ALLOCATION PROBLEMS WITH INDIVISIBILITIES WHEN PREFERENCES ARE SINGLE-PEAKED*

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ABSTRACT

We consider allocation problems with indivisible goods when agents' preferences are single-peaked. Two natural procedures (up methods and temporary satisfaction methods) are proposed to solve these problems. They are constructed by using priority methods on the cartesian product of agents and integer numbers, interpreted either as peaks or opposite peaks. Thus, two families of solutions arise this way. Our two families of solutions satisfy properties very much related to some well-known properties studied in the case of perfectly divisible goods, and they have a strong relationship with the continuous uniform and equal-distance rules, respectively.

Codes JEL : D61, D63.

Keywords: Allocation problem, indivisibilities, single-peaked preferences, temporary satisfaction method, up method.

1 Introduction

We face up in this paper the problem of allocating indivisible units of an homogeneous good among a group of agents whose preferences are single-peaked. One of the examples used to illustrate this kind of problems is that in which a group of people must supply a certain amount of labor paid hourly to complete a common task. These agents' preferences over worked hours, and thus over earned money, are assumed to be single-peaked. This means that each agent has a most preferred amount of hours (equivalently money) to work, and if it happens that he has to work more than this preferred amount he wishes to deviate as less as possible. Similarly, if it happens that he has to work less than this preferred amount he wishes to be as close as possible to his preferred amount.

Consider now the following situation. At a university department, there is a certain amount of extra hours to be covered by the faculty members. Here the agents are the members of department faculty; each of whom, given the salary per hour, has an "ideal" amount of time to work. Usually, each hour corresponds to a complete class of a particular subject, that is, it is not possible to allocate fractions of hours. In this case, thus, the task is made out of a certain number of indivisible units (hours). Similar situations appear, for instance, in the allocation of shifts in hospitals or hotels.

The above situation is a particular instance of a general set of *problems* called allocation problems under *single-peaked preferences* with indivisible goods. These problems come described by three elements. First, a set of *agents*. Second, an amount of indivisible units of a certain good to be distributed, called the *task*. Finally, a profile of the agents' *preferences* over the number of units involved in the task. A *rule*, or *solution*, is a function that distributes the task among the agents.

The literature related with allocation problems when preferences are single-peaked has focused so far on the continuous case (when the task is perfectly divisible or when monetary compensations are allowed). The traditional way of supporting rules to solve the problem is by applying the so-called *axiomatic method.* Rules are then defended on the basis of the properties they fulfil, and, in general, suitable combinations of different appealing properties are used to differentiate among rules. The most appealing properties in this case have to do with *efficiency*, *equity*, and *incentive compatibility* constrains. By far, the best-known rule in the continuous case, when the task is perfectly divisible, is the *uniform rule*, introduced and characterized by Sprumont (1991). It proposes to treat all agents as equally as possible, subject to efficiency. Characterizations of this rule also appear in Ching (1994), Sönmez (1994), Thomson (1994a,b), and Dagan (1996), among others. An alternative, also well-known rule in the continuous case is the *equal distance rule*, introduced by Thomson (1994a), that proposes to select the allocation at which all agents are equally far from their preferred consumptions, subject to efficiency and boundary conditions, in which case those agents whose consumptions would be negative are given zero instead. Characterizations of this rule appear in Herrero and Villar (1999, 2000) and Herrero (2002). Both the uniform and the equal distance rules are efficient and equitable, and the uniform rule is also incentive compatible, while the equal distance rule is not.

When the good comes in indivisible units, some of the aforementioned properties cannot be met. Whereas efficiency and strategy-proofness can be satisfied in the indivisible case. Equity properties have to be accommodated to this case. The traditional requirement of *equal treatment of equals,* for example, can only be partially reached, and we should allow equal agents to be allotted different amounts. The only way of keeping equality as far as possible is to forbid equal agents allotments to differ in more than one unit.

A natural way to solve rationing problems when the good comes in indivisible units consists of applying priority methods (see Young (1994), Moulin (2000); Moulin and Stong (2002), and Herrero and Martínez (2004)). When there is a pure priority relationship on the set of agents, the easiest way of solving the problem is by asking agents to choose the amount of the task to consume, following the priority ordering, and forcing the last agents to get whatever is left. These pure priority methods are efficient and incentive compatible, but they are far from being minimally equitable. A different method consists of applying priority orderings on the cartesian product of agents and integer numbers, the so-called *standards of comparison* by Young (1994). When using standards of comparison to solve rationing problems, the most natural and equitable procedure consists of starting from some predetermined allocation, and then move out of it, unit by unit by using the standard. This procedure was used in Herrero and Martínez (2004) to solve claims problems.

In this paper, we analyze two methods to solve allocation problems when preferences are single-peaked when the good comes in indivisible units, by using standards of comparison. The first procedure (*up methods*) allocates the task unit by unit, according to the standard, when the numbers paired with the agents are interpreted as agents' peaks. The second procedure (*temporary satisfaction methods*) starts by giving all agents their preferred consumptions, and then move away from this provisional allocation, unit by unit, by using the standard. Here, the numbers paired with the agents are interpreted either as agents' peaks or the opposite peaks, depending upon the type of problem at hand (either an excess demand or excess supply problem).

Then we explore the properties our families of discrete rules may satisfy. As it happens in Herrero and Martínez (2004), in order to approach equality we should consider a subfamily of standards, those called *monotonic standards,* that always give priority to larger numbers. Then it happens that *monotonic up methods* provide allocations very similar to those prescribed by the *equal-distance rule* when the good is perfectly divisible. Similarly, the allocations prescribed by *monotonic temporary satisfaction methods* are as close as possible to those provided by the continuous *uniform rule.* Then we obtain that our discrete families can be characterized by sets of properties very similar to those supporting some of the characterizations of the continuous versions of the respective rules.

The rest of the paper is structured as follows: In Section 2 we set up the problem of allocating indivisible units of a good when preferences are single-peaked. In Section 3 we introduce standards of comparison and use them to construct two allotment procedures: the *up* and the *temporary satisfaction methods*, that convey to construct two families of discrete rules. Section 4 analyzes the properties our families of rules may fulfil. In Section 5 we present our characterization results. Section 6 sets the connections between the divisible and indivisible case. Finally, Section 7 concludes with final comments and remarks. Examples providing the tightness of the characterizations are relegated to Appendix A, while the proofs of our results are relegated to Appendix B.

