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Some game-theoretic grounds for meeting people half-way [*]

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Abstract

It is well known that, in distributions problems, *"Fairness"* rarely leads to a single viewpoint (see Young (1994) and Moulin (1988) among many others). This paper provides, in this context, interesting basis in defense of intermediate agreements when two prominent proposals, representing different sets of *"Equity Principles"*, highlight a discrepancy in sharing resources. Specifically, we formalize such a conflicting situation by associating it with a *"natural"* cooperative game, called the *Bifocal Distribution game*, to show that both the *Nucleolus*, introduced by Schmeidler (1969), and the *Shapley value*, proposed by Shapley (1953), agree on recommending the *"average of the two focal solutions"*. Finally, applying our analysis to bankruptcy problems, which have been analyzed extensively by Thomson (2003) and Moulin (2002), provides new *"reasonable"* solutions.

Keywords: distribution problems, bankruptcy, cooperative games, nucleolus, Shapley value, Lorenz criterion.
JEL Classification: C71, D63, D71.

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1. Introduction

Several months ago, during an International Meeting on Game Theory was held, we had a very interesting conversation, with an expert on this subject, about the cost allocation that a major company undertook after receiving a detailed report carried out by an economics research group. As a conclusion, the report provided two possible cost distributions, corresponding to the Nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953)) of the TU-game associated with the real distribution problem. Furthermore, with the aim of making the choice between the two proposals easier, the properties supporting each one were specified. Surprisingly enough, the company's final decision was to distribute the cost according to the average of them.

This paper attempts to find an explanation to the previous conduct, which in our opinion is not merely anecdotal, it is in fact quite the opposite. We think that this action reflects a way of thinking applied in so many and so different situations that it could represent the popular proverb '*Virtue lies in the middle ground*'.

We were particularly interested in the normative approach to sharing problems, which rarely leads to a single proposal. In fact,a trade-off can usually be found between properties, interpreted as different *'equity principles'*, which are fulfilled by the various solution concepts.

This idea was superbly expressed by Young (1994): "*Fairness* does not boil down to a single formula, but represents a balance between competing principle of need, desert and social utility".

In this context, we have concentrated on transferable utility distribution problems with two different proposals that highlight discrepancy, i.e., problems that involve sharing a given amount of a perfectly divisible 'good' among a group of agents with two focal viewpoints.

Firstly, we introduce the *bifocal distribution problems* by adding, to a generic distribution problem, two solution concepts interpreted as prominent proposals for solving them. We then model these kinds of problems as transferable utility cooperative games (referred to hereinafter as TU-games) as follows. We associate with each coalition the smallest quantity of the 'good' that such a coalition would receive according to the two proposed allocations.

The analysis of these games, known as *bifocal distribution games*, provides 'solid' theoretical grounds in defense of intermediate compromises.

Specifically, we provide a necessary condition for sharing to be in the Core of these games: a quantity belonging to the interval defined by the extremes corresponding to the focal proposals must be recommended for each agent. Furthermore, although these games are not convex in general, we find that not only is the Shapley value a Core selection, it also coincides with the Nucleolus. We also show that the recommendations made by these two solution concepts is the '*average of the two focal distributions*'.

Unfortunately, our main results can not be generalized for distribution problems with more than two focal proposals, but our modeling is of great interest when applied to some certain problem types in which bipolarity may usually take place.

In this regard, the previous general results are then applied to bankruptcy problems: a particular kind of distribution problems in which individuals have different incompatible rights, summarized in a claims vector, so the available amount of the 'good' should be rationed. In bankruptcy problems two significant viewpoints naturally arise as any distribution can be observed by focusing either on gains or on losses. This fact together with the idea that the general desirable social goal is to treat everybody as evenly as possible, captured by the Lorenz criterion, (Lorenz (1905)), provide a new basis for the '*average of the Constrained Equal Awards and the Constrained Equal Losses bankruptcy solutions*', two proposals put forward by Maimonides in the twelfth century.

Finally, we establish the connections between *bifocal distributions games* and other well-known types of games.

The paper is organized as follows. Section 2 formally introduces *bifocal distribution problems*. Section 3 provides the interpretation of problems such as TU-games and presents their main characteristics. Section 4 applies previous ideas to bankruptcy problems. Section 5 summarizes our conclusions. The Appendices contains our technical proofs.

2. Distribution problems with transferable utility

In order to analyze distribution problems in which two prominent proposals are considered equally fair from different reasonable viewpoints, this section introduces the concepts of bifocal distribution problems and bifocal distribution rules.

A **distribution problem with transferable utility**, D, (referred to hereinafter as a **distribution problem)** is formally described by a pair $D = (M, C^N)$ where $M \in \mathbb{R}_{++}$ represents a given amount of a perfectly divisible 'good', a valuable resource that should be distributed among the agents in $N = \{1, \ldots, i, \ldots, n\}$. And C^N is a set of relevant information, concerning the agents, which should somehow be taken into account when solving the problem. Let $\mathcal{D}$ denote the type of all distribution problems. In this general context a **distribution rule** is a function, $f : \mathcal{D} \to \mathbb{R}^n$, which proposes an efficient allocation of the resource for each distribution problem $D \in \mathcal{D}$, that is, $\sum\limits_{i \in N} f_i(D) = M$. Let $\mathcal{F}$ be a family of rules.

A wide range of real situations can be modeled in this way, such as bankruptcy problems (Thomson (2003)) and certain important types of TU-games: market games (Shapley and Shubik (1969)), cost allocation games (Young (1985)) and simple games (Shapley (1962)). In a bankruptcy problem, M is the estate, usually denoted by E, and C^N is a vector $c \in \mathbb{R}^n_{++}$ representing the agents' rights that are incompatible, that is, $E < \sum_{i\in N} c_i$ (see Section 4). In a cooperative TU-game M is the worth of the grand coalition and C^N is the characteristic function (see Section 3).

