

# ***A discusión***

## **CAPITAL-SKILL COMPLEMENTARITY AND STEADY-STATE GROWTH\***

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## ABSTRACT

We construct a general-equilibrium version of Krusell, Ohanian, Ríos-Rull and Violante's (2000) model with capital-skill complementarity. To account for growth patterns observed in the data, we assume several sources of growth simultaneously, specifically, exogenous growth of skilled and unskilled labor, equipment-specific technological progress, skilled and unskilled labor-augmenting technological progress and Hicks-neutral technological progress. We derive restrictions that make our model consistent with steady-state growth. A calibrated version of our model is able to account for the key growth patterns in the U.S. data, including those for capital equipment and structures, skilled and unskilled labor and output, but it fails to explain the long-run behavior of the skill premium.

**JEL Classification:** C73, D90, E21.

**Key words:** capital-skill complementarity, steady state growth, skill premium, growth model.

# 1 Introduction

Krusell, Ohanian, Ríos-Rull and Violante (2000) show that a Constant Elasticity of Substitution (CES) production function with four production inputs, capital structures, capital equipment, skilled and unskilled labor, is consistent with the key features of the U.S. economy data. In the data, the growth patterns over the 1963-1992 period appear to be highly unbalanced: output and the stock of structures increased by a factor of two; the stock of equipment increased by more than seven times; the number of unskilled workers slightly decreased, whereas the number of skilled workers nearly doubled; the price of equipment relative to consumption (structures) went down by more than four times; and the skill-premium was roughly stationary. All the above regularities are matched in Krusell et al. (2000), by construction, under the appropriate degrees of capital-skill complementarity.<sup>1</sup>

In this paper, we extend the analysis of Krusell et al. (2000) to a general equilibrium case. We restrict our attention to the standard class of models that are consistent with steady-state growth. A convenient property of such models is that they can be converted into stationary ones, so that their equilibria can be studied with standard numerical methods. We ask: "Is a general-equilibrium steady-state growth model parameterized by Krusell's et al. (2000) CES production function still consistent with the U.S. data?"

The standard way to introduce steady-state growth in macroeconomic models is to assume labor-augmenting technological progress (see, e.g., King, Plosser and Rebelo, 1988). However, this assumption is not sufficient for our purpose since it implies that all variables (except labor) grow at the same rate, which does not agree with the empirical facts listed above. As shown in Greenwood, Hercowitz and Krusell (1997), it is possible to account for the empirical observation that equipment grows at a higher rate than output by introducing two other kinds of technological progress, such as equipment-specific and Hicks-neutral ones. However, these two kinds of progress alone are consistent with steady-state growth only under the assumption of the Cobb-Douglas production function (see Greenwood et al., 1997, p. 347) and not under our assumption of the CES production function.

It turns out, however, that we can make the CES production function consistent with steady-state growth by combining the standard labor-augmenting

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<sup>1</sup>Lindquist (2005) uses Krusell's et al. (2000) model to study long-run trends in the skill premium in Sweden.

technological progress with two kinds of progress introduced in Greenwood et al. (1997). To be specific, we simultaneously introduce equipment specific technological progress, skilled and unskilled labor-augmenting technological progress, Hicks-neutral technological progress as well as exogenous growth of skilled and unskilled population.<sup>2</sup> We impose the assumption of complete markets, which allows us to analyze equilibrium by considering the corresponding planner's problem. A distinctive feature of our setup is that skilled and unskilled population grow at different rates. We show that in spite of this feature, welfare weights assigned by the planner to the two subpopulations do not depend on their growth rates but only on their initial sizes. With this result and as with some additional restrictions on preferences and the rates of progress, there exists a stationary economy associated with our growing economy.

We calibrate the model to match a set of relevant observations about the U.S. economy. We find that the calibrated version of our model can account remarkably well for the key growth patterns in the data including those for capital equipment and structures, skilled and unskilled labor and output. Specifically, the above variables in our model grow at different rates, which are close to those in the data. Nonetheless, our model has an important drawback: it dramatically fails on the growth pattern of the skill premium predicting that the skill premium falls, while in the data, the skill premium exhibits a roughly stationary behavior. We argue that the above drawback is a generic feature of our model, and it is difficult to correct it without relaxing our restriction of steady-state growth.

As far as the business-cycle properties of our model are concerned, it turned out that the stationary version of our model is virtually identical to the one considered in Lindquist (2004) where there is no growth, by construction. Lindquist (2004) performs an extensive study of the business cycle predictions of a stochastic general-equilibrium version of Krusell's et al. (2000) model. The implications of our model are very similar and hence, are not reported.

The rest of the paper is organized as follows. Section 2 describes a competitive-equilibrium economy, presents the associated social planner's economy, introduces growth and derives the corresponding stationary model. Section 3 describes the calibration and the solution procedures. Section 4

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<sup>2</sup>In the context of endogenous growth models, Acemoglu (2003), and Acemoglu and Guerrieri (2004) also allow for several kinds of technological progress such as labor-augmenting, capital-augmenting and neutral ones.

presents the results from simulations, and finally, Section 4 concludes.

## 2 The economy

In this section, we construct a general-equilibrium model with the production function considered in Krusell et al. (2000). We first describe the environment, we then introduce technological progress, and we finally provide analytical results on the existence of a stationary equilibrium in our economy.

### 2.1 The environment

Time is discrete and the horizon is infinite,  $t = 1, 2, \dots, \infty$ . There are two types of agents, skilled and unskilled; their variables are denoted by superscripts "s" and "u", respectively. There are two types of capital stocks, capital structures and capital equipment. The economy has two sectors: one sector produces consumption goods and capital structures and the other sector produces capital equipment. Both sectors use the same technology, however, there is a technology factor specific to the capital-equipment sector. We aggregate the production of the two sectors by introducing an exogenous relative price between consumption (structures) and equipment,  $q_t$ .

Let us denote by  $B_t$  a collection of all possible exogenous states in period  $t$ . We assume that  $B_t$  follows a stationary first-order Markov process. Specifically, let  $\mathfrak{R}$  be the Borel  $\sigma$ -algebra on  $\mathfrak{S}$ . Define a transition function for the distribution of skills  $\Pi : \mathfrak{S} \times \mathfrak{R} \rightarrow [0, 1]$  on the measurable space  $(\mathfrak{S}, \mathfrak{R})$  such that: for each  $z \in \mathfrak{S}$ ,  $\Pi(z, \cdot)$  is a probability measure on  $(\mathfrak{S}, \mathfrak{R})$ , and for each  $Z \in \mathfrak{R}$ ,  $\Pi(\cdot, Z)$  is a  $\mathfrak{R}$ -measurable function. We shall interpret the function  $\Pi(z, Z)$  as the probability that the next period's distribution of skills lies in the set  $Z$  given that the current distribution of skills is  $z$ , i.e.,  $\Pi(z, Z) = \Pr\{B_{t+1} \in Z \mid B_t = z\}$ . The initial state  $B_0 \in \mathfrak{S}$  is given. We assume that there is a complete set of markets, i.e., that the agents can trade state-contingent Arrow securities. The agent's  $i \in \{s, u\}$  portfolio of securities is denoted by  $\{m_t^i(B)\}_{B \in \mathfrak{R}}$ . The claim of type  $B \in \mathfrak{R}$  pays one unit of  $t + 1$  consumption good in the state  $B$  and nothing otherwise. The price of such a claim is  $p_t(B)$ .