2 Statement of the problem

A preference relation, R_i, defined over $\mathbb{Z}_+$ is said to be **single-peaked** if there exists an integer number, $p(R_i) \in \mathbb{Z}_+$, called the **peak** of R_i, such that, for each $a, b \in \mathbb{Z}_+$,

$$aP_ib \Leftrightarrow [(b < a < p(R_i)) \text{ or } (p(R_i) < a < b)],$$

where P_i is the strict preference relation induce by R_i. Let $\mathbb{S}$ denote the class of all single-peaked preferences defined over $\mathbb{Z}_+$. Let $\mathbb{N}$ be the set of all potential **agents** and $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. An allocation problem with single-peaked preferences, or simply a **problem**, is a triple $\boldsymbol{e = (N, T, R)}$ in which a fixed number of units, T (called task) has to be distributed among a group of **agents**, $N \in \mathcal{N}$, whose **preferences** over consumption are **single-peaked**, $R = (R_i)_{i \in N} \in \mathbb{S}^N$. Let $\mathbb{A}^N$ denote the class of problems with fixed-agent set N, and $\mathbb{A}$ the class of all problems, that is,

$$\mathbb{A}^N = \left\{e = (N, T, R) \in \{N\} \times \mathbb{Z}_+ \times \mathbb{S}^N\right\}$$

and

$$\mathbb{A} = \bigcup_{N \in \mathcal{N}} \mathbb{A}^N.$$

For each problem, we face the question of finding a division of the task among the agents. An **allocation** for $e \in \mathbb{A}$ is a list of integer numbers, $\boldsymbol{x} \in \mathbb{Z}_+^N$ satisfying the condition of being a complete distribution of the task, i.e. $\sum_{i \in N} x_i = T$. Let $\boldsymbol{X(e)}$ be the set of all allocations for $e \in \mathbb{A}$. A **rule** is a function, $\boldsymbol{F :} \mathbb{A} \longrightarrow \mathbb{Z}_+^{\boldsymbol{N}}$, that selects, for each problem $e \in \mathbb{A}$, a unique allocation $F(e) \in X(e)$.

3 Standards of comparison, and up and temporary satisfaction methods

A standard of comparison is a linear order (complete, antisymmetric and transitive) over the cartesian product $\mathbb{N} \times \mathbb{Z}$ such that for each agent, larger integer numbers have priority over smaller integer numbers.

Standard of Comparison, or simply standard, (Young, 1994): $\sigma : \mathbb{N} \times \mathbb{Z} \longrightarrow \mathbb{Z}_+$ such that for each $i \in \mathbb{N}$, and each $a \in \mathbb{Z}$, $\sigma(i, a+1) < \sigma(i, a)$. Let $\boldsymbol{\Sigma}$ denote the class of all standards of comparison.

Consider a problem with only one unit of task to allocate. The standard of comparison determines the agent who receives this unit. Alternatively, if the task differs from the sum of the peaks (aggregate demand) by just one unit, then all agents, but one, are fully satisfied. In this case, the standard of comparison determines who that agent is.

This class of orders have been applied by Moulin and Stong (2002) and Herrero and Martínez (2004) in the context of claims problems with indivisiblilities.[1]

By using the standards of comparison, we can construct rules to solve allocation problems. The first option consists of an algorithm to allocate all units of the task one by one. The second one consists of accommodating all units of either excess demand or excess supply one by one, after giving (temporarily) all agents their peaks. We shall call these methods *up methods* and *temporary satisfaction methods* respectively.

In order to define our methods, we introduce the notion of agent with the strongest number. Let $M \in \mathcal{N}$ be an agent set. For each list of pairs agent-integer number $\{(i, a_i)\}_{i \in M}$, the **agent with the strongest number** according to the standard of comparison σ is the agent $k \in M$ such that the pair (k, a_k) has the highest priority among all the pairs (i, a_i) according to σ. That is, k is the agent with the strongest number according to σ if for each $i \in M \setminus \{k\}$, then $\sigma(k, a_k) < \sigma(i, a_i)$.

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^{\sigma}}$: Let $e \in \mathbb{A}$. Start by associating to each agent his peak, and then identifying the agent with the strongest number (peak) according to σ. Then give one unit of the task, T, to this agent. Reduce his number (peak) by one unit. Now identify the agent with the new strongest number for σ, and proceed in the same way. Repeat this process until the task runs out.

Temporary satisfaction method associated to $\boldsymbol{\sigma}$, $\boldsymbol{TS^{\sigma}}$: Let $e \in \mathbb{A}$. Start by giving all agents their peaks. Now we distinguish two cases. (1) If the task is not enough, i.e., $\sum_{i \in N} p(R_i) \geq T$. In this case we have to remove some units from the temporary allocation. Associate to each agent his peak, and identify the agent with the strongest number (peak) according to σ. Subtract one unit from this agent (allocation), and reduce his number accordingly. Identify again the agent with the new strongest number according to σ, and proceed in the same way until reaching the task. (2) If the task is too large, i.e., $\sum_{i \in N} p(R_i) \leq T$, we have to allocate extra units to the agents, $T' = T - \sum_{i \in N} p(R_i)$. We shall proceed in the following way. Associate to each agent the opposite of his peak, that is, let $a_i = -p(R_i)$. Identify the agent with the strongest number (-peak) according to σ. Then assign one unit of the remaining task, T', to this agent. Reduce the number of this agent by one unit. Now identify the agent with the new strongest number for σ, and proceed in the same way. Repeat this process until the task T' runs out.

We present know a collection of examples to illustrate the functioning of the two aforementioned methods. Different types of standards have been used.

Example 3.1. Assume that the standard of comparison is such that, restricted to agents in $N = \{1, 2, 3\}$, it happens that $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}_+$. Now, consider the allocation problem where $N = \{1, 2, 3\}$, $T = 6$ and $R = (R_1, R_2, R_3)$ such that $p(R) = (1, 3, 5)$. Note that, in this case, $\sum_{i \in N} p(R_i) > T$. For the pairs involved in

[1]The reader is referred to the survey by Thomson (2003) for a widely exposition of claims problems when the good is perfectly divisible.

the aforementioned problem, we have

$$\sigma(2,3) < \sigma(2,2) < \sigma(2,1) < \sigma(1,1) < \sigma(3,5) < \sigma(3,4) < \sigma(3,3) < \sigma(3,2) < \sigma(3,1).$$

The next table shows how up methods work for this problem and standard. The first column shows the kth unit of the task. The second column shows the allocation up to that unit, $x^{(k)}$. The third column shows the updated vector of numbers, $p^{(k)}$.

