SECOND BEST EFFICIENCY IN AUCTIONS*

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ABSTRACT

We characterize the incentive compatible allocation that maximizes the expected social surplus in a single-unit sale when the efficient allocation is not implementable. This allocation may involve no selling when it is efficient to sell. We then show that the English auction always implements the second best allocation when there are only two bidders, but not with more than two. Our model employs a unidimensional type space with independent types and allocative externalities, but captures some features of models with multidimensional types.

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1 Introduction

In this paper we study mechanisms that maximize the expected social surplus generated by the sale of an indivisible unit subject to the buyers' incentive compatibility constraints. This has been a fundamental question in the auction literature since Vickrey (1961)'s seminal paper.

Previous papers that have addressed this question, e.g. Maskin (1992), Maskin (2000), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), have focused on conditions that guarantee that the incentive compatibility constraints are not binding. For instance, when buyers have interdependent values, unidimensional type spaces, independent types and we consider Bayesian incentive compatibility constraints, Dasgupta and Maskin (2000) has shown that the condition takes the form of a particular single crossing condition.[1] This condition basically says that if it is efficient to allocate the good to one buyer for a given profile of types, it must also be efficient to allocate the good to this buyer when we increase her type keeping constant the types of the other buyers.[2]

We analyze a variation of this framework but depart from the existing auction literature and study efficiency when the incentive compatibility constraints are binding. Under our assumptions, this happens when the single crossing condition does not hold.

Maskin (2000) gives a realistic example in which the single crossing condition may fail. Suppose the sale of the right to drill for oil between two wildcatters. The first one has a high marginal cost and a low fixed cost, whereas the second one has a low marginal cost and a high fixed cost. In this case, it is efficient to allocate the good to the first wildcatter if there is little oil and to the second one if there is much oil. Thus, the single crossing condition will fail if the amount of oil is exclusive information of the first wildcatter and this is the only source of asymmetric information.

We argue in the paper that the single crossing condition fails with some generality in auction models with a (potentially) inefficient incumbent and some fixed costs as

[1]Krishna and Perry (1998) and Williams (1999) also characterize conditions for efficient Bayesian implementation with independent types.

[2]Assuming that types are ordered so that the value function of a bidder is increasing in her type.

suggested by the example in Maskin (2000). We also consider other applications of economic interest: in particular, models in which the underlying private information refers to common and private values or to allocative externalities. We illustrate these two cases using an insider model and a model of entry in markets. The proper framework for the general analysis of these cases is a multidimensional private uncertainty model that we study in the Appendix.

We provide two types of results. First, we characterize the *second best* allocation when the single crossing condition fails. This allocation may imply that the seller keeps the object even when all bidders always value the object more than the seller ex post. Our characterization is not complete in this last case.

Second, we analyze whether the second best can be implemented as the equilibrium of an English auction. We start noting a negative result when the second best allocation implies no selling, even if we allow for reserve prices or entry fees. Thus, we restrict to the case in which the good is always sold in the second best. In this case, our results are mixed and depend on the number of bidders.

The results are positive when there are only two bidders. The English auction possesses an equilibrium that implements the second best. We also discuss multiplicity of equilibria and robustness, and find that for a class of examples motivated by economic applications, any equilibrium in non-weakly dominated strategies of the English auction implements the second best.

However, the results are partially negative when there are more than two bidders. The English auction does not always have an equilibrium (in non-weakly dominated strategies) that implements the second best. In fact, we show that there is no such equilibrium for the particular class of examples referred above. The reason is that in equilibrium there are "rushes" with positive probability and standard rationing rules do not ensure efficiency in case of a tie.

Note that when the single crossing is violated the strategic decision of whether to remain active at a certain price p is potentially more complex than otherwise. To see why consider the wildcatter example when the first wildcatter knows the amount of oil. Recall that in this example the value of the first wildcatter is higher than the value of the second wildcatter when there is little oil, i.e. when values are low, while the reverse holds true if there is much oil, i.e. when values are high. Notice also that (as in a private value auction) the first wildcatter has a weakly dominant strategy: to stay active until her value is reached. Thus, the price that the second wildcatter pays when winning equals the first wildcatter's valuation. This means that winning at low prices implies a loss and winning at high prices a profit. Hence, the equilibrium strategy of the second wildcatter must trade off such expected profits and losses.

More generally, when the single crossing condition fails, a bidder may find it profitable to stay active at prices at which she makes a loss when she wins because she anticipates that the expected profits of winning at higher prices more than outweighs such potential losses. But this means that bidders may regret winning ex-post in equilibrium, i.e. there may be ex-post regret. It is precisely this feature that explains why there may be "rushes" with more than two bidders.

The assumption that types are independent, albeit restrictive, is reasonable for many interesting examples with common values, see our discussion in Section 2. Note that when we relax this assumption, the results of Cremer and McLean (1985) imply that the efficient allocation can be Bayesian implemented generically.[3] But, the mechanism requires arbitrarily large payments that seem unrealistic. More recently, McLean and Postlewaite (2004) have shown that this critic does not apply when agents are "informationally small". However, we expect agents to be "informationally small" in general only when they are sufficiently many.

From a different perspective, Mezzetti (2004) has shown that Bayesian implementation of the efficient outcome can always be achieved in a two-stage mechanism if the mechanism can condition on the realized outcome-payoffs. However, as Jehiel and Moldovanu (2003) have already pointed out Mezzetti's mechanism displays no incentives to reveal truthfully in the second stage, and requires that payments are made when all information is available, which makes it sensitive to renegotiation and moral hazard.

[3]Neeman (2004) and Heifetz and Neeman (2006) also cast some doubts on the genericity of these results.

Some of the above papers, in particular Maskin (1992), Maskin (2000), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001), have also studied the set of implementable allocations when bidders have multidimensional private information. They show that an implementable allocation cannot depend on the type beyond a particular one-dimensional reduction. In general, the efficient allocation does not correspond to this one-dimensional reduction and thus it is not implementable. They, however, note that we can always define a constrained efficient allocation that maximizes expected social surplus subject to the one-dimensional reduction.

Although we assume for our main results a one-dimensional type space, we note in Appendix B that our results may be useful in the efficiency analysis based on the one-dimensional reduced types. The reason is that there are no a priori arguments that ensure that the one-dimensional reduction satisfies the single crossing condition and thus the constrained efficient allocation may not be implementable.

Another related branch of the literature, in particular Maskin (1992), Krishna (2003), Birulin and Izmalkov (2003), Dubra, Echenique, and Manelli (2008), and Izmalkov (2003), analyzes whether there is an equilibrium of the English auction that allocates the good efficiently when the efficient allocation is implementable. This literature shows that the answer, as in our model, depends on the number of bidders. If there are only two bidders, there is always an efficient equilibrium, whereas with more than two bidders this is not always the case and stronger conditions are required.

However, the reason that leads to this result in the first best analysis in unrelated to ours. In fact, under the assumptions of our paper (and with no externalities) the English auction has an equilibrium that implements the first best efficient allocation (whenever it is implementable) independently of the number of bidders. We complement our results showing with some examples that under the presence of externalities (and when there are more than two bidders) the English auction may fail to deliver the efficient allocation even when it is feasible.

On the technical side, our work is related to Mussa and Rosen (1978) and Myerson (1981). Basically, they analyze the allocation that maximizes the expected profits using a technique called *ironing*. We use this technique to characterize the allocation that maximizes the expected social surplus. In a recent paper, Boone and Goeree (2008) have used a simplified version of the ironing technique in an environment closely related to our motivating example in Section 4.2. Their focus, as in Myerson (1981), is on the revenue maximizing auction rather than on the maximum expected social surplus. One of their findings is that a qualifying auction is revenue maximizing. Finally, we show in the paper that the ironing technique is not sufficient to characterize second best efficiency when the second best efficient allocation requires no selling ex post.

The rest of the paper is organized as follows. We define the formal set-up in Section 2. In Section 3 we study the implementation of the first best allocation. Section 4 includes some motivating examples in which the first best is not implementable. The second best efficient allocation is characterized in Section 5. Section 6 discusses the implementability of the second best through an English Auction and Section 7 concludes. We include two appendixes: Appendix A with the most technical proofs and Appendix B with an extension of our model to multidimensional types. Some arguments that are omitted or not fully carried over in the main appendixes are included in the supplementary material that ends the paper.

2 The Model

One unit of an indivisible good is put up for sale to a set $N \equiv \{1, 2, ..., n\}$ of n bidders. Let $s = (s_1, ..., s_n) \in \mathbb{R}^n$ be a vector where s_i corresponds to the realization of an independent random variable with distribution F_i with a strictly positive density[4] in a bounded support $S_i \subset \mathbb{R}$. Bidder $i \in N$ observes privately s_i and gets a von Neumann-Morgenstern utility $v_i(s) - p$ if she gets the good for sale at price p, and utility $-e_j(s_j) - p$ if Bidder j, $j \neq i$, gets the good and i pays a price p. Thus, e_j denotes a negative externality[5] produced by j on each of the other bidders. To make the analysis simpler, we assume that the seller neither derives utility from getting the good nor suffers any externality. We also assume that it is efficient to allocate to one

[4]Monteiro and Svaiter (2007) and Skreta (2007) have recently shown how to extend Myerson's (1981) analysis to distribution functions without density.

[5]Note that we also allow for $e_j(s_j) < 0$ and thus for positive externalities.

of the bidders, i.e. that $max_i \{v_i(s) - (n-1)e_i(s_i)\} \geq 0$ for any s.

We assume additive separability of the bidders' value functions plus a symmetry assumption on the common value component. Formally,[6] $v_i(s) = t_i(s_i) + \sum_{i \in N} q_j(s_j)$ for any $i \in N$, where $t_i(s_i)$ (t_i stands for taste) is the private value and $\sum_{i \in N} q_j(s_j)$ (q_j stands for quality) is the common value. We also assume that t_i, q_i and e_i are bounded, that $v_i(s)$ is a strictly increasing function of s_i, i.e. that $\phi_i(s_i) \equiv t_i(s_i)+q_i(s_i)$ is strictly increasing, and that $h_i(s_i) \equiv t_i(s_i)-(n-1)e_i(s_i)$ is measurable and at any point either right or left continuous.

Our additive separability assumptions are restrictive. We can interpret them as an approximation. The independency assumption may sound unrealistic for a model with common values. Note, however, that this critic only applies when two or more players have private information about the common value, whereas this is not the case in our motivating examples, see Section 4. Moreover, there are other real life examples in which the independency assumption is reasonable, see Bergemann and Välimäki (2002).

3 First Best Efficiency

Let an *allocation* be a measurable function $p : S \rightarrow [0,1]^n$, where $S \equiv \prod_{i=1}^n S_i$, such that $\sum_{i=1}^n p_i(s) \leq 1$ for any $s \in S$. We are interested in the set of allocations that can be implemented. By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair of measurable functions (p, x) where p is an allocation and $x : S \rightarrow \mathbb{R}^n$ a payment function. In the direct mechanism (p, x), each bidder announces a type, and $p_i(s)$ denotes the probability that i gets the good and $x_i(s)$ her transfers to the auctioneer when the vector of announced types is $s \in S$.

The expected utility of Bidder i with type s_i who reports s_i' when all the other

[6] In the text, we usually give as primitives the v_i's functions for simplicity. A simple way to recover the t_i's and q_i's from the v_i's when $S = [0,1]^n$ is as follows: $t_i(s_i) = v_i(0,...,0,s_i,0,...,0) - v_j(0,...,0,s_i,0,...,0) + v_j(0)$, for a $j \neq i$, and $q_i(s_i) = v_j(0,...,0,s_i,0,...,0) - v_j(0)$. The functions t_i and q_i deduced in this way correspond to the normalization that $q_i(0) = 0$ for all i.

bidders report truthfully is equal to:

$$U_i(s_i, s_i') \equiv Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x),$$

where[7]

$$Q_i(s_i', p) \equiv \int_{S_{-i}} p_i(s_i', s_{-i}) f_{-i}(s_{-i})\, ds_{-i},$$

and,

$$\Psi_i(s_i', p, x) \equiv \int_{S_{-i}} \left(\sum_{j \neq i} \left(p_i(s_i', s_{-i}) q_j(s_j) - e_j(s_j) p_j(s_i', s_{-i}) \right) - x_i(s_i', s_{-i}) \right) f_{-i}(s_{-i})\, ds_{-i},$$

for $S_{-i} \equiv \prod_{j \neq i} S_j$ and $f_{-i}(s_{-i}) \equiv \prod_{j \neq i} f_j(s_j)$.

Thus, we say that an allocation $p : S \to [0,1]^n$ is feasible if there exists a direct mechanism (p, x) that satisfies the following Bayesian incentive compatibility constraint:[8]

$$U_i(s_i, s_i) = \sup_{s_i' \in S_i} \{U_i(s_i, s_i')\},$$

for all $s_i \in S_i$ and $i \in N$.

The following lemma characterizes the feasible allocation using a standard argument in mechanism design, see for instance Myerson (1981), Rochet (1985) and McAfee and McMillan (1988):

Lemma 1. *An allocation p is feasible if and only if $Q_i(s_i, p)$ is weakly increasing in s_i for all $s_i \in [0,1]$ and $i \in N$.*

See proof in the Appendix.

We use the following natural definition:

[7]With some abuse of notation, we denote by $p_i(s_i, s_{-i})$ and $p_j(s_i, s_{-i})$ the function p_i and p_j, respectively, evaluated at a vector whose l-th component is equal to the l-th component of s_{-i} if $l < i$, it is equal to s_i if $l = i$ and it is equal to the $l-1$-th component of s_{-i} if $l > i$. We adopt the same convention for $x_i(s_i, s_{-i})$.

[8]We do not impose individual rationality constraints. They are trivially satisfied in our set-up. For instance, note that the mechanism proposed in Lemma 1 verifies that all bidders' types get non-negative utility.

Definition: We say that an allocation p is *first best efficient* when $p_i(s) > 0$ only if:

$$v_i(s) - (n-1)e_i(s_i) = \max\{v_j(s) - (n-1)e_j(s_j)\}_{j=1}^n,$$

and $\sum_{j=1}^n p_j(s) = 1$, for all $s \in S$.

Simple algebraic transformations show that first best efficiency requires allocating to the bidder with highest $h_i(s_i)$.

We adapt the following definition to our framework:[9]

Definition: We say that the *single crossing condition* is satisfied for bidder i if,

$$v_i(s) - (n-1)e_i(s_i) > \max\{v_j(s) - (n-1)e_j(s_j)\}_{j \neq i}$$

implies that

$$v_i(s') - (n-1)e_i(s'_i) \geq \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j \neq i},$$

for any $s, s' \in S$ such that $s'_i > s_i$ and $s_j = s'_j$ for $j \neq i$.

The interpretation of the single crossing condition is that if it is (first best) efficient to allocate to Bidder i for some signal profile, it cannot be the case that increasing Bidder i's type (keeping the other types constant) makes it efficient to allocate to Bidder $j \neq i$.

Our additive separability assumptions allow for a condition simpler to check in applications.

[9] The single crossing condition usually corresponds to the following alternative condition:

$$v_i(s) - (n-1)e_i(s_i) \geq \max\{v_j(s) - (n-1)e_j(s_j)\}_{j \neq i}$$

implies that

$$v_i(s') - (n-1)e_i(s'_i) > \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j \neq i},$$

for any $s'_i > s_i$ and $s'_j = s_j$ for $j \neq i$.

This alternative condition is sufficient for feasibility of the first best. If we add differentiability (and our assumption of additive separability), it also implies the single crossing condition of Dasgupta and Maskin (2000), the pairwise single crossing condition, the average crossing condition and the cyclical crossing condition of Krishna (2003) and the generalized single crossing condition of Birulin and Izmalkov (2003). We have used instead our definition to get also a necessary condition.

Lemma 2. *The single crossing condition for Bidder i is satisfied if and only if for any $s_i, s_i' \in S_i$ such that $s_i' > s_i$, the set,*

$$\{s_{-i} \in S_{-i} : \max\{h_j(s_j)\}_{j \neq i} \in (h_i(s_i'), h_i(s_i))\}$$

is empty.

See proof in the Appendix.

Note that the condition in Lemma 2 basically says that $h_i(.)$ must be an increasing function at any point at which its value may determine the first best efficient allocation.

Proposition 1. *A necessary and sufficient condition for the first best to be feasible is that the single crossing condition is satisfied for all bidders.*

See proof in the Appendix.

Intuitively, if the single crossing condition fails, the first best allocation requires that we move away the allocation of the object from Bidder i to some other bidder as we increase Bidder i's type around a given vector of types. Under our additive separability assumption, this implies that Bidder i's probability of winning conditional on her type must decrease at some point violating the feasibility conditions in Proposition 1.

That a version of our single crossing condition is sufficient for feasibility of the first best is well known. The necessary part is a consequence of the additive structure of our model. Dasgupta and Maskin (2000) have also proved that a single crossing condition is necessary for a more demanding definition of feasibility of the first best.

