

A discusión

CODIFICATION SCHEMES AND FINITE AUTOMATA*

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ABSTRACT

This paper is a note on how Information Theory and Codification Theory are helpful in the computational design both of communication protocols and strategy sets in the framework of finitely repeated games played by boundedly rational agents. More precisely, we show the usefulness of both theories to improve the existing automata bounds of Neyman's (1998) work on finitely repeated games played by finite automata.

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1 Introduction

This paper is a note on how Information Theory and Codification Theory are helpful in the computational design of both communication protocols and strategy sets in the framework of finitely repeated games played by boundedly rational agents. More precisely, we will show their usefulness for improving the existing automaton bounds of Neyman's [12] work on finitely repeated games played by finite automata.

Until quite recently, Economics and Game Theory have not yet considered computational complexity as a valuable resource. Therefore, it has sometimes been predicted that rational agents will invest a large amount of computation even for small payoffs and for all kind of deviations, in particular, for those from cooperative behavior. However, if we limit players' computational power, then new solutions might emerge, thus providing one of the main motivations for theories of bounded rationality in Economics (see, Simon [18]). Theories of bounded rationality assume that decision makers are rational in a broad sense, but not omniscient: they are rational but with limited abilities. One way to model computational capacity bounds is by assuming that decision makers use machines (automata) to implement their strategies. The machine size, or the number of states of the automaton, determines the bounds on the agents' rationality.

From the eighties onwards, there have been several papers in the repeated games literature, which have studied conditions under which the set of feasible and rational payoffs are equilibrium outcomes, when there are bounds (possibly very large) in the number of strategies that players may use. In the context of strategies implemented by finite automata, these bounds have been related to the complexity of the automata playing the equilibrium strategy (see Neyman [10]; Rubinstein [17]; Abreu and Rubinstein [1]; Ben-Porath [2]; Neyman [10]; Papadimitriou and Yannakakis [16]; Neyman and Okada [13], [14], [15], Gossner and Hernández [4]; Zemel [19], among others).

The number of strategies in finitely repeated games is finite but huge, hence full rationality in this context can be understood as the ability to implement any strategy of such games. In the context of finite automata, this implies that if an agent used an automaton of exponential size with respect to the length of the game, then he could implement all the strategies and, in particular, the one entailing the knowledge of the last stage of the game. Alternatively, if the automaton size is polynomial on the length of the game, then there will be several strategies which could not be implemented, for instance, defection at the last stage of the game. Bounds on automata complexity are important for they give a measure of how far agents are from rational behavior. The bigger the bounds the closer to full rationality. Therefore, improving the automata bounds means getting closer to a fully rational behavior yet achieving the Folk Theorem payoffs.

Neyman [12] has shown that players of a long but finite interaction can get arbi-

trarily close to the cooperative payoffs even if they chose large automata, provided that they are allowed to randomize on their choices of machines. Specifically, the key idea is that if a player randomizes among strategies, then her opponent will be forced to *fill up his complexity* in order to be able to answer to all the strategies in the support of such a random strategy. The structure of Neyman’s equilibrium strategy is twofold: first, the stronger player (the one with the bigger automaton) specifies the finite set of plays which forms her mixed strategy (uniformly distributed over the above set); and second, she communicates a play through a message to the weaker player.

The proof of the existence of such an equilibrium strategy is by construction and the framework of finite automata imposes two additional requirements, apart from strategic considerations. One refers to the computational work, which entails measuring the number of plays and building up a communication protocol between the two machines. The other is related to the problem of the automata design in the framework of game theory. These requirements jointly determine the complexity needed to support the equilibrium strategy in games played by finite automata. Specifically, the complexity of the set of plays is related to the complexity of the weaker player and the design of this set can be understood as a codification problem where the weaker player’s complexity is what is codified. These equilibrium strategies can be implemented by finite automata whose size is a subexponential function of the approximation to the targeted payoff and of the length of the repeated game. This is a very important result for two reasons. The first is that it suggests that *almost* rational agents may follow a cooperative behavior in finitely repeated games as opposed to their behavior under full rationality. The second is that it allows to quantify the distance from full rationality through a bound on the automaton size.

We focus on the computational work and apply Information Theory to construct the set of equilibrium plays, where plays differ from each other in a sequence of actions called the verification phase. In our approach the *complexity* of the mixed strategy is related to the *entropy* of the empirical distribution of each verification sequence, which captures, from an Information Theory viewpoint, the cardinality of the set of equilibrium plays and hence its complexity. Second, by Codification Theory we offer an optimal communication scheme, mapping plays into messages, which produces “short words” guaranteeing the *efficiency* of the process. We propose a set of messages with the properties that each of them has the same empirical distribution and all of them exhibit the minimal length. The former property implies the same payoff for all messages (plays) and the latter ensures the efficiency of the process. Since the complexity of the weaker player’s automaton implementing the equilibrium strategy is determined by that of the set of plays, we expand it as much as possible while maintaining the same payoff per play. Moreover, the length of communication is the shortest one given the above set. Both procedures allow us to obtain an equilibrium condition which improves that of Neyman. Under his

strategic approach, our result offers the biggest upper bound on the weaker player's automaton implementing the cooperative outcome: this bound is an exponential function of both the entropy of the approximation to the targeted payoff and the number of repetitions.

The paper is organized as follows. The standard model of finite automata playing finitely repeated games is refreshed in Section 2. Section 3 offers our main result and states Neyman's Theorem, with Section 4 explaining the key features behind his construction. The tools of Information Theory and Codification Theory needed for our approach are presented in Section 5. This section also shows the construction of the verification and communication sets as well as the consequences of our construction on the measure of the weaker player's complexity bound. Concluding remarks close the paper.

