# A discusión

# THE CORE-PERIPHERY MODEL: EXISTENCE AND UNIQUENESS OF SHORT-RUN EQUILIBRIUM

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# THE CORE-PERIPHERY MODEL: EXISTENCE AND UNIQUENESS OF SHORT-RUN EQUILIBRIUM

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## ABSTRACT

We consider the core-periphery model by Krugman (1991). The nature and stability of the possible steady states of the model have been made progressively precise, see Fujita et al. (1999) and Baldwin et al. (2003). In that model as well as in all the new economic geography models that have been derived from it, the short-run (instantaneous) equilibrium is implicitly determined by the current labor distribution across regions. The numerical computations used so far to determine the short-run equilibrium, tend to suggest its existence. In this work, an existence and uniqueness proof of short-run equilibrium is provided.

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# 1 Introduction

We consider the core-periphery model by Krugman (1991). This seminal work has led to the emergence of the so-called New Economic Geography literature. Since the early 90s, the interest in the field has attracted many scientists from various disciplines ranging from economics to regional science and geography. As an illustration of this increasing interest, publications in the field have risen dramatically, see the surveys by Ottaviano and Puga (1988) or Fujita and Thisse (1996), and the recent monographs by Fujita *et al.* (1999), Fujita and Thisse (2002), and Baldwin *et al.* (2003).

The core-periphery model shows how labor mobility leads the economic activity to concentrate in a single region provided that the taste for product variety and the share of manufacturing expenditure are large enough, and transportation costs low enough. This spatial configuration corresponds to the core-periphery equilibrium. Another possible spatial configuration is the symmetric equilibrium in which the economic activity is equally distributed among the two regions. These two spatial configurations are steady states of the spatial economy meaning that when starting from such a configuration, the economy remains in that particular state.

On the other hand, the short-run (instantaneous) equilibrium is implicitly determined by the current labor distribution across regions. The numerical computations used so far to detemine it, tend to suggest its existence. However, even though the conditions for the existence and stability of the symmetric and core-periphery equilibria have been made progressively precise, see Fujita *et al.* (1999), Baldwin *et al.* (2003),

we are not aware of any existence proof of short-run equilibrium.

This works aims at filling up this gap. In Section 1, we consider the reduced form of the core-periphery model and describe its short-run equilibrium. In Section 2 an existence proof of short-run equilibrium is provided. Finally its uniqueness is proved in Section 3.

## 2    Short-Run Equilibrium

We consider the reduced form of the core-periphery model, see Krugman (1991), or Fujita *et al.* (1999). There are two regions $i = 1, 2$. The proportions of the labor force in regions 1 and 2 are given respectively by $\lambda \in [0, 1]$ and $(1 - \lambda)$. The taste for product variety, the share of manufacturing expenditure, and the transportation cost are denoted by $\sigma > 1$, $0 < \mu < 1$, and $T > 1$. In the short-run the description of the economy is described by the variables $Y_i$, $\theta_i$, $W_i$, and $U_i$ which denote respectively the income level, the manufacturing price index, the nominal wage, and the indirect utility level in region $i$

$$
\begin{aligned}
Y_1 &= \frac{1 - \mu}{2} + \mu \lambda W_1 \\
Y_2 &= \frac{1 - \mu}{2} + \mu (1 - \lambda) W_2
\end{aligned}
\tag{1}
$$

$$\theta_1 = \left[\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}\right]^{-\frac{1}{(\sigma-1)}}$$

$$\theta_2 = \left[\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}\right]^{-\frac{1}{(\sigma-1)}} \quad (2)$$

$$W_1 = [Y_1\theta_1^{\sigma-1} + Y_2(\frac{\theta_2}{T})^{\sigma-1}]^{1/\sigma}$$

$$W_2 = [Y_1(\frac{\theta_1}{T})^{\sigma-1} + Y_2\theta_2^{\sigma-1}]^{1/\sigma} \quad (3)$$

$$U_1 = \theta_1^{-\mu}W_1$$

$$U_2 = \theta_2^{-\mu}W_2 \quad (4)$$

The issue of the existence of a short-run equilibrium is about whether there exists $Y_i$, $\theta_i$, $W_i$, and $U_i$ satisfying Eqs. (1), (2), (3), and (4) given some labor force distribution $\lambda$.

