ be a finite set of alternatives, with cardinal m, and $N = \{1, ..., n\}$ a finite set of agents, with n elements. The only requirement over agents' preferences is asymmetry, namely if $a \succ b$, then it cannot be that $b \succ a$. That is, they could be incomplete, and even can violate transitivity. A preferences profile is a combination of preferences, for each agent. When set A is fixed, a problem is specified by the set of agents and the profile of preferences of these agents over set A.

As before, we want to find a procedure to socially order the alternatives, taking into account the preferences of the agents, so that such a procedure functions for any preferences profile. And what we want is to extend the eigenvector procedure to this general case.

Given a problem, for any two alternatives, i, j, let us call n_{ij} the number of agents that strictly prefer i to j; and let n_{ji} stand for the number of agents that strictly prefer j to i, let e_{ij} the number of agents that are indifferent between i and j, and, finally, let u_{ij} the number of agents that are not able to compare a and b.It is clear that, for any profile, it always happens that

$$n_{ij} + n_{ji} + e_{ij} + u_{ij} = n$$

We are now using previous data to generate an extension of the Condorcet-Borda matrix.

Our matrix will be a square matrix, with m rows and colums, where m is the number of alternatives. Off-diagonal elements elements in the location ij, with $i \neq j$ are simply n_{ij}. Elements in the diagonal, i.e., elements in the location ii in the matrix are, for any alternative i, the aggregate number of individuals that do consider alternative i to be not strictly worse than any of the others, namely, $\sum_{j \neq i}(n_{ij} + e_{ij} + u_{ij})$. This element can be interpreted as an extension of the Borda score of each alternative. It measures the aggregate (weakly) support of alternative i. We call the matrix constructed this way the *Extended Condorcet-Borda matrix associated to problem p, ECB(p).*

Matrix $ECB(p)$ has a number of interesting properties:

- It is a non-negative matrix
- All its colums add up to $n(m-1)$
- $ECB(p)$ has a positive eigenvalue, precisely $n(m-1)$, and all other eigenvalues are (in modulus) smaller than this one
- Associated to the dominant eigenvalue there is a positive eigenvector $v(p) > 0$, unique up to normalization
- Furthermore, the components of this eigenvector fulfill the following relationships, for all $i \in A$
$$v_i = \frac{\sum_{j \neq i} n_{ij} v_j}{\sum_{k \neq i} n_{ki}}$$
- Due to previous property, it is natural to interpret the components of vector v as a measure of the *relative support* of the alternatives in problem p. Actually, we did not lose any information, and the format of this matrix is very much the same as before.

Consequently, we may think of $v(p)$ as a way of generating an endogeneous ordering of the alternatives in A, made out of a certain combination of the number of times the different alternatives weakly or strongly dominates each other. This way, and exploiting a Condorcet-type information, we end up with a sort of "scoring" of the alternatives that allow us to completely order them.

The probabilistic interpretation in this case is as before, with one proviso: an alternative is changed by another only when the second one is strictly preferred to the first one.

4 A Characterization

A problem is given by a set of alternatives, A,a set of agents, M, and a preferences profile, p.Let $P = (A, M, p)$ be a problem involving m agents, a alternatives, and a profile p that indicates the preferences of the agents over the alternatives. $P(a)$ stand for the set of problems (A, M, p) with a alternatives and any given (finite) number of agents. An evaluation function over the domain of all problems (A, M, p) is denoted by Ω, whereas Ω^a describes the restriction of Ω over $P(a)$. More formally:

Definition 1 *An* ***evaluation function*** Ω *is a mapping defined over all admissible problems* $P = (A, M, p)$, *such that, that for any* $a \in \mathbb{N}$, $\Omega^a : M(a) \to \mathbb{R}_+^a$ *is a vector such that* $\sum_i \Omega_i^a(M) = 1$.

The interpretation of $\Omega_i^a(M)$ is simply the value we attach to alternative i in problem P.

We present now two properties that characterize the evaluation function described in the former section. Then we shall comment on some other relevant properties that this evaluation function satisfies but are not required for the characterization result.

The first property refers to symmetric problems, namely those in which for any pair of alternatives, i, j, it happens that $n_{ij} = n_{ji}$. In such a case, all alternatives are evaluated equally.