2 The model

For $k \in \mathbb{R}$, we let $\lceil k \rceil$ denote the superior integer part of k , i.e. $k \leq \lceil k \rceil < k + 1$. Given a finite set Q , $|Q|$ denotes the cardinality of Q .

Let $G = (\{1, 2\}, (A_i)_{i \in \{1, 2\}}, (r_i)_{i \in \{1, 2\}})$ be a game where $\{1, 2\}$ is the set of players. A_i is a finite set of actions for player i (or pure strategies of player i) and $r_i : A = A_1 \times A_2 \rightarrow \mathbb{R}$ is the payoff function of player i .

We denote by $u_i(G)$ the individually rational payoff of player i in pure strategies, i.e., $u_i(G) = \min \max r_i(a_i, a_{-i})$ where the max ranges over all pure strategies of player i , and the min ranges over all pure strategies of the other player. For any finite set B we denote by $\Delta(B)$ the set of all probability distributions on B . An equilibrium of G is a pair $\sigma = (\sigma^1, \sigma^2) \in \Delta(A_1) \times \Delta(A_2)$ such that for every i and any strategy of player i , $\tau^i \in A_i$, $r_i(\tau^i, \sigma^{-i}) \leq r_i(\sigma^1, \sigma^2)$, where $r(\sigma) = E_\sigma(r(a_i, a_{-i}))$. If σ is an equilibrium, the vector payoff $r(\sigma)$ is called an equilibrium payoff and let $E(G)$ be the set of all equilibrium payoffs of G .

From G we define a new game in strategic form G^T which models a sequence of T plays of G , called stages. By choosing actions at stage t , players are informed of actions chosen in previous stages of the game. We denote the set of all pure strategies of player i in G^T by $\Sigma^i(T)$.

Any 2-tuple $\sigma = (\sigma^1, \sigma^2) \in \times \Sigma^i(T)$ of pure strategies induces a play $\omega(\sigma) = (\omega^1(\sigma), \dots, \omega^T(\sigma))$ with $\omega^t(\sigma) = (\omega_1^t(\sigma), \omega_2^t(\sigma))$ defined by $\omega^1(\sigma) = (\sigma^1(\emptyset), \sigma^2(\emptyset))$ and by the induction relation $\omega_i^t(\sigma) = \sigma^i(\omega^1(\sigma), \dots, \omega^{t-1}(\sigma))$.

Let $r_T(\sigma) = \frac{1}{T} \sum_{t=1}^T r(\omega^t(\sigma))$ be the average vector payoff during the first T stages induced by the strategy profile σ .

A finite automaton M_i of player i , is a finite machine which implements pure strategies for player i in strategic games, in general, and in repeated games, in particular. Formally, $M_i = \langle Q_i, q_i^0, f_i, g_i \rangle$, where, Q_i is the set of states; q_i^0 is the initial state; f_i is the action function, $f_i : Q_i \rightarrow A_i$ and g_i is the transition function

from state to state, $g_i : Q_i \times A_{-i} \rightarrow Q_i$. The size of a finite automaton is the number of its states, i.e. the cardinality of $|Q_i| = m_i$.

Finally, we define a new game in strategic form $G^T(m_1, m_2)$ which denotes the T stage repeated version of G , with the average payoff as evaluation criterion and with all the finite automata of size m_i as the pure strategies of player i , $i = 1, 2$. Let $\Sigma^i(T, m_i)$ be the set of pure strategies in $G^T(m_1, m_2)$ that are induced by an automaton of size m_i .

A finite automaton for player i can be viewed as a prescription for this player to choose her action in each stage of the repeated game. If at state q the other player chooses the action tuple a^{-i} , then the automaton's next state is $q^i = g^i(q, a^{-i})$ and the associated action is $f^i(q^i) = f^i(g^i(q, a^{-i}))$. More generally, define inductively, $g^i(q, b_1, \dots, b_t) = g^i(g^i(q, b_1, \dots, b_{t-1}), b_t)$, where $a_j^{-i} \in A^{-i}$, the action prescribed by the automaton for player i at stage j is $f^i(g^i(q^i, a_1^{-i}, \dots, a_{t-1}^{-i}))$.

For every automaton M_i , we consider the strategy σ_i^M , of player i in $G^T(m_1, m_2)$, such that the action at stage t is the one dictated by the action function of the automaton, after the updating of the transition function for the history of length $t-1$. Namely, $\sigma_i^M(a_1, \dots, a_{t-1}) = f_i(g_i(q, a_{-i}^1, \dots, a_{-i}^{t-1}))$. The strategy σ_i for player i in G^T is implementable by the automaton M_i if both strategies generate the same play for any strategy τ of the opponent, i.e., σ_i is equivalent to σ_i^M for every $\tau \in \Sigma_{-i}(T)$: $\omega(\sigma_i, \tau) = \omega(\sigma_i^M, \tau)$. Denote by $E(G^T(m_1, m_2))$ the set of equilibrium payoffs of $G^T(m_1, m_2)$.

Define the complexity of σ_i denoted by $comp(\sigma_i)$, as the size of the smallest automaton that implements σ_i . See Kalai and Stanford [9] and Neyman [12], for a deeper treatment.