4 Economic Applications

In this section, we provide some economic models in which the single crossing condition will typically fail.

4.1 An Incumbent's Model

This model formalizes a version of the wildcatters' example mentioned in the Introduction. Suppose the sale of a license to become a monopolist of a market with an inverse demand function $P(Q) = 1 - \frac{Q}{s_1}$. Suppose there is a set N of firms interested in the license. Firm $1 \in N$ is an incumbent that has zero set-up costs to start to operate the license and a constant marginal cost c_1. The other firms are potential entrants. They incur in a set-up cost to start operating the license. We denote by $-s_i$, $i \neq 1$ the set-up cost of Firm i. We assume that all the entrants have the same marginal cost c. We also assume that $c < c_1 < 1$.

We assume that s_1 is the realization of a random variable with a distribution function F_1 and a density in the support $[\underline{s}, \overline{s}]$, $0 < \underline{s} < \overline{s}$. We also assume that each s_i, $i \neq 1$, is the realization of a random variable with a distribution function F_i and a density in the support $[-\underline{s}\frac{(1-c)^2}{4}, 0]$. The lower bound of the support implies that an entrant always finds it profitable to buy the license at zero price, whereas the upper bound ensures that there is an entrant type that values the license more than the incumbent. Finally, we assume that all the above random variables are independent, and that s_i is private information of Firm i.

This model corresponds in terms of the notation of Section 2 to:

$$v_1(s) = s_1 \frac{(1-c_1)^2}{4} \text{ and } e_1(s_1) = 0,$$

and

$$v_i(s) = s_1 \frac{(1-c)^2}{4} + s_i \text{ and } e_i(s_i) = 0 \text{ for } i \neq 1.$$

Thus, $t_1(s_1) = h_1(s_1) = s_1 \left(\frac{(1-c_1)^2}{4} - \frac{(1-c)^2}{4} \right)$, $q_1(s_1) = s_1 \frac{(1-c)^2}{4}$, $e_1(s_1) = 0$, and $t_j(s_j) = h_j(s_j) = s_j$, $q_j(s_j) = 0$ and $e_j(s_j) = 0$, for $j \neq 1$.

To see why the single crossing condition is violated in this example, note that $h_1(\underline{s}) > h_1(\overline{s})$, $h_j(-\underline{s}\frac{(1-c)^2}{4}) < h_1(\underline{s})$ and $h_j(0) > h_1(\overline{s})$, $j \neq 1$. Hence, by continuity of h_j there exists an $s_j \in [-\underline{s}\frac{(1-c)^2}{4}, 0]$ such that $h_j(s_j) \in (h_1(\overline{s}), h_1(\underline{s}))$. Thus, the application of Lemma 2 for $s_i = \underline{s}$ and $s_i' = \overline{s}$, and $i = 1$ means that the single crossing condition fails for Bidder 1.

4.2 An Insider's Model

Suppose the sale of a painting to a set N of risk neutral bidders. The painting may be an original painting of a well-known (and priced) artist. Bidder i puts a value on the painting of $\tau_i + \rho$ if the painting is original and otherwise a value of τ_i. We assume that each τ_i is equal to an independent drawn of a random variable with a distribution function G_i and a density in the support $[\underline{t}, \bar{t}]$. We assume that τ_i is private information of Bidder i. One of the bidders, Bidder 1, is an expert art dealer and she is the only one knowing whether the painting is original. The other bidders know only that the ex ante probability that the picture is original is equal to $\alpha \in (0, 1)$.

Under the assumption that $\rho + \underline{t} > \bar{t}$, it is easy to see that this model may be written in terms of the notation of Section 2 as follows: $v_1(s) = s_1$, $e_1(s_1) = 0$, $v_i(s) = s_i + \mathbf{1}_{[\underline{t}+\rho, \bar{t}+\rho]}(s_1)$ and[10] $e_i(s_i) = 0$, $i \neq 1$, where s_1 has a distribution

$$F_1(s_1) = \begin{cases} \alpha G_1(s_1) & \text{if } s_1 < \bar{t} \\ \alpha & \text{if } s_1 \in [\bar{t}, \rho + \underline{t}) \\ \alpha + (1-\alpha) G_1(s_1 - \rho) & \text{otherwise,} \end{cases}$$

with support $[\underline{t}, \bar{t}] \cup [\underline{t} + \rho, \bar{t} + \rho]$ and s_i, $i \neq 1$, is distributed according to $G_i(.)$. Note that according to this convention, $s_1 \in [\underline{t} + \rho, \bar{t} + \rho]$ indicates that the painting is original.

In this application we have that $t_1(s_1) = h_1(s_1) = s_1 - \mathbf{1}_{[\underline{t}+\rho, \bar{t}+\rho]}(s_1)$, $q_1(s_1) = \mathbf{1}_{[\underline{t}+\rho, \bar{t}+\rho]}(s_i)$, $t_i(s_i) = h_i(s_i) = s_i$ and $q_i(s_i) = 0$ for $i \neq 1$, and it is easy to verify that the single crossing condition is violated for Bidder 1. To see why, apply Lemma 2 to $s_1 = \bar{t} - \epsilon$ and $s_1' = \underline{t} + \rho + \epsilon$ for $\epsilon > 0$ and small enough.

The analysis of the case $\rho + \underline{t} \leq \bar{t}$ is done in Appendix B. Such case motivates the extension that covers a special class of games with multidimensional information carried over in that Appendix.

[10] $\mathbf{1}_X(x)$ is an indicator function that takes value 1 when $x \in X$ and otherwise takes value 0.

4.3 A Model with Negative Externalities

Suppose n local markets, each with a unit mass of consumers with reservation value 1 for the consumption of the good. Suppose also a set N of n firms. Each firm starts with a branch in a local market. Initially, no two firms have a branch at the same local market. Firms can open new branches at a fixed cost $C < 1$ and serve any local market in which they have a branch at a marginal cost c.

Suppose that a seller puts up for sale a technology that reduces the marginal costs of firm i by an amount s_i. Suppose that s_i is drawn from an independent distribution F_i with support $[0, c]$. If only one firm serves a market, its profits are equal to $1 - c$. When more than one firm serves a local market, we assume an outcome consistent with Bertrand competition: the firm with the lowest marginal cost serves the market at a price equal to the second lowest marginal cost. In case of more than one firm with the lowest marginal cost, we assume that they split equally the demand at a price equal to their common marginal cost. Thus, a firm finds it profitable to open a branch in each of the other markets only if she has won the technology and the reduction in the marginal cost is sufficiently large, in particular, if and only if $s_i > C$.

We can write this model in terms of the notation of Section 2 as follows: $v_i(s) = t_i(s_i) = s_i$, and $e_i(s_i) = 0$ if $s_i \leq C$, and $v_i(s) = t_i(s_i) = s_i + (n-1)(s_i - C)$ and $e_i(s_i) = 1 - c$, otherwise, and $q_i(s_i) = 0$. As consequence, $h_i(s_i) = s_i$ for $s_i \leq C$ and $h_i(s_i) = s_i - (n-1)(C + 1 - c - s_i)$, otherwise. In this case, the single crossing condition is violated for any bidder. To see why, apply Lemma 2 to $s_i = C - \epsilon$ and $s_i = C + \epsilon$, and $s_j \in (C - \epsilon, C)$ for all $j \neq i$, and $\epsilon > 0$ and small enough.

5 Second Best Efficiency

In light of Proposition 1, it is natural to define second best efficiency.

Definition: We say that an allocation is *second best efficient* if it maximizes

$$\int_S \sum_{i=1}^{n} \left(v_i(s) - (n-1)e_i(s_i)\right) p_i(s) f(s)\, ds,$$

subject to p feasible and for $f(s) \equiv \prod_{i \in N} f_i(s_i)$.

Certainly, the set of second best allocations includes the first best allocation when the single crossing condition is satisfied.

It also turns out to be useful to define the following concept.

Definition: We say that an allocation p is *second best efficient subject to always selling* if it maximizes

$$\int_S \sum_{i=1}^{n} (v_i(s) - (n-1)e_i(s_i))\, p_i(s) f(s)\, ds,$$

subject to p feasible and $\sum_{i=1}^{n} p_i(s) = 1$ for any $s \in S$.

An equivalent characterization is that the allocation maximizes,

$$\int_S \sum_{i=1}^{n} h_i(s_i) p_i(s) f(s)\, ds,$$

subject to p feasible and $\sum_{i=1}^{n} p_i(s) = 1$ for any $s \in S$.

To simplify the notation in what follows, we shall assume without loss of generality that each F_i is uniform on $[0,1]$.[11] To see why this assumption is without loss of generality suppose that the F_i's were not uniform. Then, we could define a new vector of signals $\tilde{s}_i \equiv F_i(s_i)$ and value functions $\tilde{v}_i(\tilde{s}) \equiv \tilde{t}_i(\tilde{s}_i) + \sum_{i \in N} \tilde{q}_j(\tilde{s}_j)$ and $\tilde{e}_j(\tilde{s}_j)$ where[12] $\tilde{t}_i(\tilde{s}_i) \equiv t_i(F_i^{-1}(\tilde{s}_i))$, $\tilde{q}_j(\tilde{s}_j) \equiv q_j(F_j^{-1}(\tilde{s}_j))$ and $\tilde{e}_j(\tilde{s}_j) \equiv e_j(F_j^{-1}(\tilde{s}_j))$, where note that each of the new signals $\tilde{s}_i$'s has a uniform distribution on $[0,1]$.[13]

Recall that it is first best efficient to allocate according to $h_i(s_i)$. However, this allocation is not implementable when h_i is not increasing. We next show how to derive from the h_i functions some functions g_i that are increasing and that determine the second best allocation like the h_i's determine the first best.

Let $H_i(s_i) \equiv \int_0^{s_i} h_i(\tilde{s}_i)\, d\tilde{s}_i$ for all $i \in n$ and $s_i \in [0,1]$, and let $G_i(s_i) : [0,1] \to \mathbb{R}$ be the convex hull of the function H_i (i.e. the highest convex function on $[0,1]$ such that

[11] Lehmann (1988) already showed that there is no loss of generality in assuming that signals have a uniform marginal distribution.

[12] As a convention, we denote by $F^{-1}(z) \equiv \min\{s_i \in [\underline{s}, \overline{s}] : F(s) \geq z\}$.

[13] To see why, note that the probability of $\{\tilde{s}_i \leq z\}$ for $z \in [0,1]$ is equal to the probability of $\{F_i(s_i) \leq z\}$, which is equal to the probability of $\{s_i \leq F_i^{-1}(z)\}$ and thus, it is equal to $F_i(F_i^{-1}(z)) = z$.

$G_i(s_i) \leq H_i(s_i)$ for all $s_i \in [0,1]$.) Formally:[14]

$$G_i(s_i) = \min\left\{wH_i(r_1) + (1-w)H_i(r_2) : w, r_1, r_2 \in [0,1] \text{ and } wr_1 + (1-w)r_2 = s_i\right\}.$$

Lemma 3. *Properties of G_i:*

(a) G_i is convex.

(b) $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$.

(c) $G_i(s_i) \leq H_i(s_i)$ for all $s_i \in [0,1]$.

(d) If $G_i(s_i) < H_i(s_i)$ in an open interval, then G_i is linear in the same open interval.

Proof. All the properties in the proposition follow from the application to the definition of convex hull of standard mathematical arguments that we do not reproduce. ■

As a convex function G_i is differentiable except at countably many points, and its derivative is a non-decreasing function. We define $g_i : [0,1] \to \mathbb{R}$ to be the differential of G_i completed by right-continuity.

Note that when h_i is an increasing function then $g_i = h_i$, but this is not the case when h_i is decreasing in some interval. Suppose, for instance, that $h_i(s_i) = \beta - s_i$. Then $H_i(s_i) = \beta s_i - s_i^2/2$, and since it is concave, its convex hull is simply a straight line connecting $(0, H_i(0))$ and $(1, H_i(1))$, i.e. $G_i(s_i) = (\beta - \frac{1}{2})s_1$. Thus, $g_i(s_i) = (\beta - \frac{1}{2})$. Note that in this case g_i is in fact the average value of h_i in $[0,1]$.

More generally, the function g_i is equal to h_i except in some intervals around the points at which h_i is not increasing. In these intervals, g_i takes the average value of h_i in the interval, i.e. the h_i function is "ironed out" in these intervals. The following example illustrates this point: $h_i(s_i) = 2s_i$ if $s_i < 1/2$, and $h_i(s_i) = 2s_i - 1$ otherwise. It can be shown after some algebra that $g_i(s_i) = h_i(s_i) = 2s_i$ if $s_i < 1/4$, $g_i(s_i) = 1/2$ (i.e. the average value of h_i in $[1/4, 3/4)$) if $[1/4, 3/4)$ and $g_i(s_i) = h_i(s_i) = 2s_i - 1$ if $s_i \geq 3/4$, see Figure 1.[15]

[14] See also Rockafellar (1970), Pag. 36.

[15] Mussa and Rosen (1978), pp. 313-314, provide a similar illustration for the case of a price discriminating monopolist that faces a non-monotonic marginal revenue.

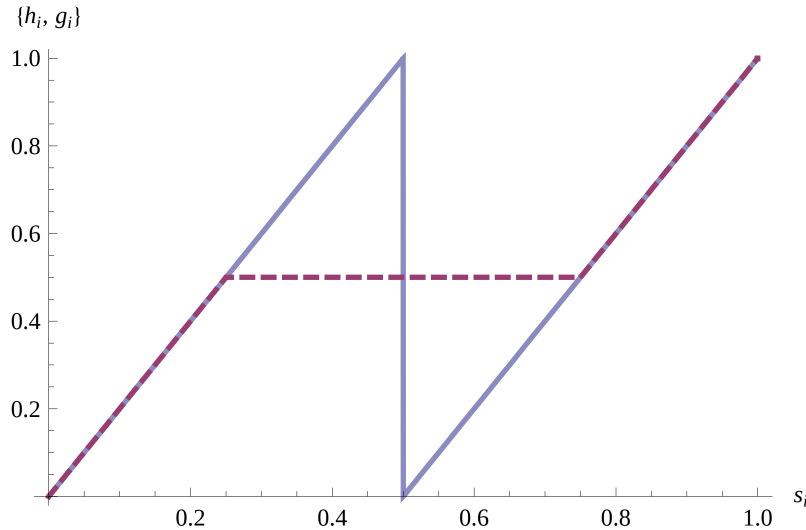

Figure 1: Ironing: the functions h_i and g_i (dashed) when h_i is not increasing.

The next proposition shows that it is second best efficient (subject to always selling) to allocate according to the g_i's plus an additional condition.

Proposition 2. *A feasible allocation p^* is second best efficient subject to always selling if and only if $\sum_{i=1}^n p_i^*(s) = 1$ for all s, and it maximizes:*[16]

$$\int_S \sum_{i=1}^n \left(g_i(s_i) + \sum_{j=1}^n q_j(s_j) \right) p_i(s)\, ds + \sum_{i=1}^n \int_{S_i} (G_i(s_i) - H_i(s_i))\; dQ_i(s_i, p). \tag{1}$$

An allocation $\sum_{i=1}^n p_i^*(s) = 1$ maximizes the above expression when $\forall i \in N$:

(i) $p_i^*(s) > 0$ only if $g_i(s_i) = \max\{g_j(s_j)\}_{j \in N}$ a.e.

(ii) $Q_i(., p^*)$ is constant in any open interval in which $G_i(s_i) < H_i(s_i)$.

See proof in the Appendix.

Note that within the set of allocations that always sells, an allocation that verifies condition (i) maximizes the first integral in Equation (1), and if it verifies condition

[16]We denote by $\int_E \varphi(x)dF(x)$ the Lebesgue-Stieljes integral of φ with respect to F in E. In particular, for any feasible allocation p, we denote by $\int_{S_i} \varphi(s_i)dQ_i(s_i, p)$ the Lebesgue-Stieljes integral of φ with respect to $Q_i(., p)$ in S_i.

(ii), it also maximizes the second integral. To see the latter, recall that by Lemma 3(c), the second integral is non-positive, whereas condition (ii) implies that it is zero.

We next illustrate the proposition with an example with two bidders. To make it simpler, in our example only Bidder 1 has private information, or equivalently, $v_1(s)$ and $v_2(s)$ are constant with respect to s_2. Although this departs from our general assumptions, the only difference is that the incentive compatibility constraints for Bidder 2 are trivially satisfied and thus the conditions of Lemma 1 only need to hold for Bidder 1.