We then introduce the type of all bifocal distribution problems, denoted by $\mathcal{B}$, where discrepancy for sharing the valuable resource is considered by means of the existence of two fixed focal proposals.

Definition 2.1. *A* ***bifocal distribution problem****, $B \in \mathcal{B}$, is a triplet $B = (D, f, g)$ where $D = \left(M, C^N\right) \in \mathcal{D}$ and both f and g are 'fixed' distribution rules representing two prominent proposals in a particular society.*

This modelling becomes even more interesting when applied to specific subclasses of distribution problems. For example, in bankruptcy problems societies could only consider admissible bankruptcy rules that meet the following requirements: Non-Negativity (for each $i \in N$, $0 \leq f_i(E, c)$) and Claims-Boundedness (for each $i \in N$, $f_i(E, c) \leq c_i$). These properties are in fact part of the definition of the bankruptcy rule in the literature. In taxation problems, which are formally identical to bankruptcy problems but assess a fixed amount of taxes, E, as a function of the agents' income, c, societies could find it *'fair'* to add the property of Progressivity in Losses (for each $i, j \in N$, if $0 < c_j \leq c_i$, $\left[c_i - f_i(E, c)\right]/c_i \geq \left[c_j - f_j(E, c)\right]/c_j$) to the previous restrictions. Therefore, as far as narrowing the data interpretation of distribution problems, consensus on the social acceptability of '*equity principles'* would be greater, which would facilitate compromises around certain proposals.

We now demand the bifocal distribution solutions to provide allocations which, for each agent, belong to the interval determined by the extremes corresponding to the two focal proposals. We intuitively think that this is a very logical requirement as it is a fact usually observed when solving any real bifocal distribution problem. In this respect, Yeh and Thomson (2006), among others (see also Yeh and Thomson (2008)) introduced the convexity operator between rules for bankruptcy problems as follows: "When two rules express opposite viewpoints on how to solve a bankruptcy problem, it is natural to compromise between them by averaging".

Our subsequent theoretical analysis provides a different rationale (apart from the fact that it corresponds with reality) to support the previous idea. Formally,

Definition 2.2. *A* ***bifocal distribution rule*** *is a function $\varphi : \mathcal{B} \to \mathbb{R}^n$, such that for each bifocal distribution problem $B = (D, f, g) \in \mathcal{B}$, φ is a distribution rule, $\sum_{i\in N} \varphi_i(D) = M$, which associates to each $i \in N$ a part of the resource satisfying*

$$\min\{f_i(D), g_i(D)\} \leq \varphi_i\left(D\right) \leq \max\{f_i(D), g_i(D)\}.$$

3. Bifocal distribution TU-games: characteristics and results

In this sections we analyze a new type of games in coalitional form, named bifocal distribution games, which are an appropriate interpretation of bifocal distribution problems in terms of TU-games. We then present the basic concept of the model.

A TU-game involving a set of agents N can be described as a function V, known as the characteristic function, which associates a real number to each subset of agents, or coalitions, S contained in N. Formally, a **TU-game** is a pair (N, V), where $V : 2^N \to \mathbb{R}$. For each coalition $S \subseteq N$, $V(S)$ is commonly called its worth and denotes the quantity that agents in S can guarantee for themselves if they cooperate. Therefore, it is assumed that $V(\varnothing) = 0$. It is also often supposed that (N, V) is **superadditive**, i.e., for any pair of coalitions $S, T \subset N$ such that $S \cap T = \varnothing$, $V(S \cup T) \geq V(S) + V(T)$, so that there is incentive for the grand coalition that N forms. Because we do not consider changes in the agents' population, we summarize a TU-game by V.

A **solution for TU-games** is a correspondence which for each TU-game selects a set of shares of the worth of the grand coalition among the agents. When a TU-game solution is single-valued, i.e., it consists of a unique sharing-out of the worth of the grand coalition, then it is called a TU-value.

Let $\mathcal{G}$ be a family of TU-games with agents set N. A **TU-value** is a function $\gamma : \mathcal{G} \to \mathbb{R}^n$, such that for each TU-game $V \in \mathcal{G}$, $\sum\limits_{i \in N} \gamma_i(V) = V(N)$.

The way we define the game corresponding to a bifocal distribution problem is by associating to the smallest quantity of the 'good' that each coalition would receive according to the two focal solution concepts.

Definition 3.1. *Given $B = (D, f, g) \in \mathcal{B}$, the corresponding **bifocal distribution game** is the TU-game V^B which associates to each coalition $S \subseteq N$, the real value*

$$V^B(S) = \min \left\{ \sum_{i \in S} f_i(D), \sum_{i \in S} g_i(D) \right\}.$$

In the bifocal distribution games type we go on to analyze three well-known solution concepts for TU-games: the Core, which is defined formally by Gillies (1953) and Shapley (1953), the Nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953)). All of these solutions have been extensively studied in the game-theoretic literature, where formal definitions and several axiomatic bases can be found (see, for instance, Kannai (1992), Peleg (1992), Maschler (1992) and Winter (2002)).