Any 2-tuple $\sigma = (\sigma_1, \sigma_2)$ of pure strategies $G^T(m_1, m_2)$ induces a play $\omega(\sigma_1, \sigma_2)$ of length T (ω , henceforth). Conversely, since a finite play of actions, ω , can be regarded as a finite sequence of pairs of actions, i.e., $\omega = (a_1, \dots, a_t)$ then, the play ω and a pure strategy σ_i are *compatible*, if for every $1 \leq s \leq T$, $\sigma_i(a_1, \dots, a_{s-1}) = a_i^s$. Now, given a play ω , define player i 's complexity of ω , $comp_i(\omega)$, as the smallest complexity of a strategy σ_i of player i which is *compatible* with ω .

$$comp_i(\omega) = \inf \{ comp(\sigma_i) : \sigma_i \in \Sigma^i \text{ is compatible with } \omega \}.$$

Let Q be a set of plays. A pure strategy σ^i of player i is conformable to Q if it is compatible with any $\omega \in Q$. The complexity of player i of a set of plays Q is defined as the smallest complexity of a strategy σ^i of player i that is conformable to Q .

$$comp^i(Q) = \inf \{ comp^i(\sigma) : \sigma \in \Sigma^i \text{ is conformable to } Q \}$$

Assume without any loss of generality that player 2 has a bigger complexity than player 1, i.e. $m_2 > m_1$, therefore player 1 will be referred to as the "weaker player" (he) and player 2 as the "stronger" player (she).

3 Main result

The main result improves upon the weaker player's automaton upper bound $-m_1$, which guarantees the existence of an equilibrium payoff of $G^T(m_1, m_2)$, ε -close to a feasible and rational payoff.

It is well known that in the context of finitely repeated games, deviations in the last stages could be precluded if players did not know the end of the game. This may be achieved if players implemented their strategies by playing with finite automata which cannot count until the last stage of the game. On the contrary, player i will deviate if she is able to implement cycles of length at least the number of repetitions. Hence, if player i has to answer to different plays of length smaller than the number of repetitions, then she could exhaust her capacity and, at the same time, she would be unable to count until the end of the game. Therefore, a player could fill up the rival's complexity by requiring her to conform with distinct plays of sufficiently large length.

With this idea in mind, Neyman (1998) establishes the existence of an equilibrium payoff of $G^T(m_1, m_2)$ which is ε -close to a feasible and rational payoff even if the complexity of player 1's automaton (the smaller automaton) is quite big, i.e, a subexponential function of ε and of the number of repetition:

Theorem (Neyman, 1998) *Let $G = (\{1, 2\}, A, r)$ be a two person game in strategic form. Then for every ε sufficiently small, there exist positive integers T_0 and m_0 , such that if $T \geq T_0$, and $x \in \text{co}(r(A))$ with $x_i > u_i(G)$ and $m_1 \leq \exp(\varepsilon^3 T)$ and $m_2 > T$ then there exists $y \in E(G^T(m_1, m_2))$ with $|y_i - x_i| < \varepsilon$.*

Basically, the structure of Neyman's equilibrium strategies is as follows:

- The stronger player selects uniformly at random a play, among a finite set of plays denoted by Q , which leads approximately to the targeted payoff.
- This player codifies the selected play into a message, transmitted to the other player in the communication phase.
- There exists a one-to-one correspondence between set Q and the set of messages M .
- The difference among the distinct plays is a small portion of each play, which is called the verification phase. The set of sequences to implement such specific part of the equilibrium play is denoted by V .

From the above description it follows that the cardinality of the set of messages M is related with the cardinality of Q (or V). This allows us to bound the weaker player's capacity. This is so because the number of states to implement such structure is at least the number of plays times the length of each play. This entails the

existence of a mapping from the set of messages to the complexity bound. Hence, the design of the set of plays can be understood as a codification problem where the weaker player's complexity is codified.

Similarly to Neyman's result, we offer the equilibrium conditions in terms of the complexity of the weaker player's automaton implementing the equilibrium play. The main difference is that our upper bound includes previous bound domains. This is due to the application of Information Theory to codify this player's complexity, in particular, to the optimal construction of the set of verification sequences and the associated communication scheme. Neyman's equilibrium structure does not change but the weaker player's equilibrium condition is relaxed: the size of m_1 increases, getting closer to full rationality.

More specifically, to establish theorem 1 below, we characterize the verification sequences as those sequences over a finite alphabet with a given empirical distribution which depends on ε -the approximation to the targeted payoff- and on a positive integer k , denoting their length. Recall that the set of such verification sequences is denoted by V . Given this set, we construct the set of communication sequences, M , described also by both the length of the sequences belonging to it, and their empirical distribution. Such length \bar{k} is about k times the entropy¹ of the empirical distribution of the sequences in V . Moreover, by the property of optimal codification (minimal expected length) the empirical distribution of the sequences in M coincides with the uniform distribution. As the finite alphabet is the binary alphabet, this implies that messages are balanced² sequences with minimal length.

We summarize the above argument in the following result 1, which is needed to show that, under our construction, the verification and communication sequences are the shortest ones to codify the weaker player's complexity. The formal statement of this result is presented in section 5 below (see lemma 1) where the tools of Information Theory needed to prove it are introduced.

Result 1: *Let V be a subset of sequences of length k over a finite alphabet $\Theta = \{0, 1\}$ with empirical rational distribution $(r, 1 - r) \in \Delta(\Theta)$. Then, the set of messages for the communication phase, codifying the uniform random variable over V , coincides with the set of sequences of length \bar{k} such that $\lceil kH(r) \rceil < \bar{k} \leq 2kH(r)$ with the uniform rational empirical distribution over Θ , where H corresponds to the entropy function.*

Our theorem is stated as follows:

Theorem 1 *Let $G = (\{1, 2\}, A, r)$ be a two person game in strategic form. Then for every ε sufficiently small, there exist positive integers T_0 and m_0 , such that if $T \geq T_0$, and $x \in co(r(A))$ with $x^i > u^i(G)$ and $m_0 \leq \min\{m_1, m_2\} \leq \exp(\frac{T}{27}H(\varepsilon))$ and $\max\{m_1, m_2\} > T$ then there exists $y \in E(G^T(m_1, m_2))$ with $|y^i - x^i| < \varepsilon$.*

¹The entropy function H is given by $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ for $0 < x < 1$.

²Sequences with the same number of zeros and ones.