Example 1. *$N = \{1, 2\}$, $v_1(s) = \beta + s_1$, $v_2(s) = \gamma + 2s_1$ and $e_i(s_i) = 0$ for all i, where $\beta, \gamma \geq 0$ and $\beta + \gamma \in (0, 1)$.*

Note that $h_1(s_1) = \beta - s_1$ and $h_2(s_2) = \gamma$, and hence the first best allocation is to give the good to Bidder 1 if $\beta - s_1 > \gamma$ and otherwise to Bidder 2. This is not feasible since it implies that the probability that Bidder 1 gets the good is decreasing in her type and thus the feasibility condition of Lemma 1 is not met. Note then that $g_1(s_1) = \beta - 1/2$, and $g_2(s_2) = \gamma$. Thus, by application of Proposition 2, the second best is to allocate to Bidder 1 if $\beta - 1/2 > \gamma$ and to allocate to Bidder 2 if $\beta - 1/2 < \gamma$.[17]

To understand why this allocation is second best efficient (subject to always selling) note that there are only two candidate allocations: to allocate the good to Bidder 1 for any s_1, or to allocate the good to Bidder 2 for any s_1. To see why, note that the greater s_1 is, the less desirable from an efficient point of view is to allocate to Bidder 1, whereas the feasibility condition requires that if we allocate to Bidder 1 for some type s_1 we must allocate to Bidder 1 also for higher types. Between these two candidates, the former is second best if Bidder 1 has greater value than Bidder 2 on average, i.e if $\beta - 1/2 > \gamma$, whereas the latter is second best otherwise. This is precisely what Proposition 2 says.

Next proposition gives sufficient conditions under which it is second best to always sell. This condition basically ensures that the set of allocations that maximizes the first integral in Equation (1) implies always selling.

[17] Note that in the case $\beta - 1/2 = \gamma$ the second best only requires that $Q_1(s_1, p^*)$ is constant in s_1 in the open interval $(0, 1)$.

Proposition 3. *If* $\max_{i\in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s) \geq 0$*, then any second best efficient allocation subject to always selling is also second best efficient.*

See proof in the Appendix.

To interpret the condition in the proposition recall that the social surplus of allocating to i is equal to $v_i(s) - (n-1)e_i(s_i) = h_i(s_i) + \sum_{j=1}^{n} q_j(s)$ and that g_i is a version of h_i in which the non-monotone parts of h_i are iron-out by taking mean values. Note that this means that when h_i is weakly increasing for one bidder and the social value of allocating the good to this bidder is greater than the seller's value, i.e. $v_i(s) - (n-1)e_i(s_i) \geq 0$, $\forall s_i$, the condition in Proposition 3 is verified. This is the case in the examples of Sections 4.1 and 4.2.

When the condition $\max_{i\in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) \geq 0$ does not hold, the second best may imply some interesting results. Consider the following example:

Example 2. $N = \{1, 2\}$*,* $v_i(s) = s_i + 2s_j$ *and* $e_i(s_i) = 0$ *for* $i, j \in \{1, 2\}$ *and* $i \neq j$*.*

In this example, $h_i(s_i) = -s_i$, $H_i(s_i) = -\frac{s_i^2}{2}$, and thus, $G_i(s_i) = -\frac{s_i}{2}$ and $g_i(s_i) = -\frac{1}{2}$. Hence, the corresponding Equation (1) to Example 2 is:

$$\int_{[0,1]^2} \sum_{i=1,2} \left(-\frac{1}{2} + 2s_1 + 2s_2\right) p_i(s)\, ds + \sum_{i=1,2} \int_0^1 \left(-\frac{s_i}{2} + \frac{s_i^2}{2}\right) dQ_i(s_i, p). \qquad (2)$$

Thus, maximizing the first integral requires that a.e. $p_1(s) + p_2(s) = 0$ for $s_1 + s_2 \leq 1/4$ and $p_1(s) + p_2(s) = 1$ otherwise, and maximizing the second integral requires that $Q_i(., p)$ is constant in $(0, 1)$ for $i = 1, 2$. The allocation described in Figure 2 satisfies both conditions[18] and it is therefore second best efficient.

In the above example it is (first best) efficient to allocate the good to Bidder 2 when her signal is low and Bidder 1's signal is high. Similarly, it is efficient to allocate the good to Bidder 1 when her signal is low and Bidder 2's signal is high. However, this allocation is difficult to implement because both bidders have very little incentives to report truthfully when their signal is high. In fact, this problem is so severe that when we restrict to always selling we cannot do better than ignoring bidders' signals

[18]Note that for this allocation $Q_1(s_1, p) = Q_2(s_2, p) = \frac{31}{64}$ for any $s_1, s_2 \in [0, 1]$.

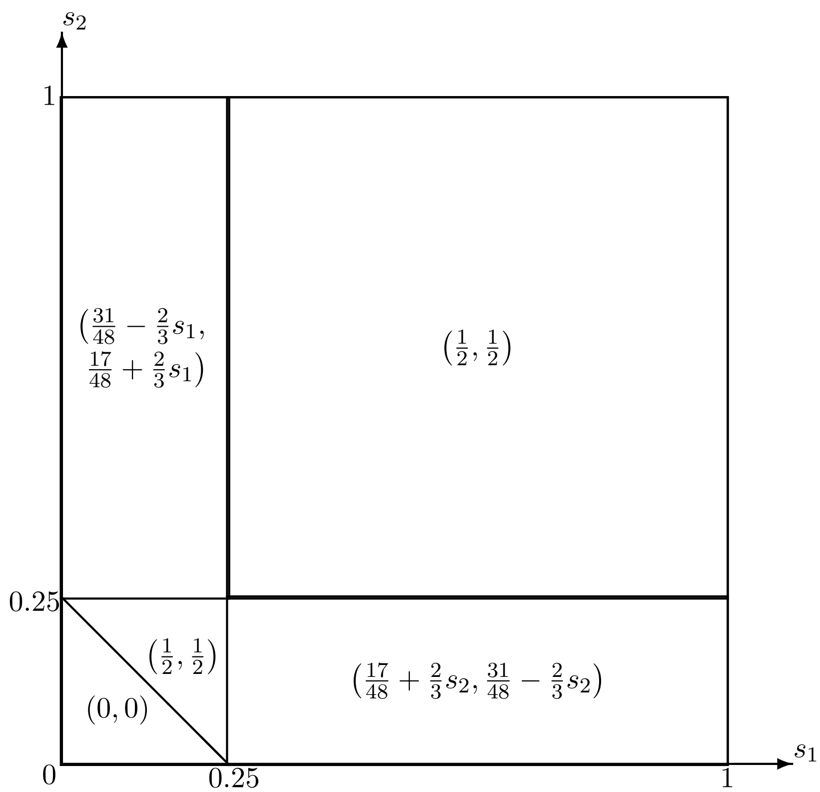

Figure 2: Second best allocation $(p_1(s), p_2(s))$ for Example 2

and allocate the object with equal probability between the bidders. This may be easily shown applying Proposition 2.

The random allocation described in the paragraph above differs from the allocation in Figure 2 in that the latter allocation does not allocate the good to any bidder in the triangle in the lower-left corner and that it allocates more often to Bidder 2 in the rectangle in the lower-right corner and more often to Bidder 1 in the rectangle in the upper-left corner. The reason why this improves expected surplus is because the efficiency loss of not allocating the good to any bidder in the triangle is small, both values are close to zero, whereas the allocation in the rectangles is closer to the first best.

Note that the second best efficient allocation cannot be always characterized by maximizing both integrals in Equation (1) simultaneously. In fact, the following slight modification of Example 2 illustrates this point:[19] $v_1(s_1, s_2) = s_1 + 2s_2 + \epsilon$ and $v_2(s_1, s_2) = s_2 + 2s_1$ with $\epsilon > 0$ and small. It may be shown that the maximization of the first integral of the corresponding Equation (1) requires that p satisfies a.e. that $p(s) = (0, 0)$ if $s_1 + s_2 < \frac{1}{4} - \frac{\epsilon}{2}$, and $p(s) = (1, 0)$, otherwise. Any such allocation verifies that $Q_1(s_1, p)$ is strictly increasing in s_1 for $s_1 \in [0, \frac{1}{4} - \frac{\epsilon}{2}]$. However, the maximization of the second integral of the corresponding Equation (1) requires that p verifies that $Q_1(s_1, p)$ is constant in s_1 in the open interval $(0, 1)$.

Finally, we show with an example that $\max_{i \in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) < 0$ does not imply that the second best requires no selling:[20]

Example 3. *$N = \{1, 2\}$, $v_1(s) = 1/4$ and $v_2(s) = s_1 - 1$, and $e_i(s_i) = 0$ for all i.*

In this example, $h_1(s_1) = 1/4 - s_1$, $q_1(s_1) = s_1$, $g_1(s_1) = -1/4$, $h_2(s_2) = g_2(s_2) = -1$ and $q_2(s_2) = 0$. Thus, $\max_{i \in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) = s_1 - 1/4$, which is negative for $s_1 < 1/4$. However, it is efficient to always allocate to Bidder 1.

6 English Auction

In this section we analyze whether the second best can be implemented with an English auction. In particular, we assume the model of the English auction described by Krishna (2003). This auction model is a variation of the Japanese auction proposed by Milgrom and Weber (1982) in which the identity of the bidders is observable.

We introduce two additional assumptions. The first one is a simplification, we assume that the functions h_i's are continuous. This assumption implies:

Lemma 4. *The functions g_i's are continuous. Moreover:*

[19] We have chosen an asymmetric counter-example because it makes the argument more transparent. However, we could also provide a symmetric counter-example. It is easy to verify that the following one does work: $N = \{1, 2\}$, $v_i(s) = 20 \cdot \mathbf{1}_{[.9,1]}(s_j) + (s_i + s_j) + \mathbf{1}_{[1/2,1]}(s_i) + \mathbf{1}_{[1/2,1]}(s_j)$ and $e_i(s_i) = 0$.

[20] To make the example more transparent, we have violated one of our assumptions, namely that $\phi_1(s_1)$ is strictly increasing. Note, however, that this can be easily fixed changing v_1 to $1/4 + \epsilon s_1$ and v_2 to $s_1 - 1 + \epsilon s_2$. For ϵ sufficiently small, this change generates the appropriate counterexample.

(a) $g_i(s_i) = h_i(s_i)$ *if* $G_i(s_i) = H_i(s_i)$ *and* $s_i \in (0,1)$.

(b) $g_i(0) \leq h_i(0)$ *with strict inequality only if* $G_i(\epsilon) < H_i(\epsilon)$ *for any* $\epsilon > 0$ *small enough.*

(c) $g_i(1) \geq h_i(1)$ *with strict inequality only if* $G_i(1-\epsilon) < H_i(1-\epsilon)$ *for any* $\epsilon > 0$ *small enough.*

See proof in the Appendix.

The second assumption is that $\zeta_i(s_i) \equiv q_i(s_i) + e_i(s_i)$ is non-decreasing. This assumption ensures that our proposed bid functions are increasing. It translates to our set-up a similar assumption used in the study of first best efficiency in English auctions, see Maskin (2000), Krishna (2003) and Birulin and Izmalkov (2003) that requires the bidders' values to be non-decreasing functions of the bidders' types.

We start remarking that in general there is very little hope that the English auction can implement the second best when it implies no selling. To see why, recall that in Example 1, second best requires no selling if $s_1 + s_2 \leq 1/4$. However, an English auction with an entry fee and/or a reserve price can only ensure no selling for sets of types $\{(s_1, s_2) \in [0,1]^2 : s_1 \geq \underline{s}_1, s_2 \geq \underline{s}_2\}$ for some $\underline{s}_i \in [0,1]$.

In what follows to save space we refer to the second best allocation subject to always selling simply as the second best. We shall show that whether the English auction implements that allocation depends on the number of bidders. We start with the two bidder's case.

6.1 An English Auction with Only Two Bidders

Suppose in this subsection that $n = 2$. We shall show that in this case, the English auction implements the second best efficient allocation. We start with an example:

Example 4. $N = \{1,2\}$, $v_1(s) = s_1 + 1$, $v_2(s) = s_2 + 2s_1$ *and* $e_i(s_i) = 0$ *for all* i.

In this example, $h_1(s_1) = 1 - s_1$, $g_1(s_1) = 1/2$ and $h_2(s_2) = g_2(s_2) = s_2$. Thus, the single crossing condition is not satisfied and it is second best to allocate to Bidder 1 if

$g_2(s_2) \geq g(s_1)$, i.e. $s_2 \leq 1/2$, and otherwise to allocate to Bidder 2.

Note that Bidder 1 has a unique weakly dominant strategy, to bid until $s_1 + 1$. The argument is the same as in private value auctions: bidding less means losing at prices below the value, and bidding higher means winning at prices greater than the value. If Bidder 1 with type s_1 bids $p \equiv s_1 + 1$ and Bidder 2 wins, Bidder 2 with type s_2 gets profits $s_2 + 2s_1 - p$, which are equal to $h_2(s_2) - h_1(s_1) = h_2(s_2) - h_1(p-1)$. Thus, since h_1 is decreasing, the greater the price, the more profitable it is for Bidder 2 to win the auction. As a consequence, Bidder 2's best response is either to submit a bid that always loses, e.g. $p = 1$, or a bid that always wins, e.g. $p = 2$. The former option is optimal if $h_2(s_2)$ is less than the average value of h_1 in $[0, 1]$, i.e. if $g_2(s_2) \leq g_1(s_1) = 1/2$, and the latter otherwise.

Consequently, Bidder 2 wins the auction if and only if $g_2(s_2) \geq g_1(s_1)$ in any equilibrium in which Bidder 1 uses her unique weakly dominant strategy, and hence the resulting allocation is second best efficient. Note that the structure of this equilibrium is more involved than in the more standard model in which the single crossing condition is satisfied. The difference is that Bidder 2 may be active at prices at which she makes a loss if she wins, i.e. there may be ex-post regret. The reason why this is profitable for Bidder 2 is that winning at higher prices is sufficiently profitable to offset the losses at lower prices.

The following lemma generalizes this example. Note that this lemma analyzes the case in which one bidder has no uncertainty with respect to her willingness to pay for the object, and it includes, as particular cases, the insider and the incumbent's models of Sections 4.1 and 4.2.

Lemma 5. *If $q_2(s_2) = e_2(s_2) = 0$ for any $s_2 \in [0, 1]$, then any equilibrium of the English auction in non-weakly dominated strategies implements the second best.*

See proof in the Appendix.

Next, we show how this result extends to the general case. As we shall see, there always exists an equilibrium that implements the second best efficient allocation but in some cases there may be other equilibria that are not second best efficient.

We follow three steps. First, we propose a bid function for each bidder; second, we prove that the good is allocated according to the second best allocation when bidders use the proposed bid functions; and finally, we show that the proposed bid functions are an equilibrium of the English auction. Next, we discuss uniqueness.

We start with some auxiliary definitions. Let $\underline{s}_i^j, \overline{s}_i^j : [0,1] \to [0,1]$, $i \neq j$, be such that $\underline{s}_i^j(s_j)$ and $\overline{s}_i^j(s_j)$ are equal to the minimum and maximum s_i, respectively, that solve $g_i(s_i) = g_j(s_j)$, if $g_i(s_i) \in [g_j(0), g_j(1)]$, and we let $\underline{s}_i^j(s_j)$ and $\overline{s}_i^j(s_j)$ be both equal to zero if $g_j(s_j) < g_i(0)$ and equal to one if $g_j(s_j) > g_i(1)$. Note that Lemma 3 (b) and (d) imply that $H_i(\underline{s}_i^j(s_j)) = G_i(\underline{s}_i^j(s_j))$ and $H_i(\overline{s}_i^j(s_j)) = G_i(\overline{s}_i^j(s_j))$.

Let $b_1^*(s_1) \equiv v_1^e(s_1, \overline{s}_2^1(s_1))$, and $b_2^*(s_2) \equiv v_2^e(\underline{s}_1^2(s_2), s_2)$, where $v_i^e(s_i, s_j) \equiv v_i(s_i, s_j) + e_j(s_j)$ (this is $v_i^e(s_i, s_j) = \phi_i(s_i) + \zeta_j(s_j)$). Thus, we propose that Bidder 1 (respectively Bidder 2) bids her maximum willingness to pay to obtain the object when the alternative is that the good goes to the other bidder conditional on the hypothetical event that the signal of the other bidder is equal to $\overline{s}_2^1(s_1)$ (respectively $\underline{s}_1^2(s_2)$.) To understand the intuitive meaning of this hypothetical event note first the following auxiliary result:

Lemma 6.

- $b_1^*(s_1) \geq b_2^*(s_2)$ *if and only if* $g_1(s_1) \geq g_2(s_2)$.
- $b_1^*(s_1) < b_2^*(s_2)$ *if and only if* $g_1(s_1) < g_2(s_2)$.

See proof in the Appendix.