The intuitive idea of a Core distribution is that no set of agents could collectively improve their share by their own cooperation. Formally, let V be a TU-game. The

Core of such a game, denoted by $\mathbb{C}(V)$, is the set

$$\mathbb{C}(V) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = V(N), \sum_{i \in S} x_i \geq V(S)\ \forall S \subset N \right\}.$$

Let us note that a bifocal distribution game, V^B with $B = (D, f, g) \in \mathcal{B}$, has a non-empty Core, since both $f(D)$ and $g(D)$ belong to it. The next proposition provides a set of necessary conditions for a proposal to be in the Core of a such game. These conditions coincide with the 'natural' requirements for a distribution rule to be considered a bifocal distribution rule (see Definition 2.2). This result, therefore, shows that the agents' behavior regarding bifocal distribution problems has, albeit unconsciously, strong theoretical support.

Proposition 3.2. *Given $B = (D, f, g) \in \mathcal{B}$, if $x \in \mathbb{C}(V^B)$ then, for all $i \in N$,*

$$\min\{f_i(D), g_i(D)\} \leq x_i \leq \max\{f_i(D), g_i(D)\}.$$

Proof. See Appendix 1 ∎

The following example shows that previous conditions do not always guarantee that an allocation belongs to the Core. it is not, therefore, a sufficient requirement.

Example 3.3. *Let us consider the bifocal distribution problem $B = (D, f, g)$, in which $M = 170$, $f(D) = (35, 45, 45, 45)$ and $g(D) = (28.75, 38.75, 48.75, 53.75)$. Then, on the one hand $V^B(\{1, 3\}) = \min\{35 + 45; 28.75 + 48.75\} = 77.5$. On the other hand, let us consider the allocation x that distributes the resources, $M = 170$, among agents as follows, $x = (30, 40.25, 46, 53.75)$. This distribution is, for each agent, between those of the two focal proposals. However,*

$$x_1 + x_3 = 76 < V^B(\{1, 3\}) = 77.5,$$

therefore x does not belongs to $\mathbb{C}\left(V^B\right)$.

Next, we informally introduce the Nucleolus and the Shapley value.

The **excess of a coalition S in a reference to an allocation** x, $\sum_{i \in S} x_i - V(S)$, is taken as a measure of satisfaction of such a coalition. The **Nucleolus**, γ^{Nu}, looks for a distribution in which the lowest excess is as high as possible. If there is more than one such distribution among them, it pays attention to the second-lowest excess and will look for a distribution in this set in which the second-lowest excess is as high as possible, and so on.

Shapley (1953) gives the following interpretation of the **Shapley value**, γ^{Sh}: "The players in N agree to play the game V in a grand coalition, formed in the following way: (i) Starting with a single member, the coalition adds one player at a time until everybody has been admitted. (ii) The order in which the players are to join is determined by chance, with all arrangements equally probable. (iii) Each player, on his admission, demands and is promised the amount which his adherence contributes to the value of the coalition (as determined by the function V). The grand coalition then plays the game 'efficiently' so as to obtain $V(N)$, exactly enough to meet all promises."

It is well known that the Nucleolus is a Core selection but the Shapley value could recommend, in general, a distribution outside the Core, although for convex games, defined below, it is also a Core selection.

Before presenting the subclass of convex TU-games additional notation is needed.

Hereafter, given a TU-game V, for each agent $i \in N$ and each coalition $S \subset N$, we call the **marginal contribution of agent i to coalition** S, denoted by $\Delta_i V(S)$, the amount which its adherence contributes to the value of the coalition, which is

$$\Delta_i V(S) = V(SU\{i\}) - V(S).$$

A TU-game is convex if agents have increasing returns to cooperation, which means that the larger the coalition that an agent joins, the larger his marginal contribution. Formally, a TU-game V is **convex** if and only if, for all $i \in N$,

$$\Delta_i V(S) \leq \Delta_i V(T) \text{ for all } S \subseteq T \subseteq N \setminus \{i\}.$$

The following example shows that bifocal distribution games need not be convex.

Example 3.4. *Let us consider the bifocal distribution problem $B = (D, f, g)$, in which $M = 30$, $f(D) = (7.5, 7.5, 7.5, 7.5)$ and $g(D) = (5, 7, 8, 10)$. It can be easily verified that $V^B(\{1\}) = 5$, $V^B(\{1,3\}) = 13$, $V^B(\{1,4\}) = 15$ and $V^B(\{1,3,4\}) = 22.5$. Thus, $\Delta_3 V^B(\{1,4\}) < \Delta_3 V^B(\{1\})$ since*

$$V^B(\{1,3,4\}) - V^B(\{1,4\}) = 7.5 \text{ and } V^B(\{1,3\}) - V^B(\{1\}) = 8.$$

Therefore, given that $\{1\} \subset \{1,4\}$, V^B is not convex.

In principle, this example would lead us to focus on the Nucleolus, given the possibility that the Shapley value does not belong to the Core. However, as shown in our main result, not only can it be guaranteed that the Shapley value is a Core selection but also that it coincides with the Nucleolus for any bifocal distribution game. The PS-game concept is used to justify this coincidence.

A TU-game is a PS-game (Kar et al. (2009)) in which the sum of a player's marginal contribution to any pair of coalitions T, T^* such that $T \cup T^* = N \setminus \{i\}$ is a player specific constant. Formally, a TU-game V is a **PS-game** if for each $i \in N$, there exists $k_i \in \mathbb{R}$ such that, for all $T \subseteq N \diagdown \{i\}$,

$$\Delta_i V(T) + \Delta_i V(N \diagdown [T \cup \{i\}]) = k_i.$$

The following result not only shows that bifocal distribution games are a subclass of PS-games, but also that the specific constant, for each agent, obtained by adding its marginal contribution to any coalition S and its complement $N \diagdown [S \cup \{i\}]$, is the sum of the recommendations for him made by the two focal proposals.