In the presence of population growth, the problem of skilled and unskilled

groups of agents,  $i \in \{s, u\}$ , can be written as

$$\max_{\{c_t^i, n_t^i, w_t^i, k_{e,t+1}^i, k_{s,t+1}^i, \{m_{t+1}^i(Z)\}^{Z \in \mathbb{R}}\}} E_0 \sum_{t=0}^{\infty} \beta^t N_t^i U^i(c_t^i, 1 - n_t^i), \quad (1)$$

$$\begin{aligned} & N_t^i c_t^i + N_{t+1}^i \left[ k_{s,t+1}^i + \frac{k_{e,t+1}^i}{q_t} + \int_{\mathbb{R}} p_t(Z) m_{t+1}^i(Z) dZ \right] \\ &= N_t^i \left[ w_t^i n_t^i + (1 - \delta_s + r_{st}) k_{st}^i + (1 - \delta_e + r_{et}) \frac{k_{et}^i}{q_t} + m_t^i(B_t) \right], \quad (2) \end{aligned}$$

where initial endowments of capital structures and equipment,  $k_{s0}^i$  and  $k_{e0}^i$ , and Arrow securities  $m_0^i(B_0)$  are given. Here,  $\beta \in (0, 1)$  is the subjective discount factor;  $E_t$  is the operator of expectation conditional on information set in period  $t$ ;  $N_t^i$  is an exogenously given number of agents of group  $i \in \{s, u\}$ ; the variables  $c_t^i$ ,  $n_t^i$ ,  $w_t^i$ ,  $k_{st}^i$  and  $k_{et}^i$  are, respectively, consumption, labor, the wage per unit of labor, the capital stock of structures and equipment of an agent of group  $i \in \{s, u\}$ ; the time endowment is normalized to one, so the term  $1 - n_t^i$  represents leisure;  $r_{st}$  and  $r_{et}$  are the interest rates paid on capital invested in structures and equipment, respectively; and  $\delta_s \in (0, 1)$  and  $\delta_e \in (0, 1)$  are the depreciation rates of capital structures and capital equipment, respectively. The period utility function  $U^i$  is continuously differentiable, strictly increasing in both arguments and concave.

The production function is of the Constant Elasticity of Substitution (CES) type:

$$y_t = A_t G(k_{st}, k_{et}, s_t, u_t) = A_t k_{st}^\alpha \left[ \mu u_t^\sigma + (1 - \mu) (\lambda k_{et}^\rho + (1 - \lambda) s_t^\rho)^{\frac{\sigma}{\rho}} \right]^{\frac{1-\alpha}{\sigma}}, \quad (3)$$

where  $y_t$  is output;  $A_t$  is an exogenously given level of technology (common to both sectors);  $k_{st}$  and  $k_{et}$  are the inputs of capital structures and capital equipment, respectively; functions  $s_t \equiv s_t(N_t^s n_t^s)$  and  $u_t \equiv u_t(N_t^u n_t^u)$  give the efficiency labor inputs of skilled and unskilled agents, respectively, and will be specified in the next section; and  $\alpha \in (0, 1)$ ,  $\mu \in (0, 1)$ ,  $\lambda \in (0, 1)$ ,  $\rho$  and  $\sigma$  are the parameters governing the elasticities of substitution between structures, equipment, skilled labor and unskilled labor. The firm maximizes period-by-period profits by hiring capital and labor

$$\max_{\{k_{st}, k_{et}, N_t^s n_t^s, N_t^u n_t^u\}} \pi_t = A_t G(k_{st}, k_{et}, s_t, u_t) - r_{st} k_{st} - r_{et} k_{et} - w_t^s N_t^s n_t^s - w_t^u N_t^u n_t^u, \quad (4)$$

taking the market prices as given.

## 2.2 Labor growth and technological progress

Krusell et al. (2000) provide time-series data for the U.S. economy over the 1963-1992 period including those for output, the stocks of structure and equipment, the numbers of skilled and unskilled workers, and the relative price between consumption (structures) and equipment. In the data, the growth patterns appear to be highly unbalanced. To be specific, over the sample period, the output and the stock of structures increased roughly by about a factor of two, while the stock of equipment increased by more than seven times; furthermore, the number of skilled workers nearly doubled, while the number of unskilled workers slightly decreased; and finally, the price of equipment relative to consumption (structures) went down by more than four times.

To make our model consistent with the above unbalanced growth patterns, we introduce several sources of exogenous growth simultaneously. First of all, we assume that skilled and unskilled population can grow at differing rates, i.e.,

$$N_t^s = N_0^s (\gamma^s)^t \quad \text{and} \quad N_t^u = N_0^u (\gamma^u)^t, \quad (5)$$

where  $\gamma^s$  and  $\gamma^u$  are the growth rates of the skilled and unskilled labor, respectively. Furthermore, we assume three different kinds of technological progress: the first one increases efficiency of both skilled and unskilled labor at possibly different rates (labor-augmenting technological progress), the second one increases the level of technology  $A_t$  (Hicks-neutral technological progress), and finally, the third one improves the technology of the equipment sector relative to that of the consumption and structure sector or, equivalently, decreases the relative price of equipment  $\frac{1}{q_t}$  (equipment-specific technological progress). We specifically assume that the aggregate labor input of skilled and unskilled agents evolves according to

$$s_t = N_t^s n_t^s (\Gamma^s)^t \quad \text{and} \quad u_t = N_t^u n_t^u (\Gamma^u)^t, \quad (6)$$

where  $\Gamma^s$  and  $\Gamma^u$  are deterministic labor-augmenting technological progress of skilled and unskilled labor, respectively. The remaining two kinds of progress have an identical structure: they include a deterministic time trend and a stochastic stationary component. In particular, the level of technology is

given by

$$A_t = A_0 (\Gamma^A)^t z_t, \quad (7)$$

where  $\Gamma^A$  is a deterministic growth rate, and  $z_t$  is a stationary process. Similarly, the relative price is given by

$$\frac{1}{q_t} = \frac{\varkappa_t}{q_0 (\Gamma^q)^t}, \quad (8)$$

where  $\Gamma^q$  is a deterministic growth rate of  $q_t$ , and  $\varkappa_t$  is a stationary process.

### 2.3 Competitive equilibrium

A competitive equilibrium in the economy (1)–(8) is defined as a sequence of contingency plans for the agents' allocation  $\{c_t^i, n_t^i, k_{s,t+1}^i, k_{e,t+1}^i, m_{t+1}^i(Z)\}_{Z \in \mathfrak{R}, t \in T}^{i \in \{s, u\}}$ , for the firm's allocation  $\{k_{st}, k_{et}, s_t, u_t\}_{t \in T}$  and for the prices  $\{r_{st}, r_{et}, w_t^s, w_t^u, p_t(Z)\}_{Z \in \mathfrak{R}, t \in T}$  such that given the prices:

- (i) the sequence of plans for the agents' allocation solves the utility-maximization problem (1), (2), (8) for  $i \in \{s, u\}$ ;
- (ii) the sequence of plans for the firm's allocation solves the profit-maximization problem of the firm (3) – (8);
- (iii) all markets clear and the economy's resource constraint is satisfied.

Moreover, the equilibrium plans are to be such that  $c_t^i > 0$  and  $0 < n_t^i < 1$  for  $i \in \{s, u\}$ ,  $k_{st}, k_{et} > 0$  and  $p_t(Z) > 0$  for all  $Z \in \mathfrak{R}$ . We assume that an equilibrium exists, it is interior and unique.

### 2.4 Pareto optimum

To simplify the analysis of equilibrium in our decentralized economy (1)–(8), we construct the associated planner's economy. The planner solves

$$\max_{\{c_t^s, c_t^u, n_t^s, n_t^u, k_{s,t+1}, k_{e,t+1}\}} \left\{ E_0 \sum_{t=0}^{\infty} \beta^t [\theta N_0^s U^s(c_t^s, 1 - n_t^s) + (1 - \theta) N_0^u U^u(c_t^u, 1 - n_t^u)] \right\}, \quad (9)$$

subject to the economy's resource constraint

$$N_t^s c_t^s + N_t^u c_t^u + k_{s,t+1} + \frac{k_{e,t+1}}{q_t} = A_t G(k_{st}, k_{et}, s_t, u_t) + (1 - \delta_s) k_{st} + (1 - \delta_e) \frac{k_{et}}{q_t}, \quad (10)$$

where initial endowments of capital structures and equipment,  $k_{s0}$  and  $k_{e0}$  are given; the production function  $G(k_{st}, k_{et}, s_t, u_t)$  is given by (3); skilled and unskilled labor grow according to (5); and the exogenous shocks are given by (6) – (8). In (9),  $\theta$  and  $(1 - \theta)$  are the welfare weights of skilled and unskilled agents, respectively, with  $\theta \in (0, 1)$ .

With the following proposition, we establish the connection between the decentralized and the planner’s economies.