T	$x^{(k)}$	$p^{(k)}$
	(0,0,0)	(1,3,5)
1	(0,1,0)	(1,2,5)
2	(0,2,0)	(1,1,5)
3	(0,3,0)	(1,0,5)
4	(1,3,0)	(0,0,5)
5	(1,3,1)	(0,0,4)
6	**(1,3,2)**	(0,0,3)

Example 3.2. The next table shows how the temporary satisfaction method works for the same standard of comparison and problem as in previous example. In this case we start by fully satisfying all agents, that is, by giving to each agent his peak amount. This implies allocating 9 units, but we only have 6 units to allocate. Thus we need to remove 3 units. The table shows the process of removing. The first column shows the kth unit of the task. We start from 9 units and we remove one by one up to reach 6 units. The second column shows the allocation up to that unit, $x^{(k)}$. The third column shows the updated vector of numbers, $p^{(k)}$.

T	$x^{(k)}$	$p^{(k)}$
9	(1,3,5)	
8	(1,2,5)	(1,3,5)
7	(1,1,5)	(1,2,5)
6	**(1,0,5)**	(1,1,5)

Previous examples illustrate the way both the up method and the temporary satisfaction method work. Additionally, they show that these methods could result in *pure priority rules*, depending upon the standard of comparison used. Given the standard of comparison in previous examples, the allocation obtained by means of the up-method is the allocation prescribed by a pure priority rule in which agent 2 is fully satisfied first, then agent 1 comes to the line and he is also fully satisfied, and, finally, any remaining units go to agent 3. As for the allocation obtained by the application of the temporary satisfaction method, it coincides with the allocation recommended by the pure priority rule with the reverse order: Now, agent 3 is the one going first to the line up to when he is fully satisfied, next, agent 1 comes to the line, and finally, any remaining units go to agent 2. Next examples consider a different type of standard of comparison.

Example 3.3. Let $N = \{1,2,3\}$, and assume that the standard of comparison is such that, restricted to agents in N, it happens that for all $i, j \in N$, and all $x, y \in \mathbb{Z}_{++}$, if

$x > y$, then $\sigma(i,x) < \sigma(j,y)$. Furthermore, $\sigma(1,x) < \sigma(2,x) < \sigma(3,x)$ if x is odd, and $\sigma(2,x) < \sigma(1,x) < \sigma(3,x)$ if x is even. Now, let be $T = 14$, and $R = (R_1, R_2, R_3)$ such that $p(R) = (1,3,5)$. The next table shows how the up method works associated to this standard of comparison, after reaching the peaks,

T	$x^{(k)}$	$p^{(k)}$
...	...	...
9	(1,3,5)	(0,0,0)
10	(1,4,5)	(0,-1,0)
11	(2,4,5)	(-1,-1,0)
12	(2,4,6)	(-1,-1,-1)
13	(3,4,5)	(-2,-1,-1)
14	**(3,5,5)**	(-2,-2,-1)

Example 3.4. This example illustrates how the temporary satisfaction method works for the same problem and standard of comparison. In this case, $T = 14 > 9 = p(R_1)+p(R_2)+p(R_3)$. Then, after fully compensating all the agents $T' = 5 = T-(p(R_1)+p(R_2)+p(R_3))$ remains. We associate to each agent his opposite peak: $(1,-1)$, $(2,-3)$, and $(3,-5)$. The next table shows the rest of the process.

T'	$x^{(k)}$	$p^{(k)}$
0	(1,3,5)	(-1,-3,-5)
1	(2,3,5)	(-2,-3,-5)
2	(3,3,5)	(-3,-3,-5)
3	(4,3,5)	(-4,-3,-5)
4	(4,4,5)	(-4,-4,-5)
5	**(4,5,5)**	(-4,-5,-5)

4 Properties

Here we look for properties our rules may fulfil. Some of the following properties have been studied in the case where the good is perfectly divisible, and their rationale and appealingness are preserved in the case of indivisible goods. For some other properties, we have to adapt the fairness principle at hand so that it becomes meaningful in the case of problems with indivisibilities.

One of the basic requirements in the literature is efficiency. An allocation is efficient if and only if there is no other allocation weakly preferred by all the agents and strictly preferred by, at least, one agent. As Sprumont (1991) points out, the principle of *efficiency* is equivalent to asking for each agent to consume no more than his preferred amount if $\sum_{i\in N} p(R_i) \geq T$, and no less if $\sum_{i\in N} p(R_i) \leq T$.

Efficiency: For each $e \in \mathbb{A}$, there is no allocation, $x \in X(e)$, such that, for each $i \in N$, $x_i R_i F_i(e)$, and for some $j \in N$, $x_j P_j F_j(e)$.

The most common and appealing requirement in the continuous case is a property of *impartiality*. In one of its forms, the so-called *equal treatment of equals*. If two agents have identical preferences, then they should be indifferent among their respective allocations. Paired with the requirement of efficiency, it simply means that agents with identical preferences should be allotted the same amount. Unfortunately, no rule can fulfill this property in the context of problems with indivisibilities. It is enough to consider a two-agent, $N = \{1, 2\}$ problem with identical preferences $R_1 = R_2$, and $T = 1$. Young (1994), and Herrero and Martínez (2004) consider a weaker version of this condition, referred to as *balancedness*: If in a problem two agents have equal preferences, then their allocations should differ, at most, by one unit.

Balancedness: For each $e \in \mathbb{A}$ and each $\{i, j\} \subseteq N$, if $R_i = R_j$, then $|F_i(e) - F_j(e)| \leq 1$.

The following property says that an agent's allocation depends only on his preferred consumption.

Peaks only: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$ such that $p(R'_i) = p(R_i)$, then $F_i(e) = F_i(e')$.

The next principle, *ar-truncation,* can be interpreted as an instance of a general principle of independence of irrelevant alternatives. Given $e \in \mathbb{A}$, let $ar(e) = \frac{\sum_{j \in N} p(R_j) - T}{n}$. The number $ar(e)$ is simply the average rationing of the task among the agents in N. This property states that any information on the agents' preferences below $ar(e)$ should be ignored. In consequence, all those problems whose preferences coincide in $[ar(e), +\infty[$ are indistinguishable.

Ar-truncation: For each $e = (N, T, R) \in \mathbb{A}$ and each $e' = (N, T, R') \in \mathbb{A}$, if for each $i \in N$, $R_i = R'_i$ on $[ar(e), +\infty[$, then, $F(e) = F(e')$.