The lemma is easy to understand for the "standard" case in which the single crossing condition holds. In this case the h_i's are typically strictly increasing. Then, $g_1(s_1) = g_2(s_2)$ is equivalent to $h_1(s_1) = h_2(s_2)$, $\overline{s}_2^1(s_1) = s_2$, and $\underline{s}_1^2(s_2) = s_1$, and these three conditions are equivalent to $b_1^*(s_1) = b_2^*(s_2)$. Thus, the monotonicity of the bid functions imply the lemma. As a matter of fact, the same argument holds true locally whenever $h_1(s_1) = g_1(s_1) = h_2(s_2) = g_2(s_2)$ and g_1 and g_2 are strictly increasing locally. However, the proof of the lemma requires a more elaborated analysis at points in which $h_i(s_i) \neq g_i(s_i)$.

One implication of the previous lemma is that $b_2^*(s_2) \in (b_1^*(s_1 - \epsilon), b_1^*(s_1 + \epsilon)]$ for $\epsilon > 0$ is equivalent to $s_2 \in (\overline{s}_2^1(s_1 - \epsilon), \overline{s}_2^1(s_1 + \epsilon)]$. Thus, in the continuity points[21] of b_1^*, $b_2^*(s_2) = b_1^*(s_1)$ is equivalent to $s_2 = \overline{s}_2^1(s_1)$, and hence, our proposed bid function b_1^* is such that Bidder 1 bids the expected value of the good conditional on tying with Bidder 2. The interpretation of Bidder 2's bid function is similar.

Corollary 1. *The allocation induced by (b_1^*, b_2^*) is second best efficient.*

See proof in the Appendix.

Finally, next proposition shows that the proposed bid functions are in fact an equilibrium:

Proposition 4. *The bid functions (b_1^*, b_2^*) form a Bayesian Nash equilibrium of the English auction.*

See proof in the Appendix.

We can thus conclude from Corollary 1 and Proposition 4,

Corollary 2. *The English auction has an equilibrium that implements the second best when there are two bidders.*

The reader may worry that the above bid functions are identity dependent. We shall argue that in general this is not a problem. First, let $K_i \equiv \{k \in \mathbb{R} : \exists(\underline{s}, \overline{s}) \neq \emptyset, g_i(s_i) = k, \forall s_i \in (\underline{s}, \overline{s})\}$ and note the following lemma:

Lemma 7. *If $K_1 \cap K_2 = \emptyset$, then $\underline{s}_j^i(s_i) = \overline{s}_j^i(s_i)$ almost everywhere.*

See proof in the Appendix.

Thus, this lemma together with the fact that asymmetries only arise when $\underline{s}_j^i(s_i) \neq \overline{s}_j^i(s_i)$ imply that when $K_1 \cap K_2 = \emptyset$ asymmetries only occur in a set of types with zero

[21] Note that b_1^* is continuous almost everywhere because it is increasing.

measure, and thus, could be removed without upsetting the equilibrium. Moreover, we expect $K_1 \cap K_2 = \emptyset$ to hold generically since the sets K_i's are countable.[22]

Nevertheless, the issue of asymmetries remains when bidders are symmetric, i.e. when $\phi_1 = \phi_2$, and $\zeta_1 = \zeta_2$. In this case, $K_1 \cap K_2 \neq \emptyset$, and in fact, the symmetric equilibrium is not second best efficient if the single crossing conditions does not hold. To see why, recall Example 2 and note that the only symmetric equilibrium is $b(s_i) = 3s_i$, $i = 1, 2$. This equilibrium allocates the good to the bidder with higher type which is not second best efficient. Instead, our proposed strategies are $b_1^*(s_1) = s_1 + 2$ and $b_2^*(s_2) = s_2$.

Indeed, the English auction has more problems of multiplicity of equilibria when the single crossing condition fails than when it holds, even when $K_1 \cap K_2 = \emptyset$. The following example provides an illustration of this point.

Example 5. *$N = \{1, 2\}$, $v_1(s_1, s_2) = s_1 + \frac{3}{2}s_2$ and $v_2(s_1, s_2) = s_2 + 2s_1$, and $e_1(s_1) = e_2(s_2) = 0$.*

In this example $h_1(s_1) = -s_1$ and $h_2(s_2) = -\frac{1}{2}s_2$. Thus, $g_1(s_1) = -\frac{1}{2}$ and $g_2(s_2) = -\frac{1}{4}$, and consequently, the second best allocation is to give the good to Bidder 2 for any realization of the bidders' types. This is the allocation that is implemented by our proposed equilibrium applied to this example: $b_1^*(s_1) = s_1$, $b_2^*(s_2) = s_2 + 2$. However, there exist other equilibria that do not implement the second best, for instance,[23] $b_1(s_1) = s_1 + \frac{3}{2}$, $b_2(s_2) = s_2$. Note that both this equilibrium and our proposed equilibrium (b_1^*, b_2^*) survive iterative elimination of weakly dominated strategies, whereas recall that Chung and Ely (2001) have shown that when the single crossing condi-

[22] A simple argument is as follows. Let $g_i^{-1}(k) \equiv \min\{s_i \in [0, 1] : g_i(s_i) = k\}$. It is easy to see that if g_i is constant and equal to k in an open interval, then the function g_i^{-1} is discontinuous at k by definition. Finally, note that the set of the discontinuities of g_i^{-1} must be countable since g_i^{-1} is increasing.

[23] Bidder 1 does not have incentives to deviate because she wins with probability one at a price less than her value, whereas Bidder 2 does not have incentives to deviate because any bid $p \in [b_1(0), b_1(1)]$, gives Bidder 2 expected payoffs $\int_0^{b_1^{-1}(p)} \left(s_2 + 2s_1 - (s_1 + \frac{3}{2})\right) ds_1 \leq 0$ for any $s_2 \in [0, 1]$. Moreover, Bidder 2 does not have incentives to bid above $b_1(1)$ because those bids give the same expected payoffs as a bid $b_1(1)$.

tion holds there is a unique equilibrium in the English auction with two bidders that survives to iterative elimination of weakly dominated strategies.

6.2 An English Auction with More than Two Bidders

In this section, we study the case in which there are more than two bidders, i.e. $n > 2$. We shall see that the English auction does not always implement the second best. We start with an example. In this example and in the rest of the section, it is important that we describe the tie-breaking rule. We shall assume the good is allocated with equal probability among the bidders that tie.[24] We conjecture that any other tie-breaking rule that does not condition on the bidders' types would imply similar results.

Example 6. *$N = \{1, 2, 3\}$, $v_1(s) = s_1 + \frac{1}{2}$, $v_2(s) = s_2 + 2s_1$, $v_3(s) = s_3 + 2s_1$ and $e_i(s_i) = 0$ for all i.*

In this example, $h_1(s_1) = \frac{1}{2} - s_1$, $g_1(s_1) = 0$, and $h_i(s_i) = g_i(s_i) = s_i$, $i \in \{2, 3\}$. Hence, it is second best efficient to allocate to Bidder 2 if $s_2 > s_3$ and to Bidder 3, otherwise. Note that, as in Example 4, Bidder 1 has a unique weakly dominant strategy: to remain in the auction until the price reaches her value $s_1 + \frac{1}{2}$. We show in the next lemma that there is no equilibrium of the English auction that implements the second best efficient allocation when Bidder 1 uses her unique weakly dominant strategy.

Lemma 8. *In Example 6, there is no equilibrium of the English auction in non-weakly dominated[25] strategies that implements the second best efficient allocation.*

[24] In our auction, the price increases continuously until one bidder or more quit. Then, the price is stopped and the following algorithm is repeated: (1) If there are no more active bidders, the good is allocated with equal probability among the bidders that last quitted at the current price. Otherwise, (2) the identity of the bidders that still remain active is announced. (3) After the announcement, bidders that still remain active declare independently and simultaneously whether they quit. If no bidder quits, the price starts again to increase from the current level. If some bidder quits, we go to (1) again.

[25] Actually, in this lemma and in Lemma 9 we only need that Bidder 1 uses her weakly dominant strategy.

Proof. To simplify, we refer in the proof to an equilibrium in non-weakly dominated strategies as an equilibrium. We shall prove the lemma in two steps. First, we show by contradiction that in any equilibrium that implements the second best, the strategies of both Bidder 2 and Bidder 3 must specify that they quit at a price less than Bidder 1's minimum bid, i.e. $\frac{1}{2}$, in information sets in which no bidder has quit yet for types less than 1/2. Second, we argue that if such is the case Bidder 3 has a profitable deviation.

Suppose Bidder 1 quits at a price $p > \frac{1}{2}$, and both Bidder 2 and 3 are still active and have types s_2 and s_3 less than 1/2. Then Bidder 2 and 3 put a value in getting the good equal to $s_2 + 2(p - 1/2)$ and $s_3 + 2(p - 1/2)$, respectively, which are less than the price p. Hence, they both quit immediately after Bidder 1 and tie. Our tie breaking rule implies that the induced allocation cannot be second best efficient.

Suppose now that Bidder 2 quits at a price $p \leq \frac{1}{2}$ when she has a type s_2 in information sets in which no bidder has quit yet. Then, second best efficiency implies that Bidders 3 with a type s_3 in $(0, s_2)$ also quits at a price strictly less than p in the same information sets. In this case, Bidder 3 has a profitable deviation. In this deviation Bidder 3 remains in the auction until either the price reaches $\frac{1}{2}$ or Bidder 2 quits. In the former case, Bidder 3 quits, and in the latter, Bidder 3 remains active until Bidder 1 quits. This deviation lets Bidder 3 win additionally when Bidder 2 has a type in the set (s_3, s_2). In this case, Bidder 3 pays Bidder 1's bid and gets strictly positive expected payoffs:

$$\int_0^1 \left(s_3 + 2s_1 - (s_1 + \frac{1}{2}) \right) ds_1 = s_3 + \int_0^1 (s_1 - \frac{1}{2})\, ds_1 = s_3 > 0.$$

■

Intuitively, if Bidders 2 and 3 have a low type and are still active when Bidder 1 quits at a price close to 1/2 a "rush" occurs since both bidders find out that their values are less than the price. Thus, the need to select the most efficient bidder between Bidder 2 and Bidder 3 before a rush may occur bounds Bidders 2 and Bidder 3 drop out prices to a level that is incompatible with their private incentives. Hence the impossibility.

The above result generalizes when Bidder 1 has no uncertainty about the value she puts in winning and under no externalities as follows:

Lemma 9. *Suppose $n > 2$ and $e_i(s_i) = q_i(s_i) = 0$ for $i \neq 1$ and $s_i \in [0,1]$. There is no equilibrium in non-weakly dominated strategies of the English auction that implements the second best efficient allocation, if:*

$$g_1(\tilde{s}_1) < \max_{j \neq 1,i}\{h_j(\tilde{s}_j)\} = h_i(\tilde{s}_i) < h_1(\tilde{s}_1). \tag{3}$$

for some vector $\tilde{s} \in (0,1)^n$.

See proof in the Appendix.

In particular, the conditions in the above lemma are the same as in Lemma 5 plus the assumption that $n > 2$ and that when the single crossing condition fails for Bidder 1 there exists a vector of types for which the first best allocates the good to Bidder 1 but the second best allocates the good to either i or j. Note that we would expect the conditions of the lemma to hold in general in the models of Section 4.1 and 4.2. However, the above impossibility result does not necessarily hold once we move away from the conditions of the above lemma. To see why, note the following generalization of Example 6.

Example 7. *$N = \{1,2,3\}$, $v_1(s) = s_1 + \frac{1}{2} + \alpha(s_2 + s_3)$, $v_2(s) = s_2 + 2s_1 + \alpha(s_2 + s_3)$, $v_3(s) = s_3 + 2s_1 + \alpha(s_2 + s_3)$, and $e_i(s_i) = 0$ for all i.*

For these value functions we have that $h_1(s_1) = -s_1 + \frac{1}{2}$, $g_1(s_1) = 0$ and $h_i(s_i) = g_i(s_i) = s_i$, $i : 2, 3$. Note that the second best allocation is as in Example 6. Moreover, for values of α close to zero, we expect that a variation of the arguments in Lemma 8 also show that the second best is not implementable with an English auction. However, if α is sufficiently large, there is some multiplicity of equilibria that allows for an equilibrium that avoids the possibility of a “rush” by making bidders bid sufficiently low in information sets in which no bidder has quit yet. As next lemma shows, this is enough to guarantee that the second best can be implemented with an English auction.

Lemma 10. *There exists a perfect Bayesian equilibrium in non-weakly dominated strategies that implements the second best allocation in Example 7 when $\alpha = 1$. In this equilibrium:*

- *Bidder 1 bids $s_1 + \frac{1}{2}$, Bidder $j \neq 1$ bids $3s_j + 2$, in information sets in which no bidder has left the auction yet.*
- *Bidder i, $i = 2, 3$, bids $3s_i + 2p - 1$ in information sets in which Bidder 1 has quit at price p.*
- *Bidder 1 bids $s_1 + \frac{1}{2} + \frac{p-2}{3}$, and Bidder $j \neq 1$ bids $2s_j + \frac{p-2}{3} + 2$ in information sets in which Bidder $i \neq 1, j$ has quit at price p.*

Proof. That the proposed strategies implement the second best is straightforward: Bidder 1 quits first, followed by the bidder with lower type between Bidder 2 and Bidder 3. To show that it is an equilibrium, note that if Bidder 1 deviates and wins, she pays a price equal to $2s_2 + s_3 + 2$ if $s_2 \geq s_3$ (the other case is symmetric), which is greater than her value $s_1 + s_2 + s_3 + 1/2$, and thus makes the deviation unprofitable. Next note that if Bidder i, $i \in \{2, 3\}$ deviates and wins the auction when the other bidders follow the proposed strategy, Bidder i pays a price $3s_j + s_1$, $j \neq 1, i$, which is less that the value of Bidder i if and only if $s_j < s_i$. Thus, Bidder i does not have incentives to deviate since our proposed strategy makes her win in these cases, and lose otherwise. ■

Note that the strategies in the lemma are such that in information sets in which no bidder has quit yet Bidder 1 bids as if the types of s_2 and s_3 were zero, whereas Bidder 2 and Bidder 3 bid as if they both had the same type and Bidder 1 had her highest possible type. Note also that Bidder 1 always quits first.

In the previous examples of this section we assumed no externalities. As we shall show next, externalities cause efficiency losses in an English auction that go beyond the problem of feasibility of the first best. To illustrate this point we study two examples in which the single crossing conditions holds and hence the first best is feasible. The first example displays positive externalities and the second one negative externalities.

Example 8. *$N = \{1, 2, 3\}$, $v_1(s) = s_1 + 1$, $v_2(s) = s_2 + 1$, $v_3(s) = s_3 + 1$, $e_2(s_2) = e_3(s_3) = 0$ and $e_1(s_1) = -1/2$.*

Bidder 1 has a weakly dominant strategy, to quit at price $b_1(s_1) = s_1 + 1$ and the first best efficient allocation is to allocate always to Bidder 1. But, this cannot occur in an equilibrium in which Bidder 1 bids $b_1(s_1) = s_1 + 1$ because Bidder 2 with a type $s_2 > 1/2$ finds it strictly profitable to outbid Bidder 1 for prices less than $s_2 + 1/2$. Clearly, Bidder 2 does not internalize the positive externality that allocating the good to Bidder 1 has on Bidder 3.

Example 9. *$N = \{1, 2, 3\}$, $v_1(s) = 2$,[26] $v_2(s) = s_2 + 1$, $v_3(s) = s_3 + 1$, $e_2(s_2) = e_3(s_3) = 0$ and $e_1(s_1) = 1$.*

Bidder 1 finds it weakly dominant to bid $b_1(s_1) = 2$ and the first best efficient allocation is that the good is allocated to Bidder 2 if $s_2 \geq s_3$, and to Bidder 3, otherwise. Note that when Bidder 1 bids $b_1(s_1) = 2$, Bidder i, $i \in \{2, 3\}$, with type s_i finds it optimal to outbid Bidder 1 in any continuation game in which only Bidder 1 and i are active. In this case, Bidder i gets payoffs $s_i + 1 - 2$, which are negative if $s_i < 1$. Thus, in any equilibrium in which Bidders 2 and Bidder 3 use this continuation strategy, they both have strict incentives to quit first when all bidders are still active. Consequently, either Bidder 2 or Bidder 3 must quit with positive probability at price 0, which is incompatible with the first best efficient allocation.

Intuitively, Bidder 2 and Bidder 3 do not want Bidder 1 to win, but they both prefer that it is the other bidder who pays the high price necessary to outbid Bidder 1.[27]

[26] To make the argument more transparent, we have deviated slightly from the general assumptions of Section 2 and we allow v_1 to be constant on s_1.

[27] A related argument was pointed out by Jehiel and Moldovanu (1996) and Hoppe, Jehiel, and Moldovanu (2006) to argue that externalities may induce strategic non participation in auctions.