- **Symmetry:** *For any symmetric problem,* $P = (A, M, p)$, $\Omega_i(P) = \Omega_j(P)$, *for all* $i, j \in A$.

A stronger assumption refers to *balanced problems,* those in which, for every alternative j, the aggregate number of dominations of j over any other alternative equals the total number of dominations of any other alternative over j, i.e, $\sum_{i \neq j} n_{ij} = \sum_{i \neq j} n_{ji}$. Indeed any symmetric problem is balanced, but not the other way around.

- **Balancedness:** *For any balanced problem* $P = (A, M, p)$, $\Omega_i(P) = \Omega_j(P)$, *for all* $i, j \in A$.

To motivate the following property, take, as a strating point a problem P, and suppose there is another problem P', such that all the Condorcet numbers remain unchanged, except for those of the form n'_{ij} for some j, such that $n'_{ij} = \lambda n_{ij}$, for some $\lambda > 0$. For example, with $A = \{1, 2, 3\}$, take the profile

num. agents	1	1	2
	1	1	2, 3
	2	3	1
	3	2	

In this case, $n_{12} = n_{21} = 2; n_{13} = n_{31} = 2$; $n_{23} = n_{32} = 1$. Thus, by symmetry, all three alternatives have the same valuation. Imagine now that there are more agents with the last profile:

num. agents	1	1	5
	1	1	2, 3
	2	3	1
	3	2	

Now, $n'_{12} = n_{12} = 2; n'_{13} = n_{13} = 2; n'_{23} = n_{23} = 1; n'_{32} = n_{32} = 1$, as before. But now, $n'_{21} = 5 = \frac{5}{2}n_{21}$; $n'_{31} = 5 = \frac{5}{2}n_{31}$; How should the valuation of the alternatives change? In this new problem, $v'_2 = v'_3 = 1$, but $v'_1 = (2/5) = (\frac{5}{2})^{-1}$. The idea is that now alternative 1 becomes less important, due to the further domination of this alternative by the other two. This example motivates the following property:

- **Inverse proportionality:** *Let* $P = (A, M, p)$ a problem, $j \in A, \lambda \in \mathbb{N}$, and $P' = (A, M', p')$ another problem such that $M' = M \cup Q$, where Q is the following profile:

num. agents	λn_{1j}	$\cdots$	λn_{aj}
	1	$\cdots$	a
	j	$\cdots$	j

  where n_{kj} are the number of agents that prefer k to j in problem P. Then, $\Omega_i(P') = \Omega_i(P)$ for all $i \neq j$, and $\Omega_j(P') = \frac{1}{\lambda}\Omega_j(P)$.

Inverse proportionality refers to two problems constructed so that their Condorcet numbers are identical except for those referring to the number of agents that prefer any other alternative to j. The agents we add to p exactly serve the purpose of multiplying n_{ij} by λ. As a consequence, only the relative valuation of j changes in sucha a way that it diminishes proportionally to λ.

Then, we have the following result:

Theorem 1: *An evaluation function Ω satisfies balancedness and inverse proportionality iff for any problem P,it happens that $\Omega(P)$ is the dominant eigenvector associated to $ECB(P)$.*

Proof: The dominant eigenvector of $ECB(P)$ satisfies balancedness and inverse proportionality.

Let now be an evaluation function that satisfies balancedness and inverse proportionality. Let us see that, for any problem, it coincides with the dominant eigenvector of $ECB(P)$.

For problem P, let v be the dominant eigenvector associated to $ECB(P)$. Then, for all $i \in A$,

$$v_i \sum_{j \neq i} n_{ji} = \sum_{j \neq i} n_{ij} v_j$$

Notice that, as all components of $ECB(P)$ are integer numbers, then all components of v arerational numbers. Then, we can find v with all its components integer numbers. Now, we can construct a new problem, P_1, so that all Condorcet numbers of P_1 coincide with those of P, except for $n^1_{i1} = n_{i1}v_1$. Then, by inverse proportionality, we know that $\Omega_i(P_1) = \Omega_i(P)$, for all $i \neq 1$, and $\Omega_1(P_1) = \frac{1}{v_1}\Omega_1(P)$. From P_1, we can now construct a new problem, P_2, so that all Condorcet numbers of P_2 coincide with those of P_1, except for $n^2_{i2} = n_{i2}v_2$.Then, by inverse proportionality, we know that $\Omega_i(P_2) = \Omega_i(P_1)$, for all $i \neq 2$, and $\Omega_2(P_2) = \frac{1}{v2}\Omega_2(P_1) = \frac{1}{v2}\Omega_2(P)$. By repeating this procedure as many times as alternatives, we get a problem P' such that $\Omega_i(P') = \frac{1}{v_i}\Omega_i(P)$. But, by construction, P' is a balanced problem, and by balancedness, $\Omega_i(P') = \Omega_j(P')$, for all i, j, and then, $\Omega_i(P) = v_i$ for all $i \in A$.□