Proposition 3.5. *Given $B = (D, f, g) \in \mathcal{B}$, the associated bifocal distribution game, V^B, is a PS-game such that for all $i \in N$ and for all coalition $T \subseteq N \diagdown \{i\}$,*

$$\Delta_i V^B(T) + \Delta_i V^B(N \diagdown [T \cup \{i\}]) = f_i(D) + g_i(D).$$

Proof. See Appendix 2 ■

Our main result, the proof of which is based on the previous proposition, states with sufficient clarity how to solve distribution bifocal games. In a way, it provides solid grounds for selecting the average of the two focal viewpoints from among all the intermediate compromises. It also ratifies the practice regularly observed in these situations, as shown by the behavior of the company mentioned in the Introduction and which motivated our research. However, we do not really believe that the company had these concepts in mind when making its decision.

Theorem 3.6. *For each bifocal distribution game, V^B, where $B = (D, f, g) \in \mathcal{B}$, the Shapley value and the Nucleolus coincide and they are obtained in the average of the two focal distributions rules, that is,*

$$\gamma^{Sh}(V^B) = \gamma^{Nu}(V^B) = 1/2\left(f(D) + g(D)\right).$$

Proof. See Appendix 3 ■

An alternative wording, though less formal, of this result could be the popular proverb '*Virtue lies in the middle ground*'.

Finally, we show through the next example that the previous result can not be extended for distribution problems with more than two focal proposals.

Example 3.7. *Let us consider the distribution problem D, in which $M = 60$ and there are three focal distribution rules, f, g and h, providing $f(D) = (10, 25, 25)$,*

$g(D) = (0, 27.5, 32.5)$ and $h(D) = (10, 22.5, 27.5)$. *If the associated TU-game is defined by*

$$V^{(D,f,g,h)}(S) = \min\left\{ \sum_{i\in S} f_i(D), \sum_{i\in S} g_i(D), \sum_{i\in S} h_i(D) \right\},$$

it is easy to verify that $V^{(D,f,g,h)}(\{1\}) = 0$, $V^{(D,f,g,h)}(\{2\}) = 22.5$, $V^{(D,f,g,h)}(\{3\}) = 25$, $V^{(D,f,g,h)}(\{1,2\}) = 27.5$, $V^{(D,f,g,h)}(\{1,3\}) = 32.5$, $V^{(D,f,g,h)}(\{2,3\}) = 50$ *and* $V^{(D,f,g,h)}(\{1,2,3\}) = 60$. *Then* $\gamma^{Sh}(V^{(D,f,g,h)}) = (5.42, 25.42, 29.16)$, $\gamma^{Nu}(V^{(D,f,g,h)}) = (6.25, 25, 28.75)$ *and* $(f+g+h)/3 = (6 + (2/3), 25, 28 + (1/3))$. *Therefore, the Nucleolus and the Shapley value do not coincide, and neither corresponds to the average of the three focal allocations. It should be noted that in this example the three focal distribution rules are involved in specifying the coalitions worth and this fact causes the loss of the PS-game quality. Furthermore, even* h *only determines the worth of player* 2, *none of the players* $i, i \in \{1,2,3\}$, *has a constant sum of his marginal contribution to all pairs of coalitions whose union is* $N\backslash\{i\}$*:*

$$\begin{aligned}
\Delta_1 V(\{2\}) + \Delta_1 V(\{3\}) &= 12.5 \neq 10 = \Delta_1 V(\{\varnothing\}) + \Delta_1 V(\{2,3\}), \\
\Delta_2 V(\{1\}) + \Delta_2 V(\{3\}) &= 52.5 \neq 50 = \Delta_2 V(\{\varnothing\}) + \Delta_2 V(\{1,3\}) \text{ and} \\
\Delta_3 V(\{1\}) + \Delta_3 V(\{2\}) &= 60 \neq 57.5 = \Delta_3 V(\{\varnothing\}) + \Delta_3 V(\{1,2\}).
\end{aligned}$$

Taking the previous example into account, our main result is somewhat limited, as, in general, there are not just two prominent solutions when facing distribution problems. In this regard, it is important to stress the continuous evolution of the numerous solutions proposed for bankruptcy problems, some of them gathered by Thomson (2003), and the interesting framework in which the most important single-valued solutions in the literature of TU-games are jointly analyzed by Arin (2007). However, the following two facts should also be noted. On the one hand, with the aim of being operative, societies establish mechanisms for reducing the number of proposals in controversial situations. On the other hand, once a society has specified the criteria to apply for solving a type of distribution problem, the number of acceptable proposals is greatly reduced and they sometimes lead to the natural form of bipolarity, as shown in the next section.

4. An application: Lorenz-bifocal bankruptcy games

This section is devoted to specifying of our previous analysis to bankruptcy problems: that is, distribution problems in which the agents' demand for a good exceeds its supply and the resources therefore need to be rationed. Formally,

A **bankruptcy problem** is a distribution problem, (M, C^N), where the set of agents, N, also called creditors or claimants, face a situation where $M = E$ is the estate and $C^N = c \in \mathbb{R}^n_{++}$ is the vector of their claims, the aggregate worth of which is greater than the estate, that is, $E < \sum_{i \in N} c_i$. Let $\mathcal{D}_* \subset \mathcal{D}$ denote the set of all bankruptcy problems.

A **bankruptcy rule** is a distribution rule (a function $f : \mathcal{D}_* \to \mathbb{R}^n$ such that for each $(E, c) \in \mathcal{D}_*$ $\sum_{i \in N} f_i(E, c) = E$) satisfying, for each bankruptcy problem $(E, c) \in \mathcal{D}_*$, the following constrains:

(a) **Non-Negativity***:* for each $i \in N$, $0 \leq f_i(E, c)$, and
(b) **Claims-Boundedness***:* for each $i \in N$, $f_i(E, c) \leq c_i$.