7 Conclusions

In this paper, we have studied mechanisms that maximize the expected social surplus deriving from the sale of a (single-unit) object subject to Bayesian incentive compatibility constraints. An alternative approach for future research is the equivalent analysis under ex post incentive compatible constraints. This alternative is especially interesting since Ledyard (1978) and Bergemann and Morris (2005) have shown it to be equivalent to Bayesian implementation for any possible prior. In this respect, it is remarkable that the set of second best allocations subject to always selling that we characterize always includes an allocation that satisfies the conditions provided by Bikhchandani, Chatterji, and Sen (2006) for ex post implementability.[28]

This extension, however, presents additional difficulties. The reason is that second best efficiency requires trading off between different inefficient allocations. Under Bayesian implementation, the common prior gives natural weights for trading-off these inefficient allocations, but this is not the case under ex post implementation.

[28]This may be shown by noting that an allocation such that for any s, $p_i(s) = 1$ if $g_i(s_i) = \max_{j\in N}\{g_j(s_j)\}$ and $i \leq \arg\max_{j\in N}\{g_j(s_j)\}$ satisfies the conditions in Proposition 2 and since the g_i's are increasing, it also satisfies the conditions for ex post implementability provided by Bikhchandani, Chatterji, and Sen (2006).

Appendix

A Proofs

Proof of Lemma 1

Proof. We first prove the "only if"-part. Suppose a feasible allocation p. Then, there exists a direct mechanism (p, x) for which:

$$\begin{aligned} V_i(s_i) \equiv U_i(s_i, s_i) &\geq U_i(s_i, s_i') \\ &= Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\ &= Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i')) \\ &= V_i(s_i') + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i')), \end{aligned}$$

for all $s_i, s_i' \in S_i$, $i \in N$. Hence, we have that for $s_i > s_i'$, $V_i(s_i) \geq V_i(s_i')$, and hence,

$$Q_i(s_i', p) \leq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')},$$

and applying the same inequality with the roles of s_i and s_i' interchanged,

$$Q_i(s_i, p) \geq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')}.$$

Thus, $Q_i(s_i, p) \geq Q_i(s_i', p)$ as desired.

To prove the "if"-part, suppose an allocation p for which $Q_i(., p)$ is increasing, for all $i \in N$. Note first that by assumption $\phi_i(.)$ is a strictly increasing function, and thus invertible in $[\phi_i(0), \phi_i(1)]$. Let $\tilde{V}_i(y) \equiv \int_{\phi_i(0)}^{y} Q_i(\phi_i^{-1}(\tilde{y}), p)\, d\tilde{y}$ for $y \in [\phi_i(0), \phi_i(1)]$ and,

$$x_i(s) \equiv Q_i(s_i, p)\phi_i(s_i) + \sum_{j \neq i} (p_i(s_i, s_{-i})q_j(s_j) - e_j(s_j)p_j(s_i, s_{-i})) - \tilde{V}_i(\phi_i(s_i)),$$

for any $s \in S$. This means that $\Psi_i(s_i, p, x) = \tilde{V}_i(\phi_i(s_i)) - Q_i(s_i, p)\phi_i(s_i)$, for any $s_i \in S_i$, and hence that $\tilde{V}_i(\phi_i(s_i)) = U_i(s_i, s_i)$ for the direct mechanism (p, x). We shall show that this direct mechanism satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $s_i, s_i' \in S_i$

$$\begin{aligned} U_i(s_i, s_i) &= \tilde{V}_i(\phi_i(s_i)) \\ &\geq \tilde{V}_i(\phi_i(s_i')) + Q_i(\phi_i^{-1}(\phi_i(s_i')), p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\ &= Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\ &= Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\ &= U_i(s_i, s_i'), \end{aligned}$$

where the inequality is a consequence of $\tilde{V}_i$ being a convex function and $Q_i(\phi_i^{-1}(y), p) \in \partial\tilde{V}_i(y)$ by definition of $\tilde{V}_i$. ■

Proof of Lemma 2

Proof. Substracting $\sum_j q_j(s_j)$ from the two sides of the inequalities that define the single crossing condition, one gets that the single crossing condition is equivalent to say that for any $i \in N$, $s \in S$ and $s_i' \in S_i$ such that $s_i' > s_i$:

$$h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i} \text{ implies } h_i(s_i') \geq \max\{h_j(s_j)\}_{j\neq i}. \tag{4}$$

This is equivalent to say that for any $i \in N$, $s_i, s_i' \in S_i$ and $s_i' > s_i$, it is verified that $A_i(s_i) \subset B_i(s_i')$ for $A_i(s_i) \equiv \{s_{-i} \in S_{-i} : h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i}\}$ and $B_i(s_i') \equiv \{s_{-i} \in S_{-i} : h_i(s_i') \geq \max\{h_j(s_j)\}_{j\neq i}\}$. This is equivalent to say that for any $i \in N$, $s_i, s_i' \in S_i$ and $s_i' > s_i$, it is verified that $A_i(s_i) \cap [S_{-i} \setminus B_i(s_i')] = \emptyset$, which corresponds to the condition in the lemma. ■

Proof of Proposition 1

Proof. For the sufficient part, note that an allocation such that $p_i(s) = 0$ if $h_i(s_i) \neq \max_{j\in N} h_j(s_j)$, and $p_i(s) = \frac{1}{m(s)}$, otherwise, where $m(s)$ denotes the cardinality of $\{k \in N : h_k(s_k) = \max_{j\in N} h_j(s_j)\}$ is first best efficient and satisfy the feasibility conditions of Lemma 1 if the condition in the Lemma 2 is satisfied.

Next, let $\overline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} \leq h_i(s_i)\}$, $\underline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} < h_i(s_i)\}$ and $\mu_i(A) \equiv \int_A \prod_{j\neq i} f_j(s_j)\, ds_{-i}$ for $A \subset S_{-i}$. Note that for any first best efficient allocation p^*:

$$\mu_i\left(\underline{J}_i(s_i)\right) \leq Q_i(s_i,p^*) \leq \mu_i\left(\overline{J}_i(s_i)\right).$$

We prove the necessary part by contradiction. Suppose that the single crossing condition does not hold. Then, by Lemma 2 there exists a bidder $i \in N$ with types $s_i' > s_i$ and a vector $s_{-i} \in S_{-i}$ such that $\max\{h_j(s_j)\}_{j\neq i} \in (h_i(s_i'), h_i(s_i))$. By either right or left continuity of the $h_j(.)$'s, there exists an open set $O \subset S_{-i}$ such that $\max\{h_j(s_j')\}_{j\neq i} \in (h_i(s_i'), h_i(s_i))$ for any $s_{-i}' \in O$. By definition, $\overline{J}_i(s_i') \cap O = \emptyset$ and $\overline{J}_i(s_i') \cup O \subset \underline{J}_i(s_i)$. Thus, for any first best efficient allocation p^*,

$$Q_i(s_i',p^*) \leq \mu_i\left(\overline{J}_i(s_i')\right) < \mu_i\left(\overline{J}_i(s_i')\right)+\mu_i(O) = \mu_i\left(\overline{J}_i(s_i') \cup O\right) \leq \mu_i\left(\underline{J}_i(s_i)\right) \leq Q_i(s_i,p^*),$$

which implies a violation of the feasibility conditions of Lemma 1. ■

Proof of Proposition 2

Proof. The second best maximizes:

$$\int_S \sum_{i=1}^n \left(v_i(t) - (n-1)e_i(s)\right) p_i(s)\, ds = \int_S \sum_{i=1}^n \left(h_i(s_i) + \sum_{j=1}^n q_j(s_j)\right) p_i(s)\, ds. \qquad (5)$$

Next note that using integration by parts (see Hewitt (1960)) and Lemma 3 (b) we can show that,

$$\begin{aligned}\int_S \left(h_i(s_i) - g_i(s_i)\right) p_i(s) ds &= \int_{S_i} \left(h_i(s_i) - g_i(s_i)\right) Q_i(s_i,p)\, ds_i = \\ &\int_{S_i} Q_i(s_i,p)\, dH_i(s_i) - \int_{S_i} Q_i(s_i,p)\, dG_i(s_i) = \\ &- \int_{S_i} \left(H_i(s_i) - G_i(s_i)\right)\, dQ_i(s_i,p).\end{aligned}$$

Consequently, the expressions in Equation (5) are equal to the expression in Equation (1) as desired.

It is easy to see that an allocation that always sells maximizes the first integral in the equation above if and only it satisfies (i). Moreover, since $Q_i(.,p)$ is increasing for any p feasible by Lemma 1, and $G_i(s_i) \leq H_i(s_i)$, by Lemma 3 (c), a feasible allocation maximizes the second integral if and only if it satisfies (ii). This completes the proof since the set of feasible allocations that always sell and satisfy (i) and (ii) is not empty. For instance, $p_i(s) = 0$ if $g_i(s_i) \neq \max_{j \in N} g_j(s_j)$, and otherwise, $p_i(s) = \frac{1}{m(s)}$, where $m(s)$ denotes the cardinality of $\{k \in N : g_k(s_k) = \max_{j \in N} g_j(s_j)\}$. The monotonicity of $g_i(s_i)$ ensures that the allocation is feasible. ■

Proof of Proposition 3.

Proof. Basically, repeat the proof of Proposition 2 noting that when $\max_{i \in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s) \geq 0$ for any s, the constraint $\sum_{i=1}^{n} p_i(s) = 1$ is not binding. ■

Proof of Lemma 4

Proof. The function g_i cannot be discontinuous at points in an open interval in which $G_i(s_i) = H_i(s_i)$ by continuity of h_i, or at points in an open interval in which $G_i(s_i) \neq H_i(s_i)$ by Lemma 3 (d). Take now a point $s_i^* \in (0, 1)$ and an open interval O that includes s_i^* and such that $H_i(s_i) = G_i(s_i)$ if $s_i \in O$ and $s_i < s_i^*$ and $H_i(s_i) > G_i(s_i)$ if $s_i \in O$ and $s_i > s_i^*$ (the other case is symmetric). Then the left derivative of G_i is equal to $h_i(s_i^*)$ and the right derivative is bounded above by $h_i(s_i^*)$. Moreover, by the convexity of G_i the left derivative of G_i must be less than or equal to the right derivative. As a consequence, G_i is differentiable at s_i^* and its differential $g_i(s_i^*)$ is equal to $h_i(s_i^*)$. Continuity at 0 and 1 together with the last two items of the lemma are direct consequences of Lemma 3 (b) and (c) and the boundedness of h_i. ■

Proof of Lemma 5

Proof. By the same argument as in Example 4, Bidder 1 has a unique weakly dominant strategy, to bid until $b_1(s_1) \equiv t_1(s_1) + q_1(s_1)$. We show next that the resulting allocation when Bidder 1 bids $b_1(s_1)$ and Bidder 2 plays a best response to $b_1(s_1)$ is second best efficient. First, note that b_1 is continuous and strictly increasing, and hence, its inverse b_1^{-1} exists. Bidder 2 wins the auction with a bid b if and only if $s_1 \leq b_1^{-1}(b)$. Thus, Bidder 2's expected payoffs when she bids $b \in [b_1(0), b_1(1)]$ are equal to:

$$\int_0^{b_1^{-1}(b)} (t_2(s_2) + q_1(\tilde{s}_1) - b_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_{b_1^{-1}(b)}^1 e_1(\tilde{s}_1)\, d\tilde{s}_1$$

$$= \int_0^{b_1^{-1}(b)} (t_2(s_2) + q_1(\tilde{s}_1) + e_1(\tilde{s}_1) - b_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1$$

$$= \int_0^{b_1^{-1}(b)} (h_2(s_2) - h_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1 =$$

$$\int_0^{b_1^{-1}(b)} (g_2(s_2) - g_1(\tilde{s}_1))\, d\tilde{s}_1 - \left(H_1(b_1^{-1}(b)) - G_1(b_1^{-1}(b))\right) - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1,$$

where in the third step we have used that h_2 is increasing under the assumptions of the lemma and thus $g_2(s_2) = h_2(s_2)$.

Let $\overline{s}_1^2(s_2)$ be the maximum of the set $\{s_1 \in [0,1] : g_1(s_1) = g_2(s_2)\}$ if non-empty; $\overline{s}_1^2(s_2) = 1$, if $g_2(s_2) > g_1(1)$; and $\overline{s}_1^2(s_2) = 0$, if $g_2(s_2) < g_1(0)$. Note that $b = b_1(\overline{s}_1^2(s_2))$ maximizes the last expression, and in particular the first two terms. That it maximizes the first term is direct from the definition of $\overline{s}_1^2$. To show that it also maximizes the second term, note that by Lemma 3 (c), this second term can only be negative or zero. Thus, it is sufficient to show that for $b = b_1(\overline{s}_1^2(s_2))$ it is equal to zero. This is direct from the definition of $\overline{s}_1^2$ and Lemma 3 (b) and (d). Consequently, any maximum to the above expression must maximize the first two terms. We shall show that this implies that the induced allocation satisfies both conditions in Proposition 2. The maximization of the first term implies directly condition (i). Condition (ii) holds trivially for $i = 2$ since $g_2 = h_2$, i.e. $G_2 = H_2$. Finally, note that the maximization of the second term implies that no type of Bidder 2 bids at points in which $H_1(b_1^{-1}(b)) > G_1(b_1^{-1}(b))$, and thus condition (ii) is also satisfied for $i = 1$. ■

Proof of Lemma 6

Proof. We only prove the first item. The second one is simply the negation of the first one. We first show the "if" part. Consider first the case $g_1(0) > g_2(1)$. By monotonicity of the bid functions, we only need to show that $g_1(0) > g_2(1)$ implies that $b_1(0) \geq b_2(1)$. Note that it is easy to see that $b_1(0) - b_2(1) = \phi_1(0) + \zeta_2(1) - (\phi_2(1) + \zeta_1(0)) = h_1(0) - h_2(1)$, which is greater than $g_1(0) - g_2(1)$ by Lemma 4, and thus non-negative as desired. Consider now the case $g_1(0) \leq g_2(1)$. In this case, $g_1(s_1) \geq g_2(s_2)$ and continuity of the g_i's, see Lemma 4, implies that there exists a $s_1' \leq s_1$ and a $s_2' \geq s_2$ such that $g_1(s_1') = g_2(s_2')$, $s_1' = \underline{s}_1^2(s_2')$ and $s_2' = \overline{s}_2^1(s_1')$. Thus:

$$
\begin{aligned}
b_1^*(s_1) - b_2^*(s_2) \geq b_1^*(s_1') - b_2^*(s_2') = \\
\left[\phi_1(s_1') - \zeta_1(\underline{s}_1^2(s_2'))\right] - \left[\phi_2(s_2') - \zeta_2(\overline{s}_2^1(s_1'))\right] = \\
\left[\phi_1(\underline{s}_1^2(s_2')) - \zeta_1(\underline{s}_1^2(s_2'))\right] - \left[\phi_2(\overline{s}_2^1(s_1')) - \zeta_2(\overline{s}_2^1(s_1'))\right] = \\
h_1(\underline{s}_1^2(s_2')) - h_2(\overline{s}_2^1(s_1')).
\end{aligned}
$$

We next argue that the last expression is weakly greater than $g_1(\underline{s}_1^2(s_2')) - g_2(\overline{s}_2^1(s_1'))$. To see why, we argue that $h_1(\underline{s}_1^2(s_2')) \geq g_1(\underline{s}_1^2(s_2'))$ and $h_2(\overline{s}_2^1(s_1')) \leq g_2(\overline{s}_2^1(s_1'))$. We only show the former inequality since the latter one has a symmetric proof. Lemma 3 (d), Lemma 4 (a) and the definition of $\underline{s}_1^2$ implies that if $\underline{s}_1^2(s_2') \in (0,1)$ then $h_1(\underline{s}_1^2(s_2')) = g_1(\underline{s}_1^2(s_2'))$. Thus, by Lemma 4, we only need to show that if $\underline{s}_1^2(s_2') = 1$ then it cannot be that $G_1(1-\epsilon) < H_1(1-\epsilon)$ for any ϵ close to zero. By contradiction, suppose that $\underline{s}_1^2(s_2') = 1$ and $G_1(1-\epsilon) < H_1(1-\epsilon)$ for any ϵ close to zero. Then, $g_2(s_2') = g_1(1)$ and $g_1(s_1)$ is flat for any s_1 in $(1-\epsilon, 1]$ by Lemma 3 (d), which contradicts that $\underline{s}_1^2(s_2') = 1$.

Thus, the "if" part follows from the fact that $g_1(\underline{s}_1^2(s_2')) - g_2(\overline{s}_2^1(s_1'))$ is equal to zero since $g_1(s_1') = g_2(s_2')$, $s_1' = \underline{s}_1^2(s_2')$ and $s_2' = \overline{s}_2^1(s_1')$.