The following two properties refer to the case in which there is a change in a problem's task, without altering agents' preferences. The first one, *one-sided resource monotonicity,* considers the case in which the change in the task does not alter the type of rationing associated to the initial problem, i.e, if initially we have to ration labor, it is still labor to be rationed after the task increasing, or else, if in the initial problem we have to ration leisure, then again, we have too much labor to allocate even after the decreasing of the task. In either case, the property states that no agent should suffer.

One-sided resource monotonicity: For all $e, e' \in \mathbb{A}$ such that $e = (N, T, R)$ and $e' = (N, T', R)$. If it happens that (a) $\sum_{j \in N} p(R_j) \geq T' > T$, or else, (b)$\sum_{j \in N} p(R_j) \leq T' < T$. Then for each $i \in N$, $F_i(e') R_i F_i(e)$.

Imagine now that when estimating the value of the task this falls short, so that the real value is larger than expected. Then two possibilities are open, either to forget about the initial allocation and just solve the new problem, or to keep the tentative allocation and then allocate the rest of the task among the agents, after adjusting the preferences by shifting them by the amount already obtained. The property of *agenda independence* requires that the final allocation should not depend on this timing.

Agenda independence For $e = (N, T, R) \in \mathbb{A}$ and each $T' \in \mathbb{Z}_{++}$, $F(e) = F(N, T', R) + F(N, T - T', R')$, where $R'_i = \pi^{F_i(N, T', R)}(R_i)$.[2]

[2]For a given $a \in \mathbb{Z}$, $\pi^a : \mathbb{S} \longrightarrow \mathbb{S}$ is defined as follows: For each $R \in \mathbb{S}$, $x \pi^a(R) y$ iff $(x + a) R (y + a)$.

The principle of *strategy-proofness* states that truthtelling should be a (weakly) dominant strategy for all agents, or, in other words, that no agent should benefit from misrepresenting his preferences.

Strategy-proofness: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$, each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$, and each $i \in N$, $F_i(e) R_i F_i(e')$.

The next group of properties refers to changes in the set of agents. Suppose that, after solving the problem $e = (N, T, R) \in \mathbb{A}$, a proper subset of agents $S \subset N$ decides to reallocate the total amount they have received, that is, they face a new allocation problem: $(S, \sum_{i \in S} a_i, R_S)$, where $R_S = (R_i)_{i \in S}$ and a is the allocation corresponding to apply the rule to the problem e. A rule satisfies *consistency* if the new reallocation is only a restriction to the subset S of the initial allocation.

Consistency: For each $e \in \mathbb{A}$, each $S \subset N$, and each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(e), R_S)$.

If the previous requirement is made only for subsets of agents of size two, then it is referred to as *bilateral consistency.*

Bilateral consistency: For each $e \in \mathbb{A}$, each $S \subset N$, such that $|S| = 2$,and each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(e), R_S)$.

Finally, as Chun (1999), we consider the possibility of recovering the solution for the general case out of the solutions in the two-agent case. Let us consider an allocation for a problem with the following feature: For each two-agents subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. Then that allocation should be the one selected by the rule for the original problem.

Let $c.con(e; F) \equiv \{x \in \mathbb{Z}_+^N : \sum_{i \in N} x_i = T$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, R_S)\}$

Converse consistency: For each $e \in \mathbb{A}$, $c.con(e; F) \neq \phi$, and if $x \in c.con(e; F)$, then $x = F(e)$.

We present now some relations among the aforementioned properties. Proofs are relegated to Appendix B.

Proposition 4.1. *One-sided resource monotonicity together with consistency imply converse consistency.*

Ching (1994) shows the relationship among efficiency, strategy-proofness, and peaks only. We present here the parallel result for the indivisible case.

Proposition 4.2. *Efficiency and strategy-proofness together imply peaks only.*

Lemma 4.1 ([Elevator lemma] Thomson, 2004)**.** *If a rule F is bilaterally consistent and coincides with a conversely consistent rule F' in the two agent case, then it coincides with F' in general.*

Given $R \in \mathbb{S}$, we call $\pi^a(R)$ the *shifting* of R by a.

5 Monotonic methods and characterizations results

As we observed in Section 3, up and temporary satisfaction methods associated to a standard of comparison may end up in pure priority methods, and thus could violate *balancedness.* In order to guarantee this minimal requirement of impartiality, we should concentrate on a particular subfamily of standards of comparison, that we call *monotonic standards.*

Monotonic standard of comparison: For each $\{i, j\} \subseteq \mathbb{N}$, and each $x, y \in \mathbb{Z}$, if $x > y$, then $\sigma(i, x) < \sigma(j, y)$. Let $\boldsymbol{\Sigma^M}$ denote the subfamily of all monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to agents with larger integer numbers.

The following result is straightforward:

Proposition 5.1. *Let $\sigma \in \Sigma$ be an standard of comparison. Then, the associated up and temporary satisfaction methods, U^σ and TS^σ, satisfy balancedness if and only if σ is monotonic.*

We shall call **up (temporary satisfaction) monotonic methods** to the up (temporary satisfaction) methods associated to monotonic standards of comparison.

We know then that, both up monotonic and temporary satisfaction methods satisfy balancedness. Moreover, they may also satisfy some of the properties introduced in Section 4 (see Table 1).

We present now a characterization of the temporary satisfaction monotonic methods. The formal statement of the result is preceded by a lemma stating properties of rules fulfilling efficiency, balancedness, and strategy proofness.

Lemma 5.1. *Let F be a rule, and two problems, e, e', involving two agents, $\{i, j\}$, such that $e = (\{i, j\}, T, (R, R))$; $e' = (\{i, j\}, T, (R', R'))$ such that either both $2p(R)$, $2p(R')$ are strictly larger or both strictly smaller than T. Then, $F(e) = F(e')$.*

Theorem 5.1. *A rule F satisfies efficiency, balancedness, strategy proofness, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = TS^\sigma$.*

Next, a characterization of the up monotonic methods is provided.

Theorem 5.2. *A rule F satisfies balancedness, peaks only, agenda independence, ar-truncation, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$.*

The following table summarizes the results in this section.