## 3    Existence of Short-Run Equilibrium

We reduce the dimensionality of the problem by eliminating the price indices and incomes. This is done by plugging the income and price index Eqs. (1) and (2) in the

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nominal wage Eqs. (3). We get then the following equations involving wages only

$$
\begin{aligned}
W_1^\sigma &= \frac{\frac{1-\mu}{2} + \mu\lambda W_1}{\left[\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}\right]} + \frac{\frac{1-\mu}{2} + \mu(1-\lambda)W_2}{T^{\sigma-1}\left[\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}\right]} \\
W_2^\sigma &= \frac{\frac{1-\mu}{2} + \mu\lambda W_1}{T^{\sigma-1}\left[\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}\right]} + \frac{\frac{1-\mu}{2} + \mu(1-\lambda)W_2}{\left[\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}\right]}
\end{aligned}
$$

$$(5)$$

These last two relationships reduce the original problem to a fixed-point problem in $(W_1, W_2)$. It turns out that it is possible to reduce this last problem to a single variable fixed-point problem by using the following lemma.

**Lemma 1.** The sum of nominal wages across regions is constant

$$\lambda W_1 + (1-\lambda)W_2 = 1 \tag{6}$$

**Proof.** See Appendix $A$.

Thus for any $\lambda \in [0, 1[$, by using the above lemma, the first relationship of Eq. (5) leads to a fixed-point problem in $W_1$

$$W_1 = g(W_1) \tag{7}$$

where the function $g$ is defined by

$$
g(W_1)^\sigma = \frac{\frac{1-\mu}{2} + \lambda\mu W_1}{W_1^{1-\sigma}\lambda + (1-\lambda)\left[T(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\right]^{1-\sigma}} + \frac{T^{1-\sigma}\left[\frac{1-\mu}{2} + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\mu\right]}{(TW_1)^{1-\sigma}\lambda + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)^{1-\sigma}}
$$

Note that in the case $\lambda \in ]0, 1]$, an analogous fixed-point problem in $W_2$ can be derived.

**Proposition 1.** For any $\lambda \in [0, 1]$, the core-periphery model admits a short-run equilibrium.

**Proof.** Without loss of generality, we consider the case $\lambda \in [0, 1[$. When $\lambda = 0$, the function $g$ is equal to a constant and a unique fixed point exists. For $\lambda \in ]0, 1[$, we show in Appendix $A$ that

$$\lim_{W_1 \longrightarrow 0} g(W_1) \;=\; 0$$

$$\lim_{W_1 \longrightarrow 0} \frac{dg}{dW_1}(W_1) \;=\; +\infty$$

$$\lim_{W_1 \longrightarrow \frac{1}{\lambda}} g(W_1) \;=\; 0$$

Since $g$ is continuous on $]0, 1/\lambda[$, this shows that $g$ admits a fixed point $W_1^* \in ]0, 1/\lambda[$. ∎

# 4　Uniqueness of Short-Run Equilibrium

We now show that the short-run equilibrium obtained in Proposition 1 is unique.

**Proposition 2.** For any $\lambda \in [0, 1]$, the short-run equilibrium of the core-periphery model is unique.

**Proof.** By Lemma 1, nominal wages are bounded. In particular $W_1$ is bounded by $1/\lambda$. This suggests the following change of variable

$$W_1 = \frac{1}{z \; \lambda}$$

where variable $z$ belongs to $[1, +\infty[$.