We prove the "only if" part by contradiction. We shall show that $g_2(s_2) > g_1(s_1)$ implies that $b_2(s_2) > b_1(s_1)$. The proof is similar to the "if" part. The case $g_2(0) > g_1(1)$ is symmetric to the case $g_1(0) > g_2(1)$ above. In the case $g_2(0) \leq g_1(1)$, $g_2(s_2) > g_1(s_1)$ implies that there exists a strictly decreasing sequence $\{s_{2,m}\}$ starting at s_2 and a strictly increasing sequence $\{s_{1,m}\}$ starting at s_1 with respective limits s_2' and s_1' that satisfy $g_2(s_2') = g_1(s_1')$, $s_2' = \overline{s}_2^1(s_1')$ and $s_1' = \underline{s}_1^2(s_2')$. Note that along the sequence $g_2(s_{2,m}) > g_1(s_{1,m})$ and hence, $s_{2,m} \geq \overline{s}_2^1(s_{1,m})$ and $s_{1,m} \leq \underline{s}_1^2(s_{2,m})$. Using these properties and the monotonicity of the bid functions and ζ_i we have that:

$$b_2^*(s_2) - b_1^*(s_1) > \lim_{m\to\infty} [b_2^*(s_{2,m}) - b_1^*(s_{1,m})] =$$
$$\lim_{m\to\infty} \left[\left(\phi_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(\phi_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right)\right] =$$
$$\lim_{m\to\infty} \left[\left(h_2(s_{2,m}) + \zeta_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(h_1(s_{1,m}) + \zeta_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right)\right] \geq$$
$$\lim_{m\to\infty} [h_2(s_{2,m}) - h_1(s_{1,m})] = h_2(\underline{s}_2^1(s_1')) - h_1(\overline{s}_1^2(s_2')),$$

which by the same arguments as in the "if" part is non-negative as desired. ■

Proof of Corollary 1

Proof. That the allocation induced by (b_1^*, b_2^*) satisfies condition (i) in Proposition 2 is direct from Lemma 6. To check condition (ii), note that for the allocation induced by (b_1^*, b_2^*), Lemma 6 implies that:

$$Q_1(s_1, p) = \int_{\{s_2: g_1(s_1) \geq g_2(s_2)\}} p_1(s_1, s_2)\, ds_2,$$

since ties occur with zero probability. Thus, in any open interval in which $H_1(s_1) > G_1(s_1)$ Lemma 3 (d) implies that $Q_1(s_1, p)$ is constant as required by condition (ii). A similar argument shows that Q_2 also satisfies (ii). ■

Proof of Proposition 4

Proof. We only show that Bidder 1 finds it optimal to bid according to b_1^* when Bidder 2 plays b_2^*. The corresponding proof for Bidder 2 is similar. Let $u_1(s_1, b)$ be the expected utility of Bidder 1 when she has a private type s_1, submits a bid b, and Bidder 2 uses the bid function b_2^*. We only show that Bidder 1 does not have incentives to deviate downwards, i.e. $u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) \geq 0$ for $b < b_1^*(s_1)$. The analysis of incentives to deviate upwards, i.e. $b > b_1^*(s_1)$, is symmetric. Downward deviations only affect payoffs when Bidder 1 wins with $b_1^*(s_1)$ and loses with b, i.e. when[29] $b_2^*(s_2) \in (b, b_1^*(s_1)]$. Recall also from Lemma 6 that $b_1^*(s_1) \geq b_2^*(s_2)$ is equivalent to $g_1(s_1) \geq g_2(s_2)$, which

[29] Note that ties occur with probability zero and thus the conclusions do not change whether considering the boundaries of the interval of bids close or open.

implies: (a) $s_1 \geq \underline{s}_1^2(s_2)$ and (b) $g_1(\underline{s}_1^2(s_2)) \geq g_2(s_2)$. Moreover, by monotonicity of the bid functions we have that $\{s_2 : b_2^*(s_2) \in (b, b_1^*(s_1)]\} = \{s_2 : s_2 \in (\tau_2(b), \overline{s}_2^1(s_1)]\}$ for some $\tau_2(b) \in [0, \overline{s}_2^1(s_1)]$. Note also that if Bidder 1 wins, she gets a good with value $v_1(s_1, s_2)$ and pays Bidder 2's bid and if Bidder 1 loses she suffers a negative externality equal to $e_2(s_2)$. Thus, the change in utility when Bidder 1 wins is equal to $v_1^e(s_1, s_2) - v_2^e(s_2, \underline{s}_1^2(s_2)) = \phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))$. Thus,

$$
\begin{aligned}
&u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left(\phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))\right) ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left[\phi_1(s_1) - \zeta_1(\underline{s}_1^2(s_2))\right] - \left[\phi_2(s_2) - \zeta_2(s_2)\right] ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left[\phi_1(\underline{s}_1^2(s_2)) - \zeta_1(\underline{s}_1^2(s_2))\right] - \left[\phi_2(s_2) - \zeta_2(s_2)\right] ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left[h_1(\underline{s}_1^2(s_2)) - h_2(s_2)\right] ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left(g_2(s_2) - h_2(s_2)\right) ds_2
\end{aligned}
$$

where we use that ϕ_1 is strictly increasing and (a) in the first inequality. As for the second inequality, we use that $h_1(\underline{s}_1^2(s_2)) \geq g_1(\underline{s}_1^2(s_2))$ and (b). To see why $h_1(\underline{s}_1^2(s_2)) \geq g_1(\underline{s}_1^2(s_2))$, we argue by contradiction. Suppose that $h_1(\underline{s}_1^2(s_2)) < g_1(\underline{s}_1^2(s_2))$, then Lemma 4 implies that $\underline{s}_1^2(s_2) > 0$ and thus by continuity there exists an interval $(a, \underline{s}_1^2(s_2)]$, with $a \neq \underline{s}_1^2(s_2)$ such that any s_1 in this interval verifies that $h_1(s_1) < g_1(s_1)$ and hence that $H_1(s_1) \neq G_1(s_1)$. By application of Lemma 3 (d) we have that $g_1(s_1)$ is constant in $(a, \underline{s}_1^2(s_2)]$, which contradicts the definition of $\underline{s}_1^2(s_2)$.

Finally, we argue that $\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} (g_2(s_2) - h_2(s_2))\, ds_2$ is non-negative. This integral is equal to

$$
\left[G_2(\overline{s}_2^1(s_1)) - H_2(\overline{s}_2^1(s_1))\right] + \left[H_2(\tau_2(b)) - G_2(\tau_2(b))\right].
$$

The second difference is non-negative by Lemma 3 (c). We next argue that the first one is equal to zero. If $\overline{s}_2^1(s_1)$ is either zero or one, this is because of Lemma 3 (b); otherwise, it is because $G_2(\overline{s}_2^1(s_1)) < H_2(\overline{s}_2^1(s_1))$ would imply that g_2 is constant around $\overline{s}_2^1(s_1)$ by Lemma 3 (d), which is a contradiction with the definition of $\overline{s}_2^1(s_1)$. ■

Proof of Lemma 7

Proof. Since any increasing function can be discontinuous in at most countably many points and $\underline{s}_j^i$ is increasing, it is sufficient to show that at any point $s_i \in [0,1]$ for which $\underline{s}_j^i(s_i) < \overline{s}_j^i(s_i)$, the function $\underline{s}_j^i$ is discontinuous. To prove so, suppose a point s_i at which $\delta \equiv \overline{s}_j^i(s_i) - \underline{s}_j^i(s_i) > 0$. Then, by definition of $\underline{s}_j^i$ and $\overline{s}_j^i$, the function g_j is constant and equal to $g_i(s_i)$ in the interval $(\underline{s}_j^i(s_i), \overline{s}_j^i(s_i))$. This means that g_i is strictly increasing at s_i since it cannot be constant because $K_1 \cap K_2 = \emptyset$. Hence, $\overline{s}_j^i(s_i) < \underline{s}_j^i(s_i + \epsilon)$, and consequently $\underline{s}_j^i(s_i) + \delta < \underline{s}_j^i(s_i + \epsilon)$, for any $\epsilon > 0$. This implies that $\underline{s}_j^i$ is discontinuous at s_i. ■

Proof of Lemma 9

Proof. As in the proof of Lemma 8, we refer to an equilibrium in non-weakly dominated strategies simply as an equilibrium. We denote by $b_1(s_1) \equiv t_1(s_1) + q_1(s_1)$ Bidder 1's unique weakly dominant strategy. Finally, note that under the assumptions of the lemma, the functions h_i's, $i \neq 1$, are strictly increasing (since $h_i = \phi_i$) and thus $h_i = g_i$ for $i \neq 1$.

The proof is also sketched in two steps. We first provide three necessary conditions that must be satisfied in an equilibrium that implements the second best:[30]

(i) For any vector of types $s \in [0,1]^n$ that satisfies:

$$g_1(s_1) < \max_{j \neq \{1,i\}} \{h_j(s_j)\} < h_i(s_i) < h_1(s_1),$$

only Bidder i and 1 can be active along the equilibrium path when the price is equal to $b_1(s_1)$. We prove the claim by contradiction. We shall argue that if Bidder 1, Bidder i, and Bidder $l \neq \{1, i\}$ are active at a price $p \equiv b_1(s_1)$ in the equilibrium path induced by the above vector of types, then both Bidder i and l quit immediately if Bidder 1 quits. Hence, there is a tie and thus a contradiction

[30] The structure of the proof generalizes the proof of Lemma 8. Basically, the necessary condition (i) corresponds to the first step in the proof of Lemma 8, the necessary conditions (ii) and (iii) only play an auxiliary role, and what follows corresponds to the second step.

since our tie-breaking rule does not ensure the second best allocation. To understand why Bidder i quits immediately after Bidder 1 in the previous argument, note first that in the equilibrium path Bidder i infers from Bidder 1 quitting at price p that Bidder 1's type is equal to s_1. Thus, Bidder i infers that her value is equal to $t_i(s_i) + q_1(s_1)$, which is strictly less than the price $p = b_1(s_1)$ since

$$t_i(s_i)+q_1(s_1)-b_1(s_1) = t_i(s_i)+q_1(s_1)-t_1(s_1)-q_1(s_1) = h_i(s_i)-h_1(s_1) < 0. \tag{6}$$

Note also that by a similar argument Bidder l also infers that her value is less than the price. This explains why both Bidder i and l must quit immediately after Bidder 1 in equilibrium.

(ii) Bidder i, $i \neq 1$, with type s_i does not win at a price $p > b_1(\overline{s}_1^i(s_i))$ in the equilibrium path when Bidder 1 bids p, and thus has a type $s_1 \equiv b_1^{-1}(p) > \overline{s}_1^i(s_i)$. The reason is that the implemented allocation would not be second best because $s_1 > \overline{s}_1^i(s_i)$ implies that $g_1(s_1) > g_i(s_i)$.

(iii) Bidder i, $i \neq 1$, with type s_i does not win in the equilibrium path at a price strictly greater than $t_i(s_i)+q_1(b_1^{-1}(p))$ when Bidder 1 quits at a price p and thus has a type $s_1 = b_1^{-1}(p)$. The reason is that Bidder i does not find it profitable to win at these prices, and hence she would have a profitable deviation, to quit at price $t_i(s_i)+q_1(b_1^{-1}(p))$ if higher than p, or immediately after Bidder 1 otherwise.

We complete the proof by showing that there is a profitable deviation when Bidder 1 uses her unique weakly dominant strategy and all the other bidders a vector of strategies that verifies conditions (i)-(iii) above and that allocates the good according to the second best.

Let $\tilde{s} \in (0,1)^n$ be a vector that verifies the conditions in the statement of the lemma. The profitable deviation exists for a Bidder i, $i \neq 1$, with type $\tilde{s}_i$. To describe it, let $s_{\inf}(s_i)$ denote the infimum of the set $\{s_1 : h_i(s_i) < h_1(s_1)\}$ if not empty and note that $s_{\inf}(s_i)$ is right-continuous since h_1 and h_i are continuous and h_i increasing. Thus, there exists an $\hat{\epsilon} > 0$ small enough such that $h_i(\tilde{s}_i) < h_1(s_1)$ for any $s_1 \in (s_{\inf}(\tilde{s}_i), s_{\inf}(\tilde{s}_i+\hat{\epsilon})]$.

The proposed deviation is that Bidder i with type $\tilde{s}_i$ plays the action prescribed by her strategy but for a type[31] $\tilde{s}_i + \hat{\epsilon}$ (rather than her true type $\tilde{s}_i$) unless either of the following two cases occur: (a) that the price reaches $b_1(\overline{s}_1^i(\tilde{s}_i))$ when Bidder 1 is active; or (b) that Bidder 1 has already quit at price p, and the price is equal or above $t_i(\tilde{s}_i) + q_1(b_1^{-1}(p))$. In either of these two cases, the deviation prescribes that Bidder i quits immediately.

Since the original strategies implemented the second best and satisfy (ii) and (iii) this deviation lets Bidder i win in all the cases in which she was already winning with the original strategy (and at the same price). Moreover, Bidder i's deviation lets her win in some additional cases. To simplify the description of these additional cases, we shall restrict in what follows to the symmetric case in which $h_l = h_k$ for any $l, k \neq 1$ (and thus $g_l = g_k$). The extension to the general case is straightforward but requires a cumbersome notation. Thus, the only additional cases in which i may win with the deviation are when the maximum of the other bidders types but 1 is in $(\tilde{s}_i, \tilde{s}_i + \hat{\epsilon})$, and Bidder 1 has a type less than $\overline{s}_1^i(\tilde{s}_i)$, see a) above. The original strategy did not let i win in these cases because it is not second best efficient. We show next what happens under the deviation in these cases depending on the value of Bidder 1's type s_1:

- If $s_1 \in [0, s_{\inf}(\tilde{s}_i)]$: then, $h_i(\tilde{s}_i) \geq h_1(s_1)$ and thus, $t_i(\tilde{s}_i) + q_1(s_1) \geq b_1(s_1)$ by a similar argument as in Equation (6). As a consequence, condition (b) above ensures that i gets non-negative payoffs with the deviation if she wins.

- If $s_1 \in (s_{\inf}(\tilde{s}_i), s_{\inf}(\tilde{s}_i + \hat{\epsilon})]$: then, $h_i(\tilde{s}_i) < h_1(s_1)$ by definition of $\hat{\epsilon}$, or equivalently $t_i(\tilde{s}_i) + q_1(s_1) - b_1(s_1) < 0$, again by a similar argument as in Equation (6). Thus condition (b) above means that i quits immediately after 1 if i is still active and as a consequence, if i wins, she pays 1's bid.

- If $s_1 \in (s_{\inf}(\tilde{s}_i + \hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$, Bidder i wins with the deviation and pays 1's bid. This is because condition (i) implies that only Bidder 1 and i are active when

[31]The reason why the proof of this lemma is more complex than the proof of Lemma 8 is that we must make sure that the game does not move to an out-of-equilibrium path after the deviation since conditions (i)-(iii) only apply in the equilibrium path.

the price goes above $b_1(s_{\inf}(\tilde{s}_i+\hat{\epsilon}))$. To see why condition (i) applies, note that $s_1 \leq \overline{s}_1^i(\tilde{s}_i)$ means that $g_1(s_1) \leq g_i(\tilde{s}_i)$ and that $g_i(s_i) = h_i(\tilde{s}_i) \leq h_i(\tilde{s}_i+\hat{\epsilon})$ since h_i is increasing. Moreover, for any s_1 arbitrarily close but above $s_{\inf}(\tilde{s}_i+\hat{\epsilon})$, we have $h_i(\tilde{s}_i+\hat{\epsilon}) < h_1(s_1)$. Putting together these facts, we get $g_1(s_1) < h_i(\tilde{s}_i+\hat{\epsilon}) < h_1(s_1)$ as required.

Denote by $\rho(s_1)$ the probability with which i wins conditional on s_1 and on the maximum of $\{s_j\}_{j\neq 1,i}$ being in $(\tilde{s}_i, \tilde{s}_i+\hat{\epsilon})$ when i plays her deviation and all the other bidders follow the proposed strategies. Note that our previous arguments imply that $\rho(s_1) = 1$ if $s_1 \in (s_{\inf}(\tilde{s}_i+\hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$. We next use ρ to show that Bidder i gets strictly positive utility with the deviation in the last two cases above:

$$
\begin{aligned}
&\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (t_i(\tilde{s}_i) + q_1(s_1) - b_1(s_1))\,\rho(s_1)\,ds_1 = \\
&\qquad \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - h_1(s_1))\,\rho(s_1)\,ds_1 \geq \\
&\qquad \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - h_1(s_1))\;ds_1 = \\
&\qquad \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (g_1(\tilde{s}_1) - h_1(s_1))\;ds_1 + \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\;ds_1 = \\
&\qquad \big(G_1(\overline{s}_1^i(\tilde{s}_i)) - H_1(\overline{s}_1^i(\tilde{s}_i))\big) - (G_1(s_{\inf}(\tilde{s}_i)) - H_1(s_{\inf}(\tilde{s}_i))) + \\
&\qquad \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\;ds_1 \geq \\
&\qquad \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\;ds_1 > 0,
\end{aligned}
$$

where in the first inequality we use that $\rho(s_1) = 1$ if $s_1 \in (s_{\inf}(\tilde{s}_i+\hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$ and that $h_i(\tilde{s}_i) < h_1(s_1)$ for $s_1 \in (s_{\inf}(\tilde{s}_i), s_{\inf}(\tilde{s}_i+\hat{\epsilon})]$ by definition of $\hat{\epsilon}$; in the second inequality we use the same arguments as in the last paragraph of the proof of Proposition 4; and in the last inequality we use that $g_i(\tilde{s}_i) \geq g_1(s_1)$ (and that $h_i = g_i$) for $s_1 \leq \overline{s}_1^i(\tilde{s}_i)$, and strictly if $s_1 \in (s_{\inf}(\tilde{s}_i), \underline{s}_1^i(\tilde{s}_i))$ by definition of $\overline{s}_1^i$ and $\underline{s}_1^i$. Note $(s_{\inf}(\tilde{s}_i), \underline{s}_1^i(\tilde{s}_i))$ is non-empty because $s_{\inf}(\tilde{s}_i) < \tilde{s}_1$ and $\tilde{s}_1 < \underline{s}_1^i(\tilde{s}_i)$. The former inequality can be proved using that the vector $\tilde{s}$ verifies Equation (3) and the definition of $s_{\inf}$, and the latter one using also that $\tilde{s}$ verifies Equation (3) and the definition of $\underline{s}_1^i$.