Property	TS^σ	U^σ
Efficiency	Y*	Y
Balancedness	Y*	Y*
Peaks only	Y	Y*
ar-truncation	N	Y*
One-sided resource monotonicity	Y	N
Agenda independence	N	Y*
Strategy-proofness	Y*	N
Consistency	Y*	Y*
Converse consistency	Y	Y

Table 1: "Y" means that the rule satisfies that property for each $\sigma \in \Sigma^M$, while "N" that it does not. On the other hand Y^* means that this property, together with the others marked with $*$ in the same column, characterize the rule.

6 Monotonic methods and continuous rules.

In the previous section we obtained characterization results for the family of up and temporary satisfaction monotonic methods. The properties used in the characterization result for the family of monotonic temporary satisfaction methods (Theorem 5.1) are very much related to those used by Ching (1994) to characterize the uniform rule,[3] with the suitable change of equal treatment of equal by balancedness. Similarly, the properties in Theorem 5.2 characterizing the family of monotonic up methods are in line with the characterization of the equal distance rule in Herrero and Villar (1999), with identical proviso. We interpret this fact as a suggestion of a strong relationship between our families of methods and the uniform and the equal distance rules, respectively. Actually, the relationship between those monotonic methods and the uniform and equal distance rules is strongest. That is, any monotonic temporary satisfaction method can be interpreted as a discrete version of the uniform rule, and, similarly, any up monotonic method could be interpreted as a discrete version of the equal distance rule. We show that, for any problem, the allocation prescribed by the uniform rule can be interpreted as the ex-ante expectations of the agents under the application of temporary satisfaction monotonic methods, if all plausible monotonic standards are equally likely. Similarly the allocation prescribed by the equal distance rule can be interpreted as the ex-ante expectations of the agents under the application of up monotonic methods, if all plausible monotonic standards are equally likely. Next proposition proves the result.

[3]Under the assumption that the task were completely divisible, two of the most widely studied rules are the so called uniform and equal distance rules. The idea underlying the first one is equality distribution of the task.

Uniform rule, $\boldsymbol{u}$: For each $e \in \mathbb{A}$, selects the unique vector $e(u) \in \mathbb{R}^N$ such that: If $\sum_{i\in N} p(R_i) \geq T$, then $u(e) = \min\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i\in N} \min\{p(R_i), \lambda\} = T$. And, if $\sum_{i\in N} p(R_i) \leq T$, then $u(e) = \max\{p(R_i), \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i\in N} \max\{p(R_i), \lambda\} = T$.

The idea of the second rule is also equality, but now focusing on losses above or below, depending on the case, with respect to the peaks.

Equal distance rule, $\boldsymbol{ed}$: For each $e \in \mathbb{A}$, selects the unique vector $ed(e) \in \mathbb{R}^N$ such that $ed(e) = \max\{0, p(R_i) + \lambda\}$ for some $\lambda \in \mathbb{R}$.

Proposition 6.1. *Let $e \in \mathbb{A}$. Let Σ_e^M denote the subset of Σ^M of the different partial standards involved in problem e.*[4] *Then*

(a) $u(e) = \frac{1}{|\Sigma_e^M|} \sum_{\sigma \in \Sigma_e^M} TS^\sigma(e)$.

(b) $ed(e) = \frac{1}{|\Sigma_e^M|} \sum_{\sigma \in \Sigma_e^M} U^\sigma(e)$.

7 Final Remarks

In this work we have considered allocation problems with indivisible goods when agents' preferences are single-peaked, that is, problems in which the task, the allocations and the preferences are only defined over the set of integer numbers. Two natural procedures, *up and temporary satisfaction methods* have been proposed to solve these problems. The construction of these methods rely on using a particular standard of comparison on the cartesian product of agents and integer numbers, interpreted either as peaks or opposite peaks. Thus, what we propose is not a pair of solutions, but else, two families of solutions, one for each method.

When we concentrate on a certain sub-family of standards, *monotonic standards,* our two families of solutions satisfy properties very much related to some well-known properties studied in the case of perfectly divisible goods, and they have a strongest relationship with the continuous uniform and equal-distance rule, respectively.

Some qualifications on the properties used in this paper are in order. The procedural properties related to changes in the set of agents, *consistency, bilateral consistency* and *converse consistency,* read exactly as in the case of a perfectly divisible good, and they maintain both their interpretation and strength in obtaining the characterization results. It is particularly interesting to note that the main incentive compatibility condition, *strategy proofness,* is not only meaningful in the case of indivisible goods, but also that all the solutions in the family of temporary satisfaction methods do satisfy this property. This means that there is a large family of allocation methods for which the agents do not have incentives to misrepresent their preferences. As for *balancedness,* this property is the best we can do to approach equal treatment, and, in this respect, we may look at our procedures as impartial as they might be, given the indivisibilities.

In the same way as the requirement of balancedness forces is to rely on a subfamily of standards of comparison, the so-called monotonic standards, we may ask whether some additional properties may also significantly reduce the family of standards. Some particular sub-families come naturally to mind, and seem to be worth studying. For instance, consider a priority relation α on the set of agents, and then, construct a monotonic standard

[4] In Σ^M we consider all possible standards over $\mathbb{N} \times \mathbb{Z}_{++}$. Notice that, for a given e, not all of them rank the pairs (i, a_i) involved in that particular problem in different ways. Σ_e^M denotes precisely the subset of those different standards.

σ, out of this priority relation in the following way

$$x > y \Longrightarrow \sigma(i,x) < \sigma(j,y) \qquad \forall i,j \in N$$
$$\alpha(i) < \alpha(j) \Longrightarrow \sigma(i,x) < \sigma(j,x) \qquad \forall x \in \mathbb{Z}$$

This family of standards always respect the priority relation in the set of agents, whenever the integer numbers coincide. We may call this standards *persistent monotonic standards.* It is an open problem to see whether persistent monotonic standards are associated to some appealing additional property for both up and temporary satisfaction methods. This and related questions are left for future research..

Appendix A. On the tightness of characterizations results.

We present now a collection of examples to illustrate the independence of properties used in Theorems 5.1 and 5.2.

Example 7.1. A rule satisfying balancedness, strategy-proofness and consistency but not efficiency. Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents such that agent labeled i has priority over agent labeled $i+1$, i.e., $i \succ i+1$. The rule F works as follows. Let $e \in \mathbb{A}$. Give to each agent the integer part of the uniform allocation,[5] that is, $F_i(e) = \lfloor \frac{T}{n} \rfloor$ for each $i \in N$. If no unit remains we have finished. If some units, $T' = T - n \cdot \lfloor \frac{T}{n} \rfloor$, remain, allotted each one of them to each one of the T' agents with the highest priority according to $\succ$.