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The fixed-point problem (7) can be rewritten in terms of variable $z$ as follows

$$\frac{1}{z^\sigma \lambda^\sigma} = \frac{\frac{1-\mu}{2} + \frac{\mu}{z}}{z^{\sigma-1}\lambda^\sigma + (1-\lambda)\left[T\left(\frac{1}{1-\lambda} - \frac{1}{z}\frac{1}{1-\lambda}\right)\right]^{1-\sigma}}$$
$$+ \frac{T^{1-\sigma}\left[\frac{1-\mu}{2} + (1-\frac{1}{z})\mu\right]}{T^{1-\sigma}z^{\sigma-1}\lambda^\sigma + (1-\lambda)^\sigma(1-\frac{1}{z})^{1-\sigma}}$$

or equivalently as

$$f(z) = 1$$

where the function $f(z)$ is defined by

$$f(z) = \frac{\frac{1}{2} + \left(\frac{1}{z} - \frac{1}{2}\right)\mu}{\frac{1}{z}\left[1 + T^{1-\sigma}(\frac{1}{\lambda} - 1)^\sigma(z-1)^{1-\sigma}\right]} + \frac{T^{1-\sigma}\left[\frac{1}{2} + (\frac{1}{2} - \frac{1}{z})\mu\right]}{\frac{1}{z}\left[T^{1-\sigma} + (\frac{1}{\lambda} - 1)^\sigma(z-1)^{1-\sigma}\right]}$$

In Appendix $A$ we show that

$$\lim_{z \longrightarrow 1^+} f(z) = 0$$
$$\lim_{z \longrightarrow +\infty} f(z) = +\infty$$
$$\frac{df}{dz}(z) > 0 \text{ for any } z > 1$$

This ensures that $f$ admits a unique $z^* \in ]1, +\infty[$ such that $f(z^*) = 1$. As a consequence, $W_1$ and $W_2$ are uniquely defined by relations (6) and (7). ■

An important issue is the robustness of the result obtained in this paper, and New Economic Geography models in general. It turns out that most results (including the determination of the price level) are very sensitive to the Dixit-Stiglitz formulation. In particular the model becomes ill behaved when $\sigma$ is not larger than 1. Alternative formulations (e.g. alternative consumer preferences) should be studied in the future so as to assess whether the implications of the core-periphery model can be extended to some general class of models.

# 5  Conclusion

In this work we have provided a proof for the existence and uniqueness of the short-run equilibrium of the core-periphery model. Despite the large number of works that have flourished during the last decade in the so-called New Economic Geography literature, and the progress made in analysing the conditions of emergence and stability of the symmetric and core-periphery equilibria, such an analysis of short-run equilibria was still missing so far.

# References

[1] Baldwin, R., Forslid, R., Martin, Ph., Ottaviano, G. and F. Robert-Nicoud (2003), *Economic Geography and Public Policy*, Princeton University Press.

[2] Fujita, M., P. Krugman and A. Venables (1999), *The Spatial Economy. Cities, Regions and International Trade*, MIT Press.

[3] Fujita M. and J.-F. Thisse (1996), Economics of Agglomerations, *Journal of the Japanese and International Economies,* 10, 339-78.

[4] Fujita M. and J.-F. Thisse (2002), *Economics of Agglomeration: Cities, Industrial Location, and Regional Growth*, Cambridge University Press.

[5] Krugman, P. (1991), Increasing Returns and Economic Geography, *Journal of Political Economy.*

[6]  Ottaviano, G. and D. Puga (1998), Agglomeration in the Global Economy: a survey of the New Economic Geography, *World Economy*, 21:6, 707-31.

**Appendix** $A$

**Proof of Lemma** 1.

By multiplying Equs. (3) respectively by $W_1^{1-\sigma}$ and $W_2^{1-\sigma}$, we get

$$W_1^{1-\sigma}W_1^{\sigma} = Y_1 W_1^{1-\sigma}\theta_1^{\sigma-1} + Y_2 W_1^{1-\sigma}(\frac{\theta_2}{T})^{\sigma-1}$$

$$W_2^{1-\sigma}W_2^{\sigma} = Y_1 W_2^{1-\sigma}(\frac{\theta_1}{T})^{\sigma-1} + Y_2 W_2^{1-\sigma}\theta_2^{\sigma-1}$$