**Proposition 1** *For any distribution of initial endowments in the decentralized economy (1) – (8), there exist welfare weights  $\theta$  and  $(1 - \theta)$  in the planner’s economy (9), (10), such that a competitive equilibrium is a solution to the planner’s problem.*

**Proof.** See Appendix A.  $\parallel$

The result of Proposition 1 might seem surprising. By assumption, the two heterogeneous groups of skilled and unskilled agents can grow at different rates. At a first glance, this feature could make the planner’s objective function non-stationary because the planner is to maximize the weighted sum of individual utilities where the weights, in particular, depend on the groups’ sizes. As follows from Proposition 1, this first-glance intuition is however not correct: the appropriate weights for the planner’s problem are those that depend on the initial sizes of the two groups; the growth rates of the skilled and unskilled groups do not enter the planner’s objective function.

## 2.5 Stationary economy

As described in Section 2.2, our economy contains several sources of growth. To be able to apply standard dynamic-programming methods, we should convert the growing economy into a stationary one. We first focus on the resource constraint (10) under the production function (3).

**Proposition 2** *Assume that  $\Gamma^s \gamma^s = \Gamma^u \gamma^u = \Gamma^q \gamma = (\Gamma^A)^{\frac{1}{\alpha-1}} \gamma$ , where  $\gamma$  is a long-run growth rate of output. Then, the stationary budget constraint that*

corresponds to (3), (10) is given by

$$N_0^s \widehat{c}_t^s + N_0^u \widehat{c}_t^u + \gamma \widehat{k}_{s,t+1} + \gamma \Gamma^q \frac{\mathcal{Z}_t \widehat{k}_{e,t+1}}{q_0} = (1 - \delta_s) \widehat{k}_{st} + (1 - \delta_e) \frac{\mathcal{Z}_t \widehat{k}_{e,t+1}}{q_0}$$

$$A_0 z_t \widehat{k}_{st}^\alpha \left[ \mu (N_0^u n_t^u)^\sigma + (1 - \mu) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) (N_0^s n_t^s)^\rho \right)^\frac{\sigma}{\rho} \right]^\frac{1-\alpha}{\sigma}, \quad (11)$$

where  $\widehat{c}_t^s = \frac{(\gamma^s)^t c_t^s}{\gamma^t}$ ,  $\widehat{c}_t^u = \frac{(\gamma^u)^t c_t^u}{\gamma^t}$ ,  $\widehat{k}_{st} = \frac{k_{st}}{\gamma^t}$  and  $\widehat{k}_{et} = \frac{k_{et}}{(\Gamma^q)^t \gamma^t}$ .

**Proof.** See Appendix A.  $\parallel$

Thus,  $c_t^s$  and  $c_t^u$  grow at the rates  $\frac{\gamma}{\gamma^s}$  and  $\frac{\gamma}{\gamma^u}$ , respectively;  $k_{st}$  and  $y_t$  grow at the rate  $\gamma$ ; and finally,  $s_t$ ,  $u_t$  and  $k_{et}$  grow at the same rate  $\Gamma^s \gamma^s = \Gamma^u \gamma^u = \Gamma^q \gamma$ .

We now turn to preferences. In terms of new variables  $\widehat{c}_t^s$  and  $\widehat{c}_t^u$ , we can re-write (9) as follows:

$$\left\{ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \theta N_0^s U^s \left( \frac{\gamma^t \widehat{c}_t^s}{(\gamma^s)^t}, 1 - n_t^s \right) + (1 - \theta) N_0^u U^u \left( \frac{\gamma^t \widehat{c}_t^u}{(\gamma^u)^t}, 1 - n_t^u \right) \right] \right\}. \quad (12)$$

King, Plosser and Rebelo (1988) shows that the standard Kydland and Prescott's (1982) model is consistent with steady-state growth only under the following two classes of preferences:

$$U(c, 1 - n) = \ln(c) + v(1 - n), \quad (13)$$

$$U(c, 1 - n) = \frac{c^{1-\varrho}}{1-\varrho} v(1 - n) \quad 0 < \varrho < 1 \text{ or } \varrho > 1, \quad (14)$$

where under the additively separable utility function (13),  $v(1 - n)$  is increasing and concave, and under the multiplicatively separable utility function (14),  $v(1 - n)$  is increasing and concave if  $0 < \varrho < 1$ , and it is decreasing and convex if  $\varrho > 1$ .

With the following proposition, we show that the above two utility functions are also consistent with steady-state growth in our heterogeneous-agent setup, however, under (14), we should impose additional restriction on the inverse of intertemporal elasticity of substitution in consumption for skilled and unskilled agents if these two groups grow at different rates.

**Proposition 3** *Preferences (12) are stationary if and only if the momentary utility function is for  $i \in \{s, u\}$  given by*

(a)  $U^i(c, 1 - n) = \ln(c) + v^i(1 - n)$ ;

(b)  $U^i(c, 1 - n) = \frac{c^{1-\varrho^i}}{1-\varrho^i} v^i(1 - n)$  with  $\varrho^s$  and  $\varrho^u$  satisfying  $\left(\frac{\gamma}{\gamma^s}\right)^{1-\varrho^s} = \left(\frac{\gamma}{\gamma^u}\right)^{1-\varrho^u}$ .

**Proof.** See Appendix A.  $\parallel$

We shall finally mention two properties of the model that are useful for our future analysis.

**Proposition 4** *In the economy that is consistent with steady-state growth,*

(a) *If  $\gamma^s \geq \gamma^u$ , then  $\Gamma^s \leq \Gamma^u$ ;*

(b) *if  $\Gamma^q \geq 1$ , then  $\Gamma^A \leq 1$ .*

**Proof.** The results (a) and (b) follow, respectively, from the restrictions  $\Gamma^s \gamma^s = \Gamma^u \gamma^u$  and  $\Gamma^q = (\Gamma^A)^{\frac{1}{\alpha-1}}$  of Proposition 2.  $\parallel$

That is, the assumption of steady-state growth requires that (a) whenever skilled labor grows at a higher (lower) rate than unskilled one, efficiency of high skilled labor should grow at a proportionally lower (higher) rate than efficiency of low skilled labor, and (b) whenever the efficiency of producing equipment relative to structures increases (decreases), Hicks-neutral technological progress is negative (positive).

### 3 Calibration and solution procedures

In this section, we describe the methodology of our numerical study. For the numerical part, we restrict attention to the additively separable utility function of the addilog type with the sub-function  $v^i(1 - n)$  being identical for two types of agents,

$$U(c, 1 - n) = \ln(c) + B \frac{(1 - n)^{1-v} - 1}{1 - v}. \quad (15)$$

Consequently, a stationary version of the planner's problem can be written

as

$$\max_{\{\widehat{c}_t^s, \widehat{c}_t^u, n_t^s, n_t^u, \widehat{k}_{e,t+1}, \widehat{k}_{s,t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta N_0^s \left[ \log(\widehat{c}_t^s) + B \frac{(1 - n_t^s)^{1-v} - 1}{1-v} \right] + (1 - \theta) N_0^u \left[ \log(\widehat{c}_t^u) + B \frac{(1 - n_t^u)^{1-v} - 1}{1-v} \right] \right\}, \quad (16)$$

subject to (11). The First Order Conditions (FOCs) of the problem (16) are derived in Appendix B.

Krusell et al. (2000) estimate the parameters in the production function (3) as well as the parameters for the stochastic shocks for the U.S. economy data over the 1963-1992 period. Since we assume the same production function, and we use the same data set, we follow the parameter choice in Krusell et al. (2000) as close as possible. However, we cannot use all their estimates because there is an important difference between our and their frameworks: Krusell et al. (2000) impose no restrictions on the growth and cyclical patterns, while we assume steady-state growth and a first-order recursive stationary Markov equilibrium. We outline the main steps of the calibration procedure below; further details are provided in Appendix C.