Both Theorems follow from conditions on: 1) a feasible payoff $x \in co(r(A))$; 2) a positive constant $\varepsilon > 0$; 3) the number of repetitions T , and 4) the bounds of the automata sizes m_1 and m_2 , that guarantee the existence of an equilibrium payoff y of the game $G^T(m_1, m_2)$ that is ε -close to x .

One of the conditions is specified in terms of the inequalities $m_i \geq m_0$, where m_0 is sufficiently large. Another condition requires the bound of one or both automata sizes to be subexponential in the number of repetitions, i.e., a condition that asserts that $(\log m_i)/T$ is sufficiently small. Neyman's condition is written in terms of $\varepsilon^3 T$. As already said, our constraint, $\frac{T}{27} H(\varepsilon)$, comes from the use of Codification tools to construct the communication and verification sets and hence, as it will be made clear later on, it is expressed in terms of the entropy function. Therefore, the improvement on the subexponential condition is connected to the codification schemes to be explained in Section 5.

4 The scheme of Neyman's equilibrium play

We present an informal description of the equilibrium strategy construction. Since the structure of the equilibrium strategies as well as their formalization are as in Neyman, we present a sketch of the construction (for more details see: Neyman [12], pages 534-549). Recall that $m_1 \leq m_2$.

Communication phase: Neyman's Theorem is stated in terms of an upper bound, $m_1 \leq exp(\varepsilon^3 T)$, of the weaker player's complexity which is related to the length of the game T , and the ε -approximation to the targeted payoff. This bound gives the constraints on a new parameter k which determines both the cardinality of the set of plays Q and the length of the communication phase. Following Neyman[12] (pag: 534), let k be the largest integer such that:

$$2^{k-1} \leq \frac{m_1 - l}{l} < 2^k \tag{1}$$

where l is the length of each play.

Consider any pair of players' actions and rename them by 0 and 1. Knowing player 1's complexity, player 2 determines a precise number of plays where each one is indicated by a sequence of the two actions. Player 1 responds properly to any message playing a fixed action (for instance 0), independently of the message (signal) sent by player 2. By equation 1, the cardinality of the set of plays Q is at most 2^k . The communication sequences consist of messages of balanced sequences in the sense that they have the same number of actions. Specifically, the first k stages coincide with an element of $\{0, 1\}^k$ and the remaining stages start with a string of zeros followed by a string of ones such that the total number of ones (or that of zeros) equals k . The set of messages M is then a subset of $\{0, 1\}^{2k}$.

Play phase: After the communication phase the equilibrium play enters into the play phase which consists of a cycle repeated along the play until T . The length of the cycle does not depend on the signal sent by player 2. Each one of the cycles has associated payoff approximately equal to the efficient and rational payoff x . The cycle has two parts: the verification play and the regular play. The regular play is common for every signal and it consists of a sequence of different action pairs ε -close to the efficient and rational targeted payoff x . Both players follow a verification play that consists of a coordinated sequence of actions of length $2k$, which coincides with the signal sent in the communication phase. This sequence of actions can be understood as a coordination process which determines each pure strategy. Each play in Q has a different verification sequence which coincides with the associated message from M . The relationship between the communication phase and the verification play is summarized as follows: 1) Set M coincides with set V , 2) both phases have the same length $2k$ and, 3) the coding rule is the identity.

The number of states needed to follow such strategies corresponds to those states for each play times the number of possible plays (the cardinality of the verification set), which yields $2^k T$ states. The upper bound of player 1 has to be at most this number in order to implement both the communication phase and the play phase, i.e.: $m_1 \leq 2^k T$. It is straightforward to check that player 1 is able to implement both phases for any k satisfying equation (1). In particular, for $k = \lceil \frac{\varepsilon^3 T}{\ln 2} - \frac{\ln T}{\ln 2} \rceil$,

$$\begin{aligned}
m_1 &\leq T 2^k \\
&= T \ln 2 \exp(k) \\
&\leq T \exp\left(\frac{\varepsilon^3 T}{\ln 2} - \frac{\ln T}{\ln 2}\right) \\
&= \exp(\varepsilon^3 T)
\end{aligned}$$

The above analysis uncovers the two *computational* features to design the equilibrium structure. The first one is the design of the *set of possible plays* given the parameters of the game. The difference among the sequences is given by the verification play which satisfies the following properties. First, each sequence has the same empirical distribution to deter player 2's deviations by selecting the best payoff sequence. Second, player 2 fills up player 1's capacity by generating enough pure strategies so that the number of remaining states is sufficiently small. In this way, player 1's deviations from the proposed play by counting up until the last stage of the game are avoided. Therefore, the size of this set gives the upper bound on the weaker player's complexity supporting the equilibrium strategies.

The second computational feature is concerned with the *communication phase*: given the set of plays, player 2 selects one of these possible plays and communicates

it by a message. Since player 2 proposes the plays, messages have to be independent of the associated payoffs to each of them³.

The above are the main points of Neyman's construction. It should be clear that not any 3-tuple of sets Q , M , and V of the equilibrium strategies can be implemented by finite automata but those implementable provide the upper bound on m_1 . Our query is whether it is possible to improve upon such a bound in order to get automata that are closer to fully rational behavior. To do that, we stick to the same equilibrium structure and the same implementation procedure and focus on the computational features of the equilibrium strategies. Namely, we focus on the design of sets Q , M and V . Since, the construct of the set of plays can be understood as a codification problem we proceed to design an optimal codification scheme.

5 Codification schemes and the equilibrium play

The complexity of player 1 (the set of states) has to be allocated in such a way that he first identifies and then plays a specific path. We can view this task as the construction of a matrix of automaton states with $O(T)$ columns (the length of the cycle) and about $\frac{m_1}{T}$ rows (the number of possible paths). Then, the rows correspond to the set of plays which player 1 has to conform with, thus filling up his complexity.