As a consequence, the proposed deviation is profitable as desired. ■

B Multidimensional Type Models

In this appendix we extend our analysis to a family of problems with a multidimensional type space. We shall show that under certain assumptions the analysis of these models can be done with an equivalent model with a one-dimensional type space. This analysis allows the extension of the models in Section 4.

Suppose that Bidder i's private information is a three dimensional vector $\hat{s}_i = (\hat{t}_i, \hat{q}_i, \hat{e}_i)$ that it is drawn according to an independent distribution $\hat{F}_i$ with support in a bounded set $\hat{S}_i \subset \mathbb{R}^3$. We shall assume that this distribution is such that the induced distribution of $\hat{t}_i+\hat{q}_i$, say $\overline{F}_i$, has a strictly positive density $\overline{f}_i$ in all the support $\overline{S}_i \subset \mathbb{R}$. Denote by $\hat{S} = \prod_{i=1}^n \hat{S}_i$ and by $\overline{S} = \prod_{i=1}^n \overline{S}_i$. We assume that Bidder i gets utility $\hat{t}_i + \sum_{j=1}^n \hat{q}_j - b$ if she wins and pays b, and utility $-\hat{e}_j - b$ if $j \neq i$ wins and Bidder i pays b.

By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair $(\hat{p}, \hat{x})$, where $\hat{p} : \hat{S} \to [0,1]^n$ and $\hat{x} : \hat{S} \to \mathbb{R}^n$ such that $\sum_{i=1}^n \hat{p}_i(\hat{s}) \leq 1$ and where $\hat{p}_i(\hat{s})$ denotes the probability that Bidder i gets the good and $\hat{x}_i(\hat{s})$ denotes the payments of i to the auctioneer when the announced vector of types is equal to $\hat{s}$. We shall refer to $\hat{p}$ as an allocation*.

The expected utility of Bidder i with type $\hat{s}_i$ that reports $\hat{s}_i'$ when all the other bidders report truthfully is equal to:

$$\hat{U}_i(\hat{s}_i, \hat{s}_i') \equiv \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}_i', \hat{p}, \hat{x}),$$

where

$$\hat{Q}_i(\hat{s}_i, \hat{p}) \equiv \int_{\hat{S}_{-i}} \hat{p}_i(\hat{s}_i, \hat{s}_{-i}) \, d\hat{F}_{-i}(\hat{s}_{-i}),$$

and,

$$\hat{\Psi}_i(\hat{s}_i, \hat{p}, \hat{x}) \equiv \int_{\hat{S}_{-i}} \left(\hat{p}_i(\hat{s}_i, \hat{s}_{-i}) \sum_{j \neq i} \hat{q}_j - \hat{x}_i(\hat{s}_i, \hat{s}_{-i}) - \sum_{j \neq i} \hat{p}_j(\hat{s}_i, \hat{s}_{-i}) \hat{e}_j \right) \, d\hat{F}_{-i}(\hat{s}_{-i}),$$

for $\hat{F}_{-i}(\hat{s}_{-i}) \equiv \prod_{j \neq i} \hat{F}_j(\hat{s}_j)$ and $\hat{S}_{-i} \equiv \prod_{j \neq i} \hat{S}_j$.

Thus, we say that an allocation* $\hat{p} : S \to [0,1]^n$ is *feasible** if there exists a direct mechanism $(\hat{p}, \hat{x})$ that satisfies the following Bayesian incentive compatibility constraint:

$$\hat{U}_i(\hat{s}_i, \hat{s}_i) = \sup_{\hat{s}'_i \in \hat{S}_i} \{\hat{U}_i(\hat{s}_i, \hat{s}'_i)\},$$

for all $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We shall show that we can study second best efficiency in the model of this section, using the results in the model of Section 2:

Definition: Let the following be the *uni-dimensional equivalent* to a model as in Section 2 in which for all $i \in N$ and $s_i \in \overline{S}_i$:

$$S_i = \overline{S}_i,\ F_i(s_i) = \overline{F}_i(s_i),$$

$$t_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{t}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},\ q_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{q}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},\ \text{and } e_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{e}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},$$

where $\hat{\mathcal{S}}_i(s_i) \equiv \{\hat{s}_i \in \hat{S}_i : \hat{t}_i + \hat{q}_i = s_i\}$.

Definition: Let the *uni-dimensional version* of an allocation* $\hat{p}$ be a function $p : \overline{S} \to [0,1]^n$ where

$$p_i(s) = \int_{\hat{\mathcal{S}}_1(s_1)} \cdots \int_{\hat{\mathcal{S}}_n(s_n)} \hat{p}(\hat{s}) \, \frac{d\hat{F}_n(\hat{s}_n)}{\overline{f}_n(s_n)} \cdots \frac{d\hat{F}_1(\hat{s}_1)}{\overline{f}_1(s_1)}.$$

Lemma 11. *The allocation* $\hat{p}$ *is feasible* if and only if its uni-dimensional version* p *is feasible and* $\hat{Q}_i(\hat{s}_i, \hat{p}) \in [Q_i(\hat{t}_i + \hat{q}_i, p)^-, Q_i(\hat{t}_i + \hat{q}_i, p)^+]$ *for any* $\hat{s}_i \in \hat{S}_i$ *and* $i \in N$.[32]

[32] We denote by $Q_i(x_0, p)^-$ and $Q_i(x_0, p)^+$ the limits

$$\lim_{\substack{x \to x_0 \\ x < x_0}} Q_i(x,p) \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ x > x_0}} Q_i(x,p)$$

respectively. To avoid problems at the infimum and supremum of $\overline{S}_i$, we shall adopt the convention that $Q_i(\inf \overline{S}_i, p)^- = Q_i(\inf \overline{S}_i, p)$ and $Q_i(\sup \overline{S}_i, p)^+ = Q_i(\sup \overline{S}_i, p)$. We adopt the same notation and conventions for the functions $\mathcal{Q}_i$ in the proof of the lemma.

Proof. Note that using the definition of the uni-dimensional version, we can show that,

$$Q(s_i,p) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{Q}_i(\hat{s}_i,\hat{p})\,\frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)}$$

where p is the uni-dimensional version of $\hat{p}$. Thus, by application of Lemma 1, p is feasible and $\hat{Q}_i(\hat{s}_i,\hat{p}) \in [Q_i(\hat{t}_i+\hat{q}_i,p)^-, Q_i(\hat{t}_i+\hat{q}_i,p)^+]$ if and only if there exists a vector of increasing functions $\mathcal{Q}_i : \overline{S}_i \to [0,1]$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i,\hat{p}) \in [\mathcal{Q}_i(t_i+q_i)^-, \mathcal{Q}_i(t_i+q_i)^+]$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$, or what is the same, if and only if there exists a set of increasing convex functions $\hat{v}_i : \overline{S}_i \to \mathbb{R}_+$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i,\hat{p}) \in \partial\hat{v}(\hat{t}_i+\hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$, see Rockafellar (1970).

Thus, to prove the lemma it is sufficient to show the following equivalent statement:

> The allocation* $\hat{p}$ is feasible* if and only if there exists a set of increasing convex functions $\hat{v}_i : \overline{S}_i \to \mathbb{R}_+$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i,\hat{p}) \in \partial\hat{v}(\hat{t}_i+\hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We first prove the "only if"-part. Suppose a feasible* allocation* $\hat{p} : \hat{S} \to [0,1]^n$, and let $V_i(\hat{s}_i) \equiv \hat{U}_i(\hat{s}_i,\hat{s}_i)$. Then,

$$\begin{aligned}
V_i(\hat{s}_i) &\geq \hat{U}_i(\hat{s}_i,\hat{s}'_i) \\
&= \hat{Q}_i(\hat{s}'_i,\hat{p})(\hat{t}_i+\hat{q}_i) + \hat{\Psi}_i(\hat{s}'_i,\hat{p},\hat{x}) \\
&= \hat{Q}_i(\hat{s}'_i,\hat{p})(\hat{t}'_i+\hat{q}'_i) + \hat{\Psi}_i(\hat{s}'_i,\hat{p},\hat{x}) + \hat{Q}_i(\hat{s}'_i,\hat{p})(\hat{t}_i+\hat{q}_i-\hat{t}'_i-\hat{q}'_i) \\
&= V_i(\hat{s}'_i) + \hat{Q}_i(\hat{s}'_i,\hat{p})(\hat{t}_i+\hat{q}_i-\hat{t}'_i-\hat{q}'_i),
\end{aligned}$$

for all $\hat{s}_i,\hat{s}'_i \in \hat{S}_i$, $i \in N$, and some $\hat{x} : \hat{S} \to \mathbb{R}^n$.

The above inequality applied twice, one with the roles of $\hat{s}_i$ and $\hat{s}'_i$ interchanged, to any two vectors $\hat{s}_i,\hat{s}'_i \in \hat{S}_i$ such that $\hat{t}_i+\hat{q}_i = \hat{t}'_i+\hat{q}'_i$, implies that $V_i(\hat{s}_i) = V_i(\hat{s}'_i)$. Consequently, there exists a function $v_i : \overline{S}_i \to \mathbb{R}$ such that $V_i(\hat{s}_i) = v_i(\hat{t}_i+\hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$. Moreover, v_i is convex because V_i is convex. Note that V_i must be convex because it is equal to the maximum of some linear functions by the incentive compatibility constraint. Finally, note that the above inequality together with the definition of v_i implies that $v_i(y) \geq v_i(\hat{t}_i+\hat{q}_i) + \hat{Q}_i(\hat{s}_i,\hat{p})(y-(\hat{t}_i+\hat{q}_i))$ for any y in $\overline{S}_i$. This means that $\hat{Q}_i(\hat{s}_i,\hat{p}) \in \partial v_i(\hat{t}_i+\hat{q}_i)$ as desired.

To prove the "if"-part, suppose a function $\tilde{v}$ that satisfies the conditions of the lemma for an allocation* $\hat{p}$, and let $\hat{x} : \hat{S} \to \mathbb{R}^n$ be such that $\hat{\Psi}_i(\hat{s}_i, \hat{p}, \hat{x}) = \hat{v}_i(\hat{t}_i + \hat{q}_i) - (\hat{t}_i + \hat{q}_i)\hat{Q}_i(\hat{s}_i, \hat{p})$ for any $i \in N$. We shall show that the direct mechanism $(\hat{p}, \hat{x})$ satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $\hat{s}_i, \hat{s}_i' \in \hat{S}_i$:

$$\begin{aligned} V_i(\hat{s}_i) = \tilde{v}_i(\hat{t}_i + \hat{q}_i) &\geq \tilde{v}_i(\hat{t}_i' + \hat{q}_i') + \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}_i' - \hat{q}_i') \\ &= \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i' + \hat{q}_i') + \hat{\Psi}_i(\hat{s}_i', \hat{p}, \hat{x}) + \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}_i' - \hat{q}_i') \\ &= \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}_i', \hat{p}, \hat{x}) = \hat{U}_i(\hat{s}_i, \hat{s}_i'), \end{aligned}$$

where the inequality is a consequence of $\hat{Q}_i(\hat{s}_i', \hat{p}) \in \partial\tilde{v}_i(\hat{t}_i' + \hat{q}_i')$. ■

Now, we can state the main result of this appendix:

Proposition 5. *An allocation** $\hat{p}^*$ *is a solution to the problem,*

$$\max \int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s})\, d\hat{F}(\hat{s}),$$

subject to $\hat{p}$ *feasible** *and* $\sum_{i \in N} \hat{p}_i(\hat{s}) = 1$ *for all* $\hat{s} \in \hat{S}$*, if and only if its uni-dimensional version* p^* *is second best efficient subject to always selling for the uni-dimensional equivalent model.*

Proof. Take any $\hat{p}$ feasible and such that $\sum_{i \in N} \hat{p}_i(\hat{s}) = 1$ for all $\hat{s} \in \hat{S}$, and denote by p its one-dimensional version. Then we can deduce the lemma from the following sequence of algebraic transformations and the fact that by Lemma 11, feasibility* of $\hat{p}$ requires feasibility of p:

$$\int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s})\, d\hat{F}(\hat{s}) =$$

$$\int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{p}_i(\hat{s})\, \hat{F}(d\hat{s}) + \int_{\hat{S}} \sum_{i=1}^{n} \hat{p}_i(\hat{s}) \sum_{j=1}^{n} \hat{q}_j\, d\hat{F}(\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\hat{S}_i} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, d\hat{F}_i(\hat{s}_i) + \int_{\hat{S}} \sum_{j=1}^{n} \hat{q}_j\, d\hat{F}(\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{\mathcal{S}}_i(s_i)} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\hat{S}} \hat{q}_j\, d\hat{F}(\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{\mathcal{S}}_i(s_i)} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\hat{S}_j} \hat{q}_j\, d\hat{F}_j(\hat{s}_j) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} \left(t_i(s_i) - (n-1)e_i(s_i)\right) Q_i(s_i, p) \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\overline{S}_j} \int_{\hat{\mathcal{S}}_j(s_j)} \hat{q}_j\, \frac{d\hat{F}_j(\hat{s}_j)}{\overline{f}_j(s_j)} \overline{f}_j(s_j)\, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) - (n-1)e_i(s_i)\right) p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j} q_j(s_j)\, \overline{f}_j(s_j)\, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) - (n-1)e_i(s_i)\right) p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j} q_j(s_j) \sum_{i=1}^{n} p_i(s)\, \overline{f}_j(s_j)\, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) - (n-1)e_i(s_i)\right) p(s) \overline{f}(s) ds + \sum_{i=1}^{n} \int_{\overline{S}} \left(\sum_{j=1}^{n} q_j(s_j) \right) p_i(s)\, \overline{f}(s)\, ds =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) + \sum_{j=1}^{n} q_j(s_j) - (n-1)e_i(s_i) \right) p_i(s) \overline{f}(s) ds,$$

where we have used: in step 2, that $\sum_{j=1}^{n} \hat{p}(\hat{s}) = 1$; in step 4, independency of the $\hat{F}_i$'s; in Step 5, that $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$ a.e., see below; and in step 7, that $\sum_{i=1}^{n} p_i(s) = 1$.

To see why $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$ a.e., note that Lemma 11 and $\hat{p}$ feasible* imply that p is feasible, and thus $Q_i(., p)$ increasing by Lemma 1, and hence continuous a.e. As a consequence, applying Lemma 11 again we get that $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$. ■

Finally, we provide as an application an example that generalizes the model in Section 4.2 to the cases not covered there, i.e. $\rho + \underline{t} \leq \overline{t}$.

Example 10. *Suppose a set $N = \{1, 2, ..., n\}$, and that $\hat{F}_1$ has full support on $\hat{S}_1 = [\underline{t}, \overline{t}] \times \{0, \rho\} \times \{0\}$ and $\hat{F}_i$ on $\hat{S}_i = [\underline{t}, \overline{t}] \times \{0\} \times \{0\}$ for $i \neq 1$. Suppose also that $\hat{t}_1$ and*

$\hat{q}_1$ are independent and $\hat{t}_1$ has a marginal distribution G with strictly positive density g over the support and $\hat{q}_1$ takes value 0 with probability $\alpha \in (0,1)$ and ρ with probability $1-\alpha$. Finally, suppose $\rho + \underline{t} \leq \bar{t}$.