The next property, *reciprocity*, introduces the idea that the evaluation of a pair of alternatives should be related to their corresponding Condorcet domination relations . More precisely, it establishes that, in those problems involving only two alternatives, the relative valuation of the alternatives should coincide with the ratio between the number of individuals that strictly prefer one to the other one. Formally:

- **Reciprocity:** *For any problem* $P = (\{i, j\}, M, p) \in M(2)$, *we have:*

$$\frac{\Omega_i^2(P)}{\Omega_j^2(P)} = \frac{n_{ij}}{n_{ji}}$$

In the next property we relate problems with m alternatives with problems of $(m-1)$ alternatives, obtained by deleting an alternative, and in a way that we keep the information on indirect dominations.

Suppose that we delete alternative k, and in the new problem, P', it happens that $n'_{ij} = n_{ij}\left(\sum_{s\neq k} n_{sk}\right) + n_{ik}n_{kj}$, then we have that the components of the eigenvector associated to the $ECB(P')$, v', and the initial vector v,it happens that $v'_i/v'_j = v_i/v_j$, for all $i, j \neq k$. That is, in this case, we keep the information after deleting alternative k. This motivates the following property:

- **Consistency:** *Let* $P = (A, M, p)$ *a problem, and* $P' = (A', M', p')$ *where* $A' = A - \{k\}$, *and* $M' = M \cup Q$, *where* Q *is the following profile*:

num. agents	$n_{1k}n_{kj}$	$\cdots$	$n_{ak}n_{kj}$
	1		a
	j		j

Then, for all $i, j \in A'$, $\Omega_i(P')/\Omega_j(P') = \Omega_i(P)/\Omega_j(P)$.

(Notice that the Condorcet numbers of P', $n'_{ij} = n_{ij}\left(\sum_{s\neq k} n_{sk}\right) + n_{ik}n_{kj}$, where n_{ij} are the Condorcet numbers of P).

Now, we have the following result:

Theorem 2: *An evaluation function* Ω *satisfies reciprocity and consistency iff for any problem* P,*it happens that* $\Omega(P)$ *is the dominant eigenvector associated to* $ECB(P)$.

Proof: The dominant eigenvector of $ECB(P)$ satisfies reciprocity and consistency.

Let now be an evaluation function that satisfies reciprocity and consistency. Let us see that, for any problem, it coincides with the dominant eigenvector of $ECB(P)$.

Let $P = (A, M, p)$. If $card(A) = 2$, by reciprocity, $\Omega(P)$ coincides with the dominant eigenvector of $ECB(P)$. Suppose now that the equality holds for problems up to $(m-1)$ alternatives, and let $P = (A, M, p)$ a problem with m alternatives.

Let $k \in A$, and take the problem $P^k = (A^k, M', p')$, where $A^k = A - \{k\}$, $M' = \left(\sum_{s \neq k} n_{sk}\right) M \cup Q$, where Q is the following profile

num. agents	$n_{1k}n_{kj}$	$\cdots$	$n_{ak}n_{kj}$
	1		a
	j		j

Now, by consistency, for all $i, j \neq k$,

$$\frac{\Omega_i(P^k)}{\Omega_j(P^k)} = \frac{\Omega_i(P)}{\Omega_j(P)} = \frac{v_i(P)}{v_j(P)} = \frac{v_i(P^k)}{v_j(P^k)}$$