Then by the substitution of the price index Equs. (2) in these relationships, we get

$$W_1 = \frac{Y_1 W_1^{1-\sigma}}{[\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}]} + \frac{Y_2 W_1^{1-\sigma}}{T^{\sigma-1}[\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}]}$$

$$W_2 = \frac{Y_1 W_2^{1-\sigma}}{T^{\sigma-1}[\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}]} + \frac{Y_2 W_2^{1-\sigma}}{[\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}]}$$

Total nominal wages can thus be written as

$$\lambda W_1 + (1-\lambda)W_2$$
$$= \frac{\lambda W_1^{1-\sigma} + (1-\lambda)W_2^{1-\sigma}T^{1-\sigma}}{\lambda W_1^{-(\sigma-1)} + (1-\lambda)(W_2 T)^{-(\sigma-1)}}Y_1 + \frac{\lambda W_1^{1-\sigma}T^{1-\sigma} + (1-\lambda)W_2^{1-\sigma}}{\lambda(W_1 T)^{-(\sigma-1)} + (1-\lambda)(W_2)^{-(\sigma-1)}}Y_2$$
$$= Y_1 + Y_2$$

Finally, by using the income relationships (1) we have

$$\lambda W_1 + (1-\lambda)W_2 = 1 - \mu + \mu(\lambda W_1 + (1-\lambda)W_2)$$

meaning that $\lambda W_1 + (1-\lambda)W_2 = 1$ since $\mu \neq 1$. ∎

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**Proof of elements of Proposition 1.**

The limit of $g^\sigma$ when $W_1$ goes to 0 is given by

$$
\lim_{W_1\to 0} g^\sigma = \lim_{W_1\to 0} \frac{\frac{1-\mu}{2} + \lambda\mu W_1}{W_1^{1-\sigma}\lambda + (1-\lambda)\left[T(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\right]^{1-\sigma}}
$$

$$
+ \lim_{W_1\to 0} \frac{T^{1-\sigma}\left[\frac{1-\mu}{2} + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\mu\right]}{(TW_1)^{1-\sigma}\lambda + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)^{1-\sigma}}
$$

$$
= \frac{\frac{1-\mu}{2}}{\lim_{W_1\to 0}\left\{W_1^{1-\sigma}\lambda + (1-\lambda)\left[T(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\right]^{1-\sigma}\right\}} +
$$

$$
+ \frac{T^{1-\sigma}\frac{1+\mu}{2}}{\lim_{W_1\to 0}\left[(TW_1)^{1-\sigma}\lambda + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)^{1-\sigma}\right]}
$$

Since the two limits in the denominators are given by

$$
\lim_{W_1\to 0} W_1^{1-\sigma}\lambda + (1-\lambda)\left[T(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\right]^{1-\sigma}
$$

$$
= \lim_{W_1\to 0} W_1^{1-\sigma}\lambda + \lim_{W_1\to 0}(1-\lambda)\left[T(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)\right]^{1-\sigma}
$$

$$
= \infty + (1-\lambda)^\sigma T^{1-\sigma} \quad\text{, as } \sigma > 1
$$

$$
= \infty
$$

$$
\lim_{W_1\to 0} (TW_1)^{1-\sigma}\lambda + (1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)^{1-\sigma}
$$

$$
= \lim_{W_1\to 0} (TW_1)^{1-\sigma}\lambda + \lim_{W_1\to 0}(1-\lambda)(\frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}W_1)^{1-\sigma}
$$

$$
= \infty + (1-\lambda)^\sigma \quad\quad\text{, as } \sigma > 1
$$

$$
= \infty
$$

we get that $\lim_{W_1\to 0} g = \lim_{W_1\to 0} g^\sigma = 0$.