We assume the depreciation rates of capital structures and capital equipment,  $\delta_s = 0.05$  and  $\delta_e = 0.125$ , and the parameters of the production function,  $\alpha = 0.117$ ,  $\sigma = 0.401$ ,  $\rho = -0.495$ , as estimated in Krusell et al. (2000). We estimate the process for  $q_t$  in (8) by assuming that the error term follows a first-order autoregressive process  $\log(\varkappa_t) = b^q \log(\varkappa_{t-1}) + \varepsilon_t^q$  with  $\varepsilon_t^q \sim N(0, \sigma^q)$ . (The estimate of Krusell et al. (2000) for  $q_t$  is not applicable to us since they assume an ARIMA process, which is not consistent with our assumption of a first-order recursive Markov equilibrium). To estimate the parameters of the production function  $\lambda$  and  $\mu$ , the parameters for shock  $A_t$  and the sizes of labor-augmenting technological progress,  $\Gamma^s$  and  $\Gamma^u$ , we employ the following iterative procedure.

1. Step 1. Fix some initial value of  $\Gamma^s$  and compute the corresponding value of  $\Gamma^u = \gamma^s \Gamma^s / \gamma^u$ , given  $\gamma^s$  and  $\gamma^u$  computed from the data.
2. Step 2. Find the parameters  $\lambda$  and  $\mu$  to reproduce two statistics in the data: the average (total) labor share of income over the period, and the average ratio of skilled labor's share of income to unskilled labor's share of income.

3. Step 3. Use the data and the obtained parameters  $\Gamma^s$ ,  $\Gamma^u$ ,  $\lambda$  and  $\mu$  to restore the process  $A_t$  according to (3) and estimate the parameters  $\Gamma^A$ ,  $b^A$  and  $\sigma^A$  in (7) by assuming that a first-order autoregressive process for the error term,  $\log(z_t) = b^A \log(z_{t-1}) + \varepsilon_t^A$  with  $\varepsilon_t^A \sim N(0, \sigma^A)$ .
4. Step 4. Given the obtained value of  $\Gamma^A$ , update the value of  $\Gamma^s$  for the next iteration by  $\Gamma^s = \frac{(0.5\Gamma^A + 0.5(\Gamma^q)^{\alpha-1})^{\frac{1}{\alpha-1}} \gamma}{\gamma^s}$ .

Repeat iterations until convergence so that  $\Gamma^s$  assumed initially is the same as the one obtained at the end of computations. Notice that the above iterative scheme simultaneously ensures that  $(\Gamma^A)^{\frac{1}{\alpha-1}} = \Gamma^q$ , which is another restriction necessary for steady-state growth. At the end, we have that  $\Gamma^s \gamma^s = \Gamma^u \gamma^u = \Gamma^q \gamma = (\Gamma^A)^{\frac{1}{\alpha-1}} \gamma$ , as required in Proposition 2.

We have to resort to this iterative procedure because our model has labor-augmenting technological progress for skilled and unskilled labor whose sizes cannot be directly estimated from the data. (This problem does not arise in the analysis of Krusell et al., 2000, since they assume no labor-augmenting technological progress).

We then calibrate the discount factor  $\beta$ , the welfare weight  $\theta$  and the utility function parameter  $B$  by using the FOCs of the problem (16), evaluated in steady state (see Appendix C). The obtained values of the parameters are summarized in *Tables 1 and 2*.

To solve the model, we use a simulation-based variant of the Parameterized Expectations Algorithm (PEA) by den Haan and Marcet (1990). To ensure the convergence of the PEA, we bound the simulated series on initial iterations, as described in Maliar and Maliar (2003b). The model has two features that complicate the computation procedure. First, there are two intertemporal FOCs, so that we must parameterize two conditional expectations. Second, there are two intratemporal conditions that cannot be resolved analytically with respect to skilled and unskilled labor. Solving numerically the two intratemporal conditions on each date within the iterative cycle is costly, so we find it easier to parameterize the intratemporal conditions in the same way as we do the intertemporal FOCs. We then solve for equilibrium by iterating on the parameters of the resulting four decision rules simultaneously. The details of the solution procedure are described in Appendix D. Once the solution to the stationary model was computed, we restore the growing variables by incorporating the corresponding deterministic trends.

## 4 Results

In Figure 1, we plot the key variables (in logarithms) of the benchmark version of our model with the elasticity of substitution of labor  $1/v = 1$  under the actual sequence of relative prices,  $1/q_t$ , and under the fitted sequence of technology levels,  $A_t$ . As we see, the model is overall successful in explaining the growth patterns observed in the data. First, by construction, it generates appropriate labor-growth patterns, namely, an increasing pattern for skilled labor and a decreasing pattern for unskilled labor. Second, it produces series for capital structures and equipment growing at different rates, which are comparable to those observed in the data. Finally, the model predicts increasing patterns for output and wages of skilled and unskilled agents, which also agrees with the data.

A striking but not surprising implication of our model is that the rate of Hicks-neutral technological progress is, on average, negative,  $\Gamma^A < 1$ . Indeed, given that in the data, equipment becomes cheaper over time in relative terms than structures (i.e.,  $\Gamma^q > 1$ ), by Proposition 4, we should necessarily have that  $\Gamma^A < 1$ . In the calibrated version of the model, this effect proved to be very large,  $\Gamma^A = 0.9586$ , as *Table 1* shows. Our finding that Hicks-neutral technological progress is, on average, negative is the same as the one of Greenwood et al. (1997) who also report a dramatic downturn in total factor productivity since the early 1970's. To explain their result, Greenwood et al. (1997) make a growth accounting exercise and demonstrate that the average growth rate of total factor productivity depends on how capital is incorporated in the model. Specifically, they show that once total capital is split between equipment and structures, the productivity downturn increases.

There is one undesirable growth feature of our model that is difficult to correct given our assumption of steady-state growth. Specifically, the model fails to generate appropriate growth rates of wages for skilled and unskilled labor: the wages of skilled agents grow more in the model than in the data, while the opposite is true for the wages of unskilled agents. As a result, the model fails to explain the time-series behavior of the skill premium,  $\pi_t \equiv \frac{w_t^s}{w_t^u}$ : in the model, the skill premium has a strong downward trend, while in the data, such a trend is absent.

In fact, the above undesirable feature is generic to our model and have

been already anticipated in Proposition 4. Specifically, we have

$$\begin{aligned} \pi_t &= \left\{ \frac{G_3(k_{st}, k_{et}, s_t, u_t)}{G_4(k_{st}, k_{et}, s_t, u_t)} \right\} \times \left[ \frac{\Gamma^s}{\Gamma^u} \right]^t = \\ &= \left\{ \frac{(1-\mu)(1-\lambda) \left[ \lambda \left( \frac{\hat{k}_{et}}{N_0^s n_t^s} \right)^\rho + (1-\lambda) \right]^{\frac{\sigma}{\rho}-1} (N_0^s n_t^s)^{\sigma-1}}{\mu (N_0^u n_t^u)^{\sigma-1}} \right\} \times \left[ \frac{\Gamma^s}{\Gamma^u} \right]^t. \quad (17) \end{aligned}$$

Since in the data, skilled labor grows at a higher rate than unskilled labor,  $\gamma^s > \gamma^u$ , the assumption of steady-state growth implies that labor-augmenting technological progress is larger for unskilled agents, than for skilled agents,  $\Gamma^u > \Gamma^s$ . As follows from *Table 1*, the difference between  $\Gamma^s$  and  $\Gamma^u$  in the calibrated version of the model is very large, i.e.,  $\Gamma^s = 1.0562$  and  $\Gamma^u = 1.0856$ . Given that the first term of the expression (17) is stationary, and that the second term has a downward growth component  $\left[ \frac{\Gamma^s}{\Gamma^u} \right]^t$ , we have a strong decreasing pattern in the skill premium. The analysis of Krusell et al. (2000) does not suffer from this shortcoming because they do not impose the restriction of steady-state growth and hence, the skill premium in their model does not have a downward growth component  $\left[ \frac{\Gamma^s}{\Gamma^u} \right]^t$ .

To check the robustness of our results, we perform the sensitivity analysis with respect to the elasticity of substitution of labor,  $1/v$ , the only parameter that is not identified by our calibration procedure and/or Krusell's et al. (2000) analysis. In *Table 3*, we report the growth rates for the key model's variables under the values of  $v \in \{0.5, 1, 5\}$ . As we can see, the specific value of  $v$  has virtually no effect on the growth properties of the model.