and, as this relationship holds for any $k \in A$, then $\Omega(P) = v$.□

5 Some illustrations

5.1 The Solar Decathlon Europe competition 2012

Consider now an illustration of our method, to the Solar Decathlon Europe competition 2012. The Solar Decathlon Europe is an architectonic competition of European universities aimed at promoting efficient houses, specially with respect to their environmental impact: consuming less energy, producing less wastes. It is called Decathlon because the competitor projects are evaluated from 10 diferent aspects: (1) *architecture; (2) engineering and building; (3) energy efficiency; (4) electrical energy balance; (5) comfort conditions; (6) house functioning; (7) communication and social awareness; (8) industrialization and market viability; (9) innovation, and (10) sustainability.* Each of the teams are valued for the 10 aspects, independently. In some cases, by using some specific monitoring techniques, in some others a jury clasifies and gives puntuation to the competitors. Let us call $T = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O\}$ the 15 teams competing In the 2012 contest The order the different teams got out of the 10 aspects were as follows:

Architecture	**En @ B**	**En Eff**	**EL Bal**	**Comfort**	**Fun**	**Comm**	**Viab**	**Inn**	**Sust**
A	*D*	*L*	*L*	*A*	*A*	*L*	*D*	*A*	*H*
B	*I*	*M*	*N*	*I*	*H*	*A*	*A*	*L*	*L*
H	*L*	*D,I*	*H*	*K*	*K*	*B,H*	*B*	*H*	*B,F*
C,D,L,N	*H*	*A,B,H*	*M*	*C*	*D*	*F*	*C*	*B*	*A,D,I*
E,I,K,O	*J*	*N*	*A,C,O*	*F*	*B*	*M,N*	*G,L*	*D*	*N*
F,G	*A*	*G*	*G*	*M*	*J*	*J,K*	*H*	*N*	*E,O*
J,M	*M*	*F,J*	*K*	*G*	*E*	*D*	*E*	*I*	*G,M*
	C	*K*	*E*	*J*	*L*	*I*	*M,O*	*J*	*C,K*
	N	*E,O*	*D*	*H*	*M*	*G*	*I*	*C*	*J*
	G	*C*	*B*	*D*	*G*	*O*	*F*	*G*	
	O		*I*	*O*	*I*	*C,E*	*N*	*K*	
	B		*F*	*L*	*C*		*K*	*E*	
	E		*J*	*E*	*N*		*J*	*O*	
	F			*N*	*O*			*F*	
	K			*B*	*F*			*M*	

As we may see, in several aspects different teams were considered as equally good. Out of these orders, we obtain the following matrix:

$$CB(P) = \begin{pmatrix} 112 & 8 & 9 & 6 & 10 & 9 & 10 & 6 & 7 & 10 & 10 & 5 & 8 & 9 & 9 \\ 1 & 87 & 7 & 4 & 8 & 8 & 7 & 2 & 7 & 9 & 7 & 3 & 6 & 7 & 7 \\ 0 & 3 & 63 & 2 & 6 & 8 & 6 & 2 & 3 & 5 & 5 & 2 & 4 & 4 & 5 \\ 3 & 6 & 7 & 96 & 9 & 7 & 8 & 3 & 7 & 8 & 6 & 4 & 6 & 7 & 9 \\ 0 & 2 & 3 & 1 & 43 & 6 & 3 & 0 & 3 & 4 & 3 & 1 & 5 & 3 & 3 \\ 1 & 1 & 2 & 3 & 4 & 45 & 3 & 1 & 2 & 6 & 5 & 1 & 5 & 4 & 4 \\ 0 & 3 & 4 & 2 & 7 & 6 & 55 & 2 & 3 & 6 & 6 & 1 & 3 & 3 & 7 \\ 3 & 6 & 8 & 7 & 10 & 9 & 9 & 110 & 7 & 9 & 9 & 4 & 8 & 9 & 10 \\ 2 & 3 & 7 & 1 & 6 & 8 & 7 & 3 & 77 & 8 & 6 & 2 & 5 & 6 & 7 \\ 0 & 1 & 5 & 2 & 6 & 3 & 4 & 1 & 2 & 44 & 3 & 2 & 3 & 3 & 6 \\ 10 & 3 & 4 & 4 & 6 & 5 & 4 & 1 & 3 & 6 & 54 & 2 & 4 & 2 & 5 \\ 5 & 7 & 7 & 5 & 9 & 9 & 8 & 6 & 8 & 8 & 8 & 111 & 9 & 9 & 9 \\ 2 & 4 & 6 & 4 & 5 & 5 & 6 & 2 & 5 & 6 & 6 & 1 & 67 & 5 & 6 \\ 1 & 3 & 5 & 2 & 7 & 6 & 7 & 1 & 4 & 7 & 8 & 0 & 4 & 67 & 8 \\ 0 & 3 & 3 & 1 & 4 & 6 & 32 & 0 & 2 & 4 & 4 & 1 & 3 & 2 & 45 \end{pmatrix}$$