Example 7.2. A rule satisfying efficiency, strategy-proofness and consistency but not balancedness. Consider the standard σ such that $\sigma(i, x) < \sigma(i+1, y)$. That is, σ is an standard in which the smaller the agent's label, the higher the priority. Let us consider now the temporary satisfaction method associated to this standard, TS^{σ}. It is easy to check that it satisfies peaks only, strategy-proofness and consistency. But, if we consider the problem $e = (N, T, R)$ where $N = \{1, 2\}$, $T = 4$ and R is such that $R_1 = R_2$ and $p(R_1) = p(R_2) = 5$. Then $TS^{\sigma}(e) = (0, 4)$, violating balancedness.

Example 7.3. A rule satisfying efficiency, balancedness and consistency but not strategy-proofness. It is enough to consider any monotonic up method and to observe Theorem 5.2.

Example 7.4. A rule satisfying efficiency, balancedness and strategy-proofness, but not consistency. This rule, F, can be defined as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotonic standards such that $\sigma_1(i, x) < \sigma_1(i+1, x)$ and $\sigma_2(i+1, x) < \sigma_2(i, x)$. Then, we define the solution $F^{(\sigma_1, \sigma_2)}$ as

$$F^{(\sigma_1,\sigma_2)}(e) = \begin{cases} TS^{\sigma_1}(e) & \text{if } |N| = 2 \\ TS^{\sigma_2}(e) & \text{otherwise} \end{cases}$$

Consider now the problem $e = (N, T, R)$ where $N = \{1, 2, 3\}$, $T = 5$, and R is such that $p(R) = (3, 5, 6)$. Then $F^{(\sigma_1,\sigma_2)}(e) = (1, 2, 2)$; but if $S = \{1, 2\}$ then $F^{(\sigma_1,\sigma_2)}(S, 3, R_S) = (2, 1)$. Therefore, this rule is not consistent.

Example 7.5. A rule satisfying balancedness, ar-truncation and agenda-independence, and consistency, but not peaks only. Let us define such a rule, F, for the two-agent problem, $N = \{i, j\}$. Let us define the order $\alpha : N \times \mathbb{Z} \times \mathbb{S}^2 \longrightarrow \mathbb{Z}_{++}$ such that, if $x > y$, then $\alpha(\cdot, x, \cdot) < \alpha(\cdot, x, \cdot)$. And, if $x = y$, then

$$\begin{aligned} R_i = R_j &\Rightarrow \alpha(i, x, (R_i, R_j)) < \delta(j, x, (R_i, R_j)) \\ R_i \neq R_j &\Rightarrow \alpha(j, x, (R_i, R_j)) < \delta(i, x, (R_i, R_j)) \end{aligned}$$

The order α determines, in case of having only one unit, the agent who gets it. It will depend on the agent, the peaks, and the preferences. To obtain the allocation prescribed

[5]We denote by $\lfloor a \rfloor$ the smallest integer number non greater than a.

by the rule associated to that order α, F^{α}, we proceed in the following way. Let us consider the problem $(\{i,j\}, T, (R_i, R_j))$. Then, identify the agent with the smallest α for the problem, let us say agent i. Give one unit of the task to i. Shift agent i's preferences by a unit to $R_i' = \pi^{F_i(\{i,j\},1,(R_i,R_j))}(R_i)$. In the new problem, $(\{i,j\}, T-1, (R_i', R_j))$, proceed in the same way. Repeat this process until the task runs out.

Example 7.6. A rule satisfying peaks only, ar-truncation, agenda-independence, and consistency, but not balancedness. Select one particular agent $i \in \mathbb{N}$ from the set of all potencial agents. For each $\sigma \in \Sigma^M$, the rule F^{σ} is defined as

$$F_j^{\sigma}(e) = \begin{cases} U_j^{\sigma}(e) & \text{if } \sum_{k\in N} p(R_k) \geq T \\ p(R_i) & \text{if } \sum_{k\in N} p(R_k) < T \text{ and } j = i \\ U_j^{\sigma}(N \smallsetminus \{i\}, T - p(R_i), R_{N\smallsetminus\{i\}}) & \text{if } \sum_{k\in N} p(R_k) < T \text{ and } j \neq i \end{cases}$$

Example 7.7. A rule satisfying peaks only, balancedness, ar-truncation, and consistency, but not agenda-independence. Let $\sigma \in \Sigma^M$, then

$$F(e) = \begin{cases} U^{\sigma}(e) & \text{if } \sum_{j\in N} p(R_j) \leq T \\ TS^{\sigma}(e) & \text{if } \sum_{j\in N} p(R_j) \geq T \end{cases}$$

Example 7.8. A rule satisfying peaks only, balancedness, agenda-independence, and consistency, but not ar-truncation. Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents such that agent labeled i has priority over agent labeled $i+1$, i.e., $i \succ i+1$. And let $\sigma \in \Sigma^M$ a monotonic standard of comparison. Both $\succ$ and σ are independent. Now, for each problem $e \in \mathbb{A}$, the rule F works as follows. If no subset of agents have equal peaks (i.e., all the peaks are different), then we give one unit of the task according to the up monotonic method associated to σ, U^{σ}, and we reduce one unit the peak of the agent who has received the unit. If a subset of agents, let say S, have equal peaks, then we give one unit of the task to the agent in S who has the smallest label (that is, the agent in S with the highest priority according to $\succ$) among all of them involved in S. If there were two or more subsets of agents, let us say S and T, with equal peaks, then we give unit of the task to the agent with the smallest label among all of them involved in S and T. After that, we reduce this agent's peak by one unit. We repeat the process until the task runs out.

Example 7.9. A rule satisfying peaks only, balancedness, ar-truncation and agenda-independence, but not consistency. This rule, F, can be defined as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotonic standards such that $\sigma_1(i,x) < \sigma_1(i+1,x)$ and $\sigma_2(i+1,x) < \sigma_2(i,x)$. Then, we define the solution $F^{(\sigma_1,\sigma_2)}$ as

$$F^{(\sigma_1,\sigma_2)}(e) = \begin{cases} U^{\sigma_1}(e) & \text{if } |N| = 2 \\ U^{\sigma_2}(e) & \text{otherwise} \end{cases}$$

Consider now the problem $e = (N, T, R)$ where $N = \{1,2,3\}$, $T = 5$, and R is such that $p(R) = (3,5,6)$. Then $F^{(\sigma_1,\sigma_2)}(e) = (2,2,1)$; but if $S = \{2,3\}$ then $F^{(\sigma_1,\sigma_2)}(S, 3, R_S) = (1,2)$. Therefore, this rule is not consistent.