Finally, we should draw attention to the fact that we do not report the business-cycle predictions of the model such as standard deviations and correlation coefficients. Our predictions are very similar to those obtained in Lindquist (2004). This is because the stationary version of our model is identical to the one considered in Lindquist (2004), up to a different choice of the utility function (he uses the Cobb-Douglas function while we use the addilog one) and up to some differences in the calibration procedure (in particular, he uses quarterly U.S. data while we use yearly U.S. data). Hence, the results of Lindquist (2004) are also valid for our model.

## 5 Conclusion

In this paper, we develop a general-equilibrium version of Krusell's et al. (2000) model of the production side of the economy. A distinctive feature of our analysis is that we allow for several kinds of technological progress simultaneously. As a result, our model is capable of generating variables that grow at different rates. A calibrated version of our model proved to be successful in matching the long-run properties of U.S. economy data on capital equipment and structures, skilled and unskilled labor and output. Nonetheless, the model has an important shortcoming, namely, it fails to explain the long-run behavior of the skill premium. Therefore, the answer to the question posed in the introduction is as follows: "Our general-equilibrium steady-state growth model parameterized by the CES production function cannot explain all features of the U.S. economy data that can do Krusell's et al. (2000) setup".

We show that the shortcoming of our analysis is the consequence of the assumption of steady-state growth. A mechanism that helps Krusell's et al. (2000) account for the skill-premium behavior is the capital-skill complementarity: equipment is a complement with skilled labor and a substitute with unskilled labor, so that an increase in equipment increases productivity of skilled labor and decreases productivity of unskilled labor. This mechanism is not consistent with the assumption of steady-state growth, which lies in the basis of our analysis. Under this assumption, the share of each input in production remains constant even though different variables grow at different rates. Therefore, it cannot happen in our model that one production input substitutes another production input over time, which is the key insight of Krusell's et al. (2000) analysis. To restore the importance of Krusell's et al. (2000) capital-skill complementarity mechanism for the long-run economy's behavior, one should develop models with unbalanced (not steady-state) growth. This modification is however not trivial since the computation of equilibrium cannot be implemented with standard numerical methods.

## References

- [1] Acemoglu, D., 2003, Labor- and capital-augmenting technical change, *Journal of European Economic Association* 1, 1-37.

- [2] Acemoglu, D. and V. Guerrieri, 2004, Non-balanced endogenous growth, MIT, manuscript.
- [3] Den Haan, W. and A. Marcet, 1990, Solving the stochastic growth model by parameterizing expectations, *Journal of Business and Economic Statistics* 8, 31-34.
- [4] Greenwood, J., Hercowitz, Z. and P. Krusell, 1997, Long-run implications of investment-specific technological change, *American Economic Review* 87, 342-362.
- [5] King, R., Plosser, C. and S. Rebelo, 1988, Production, growth and business cycles, *Journal of Monetary Economics* 21, 195-232.
- [6] Krusell, P., Ohanian, L., Ríos-Rull, V., and G. Violante, 2000, Capital-skill complementarity and inequality, *Econometrica* 68, 1029-1053.
- [7] Kydland, F. and E. Prescott, 1982, Time to build and aggregate fluctuations, *Econometrica* 50, 1345-1370.
- [8] Lindquist, M., 2004, Capital-skill complementarity and inequality over the business cycle, *Review of Economic Dynamics* 7, 519-540.
- [9] Lindquist, M., 2005, Capital-skill complementarity and inequality in Sweden, forthcoming in *Scandinavian Journal of Economics* 107.
- [10] Maliar, L. and S. Maliar, 2003a, The representative consumer in the neo-classical growth model with idiosyncratic shocks, *Review of Economic Dynamics* 6, 362-380.
- [11] Maliar, L. and S. Maliar, 2003b, Parameterized expectations algorithm and the moving bounds, *Journal of Business and Economic Statistics* 21, 88-92.

## 6 Appendices

This section presents the supplementary results. Appendix A proves Propositions 1, 2 and 3 in the main text. Appendix B derives the FOCs of the problem (16). Finally, Appendices C and D elaborate the calibration and the solution procedures, respectively.

## 6.1 Appendix A

**Proof of Proposition 1.** Consider the problem of a representative agent of type  $i \in \{s, u\}$ , given by (1) and (2). Dividing by the number of agents  $N_t^i$ , we get

$$\max_{\{c_t^i, n_t^i, k_{e,t+1}^i, k_{s,t+1}^i, \{m_{t+1}^i(Z)\}^{Z \in \mathfrak{R}}\}} E_0 \sum_{t=0}^{\infty} \beta^t U^i(c_t^i, 1 - n_t^i), \quad (18)$$

subject to

$$\begin{aligned} c_t^i + \gamma^i k_{s,t+1} + \frac{\gamma^i k_{e,t+1}}{q_t} + \gamma^i \int_{\mathfrak{R}} p_t(Z) m_{t+1}^i(Z) dZ \\ = w_t^i n_t^i + (1 - \delta_s + r_{st}) k_{st} + (1 - \delta_e + r_{et}) \frac{k_{et}}{q_t} + m_t^i(B_t). \end{aligned} \quad (19)$$

The First Order Condition (FOC) of the problem (18), (19) with respect to holdings of Arrow securities is

$$\phi_t^i p_t(B) \gamma^i = \beta \lambda_{t+1}^i(B') \cdot \Pi \{B_{t+1} = B' \mid B_t = B\}_{B', B \in \mathfrak{R}}, \quad (20)$$

where  $\phi_t^i$  is the Lagrange multiplier associated with the budget constraint (19). By taking the ratio of FOC (20) of a skilled agent  $s$  to that of an unskilled agent  $u$ , we obtain

$$\frac{\phi_0^s}{\phi_0^u} = \frac{\phi_1^s / \gamma^s}{\phi_1^u / \gamma^u} = \dots = \frac{\phi_t^s / (\gamma^s)^t}{\phi_t^u / (\gamma^u)^t} \dots \equiv \frac{\phi^u}{\phi^s}, \quad (21)$$

$\phi^s$  and  $\phi^u$  are some constants. Given that  $\phi_t^i = U_1^i(c_t^i, 1 - n_t^i)$ , we have that the ratio of marginal utilities of consumption of two heterogeneous consumers, adjusted to the corresponding growth rates of population, is constant across time and states of nature

$$\frac{U_1^i(c_0^s, 1 - n_0^s)}{U_1^i(c_0^u, 1 - n_0^u)} = \frac{U_1^i(c_1^s, 1 - n_1^s) / \gamma^s}{U_1^i(c_1^u, 1 - n_1^u) / \gamma^u} = \dots = \frac{U_1^i(c_t^s, 1 - n_t^s) / (\gamma^s)^t}{U_1^i(c_t^u, 1 - n_t^u) / (\gamma^u)^t} = \frac{\phi^u}{\phi^s}. \quad (22)$$

This is a consequence of the assumption of complete markets. The FOCs with respect to physical hours worked, capital structures and equipment of a representative agent of type  $i$ , respectively, are

$$U_2^i(c_t^i, 1 - n_t^i) = U_1^i(c_t^i, 1 - n_t^i) w_t^i (\Gamma^i)^t, \quad (23)$$

$$\gamma^i U_1^i (c_t^i, 1 - n_t^i) = \beta E_t [U_1^i (c_{t+1}^i, 1 - n_{t+1}^i) \cdot (1 - \delta_s + r_{st+1})], \quad (24)$$

$$\gamma^i U_1^i (c_t^i, 1 - n_t^i) / q_t = \beta E_t [U_1^i (c_{t+1}^i, 1 - n_{t+1}^i) / q_{t+1} \cdot (1 - \delta_e + r_{et+1})]. \quad (25)$$

Thus, (22) – (25) are the FOCs of the competitive equilibrium economy.