In the above example $q_1(s_1) = \rho \frac{g(s_1-\rho)(1-\alpha)}{g(s_1)\alpha + g(s_1-\rho)(1-\alpha)}$, if $s_1 > \rho$ and zero, otherwise; $t_1(s_1) = s_1 - q_1(s_1)$, $e_1(s_1) = 0$, $t_i(s_i) = s_i$, $q_i(s_i) = e_i(s_1) = 0$ for $i \neq 1$. Thus, its uni-dimensional version violates the single crossing condition. To see why, apply Lemma 2 to $s_1 = \rho + \underline{t} - \epsilon$ and $s_1' = \rho + \underline{t} + \epsilon$ noting that $h_1(s_1) = t_1(s_1)$, and thus that $h_1(\rho + \underline{t} - \epsilon) = \rho + \underline{t} - \epsilon > \rho + \underline{t} + \epsilon - q_1(\rho + \underline{t} + \epsilon) = h_1(\rho + \underline{t} + \epsilon)$ for $\epsilon > 0$ and small enough.

Supplementary Material

Supplementary material to Footnote 17.

Example: $N = \{1, 2\}$, $v_i(s) = 20 \cdot \mathbf{1}_{[.9,1]}(s_j) + (s_i + s_j) + \mathbf{1}_{[1/2,1]}(s_i) + \mathbf{1}_{[1/2,1]}(s_j)$ and $e_i(s_i) = 0$.

In this example, note that $h_i(s_i) = -20 \cdot \mathbf{1}_{[.9,1]}$ and $q_i(s_i) = 20 \cdot \mathbf{1}_{[.9,1]}(s_i) + s_i + \mathbf{1}_{[1/2,1]}(s_i)$. Thus, $H_i(s_i) = -20 \cdot \mathbf{1}_{[.9,1]}(s_i - .9)$. Consequently, $G_i(s_i) = -2 \cdot s_i$ and $g_i(s_i) = -2$. Hence, it is easy to see that $g_i(s_i) + q_i(s_i) + q_j(s_j) > 0$ if $s_i + s_j > 1$ and $g_i(s_i) + q_i(s_i) + q_j(s_j) < 0$ if $s_i < .1$ and $s_j < .9$. The former implies that $\int_0^1 \int_0^1 (p_1(s) + p_2(s))\, ds_1\, ds_2 > 1/2$ and the latter that $Q_i(s_i, p) \leq .1$ if $s_i < .1$ for any an allocation that maximizes the first integral in Equation (1). Moreover, it is also easy to see that any allocation that maximizes the second integral in Equation (1) requires that the $Q_i(., p)$'s are constant. Hence any allocation that maximizes both integrals must satisfy that $1/2 < \int_0^1 \int_0^1 (p_i(s) + p_i(s))\, ds_i\, ds_j = \int_0^1 Q_i(s_i)\, ds_i + \int_0^1 Q_j(s_j)\, ds_j \leq .1 + .1$, which is a contradiction.

Omitted steps in the Proof of Proposition 4

We provide here the three omitted steps in the Proof of Lemma 4 in the Appendix.

1) No upward deviation from the equilibrium strategy is profitable for Bidder 1.

Proof. We show here that Bidder 1 does not have incentives to deviate upwards, i.e. $u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) \geq 0$ for $b > b_1^*(s_1)$. Upward deviations only affect payoffs when Bidder 1 loses with $b_1^*(s_1)$ and wins with b, i.e. when [33] $b_2^*(s_2) \in (b_1^*(s_1), b)$. Recall also from Lemma 6 that $b_2^*(s_2) > b_1^*(s_1)$ is equivalent to $g_2(s_2) > g_1(s_1)$, which implies: (a) $s_1 < \underline{s}_1^2(s_2)$ and (b) $g_1(\underline{s}_1^2(s_2)) \leq g_2(s_2)$. Moreover, by monotonicity of the bid functions we have that $\{s_2 : b_2^*(s_2) \in (b_1^*(s_1), b)\} = \{s_2 : s_2 \in (\overline{s}_2^1(s_1), \tau_2(b))\}$ for some $\tau_2(b) \in (\overline{s}_2^1(s_1), 1]$. Note also that if Bidder 1 wins, she gets a good with value $v_1(s_1, s_2)$ and pays Bidder 2's bid and if Bidder 1 loses she suffers a negative

[33] Note that ties occur with probability zero and thus the conclusions do not change whether considering the boundaries of the interval of bids close or open.

externality equal to $e_2(s_2)$. Thus, the change in utility when Bidder 1 wins is equal to $v_1^e(s_1, s_2) - v_2^e(s_2, \underline{s}_1^2(s_2)) = \phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))$. Thus,

$$
\begin{aligned}
u_1(s_1, b) - u_1(s_1, b_1^*(s_1)) = &\\
&\int_{\overline{s}_2^1(s_1)}^{\tau_2(b)} \left(\phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))\right) ds_2 = \\
&\int_{\overline{s}_2^1(s_1)}^{\tau_2(b)} \left[\phi_1(s_1) - \zeta_1(\underline{s}_1^2(s_2))\right] - \left[\phi_2(s_2) - \zeta_2(s_2)\right] ds_2 < \\
&\int_{\overline{s}_2^1(s_1)}^{\tau_2(b)} \left[\phi_1(\underline{s}_1^2(s_2)) - \zeta_1(\underline{s}_1^2(s_2))\right] - \left[\phi_2(s_2) - \zeta_2(s_2)\right] ds_2 = \\
&\int_{\overline{s}_2^1(s_1)}^{\tau_2(b)} \left[h_1(\underline{s}_1^2(s_2)) - h_2(s_2)\right] ds_2 \leq \\
&\int_{\overline{s}_2^1(s_1)}^{\tau_2(b)} \left(g_2(s_2) - h_2(s_2)\right) ds_2
\end{aligned}
$$

where we use that ϕ_1 is strictly increasing and (a) in the first inequality. As for the second inequality, we use that $h_1(\underline{s}_1^2(s_2)) \leq g_1(\underline{s}_1^2(s_2))$ and (b). To see why $h_1(\underline{s}_1^2(s_2)) \leq g_1(\underline{s}_1^2(s_2))$, we argue by contradiction. Suppose that $h_1(\underline{s}_1^2(s_2)) > g_1(\underline{s}_1^2(s_2))$, then Lemma 4 implies that $\underline{s}_1^2(s_2) < 1$ and thus by continuity there exists an interval $[\underline{s}_1^2(s_2), a)$ with $a \neq \underline{s}_1^2(s_2)$ such that any s_1 in this interval verifies that $h_1(s_1) < g_1(s_1)$ and hence that $H_1(s_1) \neq G_1(s_1)$. By application of Lemma 3 (d) we have that $g_1(s_1)$ is constant in $(a, \underline{s}_1^2(s_2)]$, which contradicts the definition of $\underline{s}_1^2(s_2)$.

Finally, we argue that the last expression is non-positive. This integral is equal to

$$\left[H_2(\overline{s}_2^1(s_1)) - G_2(\overline{s}_2^1(s_1))\right] + \left[G_2(\tau_2(b)) - H_2(\tau_2(b))\right].$$

The second difference is non-positive by Lemma 3 (c). We next argue that the first one is equal to zero. If $\overline{s}_2^1(s_1)$ is either zero or one, this is because of Lemma 3 (b); otherwise, it is because $G_2(\overline{s}_2^1(s_1)) < H_2(\overline{s}_2^1(s_1))$ would imply that g_2 is constant around $\overline{s}_2^1(s_1)$ by Lemma 3 (d), which is a contradiction with the definition of $\overline{s}_2^1(s_1)$. ■

2) No downward deviation from the equilibrium strategy is profitable for Bidder 2.

Proof. We show here that Bidder 2 does not have incentives to deviate downwards, i.e. $u_2(s_2, b_2^*(s_2)) - u_2(s_2, b) \geq 0$ for $b < b_2^*(s_2)$ Downward deviations only affect payoffs when Bidder 2 wins with $b_2^*(s_2)$ and loses with b, i.e. when[34] $b_1^*(s_1) \in (b, b_2^*(s_2))$. Recall also from Lemma 6 that $b_1^*(s_1) < b_2^*(s_2)$ is equivalent to $g_1(s_1) < g_2(s_2)$, which implies: (a) $s_2 > \overline{s}_2^1(s_1)$ and (b) $g_2(\overline{s}_2^1(s_1)) \geq g_1(s_1)$. Moreover, by monotonicity of the bid functions we have that $\{s_1 : b_1^*(s_1) \in (b, b_2^*(s_2))\} = \{s_1 : s_1 \in (\tau_1(b), \underline{s}_1^2(s_2))\}$ for some $\tau_1(b) \in [0, \underline{s}_1^2(s_2))$. Note also that if Bidder 2 wins, she gets a good with value $v_2(s_2, s_1)$ and pays Bidder 1's bid and if Bidder 2 loses she suffers a negative externality equal to $e_1(s_1)$. Thus, the change in utility when Bidder 2 wins is equal to $v_2^e(s_2, s_1) - v_1^e(s_1, \overline{s}_2^1(s_1)) = \phi_2(s_2) + \zeta_1(s_1) - \phi_1(s_1) - \zeta_2(\overline{s}_2^1(s_1))$. Thus,

$$
\begin{aligned}
&u_2(s_2, b_2^*(s_2)) - u_2(s_2, b) = \\
&\int_{\tau_1(b)}^{\underline{s}_1^2(s_2)} \left(\phi_2(s_2) + \zeta_1(s_1) - \phi_1(s_1) - \zeta_2(\overline{s}_2^1(s_1))\right) ds_1 = \\
&\int_{\tau_1(b)}^{\underline{s}_1^2(s_2)} \left[\phi_2(s_2) - \zeta_2(\overline{s}_2^1(s_1))\right] - \left[\phi_1(s_1) - \zeta_1(s_1)\right] ds_1 > \\
&\int_{\tau_1(b)}^{\underline{s}_1^2(s_2)} \left[\phi_2(\overline{s}_2^1(s_1)) - \zeta_2(\overline{s}_2^1(s_1))\right] - \left[\phi_1(s_1) - \zeta_1(s_1)\right] ds_1 = \\
&\int_{\tau_1(b)}^{\underline{s}_1^2(s_2)} \left[h_2(\overline{s}_2^1(s_1)) - h_1(s_1)\right] ds_1 \geq \\
&\int_{\tau_1(b)}^{\underline{s}_1^2(s_2)} \left(g_1(s_1) - h_1(s_1)\right) ds_1
\end{aligned}
$$

where we use that ϕ_2 is strictly increasing and (a) in the first inequality. As for the second inequality, we use that $h_2(\overline{s}_2^1(s_1)) \geq g_2(\overline{s}_2^1(s_1))$ and (b). To see why $h_2(\overline{s}_2^1(s_1)) \geq g_2(\overline{s}_2^1(s_1))$, we argue by contradiction. Suppose that $h_2(\overline{s}_2^1(s_1)) < g_2(\overline{s}_2^1(s_1))$, then Lemma 4 implies that $\overline{s}_2^1(s_1) > 0$ and thus by continuity there exists an interval $(a, \overline{s}_2^1(s_1)]$ with $a \neq \overline{s}_2^1(s_1)$ such that any s_2 in this interval verifies that $h_2(s_2) < g_2(s_2)$ and hence that $H_2(s_2) \neq G_2(s_2)$. By application of Lemma 3 (d) we have that $g_2(s_2)$ is constant in $(a, \overline{s}_2^1(s_1)]$ which contradicts the definition of $\overline{s}_2^1(s_1)$.

Finally, we argue that the last expression is non-negative. This integral is equal to

$$
\left[G_1(\underline{s}_1^2(s_2)) - H_1(\underline{s}_1^2(s_2))\right] + \left[H_1(\tau_1(b)) - G_1(\tau_1(b))\right].
$$

[34]Note that ties occur with probability one and thus the conclusions do not change whether considering the boundaries of the interval of bids close or open.

The second difference is non-negative by Lemma 3 (c). We next argue that the first one is equal to zero. If $\underline{s}_1^2(s_2)$ is either zero or one, this is because of Lemma 3 (b); otherwise, it is because $G_1(\underline{s}_1^2(s_2)) < H_1(\underline{s}_1^2(s_2))$ would imply that g_1 is constant around $\underline{s}_1^2(s_2)$ by Lemma 3 (d), which is a contradiction with the definition of $\underline{s}_1^2(s_2)$. ■

3) No upward deviation from the equilibrium strategy is profitable for Bidder 2.

Proof. We show here that Bidder 2 does not have incentives to deviate upwards, i.e. $u_2(s_2, b_2^*(s_2)) - u_2(s_2, b) \geq 0$ for $b > b_2^*(s_2)$. Upward deviations only affect payoffs when Bidder 2 loses with $b_2^*(s_2)$ and wins with b, i.e. when [35] $b_1^*(s_1) \in [b_2^*(s_2), b)$. Recall also from Lemma 6 that $b_1^*(s_1) \geq b_2^*(s_2)$ is equivalent to $g_1(s_1) \geq g_2(s_2)$, which implies: (a) $s_2 \leq \overline{s}_2^1(s_1)$ and (b) $g_2(\overline{s}_2^1(s_1)) \leq g_1(s_1)$. Moreover, by monotonicity of the bid functions we have that $\{s_1 : b_1^*(s_1) \in [b_2^*(s_2), b)\} = \{s_1 : s_1 \in [\underline{s}_1^2(s_2), \tau_1(b))\}$ for some $\tau_1(b) \in (\underline{s}_1^2(s_2), 1]$. Note also that if Bidder 2 wins, she gets a good with value $v_2(s_2, s_1)$ and pays Bidder 1's bid and if Bidder 2 loses she suffers a negative externality equal to $e_1(s_1)$. Thus, the change in utility when Bidder 2 wins is equal to $v_2^e(s_2, s_1) - v_1^e(s_1, \overline{s}_2^1(s_1)) = \phi_2(s_2) + \zeta_1(s_1) - \phi_1(s_1) - \zeta_2(\overline{s}_2^1(s_1))$. Thus,

$$
\begin{aligned}
&u_2(s_2, b) - u_2(s_2, b_2^*(s_2)) = \\
&\int_{\underline{s}_1^2(s_2)}^{\tau_1(b)} \left(\phi_2(s_2) + \zeta_1(s_1) - \phi_1(s_1) - \zeta_2(\overline{s}_2^1(s_1))\right) ds_1 = \\
&\int_{\underline{s}_1^2(s_2)}^{\tau_1(b)} \left[\phi_2(s_2) - \zeta_2(\overline{s}_2^1(s_1))\right] - \left[\phi_1(s_1) - \zeta_1(s_1)\right] ds_1 \leq \\
&\int_{\underline{s}_1^2(s_2)}^{\tau_1(b)} \left[\phi_2(\overline{s}_2^1(s_1)) - \zeta_2(\overline{s}_2^1(s_1))\right] - \left[\phi_1(s_1) - \zeta_1(s_1)\right] ds_1 = \\
&\int_{\underline{s}_1^2(s_2)}^{\tau_1(b)} \left[h_2(\overline{s}_2^1(s_1)) - h_1(s_1)\right] ds_1 \leq \\
&\int_{\underline{s}_1^2(s_2)}^{\tau_1(b)} \left(g_1(s_1) - h_1(s_1)\right) ds_1
\end{aligned}
$$

where we use that ϕ_2 is strictly increasing and (a) in the first inequality. As for the second inequality, we use that $h_2(\overline{s}_2^1(s_1)) \leq g_2(\overline{s}_2^1(s_1))$ and (b). To see why $h_2(\overline{s}_2^1(s_1)) \leq$

[35]Note that ties occur with probability zero and thus the conclusions do not change whether considering the boundaries of the interval of bids close or open.

$g_2(\overline{s}_2^1(s_1))$, we argue by contradiction. Suppose that $h_2(\overline{s}_2^1(s_1)) > g_2(\overline{s}_2^1(s_1))$, then Lemma 4 implies that $\overline{s}_2^1(s_1) < 1$ and thus by continuity there exists an interval $[\overline{s}_2^1(s_1), a)$ with $a \neq \overline{s}_2^1(s_1)$ such that any s_2 in this interval verifies that $h_2(s_2) < g_2(s_2)$ and hence that $H_2(s_2) \neq G_2(s_2)$. By application of Lemma 3 (d) we have that $g_2(s_2)$ is constant in $(a, \overline{s}_2^1(s_1)]$, which contradicts the definition of $\overline{s}_2^1(s_1)$.

Finally, we argue that the last expression is non-positive. This integral is equal to

$$\left[H_1(\underline{s}_1^2(s_2)) - G_1(\underline{s}_1^2(s_2))\right] + \left[G_1(\tau_1(b)) - H_1(\tau_1(b))\right].$$

The second difference is non-positive by Lemma 3 (c). We next argue that the first one is equal to zero. If $\underline{s}_1^2(s_2)$ is either zero or one, this is because of Lemma 3 (b); otherwise, it is because $G_1(\underline{s}_1^2(s_2)) < H_1(\underline{s}_1^2(s_2))$ would imply that g_1 is constant around $\underline{s}_1^2(s_2)$ by Lemma 3(d), which is a contradiction with the definition of $\underline{s}_1^2(s_2)$. ■

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