A solution to this problem is equivalent to solving a codification problem in Information Theory, since the verification sequences have to be codified in the communication phase. To codify means to describe a phenomenon. The realization of this phenomenon can be viewed as the representation of a random variable. Then, a codification problem is just a one-to-one mapping (the source code) from a finite set (the range of a random variable or input) to another set of sequences of finite length (output sequences) with specific properties. The most important one is that the expected length of the source code of the random variable is as short as possible. With this requirement we achieve an optimal data compression.

We proceed to construct the set of verification sequences and the associated communication scheme under the framework of finite automata. The key points of the construction are: 1) the characterization of such sequences by their empirical distribution and 2) the design of the set of communication sequences by the optimal codification of the verification set. Nevertheless, codification results are not enough to guarantee the improvement of automata bounds.

These two features have to be combined with the implementation requirements of finite automata, specially for player 1. On the one hand, player 1 has to follow

³These two features have to be combined with the implementation requirements of finite automata. To avoid repetition, this issue is explained in more detail in the next section.

the specific play selected by player 2, therefore he stores all possible cycles. On the other hand, he wants to design a “smart” automaton making a good use of his complexity. Player 1’s information processing is minimized by using the same states to process the signal and to follow the regular part of the different cycles. However, this introduces a difficulty since these states of player 1’s automaton admit more than one action, giving rise to the existence of deviations by player 2 that might remain unpunished. To avoid this problem, player 1 uses a mixed strategy whose support consists of a subset of pure strategies, conformable with all the proposed plays. The mixed strategy generates enough randomization to obscure the location of the reused states.

Namely, in the framework of repeated games with finite automata the corresponding set of sequences of verification and of communication have to satisfy the following requirements:

- **R1:** Each message determines a unique play, therefore a unique verification sequence.
- **R2:** Each communication sequence and each verification sequence have the same associated payoff in order to preclude any strategic deviation.
- **R3:** The length of each signal is the smallest positive integer such that it can be implemented by a finite automaton and that generates the lowest distortion from the targeted payoff.
- **R4:** It is required to signal the end of both the communication and the verification phases in order to properly compute the complexity associated to any equilibrium strategy.

In our setting the set of verification sequences is the input set, the set of messages corresponds to the output set and the random variable is the uniform distribution over the input set. The verification phase is composed of the set of sequences with the same empirical distribution of finite length k and with a fixed last component to signal the end of such a phase. This set satisfies all of the above requirements. The codification of this set results in the set of messages for the communication phase. This is the output set consisting of strings of finite length from the binary alphabet and with the optimal length. The communication sequences have the same empirical rational distribution to deter payoff deviation and minimal length by optimal codification theory results.

The formal details of our construction are presented next. Firstly, by the methodology of types, we consider sequences with the same empirical distribution. Secondly, we analyze the information properties of these sequences by using the entropy concept. Finally, we state the codification minimal length of the verification sequences to become those of the communication phase.

5.1 Information theory and codification

We present here some basic results from Information Theory. For a more complete treatment consult Cover and Thomas [3].

Let X be a random variable over a finite set Θ , whose distribution is $p \in \Delta(\Theta)$, i.e.: $p(\theta) = \Pr(X = \theta)$ for each $\theta \in \Theta$. The entropy $H(X)$ of X is defined by $H(X) = -\sum_{\theta \in \Theta} p(\theta) \log(p(\theta)) = -E_X [\log p(X)]$, where $0 \log 0 = 0$ by convention. With some abuse of notation, we denote the entropy of a binary random variable X by $H(p)$, where $p \in [0, 1]$.

Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of n symbols from a finite alphabet Θ . The type $P_{\mathbf{x}}$ (or empirical probability distribution) of \mathbf{x} is the relative proportion of occurrences of each symbol of Θ , i.e.: $P_{\mathbf{x}}(a) = \frac{N(a|\mathbf{x})}{n}$ for all $a \in \Theta$, where $N(a|\mathbf{x})$ is the number of times that a occurs in the sequence $\mathbf{x} \in \Theta^n$. The set of types of sequences of length n is denoted by $\mathbf{P}_n = \{P_{\mathbf{x}} \mid \mathbf{x} \in \Theta^n\}$. If $P \in \mathbf{P}_n$, then the set of sequences of length n and type P is called the type class of P , denoted by $S_n(P) = \{\mathbf{x} \in \Theta^n : P_{\mathbf{x}} = P\}$.

The cardinality of a type class $S_n(P)$ is related to the type entropy:

$$\frac{1}{(n+1)^{|\Theta|}} 2^{nH(P)} \leq |S_n(P)| \leq 2^{nH(P)} \quad (2)$$

Now we present the definitions of codification and data compression.

A source code C for a random variable X is a mapping from Θ , the range of X , to \mathfrak{D}^* the set of finite length strings of symbols from a D -ary alphabet. Denote by $C(x)$ the codeword corresponding to x and let $l(x)$ be the length of $C(x)$. A code is said to be non-singular if every element of the range of X maps into a different string in \mathfrak{D}^* , i.e.: $x_i \neq x_j \Rightarrow C(x_i) \neq C(x_j)$.

The expected length $L(C)$ of a source $C(x)$ for a random variable X with probability mass function $p(x)$ is given by $L(C) = \sum_{x \in \Theta} p(x)l(x)$, where $l(x)$ is the length of the codeword associated with x . An optimal code is a source code with the minimum expected length.