Then, our valuation vector is $\begin{pmatrix} 5,66812588 \\ 1,88976493 \\ 0,85472137 \\ 2,75377542 \\ 0,41011516 \\ 0,57512641 \\ 0,70227095 \\ 4,62373343 \\ 1,39885445 \\ 0,51178203 \\ 1,34510662 \\ 5,22681991 \\ 1,1462859 \\ 0,88733307 \\ 0,40895656 \end{pmatrix}$

and,consequently, the final order of the teams is as follows: *A,L,H,D,B,I,K,M,N,C,G,F,J,E,O*

The order in the real competition was: *A,L,H,D,B,I,N,C,M,G,K,E,O,F,J.* We see that even though in the first 5 positions there are no changes, the order differs from then on.

5.2 The distribution of support among political parties

Suppose that there is a country with four parties, a Parliament of 135 seats, and such that the preferences of the citizens are distributed as follows:

parties/percentages	30%	40%	30%
	A	D, C	B, A
	B, C	B	D, C
	D	A	

If citizens only can vote for their best preferred party, the result of the elections will be: $A: 45\%$, $B: 15\%$, $C: 20\%$; $D: 20\%$,

Under approval voting, in which citizens can vote for more than one acceptable candidate, the result would be: $A : 45\%$, $B : 23\%, C : .31\%$; $D : 31\%$. Finally, if we consider preferences in full, we may construct our CB-Matrix, as follows:

$$\begin{pmatrix} 18 & 3 & 6 & 6 \\ 4 & 19 & 3 & 6 \\ 4 & 4 & 21 & 3 \\ 4 & 4 & 0 & 15 \end{pmatrix}$$

The eigenvector associated to the dominant eigenvalue (30) turns out to be the following: $\begin{pmatrix} \frac{39}{20} \\ \frac{9}{5} \\ 2 \\ 1 \end{pmatrix}$

Then the distribution of seats among the parties under the different voting systems would be

Seats	**Plurality**	**Approval**	**Support**
A	60	47	39
B	21	24	36
C	27	32	40
D	27	32	20

6 References

Chevotarev, P.Y., E. Shamis (1998), Characterizations of scoring methods for preference aggregation, *Annals of Operations Research* 80: 299-332

Freeman, L.C. (1977), A set of measures of centrality based on betweenness, *Sociometry,* 40: 35-41

Keener, J.P. (1993), The Perron-Frobenius Theorem and the ranking of football teams, *SIAM Review* 35(1): 80-93

Laslier, J. (1997), *Tournament solutions and Majority voting.* Springer, Berlin, Heidelberg, New York.

Liebowitz, S., J. Palmer (1984), Assesing the relative impact of Economic Journals, *Journal of Economic Literature* 22: 77-88

Morales, J.I. (1797), *Memoria Matemática sobre el cálculo de la opinión en las elecciones,* Imprenta Real, Madrid.

Morales, J.I. (1805), *Apéndice a la Memoria Matemática sobre el cálculo de la opinión en las elecciones,* Imprenta de Sancha, Madrid.

Moon, J.W., N.J. Pullman, (1970), On Generalized Tournament Matrices, *SIAM Review,* 12(3): 384-399

Newmann, MEJ, (2003), The structure and function of complex networks, *SIAM Review*, 45: 167-256

Palacios-Huerta, I., O. Volij, The Measurement of Intellectual Influence, *Econometrica* 72(3): 963-977

Pinsky, G., F. Narin (1976), Citation Influence for Journal Aggregates of Scientific Publications: Theory, with applications to Physics, *Information Processing and Management* 12: 297-312

Slutzki, G, O. Volij, (2006), Scoring of web pages and tournaments: axiomatizations, *Social Choice and Welfare* 26 (1): 75-92

Wasserman, S., K. Faust (1994), *Social Network Analysis: Methods and Applications.* Cambridge, Cambridge University Press

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