Let $\mathcal{F}_*$ denote the family of bankruptcy rules.

At this point, it is worth emphasizing that for each bankruptcy problem, $(E, c) \in \mathcal{D}_*$, given that there is no enough to share, any recommendation of a bankruptcy rule f could be interpreted as both a gains allocation, $f(E, c)$, and a losses allocation, $c - f(E, c)$. These two perspectives arise in a natural way when solving these problems.

We then present formal definitions of the Constrained Equal Awards and the Constrained Equal Losses bankruptcy rules, two proposal focusing respectively on the mentioned interpretations, and introduced by Maimonides in the twelfth century.

The **Constrained Equal Awards** bankruptcy rule, CEA, (Maimonides, 12th century, among others) recommends equal awards to all claimants subject to no-one receiving more than their claim. That is, for each $(E, c) \in \mathcal{D}_*$ and each $i \in N$,

$$CEA_i(E, c) \equiv \min \{c_i, \mu\},$$

where μ is chosen so that $\sum_{i \in N} \min \{c_i, \mu\} = E$.

Given a bankruptcy rule f, its **dual**, denoted by $[f]^*$, which, as is known, is well-defined, divides the aggregate loss in the same way as f divides the endowment (Aumann and Maschler (1985)): for each $(E, c) \in \mathcal{D}_*$ and each $i \in N$, $[f]_i^* (E, c) = c_i - f_i(\sum_{i \in N} c_i - E, c)$.

The **Constrained Equal Losses** rule, CEL, discussed by Maimonides (Aumann and Maschler (1985)), is the dual of the Constrained Equal Awards bankruptcy rule (Herrero (1)). It chooses the awards vector at which losses from the claims vector are the same for all agents, subject to no-one receiving a negative amount. That is, for each $(E, c) \in \mathcal{D}_*$ and each $i \in N$,

$$CEL_i(E, c) \equiv \max \{0, c_i - \mu\},$$

where μ is chosen so that $\sum_{i \in N} \max \{0, c_i - \mu\} = E$.

Our next analysis is somewhat in the line of Dutta and Ray (1989). With the aim of determining the two focal distribution rules, which by definition respect individual rights, we consider a society that applies the Lorenz criterion on bankruptcy rules but that has found no compelling reasons for focusing exclusively neither on gains nor on losses (Lorenz comparisons of bankruptcy rules from the gains viewpoint can be found in Bosmans and Lauwers (2007) and in Thomson (2007b)).

Given $x \in \mathbb{R}^N$, we denote by $\theta(x)$ the vector that results from x by permuting the coordinates in such a way that $\theta_1(x) \leq \theta_2(x) \leq ... \leq \theta_n(x)$.

Let $x, y \in \mathbb{R}^N$, we say that x **Lorenz dominates** y, denoted by $x \succ_L y$, if $\theta_1(x) \geq \theta_1(y)$, $\theta_1(x) + \theta_2(x) \geq \theta_2(x) + \theta_2(x)$, and so on, with at least one strict inequality[1].

If the partial sums are equal for $\theta(x)$ and $\theta(y)$, the two vectors x and y are **Lorenz equivalent**, denoted by $x \sim_L y$.

Given a set A, we define the set of **Lorenz undominated** elements of A as follows:

$$L(A) = \{x \in A \mid \text{there is no } y \in A \text{ such that } y \succ_L x\}.$$

Elements of $L(A)$ are called **Lorenz maximal** elements on A.

Given two bankruptcy rules f and g, f **Lorenz-gains dominates** g if for each $(E, c) \in \mathcal{D}_*$, $f(E, c) \succeq_L g(E, c)$. A bankruptcy rule f is **Lorenz-gains maximal**, LM_g, if there is no other g such that, for each $(E, c) \in \mathcal{D}_*$, $g(E, c) \succeq_L f(E, c)$.

Similarly, f **Lorenz-losses dominates** g if for each $(E, c) \in \mathcal{D}_*$, $c - f(E, c) \succeq_L c - g(E, c)$. A bankruptcy rule f is **Lorenz-losses maximal**, LM_l, if there is no other g such that, for each $(E, c) \in \mathcal{D}_*$, $c - g(E, c) \succeq_L c - f(E, c)$.

For the considered society, if there exist, previous reasoning provides the two focal bankruptcy solutions, f and g, which are the only ones Lorenz-gains maximal and Lorenz-losses maximal, respectively, in the set of all bankruptcy rules. In this regard, Schummer and Thomson (1997) showed that the Constrained Equal Awards bankruptcy rule is the only Lorenz-gains maximal. Furthermore, by starting with this result and using the concept of dual bankruptcy rule, it can be directly deduced that the Constrained Equal Losses bankruptcy rule is the only Lorenz-losses maximal.

As a result, this subclass of bifocal distributions problems, denoted by $\mathcal{B}_*^L \subset \mathcal{B}$ and called **Lorenz-bifocal bankruptcy problems**, is well-defined, its elements being triplets $B_*^L = ((E, c), CEA, CEL)$ where $(E, c) \in \mathcal{D}_*$. Within this subclass, we then present the concepts of the bifocal distribution rule and the bifocal distribution game, which we call the Lorenz-bifocal bankruptcy rule and the Lorenz-bifocal bankruptcy game, respectively.