The sufficient condition for the existence of a codeword set with the specified set of codeword lengths is known as the Kraft inequality: for any code over an alphabet of size D , the codeword lengths l_1, l_2, \dots, l_m must satisfy the inequality $\sum D^{-l_i} \leq 1$. Given this condition we ask what is the optimal coding among a feasible code. The optimal coding is found by minimizing $L = \sum p_i l_i$ subject to $\sum D^{-l_i} \leq 1$. By the Lagrangian multipliers, the optimal code lengths are $l_i^* = -\log_D p_i$. Therefore, the expected length is $L^* = \sum p_i l_i^* = -\sum p_i \log_D p_i$ which coincides with $H_D(X)$. Since the codeword length must be integers, it is not always possible to construct codes with minimal length exactly equal to $l_i^* = -\log_D p_i$. The next remark summarizes these issues.

Remark 1 *The expected length of a source $C(X)$ for a random variable X is greater than or equal to the entropy of X . Therefore, the optimal expected length of $C(X)$ coincides with the entropy of X .*

5.2 Construction of the verification and communication sets

We first construct the communication set M . Let k be a positive integer. Consider a source code of a uniformly distributed random variable X over a finite alphabet of size 2^k (k -adica distribution). The associated entropy of X is $H = -\sum_{i=1}^{2^k} 2^{-k} \log 2^{-k} = k$. By the above bound on L^* , such a source is optimally codified by all codewords with length k . Hence, if the random variable has a k -adica (uniform) distribution over a binary alphabet, then the optimal code length is equal to k since the entropy is an integer number.

In the framework of repeated games with finite automata, let V be the set of verification sequences of length k belonging to the same type set over a binary alphabet and the last component equals 1; i.e.: $V \subset S_k(P) \times \{1\}$ for some⁴ $P \in \mathbf{P}_k$. Let $\hat{k} = \log_2 |V|$ and X be a \hat{k} -adica uniform random variable and a source code C for X to $\{0, 1\}^*$. By remark 1 above, the optimal length is equal to the entropy of X , which coincides with the logarithm of the cardinality of V , i.e. \hat{k} , and it also provides the shortest length of the communication sequences. Notice that \hat{k} need not be an integer. Let \bar{K} be the set of integers greater than or equal to \hat{k} .

Given the above remark and the four requirements summarized in section 5, we can construct the optimal set of messages M , implementing the communication phase:

- By remark 1 and requirement **R3**, $M \subseteq \{0, 1\}^k$ for some $k \in \bar{K}$
- By **R2** and **R4**, $M \subset S_k(r) \times 1$ for some $r \in \mathbf{P}_k$
- By requirement **R1**, $|M| \geq |V|$.

Therefore, to completely describe the set of signals for the communication phase, we have to establish both the length $k \in \bar{K}$ and the empirical distribution r of the sequences belonging to M . With abuse of notation we denote the empirical distribution R as $r = Pr(0)$. Let $G(k, r) = \{(k, r) \in (\bar{K} \times \mathbb{P}_k) \mid \frac{1}{(k+1)^2} 2^{kH(r)} > 2^{\hat{k}}\}$ be the set of pairs (k, r) which determines the communication set. Define \bar{k} as the smallest positive even integer such that there exists some $r \in [0, \frac{1}{2}]$ which verifies that $(\bar{k}, r) \in G(k, r)$. Fixing k , let $r_k \in [0, \frac{1}{2}]$ be the smallest r such that $(k, r_k) \in G(k, r)$.

The next Lemma, that formalizes the previously presented Result 1, states that fixing \bar{k} , the corresponding $r_{\bar{k}}$ is unique and equal to $\frac{1}{2}$. It also provides the bounds

⁴An extra 1 is added in order to signal the end of the verification phase.

of \bar{k} . As a consequence, the set of signals for the communication phase consists of the set of sequences with the same number of zeros and ones⁵ with length \bar{k} which might be shorter than the length of the sequences for the verification play.

Lemma 1 • $(\bar{k}, r) \notin G(k, r)$ for any $r < \frac{1}{2}$.

- $r_{\bar{k}} = \frac{1}{2}$.
- $\lceil \hat{k} \rceil < \bar{k} \leq 2\hat{k}$ for $\hat{k} > 6.694$.

Proof. Suppose that $(\bar{k}, r) \in G(k, r)$ for $r \in [0, \frac{1}{2})$.

Let $\tilde{k} = \lceil \bar{k}H(r) \rceil$. We can assume without any loss of generality that $\tilde{k} < \hat{k}$.

If \tilde{k} is even, consider the pair $(\tilde{k}, \frac{1}{2})$ and the corresponding type set $S_{\tilde{k}}(\frac{1}{2})$. By property 2 we get:

$$\frac{1}{(\tilde{k}+1)^2} 2^{\tilde{k}H(\frac{1}{2})} \leq |S_{\tilde{k}}(r)| \leq 2^{\tilde{k}H(\frac{1}{2})}, \text{ and then}$$

$$\begin{aligned} \frac{1}{(\tilde{k}+1)^2} 2^{\tilde{k}H(\frac{1}{2})} &= \frac{1}{(\tilde{k}+1)^2} 2^{\lceil \bar{k}H(r) \rceil} H(\frac{1}{2}) \\ &> \frac{1}{(\tilde{k}+1)^2} 2^{\bar{k}H(r)} \\ &> \frac{1}{(\bar{k}+1)^2} 2^{\bar{k}H(r)} \\ &> 2^{\hat{k}} \end{aligned}$$

Therefore $(\tilde{k}, \frac{1}{2}) \in G(k, r)$. This is a contradiction because $\tilde{k} < \bar{k}$ since $H(r) < 1$ for $r \in [0, \frac{1}{2})$.