Let us consider now the planner's problem (9), (10). The FOC with respect to consumption of the skilled and the unskilled agents, respectively, are

$$\theta U_1^s (c_t^s, 1 - n_t^s) = \eta_t (\gamma^s)^t, \quad (26)$$

$$(1 - \theta) U_1^u (c_t^u, 1 - n_t^u) = \eta_t (\gamma^u)^t, \quad (27)$$

where  $\eta_t$  is the Lagrange multiplier associated with the economy's resource constraint (10). Dividing (26) by (27) and setting the value of  $\theta$  so that  $\frac{\phi^u}{\phi^s} = \frac{1-\theta}{\theta}$ , we obtain condition (22) of the competitive equilibrium economy. The FOC with respect to capital structures is

$$\eta_t = \beta E_t [\eta_{t+1} (1 - \delta_s + r_{st+1})]. \quad (28)$$

Combining (26) and (27) with (28), we get condition (24) of the competitive equilibrium economy. Similarly, the FOC with respect to equipment is

$$\eta_t / q_t = \beta E_t [\eta_{t+1} / q_{t+1} (1 - \delta_e + r_{et+1})]. \quad (29)$$

After substituting conditions (26) and (27) into (29), we obtain condition (25) of the competitive equilibrium economy. From the firm's problem (4), equilibrium wages are given by  $w_t^s = A_t G_3 (k_{st}, k_{et}, s_t, u_t) (\Gamma^s)^t$  and  $w_t^u = A_t G_4 (k_{st}, k_{et}, s_t, u_t) (\Gamma^u)^t$ . By substituting these wages into a FOC with respect to physical hours worked of the planner's problem, we get (23). Finally, the resource constraint (10) should be satisfied in competitive equilibrium by definition. The fact that the optimality conditions of the planner's problem are necessary for competitive equilibrium proves the statement of Proposition 1.<sup>3</sup>

**Proof of Proposition 2.** Let us introduce a new variable  $\tilde{k}_{et} \equiv \frac{k_{et}}{(\Gamma^q)^t}$ . In terms of this new variable, the budget constraint (10) combined with the

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<sup>3</sup>Strictly speaking, we also need to show that the individual transversality conditions in the decentralized economy imply the aggregate transversality condition in the planner's economy. This can be shown as in Maliar and Maliar (2003a).

production function (3) becomes

$$\begin{aligned}
& N_t^s c_t^s + N_t^u c_t^u + k_{s,t+1} + \Gamma^q \frac{\varkappa_t \tilde{k}_{e,t+1}}{q_0} = \\
& (1 - \delta_s) k_{st} + (1 - \delta_e) \frac{\varkappa_t \tilde{k}_{et}}{q_0} + A_0 (\Gamma^A)^t z_t k_{st}^\alpha \times \\
& \left\{ \mu [N_t^u n_t^u (\Gamma^u)^t]^\sigma + (1 - \mu) \left( \lambda \tilde{k}_{et}^\rho [(\Gamma^q)^t]^\rho + (1 - \lambda) [N_t^s n_t^s (\Gamma^s)^t]^\rho \right)^{\frac{\sigma}{\rho}} \right\}^{\frac{1-\alpha}{\sigma}}.
\end{aligned} \tag{30}$$

Let us introduce  $\gamma$ , which is defined as a common long-run growth rate of output,  $y_t$ , structures  $k_{st}$  and adjusted equipment  $\tilde{k}_{et}$ . We divide (30) by  $\gamma^t$  to obtain

$$\begin{aligned}
& \frac{N_0^s (\gamma^s)^t c_t^s}{\gamma^t} + \frac{N_0^u (\gamma^u)^t c_t^u}{\gamma^t} + \gamma \frac{k_{s,t+1}}{\gamma^{t+1}} + \gamma \Gamma^q \frac{\varkappa_t k_{e,t+1}}{\gamma^{t+1}} = \\
& (1 - \delta_s) \frac{k_{st}}{\gamma^t} + (1 - \delta_e) \frac{\varkappa_t k_{et}}{q_0 \gamma^t} + A_0 z_t \left( \frac{k_{st}}{\gamma^t} \right)^\alpha \times \\
& \left\{ \frac{\mu [N_0^u n_t^u (\Gamma^u \gamma^u)^t]^\sigma + (1 - \mu) \left[ \lambda \tilde{k}_{et}^\rho [(\Gamma^q)^t]^\rho + (1 - \lambda) [N_0^s n_t^s (\Gamma^s \gamma^s)^t]^\rho \right]^{\frac{\sigma}{\rho}}}{[(\Gamma^A)^t]^{\frac{\sigma}{\alpha-1}} (\gamma^t)^\sigma} \right\}^{\frac{1-\alpha}{\sigma}},
\end{aligned} \tag{31}$$

where we take into account that skilled and unskilled labor grow at constant rates  $\gamma^s$  and  $\gamma^u$ , as is assumed in (5). By imposing the restrictions  $\Gamma^s \gamma^s = \Gamma^u \gamma^u = \Gamma^q \gamma = (\Gamma^A)^{\frac{1}{\alpha-1}} \gamma$  and by introducing notation  $\hat{c}_t^s$ ,  $\hat{c}_t^u$ ,  $\hat{k}_{st}$  and  $\hat{k}_{et}$ , as is shown in Proposition 2, we get the budget constraint (11).

**Proof of Proposition 3.** The necessity part can be shown following the steps outlined in King, Plosser and Rebelo (1988). The sufficiency part can be shown as follows. Under the additively-separable addilog preferences

of type (a), the stationary version of the planner's preferences is

$$\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta N_0^s \left[ \log \left( \frac{(\gamma^s)^t \widehat{c}_t^s}{\gamma^t} \right) + v^s (1 - n_t^s) \right] + \right. \\
& \quad \left. + (1 - \theta) N_0^u \left[ \log \left( \frac{(\gamma^u)^t \widehat{c}_t^u}{\gamma^t} \right) + v^u (1 - n_t^u) \right] \right\} = \\
& = E_0 \sum_{t=0}^{\infty} \beta^t \{ \theta N_0^s [\log (\widehat{c}_t^s) + v^s (1 - n_t^s)] + (1 - \theta) N_0^u [\log (\widehat{c}_t^u) + v^u (1 - n_t^u)] \} + \Upsilon,
\end{aligned} \tag{32}$$

where  $\Upsilon \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta N_0^s \left[ \log \left( \frac{(\gamma^s)^t}{\gamma^t} \right) \right] + (1 - \theta) N_0^u \left[ \log \left( \frac{(\gamma^u)^t}{\gamma^t} \right) \right] \right\}$  is a finite additive term, which has no effect on equilibrium allocation.

Under the multiplicatively-separable Cobb-Douglas preferences (b), restricted to satisfy  $\left(\frac{\gamma}{\gamma^s}\right)^{1-\varrho^s} = \left(\frac{\gamma}{\gamma^u}\right)^{1-\varrho^u}$ , the stationary planner's preferences are given by

$$\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t \left[ \theta N_0^s \frac{\left(\frac{(\gamma^s)^t \widehat{c}_t^s}{\gamma^t}\right)^{1-\varrho^s}}{1 - \varrho^s} v^s (1 - n_t^s) + (1 - \theta) N_0^u \frac{\left(\frac{(\gamma^u)^t \widehat{c}_t^u}{\gamma^t}\right)^{1-\varrho^u}}{1 - \varrho^u} v^u (1 - n_t^u) \right] \\
& = E_0 \sum_{t=0}^{\infty} \widehat{\beta}^t \left[ \theta N_0^s \frac{(\widehat{c}_t^s)^{1-\varrho^s}}{1 - \varrho^s} v^s (1 - n_t^s) + (1 - \theta) N_0^u \frac{(\widehat{c}_t^u)^{1-\varrho^u}}{1 - \varrho^u} v^u (1 - n_t^u) \right],
\end{aligned} \tag{33}$$

where  $\widehat{\beta} \equiv \beta \left(\frac{\gamma}{\gamma^s}\right)^{1-\varrho^s} = \beta \left(\frac{\gamma}{\gamma^u}\right)^{1-\varrho^u}$ .