A **Lorenz-bifocal bankruptcy rule** is a function $\varphi : \mathcal{B}_*^L \to \mathbb{R}^n$, such that for each Lorenz-bifocal bankruptcy problem $B_*^L = ((E, c), CEA, CEL) \in \mathcal{B}_*^L$, φ is a bankruptcy rule and for all $i \in N$,

$$\min\{CEA_i(E, c), CEL_i(E, c)\} \leq \varphi_i\left(B_*^L\right) \leq \max\{CEA_i(E, c), CEL_i(E, c)\}.$$

[1]Given $x, y \in \mathbb{R}^N$, we do not impose the condition $\sum_{i \in N} x_i = \sum_{i \in N} y_i$ on these vectors to apply the Lorenz domination criterion (see Arin (2007)).

For each Lorenz-bifocal bankruptcy problem, $B^L_* \in \mathcal{B}^L_*$, the associated **Lorenz-bifocal bankruptcy game**, denoted by $V^{B^L_*}$, is the TU-game defined by the function $V^{L_B} : 2^N \to \mathbb{R}_+$ which associates with each coalition $S \subseteq N$, the minimum worth that it would receive according to the two focal proposals, i.e., the real value

$$V^{B^L_*}(S) = \min \left\{ \sum_{i \in S} CEA_i(E, c), \sum_{i \in S} CEL_i(E, c) \right\}.$$

Because the class of Lorenz-bifocal bankruptcy games is a subclass of the general class of bifocal distribution games and Examples 3.3 and 3.4 have been constructed so that they corresponds to Lorenz-bifocal distribution problems with $(E, c) = (170, (35, 45, 55, 60))$ and $(E, c) = (30; (10, 12, 13, 15))$ respectively, all the characteristics and results set out in the previous section still hold.

In particular, our main result, Theorem 3.6, on Lorenz-bifocal bankruptcy games is dealt with in the following corollary. It represents a game-theoretic support of a new bankruptcy rule, partially analyzed by Thomson (2007a): the average of two well-known old solutions representing opposite viewpoints.

Corollary 4.1. *For each bifocal bankruptcy game, $V^{B^L_*}$, the Shapley value and the Nucleolus coincide. They are also defined by the Average of the Constrained Equal Awards and the Constrained Equal Losses rules, that is,*

$$\gamma^{Sh}(V^{B^L_*}) = \gamma^{Nu}(V^{B^L_*}) = 1/2 \left[CEA\,(E, c) + CEL\,(E, c) \right].$$

At this point it should be noted that the compromise reached by the society that we have considered could be classed as coherent, at least, for two reasons. The first of these is that the properties that the society has applied to restrict the set of admissible distribution rules, Non-Negativity and Claims-Boundedness, are also satisfied by the final agreement. The second reason is that the '*equity criterion*' on which the society bases its choices, Lorenz domination, is preserved, although the application of such a criterion is carried out in different domains, each one corresponding to the problem facing society at each time (allocations for bankruptcy problems and excesses of coalitions for bankruptcy games). In this regard, it is well known that, for each TU-game V, the Nucleolus selects the allocation of the resource providing the maximal Lorenz element on the set of vectors the coordinates of which are the excesses of all coalitions referring to efficient and individually rational distributions, $x = V(N)$ and $x_i \geq V(\{i\}$ for all $i \in N$, (see Arin (2007)).

Our modeling could be applied to other societies, represented either by different desirable social goals or by different requirements to the admissible distribution rules.

By way of illustration, let us consider a society which to solve bankruptcy problems applies the Lorenz criterion but imposes to bankruptcy rules Non-Negativity, Claims-Boundedness, Midpoint Property and Resource Monotonicity (see Thomson (2003) for formal definitions of these properties). It can be shown straightforwardly that our model provides a new bankruptcy solution: the Average of the Constrained Egalitarian rule and its dual (see Chun, Schummer and Thomson (2001) for the identification of the Lorenz-gains maximal rule in this context). Similarly, adding the property of Super-Modularity to the previous restrictions on solutions, the average of Piniles rule and its dual will arise by means of our previous analysis when the Lorenz criterion applies (see Thomson (2003) and Bosmans and Lauwers (2007)).

It is clear that coherence, in the previously mentioned sense, is a characteristic that cannot be generalized for any society[2]. Therefore, a trade-off between '*fairness*' and '*resoluteness*' could arise. However, the following two facts, which in our view reduce such a conflict, should be noted. Firstly, Yeh and Thomson (2006) write: "Concerning the extent of preserving properties, the convexity operator tend to be the least disruptive". Secondly, our analysis states that: "Concerning the extent of preserving '*equity criteria*', at several of them meet the reached compromise: those ones corresponding to the Shapley value and the Nucleolus".

5. Conclusions

Firstly, this section clarifies some of the relationship between *bifocal distributions games* and other well-known classes of TU-games.

It is straightforward to verify that *bifocal distributions games* are *minimum games* of two *additive games* with equal worths for the grand coalition, so that they are a very specific subclass of *flow games* (see Kalai and Zemel (1982)). They are also a subclass of *exact games* (see Smeidler (1972)). As far as we know, no subclasses of these classes of games have been identified for which the two prominent single-valued TU-games solutions, the Shapley value and the Nucleolus, coincide. In fact, until the recent paper by Kar et al. (2009), we have found no on their coinciding apart from work on the so-called *2-games*, a special class of the *k-games* defined by Deng and Papadimitriou (1994) (see also van den Nouweland et al. (1996) and Chun and Hokari (2007)). In this regard, an inclusion relation between *2-games* and *bifocal distributions games* cannot be established, although the intersection of these two classes of games is non-empty. Finally, we remark that both *2-games* and *bifocal distribution games* are *PS-games*, but not all of the *2-games* nor all of the *bifocal distribution games* are *PS-games*.