Appendix B. Proofs of the results

Proof of Proposition 4.1.

Let $e \in \mathbb{A}$. By *consistency* the set $c.con(e;F) \neq \phi$. Let $x, y \in c.con(e;F)$ with $x \neq y$. We distinguish two cases.

Case 1. If $\sum_{i\in N} p(R_i) \geq T$. There exists $k \in N$ such that $x_k > y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq k$. Since $x, y \in c.con(e;F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *one-sided resource monotonicity*, $x_j \geq y_j$. This fact, join with $x_k > y_j$, and $\sum_{i\in N} x_i = T = \sum_{i\in N} y_i$ yields a contradiction.

Case 2. If $\sum_{i\in N} p(R_i) \leq T$. There exists $k \in N$ such that $x_k < y_k$. Consider each two-agent set $S = \{k, j\}$ with $j \in N$ and $j \neq k$. Since $x, y \in c.con(T, R; F)$, $x_S = F(S, x_j + x_k, R_S)$ and $y_S = F(S, y_j + y_k, R_S)$. By *one-sided resource monotonicity*, $x_j \leq y_j$. This fact, join with $x_k < y_j$, and $\sum_{i\in N} x_i = T = \sum_{i\in N} y_i$ yields a contradiction.

Proof of Proposition 4.2.

Let F be a rule fulfilling *strategy-proofness*. Let $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$ such that $p(R'_i) = p(R_i)$. Let us show that $x_i = F(e) = F(e') = x'_i$ when $T \leq \sum_{j\in N} p(R_j)$. Let us suppose that this is not true, and $x_i \neq x'_i$. We can assume without loss of generality that $x_i < x'_i$. If this is the case, *efficiency* implies that $x_i < x'_i \leq p(R_i) = p(R'_i)$. Then, $x'_i P_i x_i$, which means that $F_i(N, T, (R'_i, R_{-i})) P_i F_i(N, T, (R_i, R_{-i}))$. This implies a contradiction with *strategy-proofness*. Therefore $x_i = x'_i$.

The case when $T \geq \sum_{j\in N} p(R_j)$ is analogous.

Proof of Lemma 5.1.

Consider first the case where $2p(R) > T$, $2p(R') > T$. Let R'' be such that $p(R'') = \frac{T+1}{2}$, and let $e'' = (\{i, j\}, T, (R'', R''))$. We shall prove that $F(e) = F(e'') = F(e')$.

If T is even, *balancedness* implies the result. Let $T = 2\lambda + 1$, for some $\lambda \in \mathbb{Z}$, and suppose, w.l.o.g., that $F(e'') = (\lambda, \lambda + 1)$, whereas $F(e) = (\lambda + 1, \lambda)$. This is the only possibility of discrepancy because of *efficiency* and *balancedness*. Since $p(R) \geq p(R'') = \lambda + 1$, agent j is happier in problem e'' than he is in problem e, and it is the other way around for agent i. Additionally, *strategy proofness* implies that

$$F_i(\{i, j\}, T, (R, R'')) \leq \lambda; \qquad F_j(\{i, j\}, T, (R, R'')) \leq \lambda$$

The first inequality follows from agent i's inability to get a better result when misrepresenting his preferences in problem e'', while the second inequality follows from agent j's inability to benefit from misrepresenting his preferences in problem e. But, if this is the case,

$$F_i(\{i, j\}, T, (R, R")) + F_j(\{i, j\}, T, (R, R")) \leq 2\lambda < T$$

which is a contradiction with F being a rule.

The case where $2p(R) < T$, $2p(R') < T$ is analogous.

Proof of Theorem 5.1.

It is easy to check that each TS^σ satisfies the four properties. Conversely, let F be a rule satisfying all the properties. We divide the rest of the proof into two steps.

Step 1. *Definition of the standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$\begin{aligned}&a > b \Rightarrow \sigma(i,a) < \sigma(j,b)\\&a = b \Rightarrow [\sigma(i,a) < \sigma(j,b) \Leftrightarrow F_i(\{i,j\}, 2a-1, (R_i, R_j)) = a-1],\end{aligned}$$

where R_i and R_j are two single-peaked preference relations such that $p(R_i) = a = b = p(R_j)$. It is straightforward to see that such a σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i,j,k\} \subseteq \mathbb{N}$ such that $\sigma(i,x) < \sigma(j,y)$, $\sigma(j,y) < \sigma(k,z)$, but $\sigma(i,x) > \sigma(k,z)$. By construction and *peaks only* (implied by *strategy proofness* according to Proposition 4.2), this can only happen when $x = y = z$. By the definition of σ, in such a case, $F_i(\{i,j\}, 2x-1, (R_i, R_j)) = x-1$, $F_j(\{j,k\}, 2x-1, (R_j, R_k)) = x-1$ and $F_k(\{k,i\}, 2x-1, (R_k, R_i)) = x-1$, where $p(R_i) = p(R_j) = p(R_k) = x = y = z$. Consider the problem $(\{i,j,k\}, 3x-2, (R_i, R_j, R_k))$. There are only three possible allocations: $(x-1, x-1, x)$, $(x-1, x, x-1)$ and $(x, x-1, x-1)$. Suppose that $F(\{i,j,k\}, 3x-2, (R_i, R_j, R_k)) = (x-1, x-1, x)$, by *consistency*, $F_k(\{i,k\}, 2x-1, (R_i, R_k)) = x$, achieving in this way a contradiction with $F_k(\{i,k\}, 2x-1, (R_i, R_j)) = x-1$. An analogous argument is applied if $F(\{i,j,k\}, 3x-2, (R_i, R_j, R_k)) = (x-1, x, x-1)$, or if $F(\{i,j,k\}, 3x-2, (R_i, R_j, R_k)) = (x, x-1, x-1)$. Therefore $\sigma(i,x) < \sigma(k,z)$, and then σ is transitive.