If \tilde{k} is odd then $\tilde{k} \leq \bar{k} - 1$. It suffices to prove the statement for $\bar{k} - 1$. Consider the pair $(\tilde{k}, \frac{\tilde{k}-1}{2\tilde{k}}) = (\lceil \bar{k}H(r) \rceil, \frac{\lceil \bar{k}H(r) \rceil - 1}{2\lceil \bar{k}H(r) \rceil})$. Then,

$$\begin{aligned} \frac{1}{(\tilde{k}+1)^2} 2^{\tilde{k}H(\frac{\tilde{k}-1}{2\tilde{k}})} &= \frac{1}{(\tilde{k}+1)^2} 2^{\lceil \bar{k}H(r) \rceil H(\frac{\lceil \bar{k}H(r) \rceil - 1}{2\lceil \bar{k}H(r) \rceil})} \\ &\geq \frac{1}{(\tilde{k}+1)^2} 2^{\lceil \bar{k}H(r) \rceil (\frac{\lceil \bar{k}H(r) \rceil - 1}{\lceil \bar{k}H(r) \rceil} + \frac{1}{\lceil \bar{k}H(r) \rceil})} \\ &= \frac{1}{(\tilde{k}+1)^2} 2^{\lceil \bar{k}H(r) \rceil} \\ &> \frac{1}{(\bar{k}+1)^2} 2^{\bar{k}H(r)} \\ &> 2^{\hat{k}} \end{aligned}$$

⁵Neyman's construction gives the same kind of balanced sequences for the communication phase but their length is not the lowest one.

where the first inequality follows from the fact that $H(\frac{[\bar{k}H(r)]-1}{2[\bar{k}H(r)]})$ is greater than $2(\frac{[\bar{k}H(r)]-1}{2[\bar{k}H(r)]} + \frac{1}{2[\bar{k}H(r)]})$; the second inequality is given by the definition of the upper integer part of a number and the last one comes from the assumption that the pair $(\bar{k}, H(r)) \in G(k, r)$. We obtain, again, a contradiction because $\tilde{k} < \bar{k}$

Moreover, from above we get that $(\bar{k}, \frac{1}{2}) \in G(k, r)$. In other words, $r_{\bar{k}} = \frac{1}{2}$.

Finally, $[\hat{k}] < \bar{k}$ since $\frac{1}{([\hat{k}+1]^2)}2^{[\hat{k}]H(r)} < \frac{1}{(\bar{k}+1)^2}2^{\bar{k}H(\frac{1}{2})}$ for any $r \in [0, \frac{1}{2}]$. In order to prove that $\bar{k} \leq 2\hat{k}$ for $\hat{k} > 6.694$ it suffices to see that

$$\frac{1}{(2\hat{k} + 1)^2}2^{2\hat{k}H(\frac{1}{2})} \geq 2\hat{k}$$

for $\hat{k} > 6.694$. ■

We now construct the set of equilibrium plays Q and the corresponding set of sequences of the verification phase V . We consider sequences of action pairs of length $l = O(T)$. This means that plays are cycles that are repeated until the end of the game. In order to determine the number of repetitions we have to take into account any possible deviation by player 1, who has to be unable to count until the end of the game. Moreover, it is necessary to obscure the location of the states to implement the communication phase. Recall that player 1's automaton can be viewed as a matrix with $|Q|$ rows and l columns. Therefore, the mixed strategy has to be constructed such that the states for the communication phase have to be allocated in a *window*⁶ inside the matrix of states. We can assume that the size of such window is $\frac{l}{6}$. Following Neyman's construction on page 534, the specification of length l could be the integer part of $[\frac{2T}{9}]$. Therefore, if the number of repetitions is larger than $\frac{9}{2}$, then the construction could be repeated. Then consider, for instance, that the number of repetitions is the integer part of $\frac{9}{2}$, i.e., $l = [\frac{2T}{9}]$, with a fixed part for the regular play of length $\frac{5T}{27}$ and $\frac{T}{27}$ stages for the verification phase. Assume that only two actions are used for such phase, that we label by 0 and 1, with $\frac{\varepsilon T}{27}$ stages for one action pair such that the different plays achieve the targeted payoff x . Then, the set of sequences for set V is defined as those sequences belonging to the type set $S_{\frac{T}{27}}(1 - \varepsilon, \varepsilon)$. By equation 2, the cardinality of V verifies:

$$\frac{1}{(\frac{T}{27} + 1)^2}2^{\frac{T}{27}H(\varepsilon)} \leq |V| \leq 2^{\frac{T}{27}H(\varepsilon)}$$

To summarize, the set of plays Q has the same cardinality than that of V . This cardinality is approximately $2^{\frac{T}{27}H(\varepsilon)}$ (i.e., the number of rows of the matrix of player

⁶A window is a submatrix of states where player 1 processes the signal sent by player 2. Notice that after the communication phase player 1 has to follow the row -in the general matrix- corresponding to the play chosen by player 2.

1's automaton). The set of messages M is the typical set with empirical distribution $(\frac{1}{2}, \frac{1}{2})$ and length \bar{k} satisfying:

$$\lceil \frac{T}{27} H(\varepsilon) \rceil < \bar{k} \leq 2 \frac{T}{27} H(\varepsilon)$$

Then,

$$\begin{aligned} m_1 &\leq \lceil \frac{2T}{9} \rceil 2^{\frac{2T}{27} H(\varepsilon)} \\ &\approx \exp\left(\frac{T}{27} H(\varepsilon)\right) \end{aligned}$$

This construction, as well as that of Neyman, relies on three key parameters: the cardinality of Q , the length of the messages belonging to M and the ε -approximation to the targeted payoff x . First, we fix one of them and examine the consequences on the other two. We use an N superscript to refer to Neyman's parameters and an HU superscript to denote ours. We begin by comparing that, for a fixed ε -approximation, m_1^{HU} exceeds m_1^N . Notice that to consider the same ε -approximation, implies that k is the same for the two constructions. Then,

Fix the ε -approximation, then $k^N = k^{HU} = k$. From Neyman's construction, m_1^N is about $T2^k$ and from ours, m_1^{HU} is about $T2^{2k}$, then $m_1^{HU} > m_1^N$.