[2] It is well known that Constrained Equal Awards and the Constrained Equal Losses bankruptcy rules satisfy Consistency but, as Thomson (2007a) points out, there is no strict convex combination of them with fixed weights for all the agents that satisfies Consistency.

By way of a conclusion, the following highlights the scope, from our viewpoint, of our main result.

(i) It offers game-theoretic grounds for meeting people half-way.

(ii) It reinforces the subclass of *PS-games* by identifying within it a broad range of real situations and bifocal distribution problems, modeled as TU-games in a very 'natural' way. Up to now, only different games underlying queueing problems had been identified as *PS-games* (see Maniquet (2003), Chun (2006) and Kar et al. (2009)).

(iii) It provides powerful theoretical solution concepts with an extremely simple calculation not only for *bifocal distribution games*, but also for other classes of games, such as certain specific *flow games*, which are useful for modeling problems of profit sharing in an integrated production system with alternative production routes.

(iv) It shows how to introduce new applications of game theory and, consequently, new solution concepts by following the line of argument of Section 4.

In summary and to return to our initial motivation, this paper uses cooperative game theory to support very simple, current and commonly observed collective decisions. They are so common as to be expressed in the popular proverb: *'Virtue lies in the middle ground'*. It therefore involves both simplicity and the matching of theory with real life, producing an interesting combination.

APPENDIX 1.

Proof of Proposition 3.2.

Let $B = (D, f, g) \in \mathcal{B}$ and let $x \in \mathbb{C}\left(V^B\right)$. Then, by definition of both the distribution rule and the Core distribution,

$$\sum_{i \in N} f_i(D) = \sum_{i \in N} g_i(D) = \sum_{i \in N} x_i = V^B(N) = M \tag{5.1}$$

and

$$x_i \geq V^B(\{i\}) = \min\{f_i(D), g_i(D)\} \text{ for all } i \in N.$$

Now, we only have to prove that $x_i \leq \max\{f_i(D), g_i(D)\}$ for all $i \in N$.

Let us suppose that there $i \in N$ exists such that $x_i > \max\{f_i(D), g_i(D)\}$ and, without loss of generality, let us assume that

$$f_i(D) \leq g_i(D).$$

Then

$$x_i > g_i(D). \tag{5.2}$$

Let $S = \{j \in N \backslash i\}$. Then, on the one hand, by Conditions 5.1 and 5.2,

$$\sum_{j \in S} x_j < \sum_{j \in S} g_j(D) \tag{5.3}$$

and on the other hand, since $f_i(D) \leq g_i(D)$, Condition 5.1 implies

$$\sum_{j \in S} f_j(D) \geq \sum_{j \in S} g_j(D). \tag{5.4}$$

Therefore, by Conditions 5.4 and 5.3,

$$V^B(S) = \min\left\{\sum_{j \in S} f_j(D), \sum_{j \in S} g_j(D)\right\} = \sum_{j \in S} g_j(D) > \sum_{j \in S} x_j,$$

in contradiction with the fact that $x \in \mathbb{C}\left(V^B\right)$. Thus,

$$x_i \leq \max\{f_i(D), g_i(D)\} \text{ for all } i \in N.$$

■

APPENDIX 2.

Proof of Proposition 3.5.

Let $B = (D, f, g) \in \mathcal{B}$. By Definition 3.1, the corresponding bifocal distribution game, V^B, is such that $V^B(\varnothing) = 0$ and, for each coalition $S, \varnothing \neq S \subseteq N$,

$$V^B(S) = \min\left\{\sum_{i\in S} f_i(D), \sum_{i\in S} g_i(D)\right\}.$$

Let $\alpha_i = g_i(D) - f_i(D)$ for all $i \in N$. Then, the worth of each coalition $S \subseteq N$ can be expressed as follows:

$$V^B(S) = \left\{\begin{array}{ll} \sum\limits_{j\in S} f_j(D) & \text{if } \sum\limits_{j\in S} \alpha_j \geq 0 \\ \sum\limits_{j\in S} g_j(D) & \text{if } \sum\limits_{j\in S} \alpha_j \leq 0 \end{array}\right\}. \tag{5.5}$$

Next, we calculate the marginal contribution of any agent $i \in N$ to any coalition $T \subseteq N \backslash \{i\}$, that is, $\Delta_i V^B(T) = V^B(T \cup \{i\}) - V^B(T)$. The following four cases exhaust all the possibilities:

Case 1 : $\sum\limits_{j\in T} \alpha_j \geq 0$ and $\alpha_i + \sum\limits_{j\in T} \alpha_j \geq 0$.

By Condition 5.5, $V^B(T) = \sum\limits_{j\in T} f_j(D)$ and $V^B(T \cup \{i\}) = \sum\limits_{j\in T} f_j(D) + f_i(D)$. Thus,

$$\Delta_i V^B(T) = f_i(D). \tag{5.6}$$

Case 2 : $\sum\limits_{j\in T} \alpha_j \geq 0$ and $\alpha_i + \sum\limits_{j\in T} \alpha_j \leq 0$

By Condition 5.5, $V^B(T) = \sum\limits_{j\in T} f_j(D)$ and $V^B(T \cup \{i\}) = \sum\limits_{j\in T} g_j(D) + g_i(D)$. Thus,

$$\Delta_i V^B(T) = g_i(D) + \sum_{j\in T} g_j(D) - \sum_{j\in T} f_j(D) = g_i(D) + \sum_{j\in T} \alpha_j. \tag{5.7}$$