## 6.2 Appendix B

Let us denote  $\widehat{s}_t = N_0^s n_t^s$  and  $\widehat{u}_t = N_0^u n_t^u$ . Optimality conditions of the problem (16), (11) with respect to  $\widehat{c}_t^s$ ,  $\widehat{c}_t^u$ ,  $n_t^s$ ,  $n_t^u$ ,  $\widehat{k}_{s,t+1}$  and  $\widehat{k}_{e,t+1}$ , respectively, are

$$\theta (\widehat{c}_t^s)^{-1} = \eta_t, \tag{34}$$

$$(1 - \theta) (\widehat{c}_t^u)^{-1} = \eta_t, \tag{35}$$

$$\theta B (1 - n_t^s)^{-v} = \eta_t A_0 z_t G_3 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right), \tag{36}$$

$$(1 - \theta) B (1 - n_t^u)^{-v} = \eta_t A_0 z_t G_4 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right), \quad (37)$$

$$\gamma \eta_t = \beta E_t \left\{ \eta_{t+1} \left[ 1 - \delta_s + A_0 z_{t+1} G_1 \left( \widehat{k}_{s,t+1}, \widehat{k}_{e,t+1}, \widehat{s}_{t+1}, \widehat{u}_{t+1} \right) \right] \right\}, \quad (38)$$

$$\frac{\gamma \Gamma^q \varkappa_t}{q_0} \eta_t = \beta E_t \left\{ \eta_{t+1} \left[ \frac{(1 - \delta_e) \varkappa_{t+1}}{q_0} + A_0 z_{t+1} G_2 \left( \widehat{k}_{s,t+1}, \widehat{k}_{e,t+1}, \widehat{s}_{t+1}, \widehat{u}_{t+1} \right) \right] \right\}, \quad (39)$$

where  $G_i$  is a first-order partial derivative of the function  $G$  with respect to the  $i$ -th argument,  $i = 1, \dots, 4$ . These derivatives are given by

$$G_1 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) = \alpha A_0 z_t \widehat{k}_{st}^{\alpha-1} \left[ \mu \widehat{u}_t^\sigma + (1 - \mu) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1-\alpha}{\sigma}}, \quad (40)$$

$$G_2 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) = A_0 z_t \widehat{k}_{st}^\alpha (1 - \alpha) (1 - \mu) \lambda \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}-1} \widehat{k}_{et}^{\rho-1} \times \left[ \mu \widehat{u}_t^\sigma + (1 - \mu) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1-\alpha}{\sigma}-1}, \quad (41)$$

$$G_3 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) = A_0 z_t \widehat{k}_{st}^\alpha (1 - \alpha) (1 - \mu) (1 - \lambda) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}-1} \widehat{s}_t^{\rho-1} \times \left[ \mu \widehat{u}_t^\sigma + (1 - \mu) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1-\alpha}{\sigma}-1}, \quad (42)$$

$$G_4 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) = A_0 z_t \widehat{k}_{st}^\alpha (1 - \alpha) \mu \times \left[ \mu \widehat{u}_t^\sigma + (1 - \mu) \left( \lambda \widehat{k}_{et}^\rho + (1 - \lambda) \widehat{s}_t^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1-\alpha}{\sigma}-1} \widehat{u}_t^{\sigma-1}. \quad (43)$$

After some algebra, conditions (34) – (39) can be rewritten as follows

$$\gamma \widehat{c}_t^{-1} = \beta E_t \left[ \widehat{c}_{t+1}^{-1} \left( 1 - \delta_s + A_0 z_t G_1 \left( \widehat{k}_{s,t+1}, \widehat{k}_{e,t+1}, \widehat{s}_{t+1}, \widehat{u}_{t+1} \right) \right) \right]. \quad (44)$$

$$\frac{\gamma \Gamma^q \varkappa_t}{q_0} \widehat{c}_t^{-1} = \beta E_t \left[ \widehat{c}_{t+1}^{-1} \left( \frac{(1 - \delta_e) \varkappa_{t+1}}{q_0} + A_0 z_t G_2 \left( \widehat{k}_{s,t+1}, \widehat{k}_{e,t+1}, \widehat{s}_{t+1}, \widehat{u}_{t+1} \right) \right) \right], \quad (45)$$

$$\widehat{s}_t = N_0^s \left[ 1 - \frac{\widehat{c}_t^{\frac{1}{v}} \theta^{\frac{1}{v}} B^{\frac{1}{v}} \left( A_0 z_t G_3 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) \right)^{-\frac{1}{v}}}{[N_0^s \theta + N_0^u (1 - \theta)]^{\frac{1}{v}}} \right], \quad (46)$$

$$\widehat{u}_t = N_0^u \left[ 1 - \frac{\widehat{c}_t^{\frac{1}{v}} (1 - \theta)^{\frac{1}{v}} B^{\frac{1}{v}} \left( A_0 z_t G_4 \left( \widehat{k}_{st}, \widehat{k}_{et}, \widehat{s}_t, \widehat{u}_t \right) \right)^{-\frac{1}{v}}}{[N_0^s \theta + N_0^u (1 - \theta)]^{\frac{1}{v}}} \right]. \quad (47)$$

Optimality conditions (44) – (47) together with the resource constraint (11) characterize the equilibrium.

### 6.3 Appendix C

To compute the values of  $\lambda$  and  $\mu$  in Step 2 of the iterative procedure described in Section 3, we use the derivatives (42) and (43) of the production function to get

$$\mu = \left[ 1 + \frac{\frac{G_{3t} s_t}{G_{4t} u_t} \left( \frac{u_t}{k_{et}} \right)^\sigma}{\left( \frac{s_t}{k_{et}} \right)^\rho (1 - \lambda) \left( \lambda + (1 - \lambda) \left( \frac{s_t}{k_{et}} \right)^\rho \right)^{\sigma/\rho - 1}} \right]^{-1}, \quad (48)$$

$$\frac{G_{3t} s_t + G_{4t} u_t}{y_t} = (1 - \alpha) \times \frac{\left[ (1 - \mu) (1 - \lambda) \left( \lambda + (1 - \lambda) \left( \frac{s_t}{k_{et}} \right)^\rho \right)^{\sigma/\rho - 1} \left( \frac{s_t}{k_{et}} \right)^\rho + \mu \left( \frac{u_t}{k_{et}} \right)^\sigma \right]}{(1 - \mu) \left( \lambda + (1 - \lambda) \left( \frac{s_t}{k_{et}} \right)^\rho \right)^{\sigma/\rho} \left( \frac{s_t}{k_{et}} \right)^\rho + \mu \left( \frac{u_t}{k_{et}} \right)^\sigma}, \quad (49)$$

where  $G_{it} \equiv G_i(k_{st}, k_{et}, s_t, u_t)$ . We compute the ratios  $\frac{G_{3t} s_t}{G_{4t} u_t}$ ,  $\frac{s_t}{k_{et}}$ ,  $\frac{u_t}{k_{et}}$  and  $\frac{G_{3t} s_t + G_{4t} u_t}{y_t}$  as time-series average of variables  $\frac{w_t^s N_t^s (\Gamma^s)^t}{w_t^u N_t^u (\Gamma^u)^t}$ ,  $\frac{N_t^s (\Gamma^s)^t}{k_{et}}$ ,  $\frac{N_t^u (\Gamma^u)^t}{k_{et}}$  and  $\frac{w_t^s N_t^s (\Gamma^s)^t + w_t^u N_t^u (\Gamma^u)^t}{y_t}$ , respectively, where the last four variables are constructed from the data in Krusell et al. (2000) under the assumed values of  $\Gamma^s$  and  $\Gamma^u$ . We then solve numerically equations (48) and (49) with respect to  $\lambda$  and  $\mu$ .

In Step 3, we use the obtained parameters to restore the process for  $A_t$  from (3), i.e.,

$$A_t = \frac{y_t}{k_{st}^\alpha \left\{ \mu [N_t^u (\Gamma^u)^t]^\sigma + (1 - \mu) \left[ \lambda k_{et}^\rho + (1 - \lambda) [N_t^s (\Gamma^s)^t]^\rho \right]^\frac{\sigma}{\rho} \right\}^\frac{1-\alpha}{\sigma}}, \quad (50)$$

where  $y_t$ ,  $N_t^u$ ,  $N_t^s$ ,  $k_{st}$  and  $k_{et}$  are the corresponding time series taken from the data in Krusell et al. (2000).