Step 2. *Let us prove now that $F = TS^\sigma$.* It is straightforward that TS^σ is one-sided resource monotonic and consistent, then, by Proposition 4.1, TS^σ is converse consistent. Therefore, in application of Lemma 4.1, it is sufficient to show the equivalence of both F and TS^σ in the two-agent case. Then, let us consider the problem $e = (S, T, R) \in \mathbb{A}$ where $S = \{i,j\}$. Without loss of generality we can assume that $p(R_i) \leq p(R_j)$. We analyze the case in which $p(R_i) + p(R_j) \geq T$. The other case is completely analogous. We distinguish the following cases:

Case 1. If $R_1 = R_2$ and T is even. By *balancedness*, $F(e) = \left(\frac{T}{2}, \frac{T}{2}\right) = TS^\sigma(e)$.

Case 2. If $R_i = R_j$ and T is odd. If $T = 2p(R_i) - 1$, by the definition of the standard of comparison, $F(e) = TS^\sigma(e)$. If $T < 2p(R_i) - 1$, by Lemma 5.1, $F(e) = F(\{i,j\}, T, (R'_i, R'_j))$, where $R'_i = R'_j$ and $p(R'_i) = p(R'_j) = \frac{T+1}{2}$. And then, $F(e) = F(\{i,j\}, T, (R'_i, R'_j)) = TS^\sigma(\{i,j\}, T, (R'_i, R'_j)) = T^\sigma(e)$.

Case 3. If $F_i(e) \leq F_j(e) \leq p(R_i) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_i(e) = F_i(S, T, (R_j, R_j))= TS_i^\sigma(S, T, (R_j, R_j)) = TS_i^\sigma(e)$.

Case 4. If $F_j(e) \leq F_i(e) \leq p(R_i) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_j(e) = F_j(S, T, (R_i, R_i)) = TS_j^\sigma(S, T, (R_i, R_i)) = TS_j^\sigma(e)$.

Case 5. If $F_i(e) \leq p(R_i) < F_j(e) \leq p(R_j)$. By *efficiency* and *strategy proofness*, $F_i(e) = F_i(S, T, (R_j, R_j)) = TS_i^\sigma(S, T, (R_j, R_j)) = TS_i^\sigma(e)$. If $F_i(e) = TS_i^\sigma(e) = p(R_i)$, then $F_j(e) = T - F_i(e) = T - TS_i^\sigma(e) = TS_j^\sigma(e)$. If $F_i(e) \leq p(R_i) - 1$, then $F_j(e) \leq p(R_i)$, which is a contradiction.

Then, F coincides with TS^σ in the two agents case, and therefore they do so in general.

Proof of Theorem 5.2.

It is easy to check that each up monotonic method satisfies the properties. Conversely, let F be a rule satisfying the five properties.

Step 1. *Definition of the standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$\begin{aligned} &a > b \Rightarrow \sigma(i, a) < \sigma(j, b) \\ &a = b \Rightarrow [\sigma(i, a) < \sigma(j, b) \Leftrightarrow F_i(\{i, j\}, 1, (R_i, R_j)) = 1], \end{aligned}$$

where R_i and R_j are two single-peaked preference relations such that $p(R_i) = a = b = p(R_j)$. It is straightforward to check that σ is an order following a similar argument to Step 1 in the previous theorem.

Step 2. *Let us prove now that $F = U^\sigma$.* It is straightforward that U^σ is one-sided resource monotonic and consistent, then, by Proposition 4.1, U^σ is converse consistent. Therefore, in application of Lemma 4.1 it is sufficient to show the equivalence of both F and U^σ in the two-agent case. Then, let us consider the problem $e = (S, T, R) \in \mathbb{A}$ where $S = \{i, j\}$. Without loss of generality we can assume that $p_i \equiv p(R_i) \leq p(R_j) \equiv p_j$. Suppose first that $p_i = p_j$. By *peaks only, balancedness, agenda independence*, and the definition of the standard, $F(e) = U^\sigma(e)$. Let now $p_i \neq p_j$. We distinguish now the following cases:

Case 1. If $p_i + p_j = T$. Let us show that $F(S, T, (R_i, R_j)) = (p_i, p_j) = U^\sigma(S, T, (R_i, R_j))$. By *ar-truncation*, $F(S, p_j - p_i, (R_i, R_j)) = (0, p_j - p_i)$. Once we have allotted the amount $p_j - p_i$, both agents have the same preference relation: $R_i' = R_j'$, and $T - (p_j - p_i) = 2p_i$ units remain to allocate. By *balancedness*, $F(S, 2p_i, (R_i', R_j')) = (p_i, p_i)$. In application of *agenda independence*, $F(S, T, (R_i, R_j)) = F(S, p_j - p_i, (R_i, R_j)) + F(S, 2p_i, (R_i', R_j')) = (0, p_j - p_i) + (p_i, p_i) = (p_i, p_j)$.

Case 2. If $p_i + p_j < T$. Let us define $T' = p_i + p_j$. Then $F(S, T', R) = (p_i, p_j) = U^\sigma(S, T', R)$ by Case 1. Once we have allotted the amount T', both agents have the same preference relation: $R_i' = R_j'$. And then $F(S, T - T', (R_i', R_j')) = U^\sigma(S, T - T', (R_i', R_j'))$. By *agenda independence*, $F(e) = F(S, T', R) + F(S, T - T', (R_i', R_j')) = U^\sigma(S, T', R) + U^\sigma(S, T - T', (R_i', R_j')) = U^\sigma(e)$.

Case 3. If $p_i + p_j > T$. If T is such that $0 \leq T \leq p_j - p_i$, then $ar(e) \leq p_i$. By *ar-truncation*, $F(e) = (0, T) = U^\sigma(e)$. If T is such that $p_j - p_i \leq T \leq p_i + p_j$, then, by *agenda independence*, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R')$, where $R'_i = R'_j$. Note that, by *ar-truncation*, $F(S, p_j - p_i, R) = (0, p_j - p_i) = U^\sigma(S, p_j - p_i, R)$. By *balancedness* and the definition of the standard, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R') = U^\sigma(S, p_j - p_i, R) + U^\sigma(S, T - (p_j - p_i), R') = U^\sigma(e)$.

Then, F coincides with U^σ in the two agents case, and therefore they do so in general.

Proof of Proposition 6.1.

Let us prove the result for the uniform rule. On one hand, it is known that the continuous uniform rule satisfies *converse consistency*. On the other hand, it is easy to check that the temporary satisfaction monotonic methods are *consistent*. Then the average given by the right hand side in the formula is also consistent (see Thomson (2004)). By using the Elevator Lemma it is enough to consider the two-agent case. But it is straightforward that in this case both the uniform rule and the average coincide. As a result, they are equal in general. An analogous argument proves Statement b.

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