Alternatively, and in order to see the efficiency of our construction, let us now consider the dual of this result. Start by fixing the number of plays, i.e., the cardinality of Q for both constructions. Namely, $|Q| = 2^k$. This means that both upper bounds coincide, i.e., $m_1^N = m_1^{HU}$. Then, Neyman's length of the messages is $k^N = 2k$ while ours is $k^{HU} = \bar{k}$, where \bar{k} satisfies $k \leq \bar{k} \leq 2k$. This implies that our length is shorter than Neyman's and therefore the number of communication stages is smaller in our case. Hence, the distortion generated by the communication phase is also smaller and then $\varepsilon^{HU} < \varepsilon^N$.

Now consider that no parameter is fixed. We compare the biggest bound of our smaller automaton, m_1^{HU} , with that of Neyman, m_1^N . Let us recall that the set of plays in Neyman's construction has cardinality 2^{k^N} where k^N satisfies equation (1) and, in particular,

$$k^N = \lceil \frac{\varepsilon^3 T}{\ln 2} - \frac{\ln T}{\ln 2} \rceil$$

and that in our construction, the set of plays has cardinality equal to $|V| = |S_{\frac{T}{27}}(1 - \varepsilon, \varepsilon)|$. Therefore, $|V| \approx 2^{k^{HU}}$ where k^{HU} is larger than

$$k^{HU} > \frac{T}{27}H(\varepsilon) - 2\log\left(\frac{T}{27} + 1\right).$$

As already shown, m_1 directly depends on the number of plays, which is an exponential number of the length of the communication phase. To stress the consequences of our construction, the next figure illustrates one of its keypoints: we increase the cardinality of the set of plays while guaranteeing that the communication length is the shortest one. This provides an “efficient” construction, in the sense of the ε -approximation. More precisely, for a T large enough, Figure 1 plots the exponents of the size of m_1 (i.e., $\frac{\log m_1}{T}$) in the y -axis, both in Neyman and ours, as functions of ε (x -axis). Neyman’s approach is represented by the dotted curve $\varepsilon^3 T$, and ours by the full-line curve $\frac{T}{27}H(\varepsilon)$. For small values of ε , the increase in m_1^N is very slow, while the increase in m_1^{HU} is very fast, strongly approaching the exponential size of m_1 (full rationality). Actually, the limit of the ratio $\frac{\log(m_1^{HU})}{\log(m_1^N)} = \frac{H(\varepsilon)}{27\varepsilon^3}$ tends to infinity as ε goes to 0. In other words, for small distortions from the targeted payoff, our exponent grows very fast, thus making m_1^{HU} very close to full rational behavior. More precisely, the concave shape of $H(\varepsilon)$ versus the convex shape of ε^3 , captures the usefulness of Information Theory and Codification Theory to solve computational designs in the framework of repeated games played by finite automata.

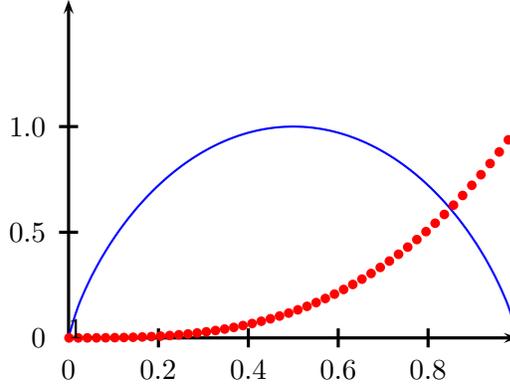


Figure 1: Range of ε with respect to $\frac{\log m_1}{T}$

6 Concluding remarks

This paper has developed a codification approach to construct the computational features of equilibrium strategies in finitely repeated games played by finite automata. We assumed that punishment levels were in pure strategies. Different punishments, either in pure strategies for player 2 and mixed strategies for player 1, or in mixed strategies for both players convey additional relationships between the automata complexity bounds, i.e., between m_1 and m_2 , that depend on the bound on m_1 , already obtained in the pure strategy construction. For example, the former case is in Theorem 2 in Neyman (1998, page 522), where $m_1 \leq \min(m_2, \exp(\varepsilon^3 T))$, and the latter case in Theorem 3 in Neyman, same page, with $m_1 \leq m_2 \leq \exp(\varepsilon^3 \min(T, m_1))$. As it is easily seen the above conditions depend on the constraint stated in Neyman's Theorem 1 (page 521), i.e., $m_1 \leq \exp(\varepsilon^3 T)$. Then, the bounds in both these Theorem can also be improved by our construction.

As already shown, the bound improvement directly depends on the design of both the verification and communication sets. There are several ways to construct both sets and characterize the equilibrium strategies, and therefore the equilibrium bounds. Neyman presented one of them and we have proven that ours is the best one, by means of optimal codification tools applied to repeated games played by finite automata and in the framework of Neyman's equilibrium design (see Lemma 1).

Codification tools and/or Information Theory have been applied to solve different questions in the framework of long strategic interactions. For instance, Gossner, Hernández and Neyman [5] design the optimal strategies in a three-player zero-sum game by the Type methodology and Gossner, Hernández and Neyman [6] apply codification theory to construct strategies in repeated games with asymmetric information about a dynamic state of nature. Their main theorem provides how to construct equilibrium strategies by means of the existence of a code-book, ensuring an appropriate communication level in order to reduce the inefficiencies due to information asymmetries between players. Tools of Information Theory, in particular, the entropy concept, the Kullback distance and typical sets have also been applied by Gossner and Vieille [8], when constructing epsilon-optimal strategies in two-person repeated zero sum games and in Gossner and Tomala [7], to characterize the set of limits of the expected empirical distribution of a stochastic process with respect to the conditional beliefs of the observer.

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