We calibrate the discount factor  $\beta$  by using FOC (44) evaluated in the steady state

$$\beta = \gamma / \left( 1 - \delta_s + \alpha \frac{y}{k_s} \right), \quad (51)$$

where  $\frac{y}{k_s}$  is the time series average of output to structures ratio in Krusell's et al. (2000) data.<sup>4</sup>

We assume that both skilled and unskilled employed agents work in the steady state 1/3 of their total time,  $n^s = n^u = 1/3$ , so that we can compute  $\widehat{s} = N_0^s n^s$  and  $\widehat{u} = N_0^u n^u$ . We then compute steady state values of capital equipment,  $\widehat{k}_e$ , and structures,  $\widehat{k}_s$ , by solving FOCs (44) and (45) numerically. Combining equations (46) and (47) and evaluating the resulting condition in the steady state, we obtain a formula for calibrating the welfare weight  $\theta$

$$\theta = \frac{(1 - \mu)(1 - \lambda) \left( \lambda \widehat{k}_e^\rho + (1 - \lambda) \widehat{s}^\rho \right)^\frac{\sigma}{\rho} \widehat{s}^{\rho-1}}{(1 - \mu)(1 - \lambda) \left( \lambda \widehat{k}_e^\rho + (1 - \lambda) \widehat{s}^\rho \right)^\frac{\sigma}{\rho} \widehat{s}^{\rho-1} + \mu \widehat{u}^{\sigma-1}}. \quad (52)$$

Finally, to calibrate the utility function parameter  $B$ , we use (46) evaluated in the steady state

$$B = \widehat{c}^{-\gamma} \widehat{G}_3 (1 - n^s)^v \frac{[N_0^s \theta + N_0^u (1 - \theta)]}{\theta}, \quad (53)$$

where the steady-state consumption  $\widehat{c} \equiv N_0^s \widehat{c}_t^s + N_0^u \widehat{c}_t^u$  is obtained from the budget constraint (11) evaluated in the steady state.

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<sup>4</sup>Here and further in the text, we use variables without time subscripts to denote the corresponding steady state values.

## 6.4 Appendix D

We shall first notice that if the expectations were parameterized in both intertemporal FOCs (44) and (45), both conditions would identify consumption. As a consequence, consumption would be overidentified, while the rest of variables would be not identified. We therefore re-write the FOCs in the way, which is more suitable for parameterization, by premultiplying (44) by  $\widehat{k}_{s,t+1}$  and by premultiplying (45) by  $\widehat{k}_{e,t+1}$ . In this way, we obtain two equations that identify two capital stocks,

$$\widehat{k}_{s,t+1} = \frac{\beta E_t [1] \widehat{k}_{s,t+1}}{\gamma \widehat{c}_t^{-\gamma}} \quad \text{and} \quad \widehat{k}_{e,t+1} = \frac{q_0 \beta E_t [2] \widehat{k}_{e,t+1}}{\gamma \gamma^q \varkappa_t \widehat{c}_t^{-\gamma}}, \quad (54)$$

where  $E_t [1]$  and  $E_t [2]$  denote the expectation terms within the brackets in FOCs (44) and (45), respectively.

As far as the intratemporal conditions (46) and (47) are concerned, they do not allow for analytical solution with respect to  $\widehat{s}_t$  and  $\widehat{u}_t$ . Finding a numerical solution to the intratemporal conditions on each date within the iterative cycle is costly, so, as we mentioned in the main text, we find it easier to parameterize the intratemporal conditions in the same way as we parameterize the intertemporal FOCs. To be specific, we parameterize the total hours worked by skilled and unskilled agents

$$\widehat{s}_t = N_0^s [3] \quad \text{and} \quad \widehat{u}_t = N_0^u [4] \quad (55)$$

where [3] and [4] are the expressions within the brackets of FOCs (46) and (47), respectively. Each of the four variables  $\widehat{k}_{s,t+1}$ ,  $\widehat{k}_{e,t+1}$ ,  $\widehat{s}_t$ ,  $\widehat{u}_t$  is parameterized by a first-order exponentiated polynomial

$$\exp \left( \beta_0 + \beta_1 \ln \widehat{k}_{s,t+1} + \beta_2 \ln \widehat{k}_{e,t+1} + \beta_3 \ln z_t + \beta_4 \ln \varkappa_t \right). \quad (56)$$

We are therefore to identify 20 unknown coefficients, five coefficients for each of the four variables parameterized. We do so by using the following iterative procedure:

- *Step 1.* Fix initial  $\beta$ s. Fix initial condition  $(\widehat{k}_{s0}, \widehat{k}_{e0}, z_0, \varkappa_0)$ . Draw and fix a random series for exogenous shocks  $\{z_t, \varkappa_t\}_{t=0}^T$ .
- *Step 2.* Use the assumed decision rules (54), (55) and the budget constraint (11) to calculate recursively  $\{\widehat{k}_{s,t+1}, \widehat{k}_{e,t+1}, \widehat{s}_t, \widehat{u}_t, \widehat{c}_t\}_{t=0}^T$ .

- *Step 3.* Run the non-linear least squares regressions of the corresponding variables on the functional form (56). Use the re-estimated coefficients  $\Phi(\beta(j))$  obtained on iteration  $j$  to update each of 20 coefficients for the next iteration  $(j+1)$  according to  $\beta(j+1) = (1 - \varpi)\beta(j) + \varpi\Phi(\beta(j))$ ,  $\varpi \in (0, 1)$ .

Iterate on  $\beta$ s, until a fixed-point is found.

As an initial guess, we set the values of  $\beta$ s equal to the deterministic steady state. The algorithm was able to systematically converge to the true solution if the coefficients were updated slowly,  $\varpi \leq 0.01$ , and if the simulated series were bounded to rule out implausible (explosive) strategies as described in Maliar and Maliar (2003b). The computational time was around a half an hour when the length of simulations was  $T = 10000$ .

Table 1. The parameters of the utility and production functions.

Parameter	$N_0^s$	$\gamma^s$	$N_0^u$	$\gamma^u$	$\lambda$	$\mu$	$\theta$	$\Gamma^s$	$\Gamma^u$
Value	4.4558	1.0224	17.9903	0.9945	0.9979	0.9197	0.3530	1.0562	1.0856

Table 2. The shock parameters.

Parameter	$q_0$	$\gamma_q$	$(\sigma^q)^2$	$b_q$	$A_0$	$\gamma_A$	$(\sigma^A)^2$	$b_A$
Value	0.9664	1.0491	0.0306	0.9352	10.2125	0.9586	0.0326	0.7143

Table 3. Growth rates for the U.S. and artificial economies.

Statistic <sup>a</sup>	Artificial economy			U.S. Economy <sup>b</sup>
	$\nu=0.5$	$\nu=1.0$	$\nu=5.0$	
$\gamma(k_{st})$	1.0271 (0.0033)	1.0272 (0.0031)	1.0275 (0.0027)	1.0244
$\gamma(k_{et})$	1.0771 (0.0065)	1.0773 (0.0063)	1.0777 (0.0056)	1.0707
$\gamma(N_t^s n_t^s)$	1.0224 (0.0022)	1.0223 (0.0018)	1.0223 (0.0007)	1.0224
$\gamma(N_t^u n_t^u)$	0.9958 (0.0012)	0.9954 (0.0008)	0.9947 (0.0003)	0.9945
$\gamma(y_t)$	1.0308 (0.0029)	1.0304 (0.0027)	1.0299 (0.0022)	1.0294
$\gamma(\pi_t)$	0.9516 (0.0005)	0.9515 (0.0009)	0.9513 (0.0017)	1.0063
$\gamma(w_{st})$	1.0249 (0.0023)	1.0250 (0.0025)	1.0251 (0.0031)	1.0628
$\gamma(w_{ut})$	(1.0771) (0.0020)	(1.0773) (0.0020)	(1.0776) (0.0019)	1.0564

Note: <sup>a</sup> $\gamma(x_t)$  denotes the growth rate of variable  $x_t$ . The growth rates in the model are sample averages computed across 500 simulations. Each simulated series has a length of 30 periods, as do time series for the U.S. economy. The numbers in brackets are sample standard deviations of the corresponding growth rates.

<sup>b</sup>The source for the U.S. data: Krusell, Ohanian, Ríos-Rull and Violante (2000).

Figure 1. The actual and the simulated paths for the US economy through 1963-1992.