Case 3 : $\sum\limits_{j\in T} \alpha_j \leq 0$ and $\alpha_i + \sum\limits_{j\in T} \alpha_j \leq 0$

By Condition 5.5, $V^B(T) = \sum\limits_{j\in T} g_j(D)$ and $V^B(T \cup \{i\}) = \sum\limits_{j\in T} g_j(D) + g_i(D)$. Thus,

$$\Delta_i V^B(T) = g_i(D). \tag{5.8}$$

Case 4 : $\sum_{j\in T}\alpha_j \le 0$ and $\alpha_i + \sum_{j\in T}\alpha_j \ge 0$

By Condition 5.5, $V^B(T) = \sum_{j\in T} g_j(D)$ and $V^B(T\cup\{i\}) = \sum_{j\in T} f_j(D) + f_i(D)$. Thus

$$\Delta_i V^B(T) = f_i(D) + \sum_{j\in T} f_j(D) - \sum_{j\in T} g_j(D) = f_i(D) - \sum_{j\in T}\alpha_j. \tag{5.9}$$

Next, we calculate the sum of the marginal contributions of any agent $i \in N$ to any pair of coalitions T, T^* such that $T\cup T^* = N\backslash\{i\}$. With this aim, let us note that, given that both f and g are distribution rules,

$$\sum_{i\in N}\alpha_i = \sum_{i\in N} g_i(D) - \sum_{i\in N} f_i(D) = 0.$$

Therefore,

$$\sum_{k\in T^*}\alpha_k = -\left(\alpha_i + \sum_{j\in T}\alpha_j\right). \tag{5.10}$$

Here again we consider four cases for coalition T, which exhaust all the possibilities:

Case 1 : $\sum_{j\in T}\alpha_j \ge 0$ and $\alpha_i + \sum_{j\in T}\alpha_j \ge 0$. Then, by Condition 5.10, $\sum_{k\in T^*}\alpha_k \le 0$ and $\alpha_i + \sum_{k\in T^*}\alpha_k \le 0$. Now, applying Condition 5.6 to coalition T and Condition 5.8 to coalition T^*, we have

$$\Delta_i V^B(T) + \Delta_i V^B(T^*) = f_i(D) + g_i(D).$$

Case 2 : $\sum_{j\in T}\alpha_j \ge 0$ and $\alpha_i + \sum_{j\in T}\alpha_j \le 0$. Then, by Condition 5.10, $\sum_{k\in T^*}\alpha_k \ge 0$ and $\alpha_i + \sum_{k\in T^*}\alpha_k \le 0$. Now, applying Condition 5.7 to both coalitions T and T^*, we have

$$\Delta_i V^B(T) + \Delta_i V^B(T^*) = g_i(D) + \sum_{j\in T}\alpha_j + g_i(D) + \sum_{k\in T^*}\alpha_k =$$

$$= g_i(D) + g_i(D) - \alpha_i = g_i(D) + g_i(D) - g_i(D) + f_i(D) =$$

$$= f_i(D) + g_i(D).$$

Case 3 : $\sum\limits_{j\in T}\alpha_j \leq 0$ and $\alpha_i + \sum\limits_{j\in T}\alpha_j \leq 0$. Then, by Condition 5.10, $\sum\limits_{k\in T^*}\alpha_k \geq 0$ and $\alpha_i + \sum\limits_{k\in T^*}\alpha_k \geq 0$. Now, applying Condition 5.8 to coalition T and Condition 5.6 to coalition T^*, we have

$$\Delta_i V^B(T) + \Delta_i V^B(T^*) = g_i(D) + f_i D).$$

Case 4 : $\sum\limits_{j\in T}\alpha_j \leq 0$ and $\alpha_i + \sum\limits_{j\in T}\alpha_j \geq 0$. Then, by Condition 5.10, $\sum\limits_{k\in T^*}\alpha_k \leq 0$ and $\alpha_i + \sum\limits_{k\in T^*}\alpha_k \geq 0$. Now, applying Condition 5.9 to both coalitions T and T^*, we have

$$\Delta_i V^B(T) + \Delta_i V^B(T^*) = f_i(D) - \sum_{j\in T}\alpha_j + f_i(D) - \sum_{k\in T^*}\alpha_k =$$

$$= f_i(D) + f_i(D) + \alpha_i = f_i(D) + f_i(D) + g_i(D) - f_i(D) =$$

$$= f_i(D) + g_i(D).$$

■

APPENDIX 3.

Proof of Theorem 3.6.

Taking into account Proposition 3.5 and applying, to V^B, with $B = (D, f, g) \in \mathcal{B}$, the main result in Kar et al. (2009), gathered below, we obtain that for all $i \in N$, $\gamma_i^{Sh}(V^B) = \gamma_i^{PNu}(V^B) = (f_i(D) + g_i(D))/2$, where γ^{PNu} denotes the Prenucleolus. Now, given that, by the definition of V^B, $\gamma^{PNu}(V^B)$ satisfies individual rationality, that is, $\gamma_i^{PNu}(V^B) \geq V^B(\{i\})$ for all $i \in N$, we have that $\gamma^{Nu}(V^B) = \gamma^{PNu}(V^B)$.

Main Result in Kar*,* **Mitra and Wutuswami (2009)***:* If a TU-game V is a PS-game, then for all $i \in N$, $\gamma_i^{Sh}(V) = \gamma_i^{PNu}(V) = k_i/2$, where γ^{PNu} denotes the Prenucleolus and k_i is the player $i's$ specific constant corresponding to the sum of his marginal contribution to any pair of coalitions T, T^* such that $T \cup T^* = N\backslash\{i\}$.

■

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