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This can also be seen by noting that the function $g^\sigma$ may be approximated asymptotically when $W_1$ is close to 0 by the following expression

$$
\begin{aligned}
g^\sigma &\sim \frac{1-\mu}{2}\frac{1}{\lambda W_1^{1-\sigma}} + \frac{T^{1-\sigma}(\frac{1-\mu}{2}+\mu)}{T^{1-\sigma}\lambda W_1^{1-\sigma}} \\
&= \frac{\frac{1-\mu}{2}+\frac{1+\mu}{2}}{\lambda}W_1^{\sigma-1} \\
&= \frac{1}{\lambda}W_1^{\sigma-1} \qquad , \quad W_1 \to 0
\end{aligned}
$$

This confirms that $\lim_{W_1\to 0} g = 0$. Moreover we deduce from the asymptotic approximation that $dg/dW_1 \sim (1/\lambda)^{1/\sigma}(\sigma-1)/\sigma W_1^{-1/\sigma}$ implying that $\lim_{W_1\to 0} dg/dW_1 = +\infty$.

Finally,

$$
\begin{aligned}
&\lim_{W_1\to\frac{1}{\lambda}} g^\sigma \\
&= \frac{\frac{1-\mu}{2}+\mu}{(1-\lambda)\frac{1}{T^{\sigma-1}\left[\lim_{W_1\to\frac{1}{\lambda}}(\frac{1}{1-\lambda}-\frac{\lambda}{1-\lambda}W_1)\right]^{\sigma-1}}} + \frac{T^{1-\sigma}\frac{1-\mu}{2}}{(1-\lambda)\frac{1}{\left[\lim_{W_1\to\frac{1}{\lambda}}(\frac{1}{1-\lambda}-\frac{\lambda}{1-\lambda}W_1)\right]^{\sigma-1}}} \\
&= 0
\end{aligned}
$$

∎

**Proof of elements of Proposition 2.**

The function $f(z)$ can be decomposed as

$$
f(z) = f_1(z) + f_2(z)
$$

where

$$
f_1(z) = \frac{\frac{1}{2}z + \left(1-\frac{1}{2}z\right)\mu}{1 + T^{1-\sigma}\frac{(\frac{1}{\lambda}-1)^\sigma}{(z-1)^{\sigma-1}}} \quad ; \quad f_2(z) = \frac{T^{1-\sigma}\left[\frac{1}{2}z + (\frac{1}{2}z-1)\mu\right]}{T^{1-\sigma} + \frac{(\frac{1}{\lambda}-1)^\sigma}{(z-1)^{\sigma-1}}} \tag{8}
$$

By the inspection of the above relation (8), we have that

$$\lim_{z \longrightarrow 1^+} f_1(z) \;=\; \lim_{z \longrightarrow 1^+} f_2(z) = \lim_{z \longrightarrow 1^+} f(z) = 0$$

$$\lim_{z \longrightarrow +\infty} f_1(z) \;=\; \lim_{z \longrightarrow +\infty} f_2(z) = \lim_{z \longrightarrow +\infty} f(z) = +\infty$$

We now show that $df_1/dz > 0$ and $df_2/dz > 0$ for any $z > 1$. This will imply that

$df/dz > 0$ for any $z > 1$.

The derivative of $f_1$ with respect to $z$ is given by

$$\frac{df_1}{dz}(z) = \frac{(z-1)^\sigma T^\sigma \left[ (z-1)^\sigma T^\sigma (1-\mu) - T(\frac{1}{\lambda}-1)^\sigma (1 - z\sigma + \mu(1 + (z-2)\sigma)) \right]}{2 \left[ (z-1)^\sigma T^\sigma + (z-1)T(\frac{1}{\lambda}-1)^\sigma \right]^2}$$

The first term $(z-1)^\sigma T^\sigma (1-\mu)$ is clearly positive while the sign of the second term

depends on the sign of the following affine function $-(1 - z\sigma + \mu(1 + (z-2)\sigma))$. This

function is strictly positive for any $z > 1$ since it has value $(\sigma - 1)(1 + \mu) > 0$ in $z = 1^+$

and its slope is $\sigma(1 - \mu) > 0$.

Similarly the derivative of $f_2$ with respect to $z$ is given by

$$\frac{df_2}{dz}(z) = \frac{(z-1)^\sigma T \left[ (z-1)^\sigma T(1+\mu) + T^\sigma (\frac{1}{\lambda}-1)^\sigma \left(-1 + z\sigma + \mu(1 + (z-2)\sigma)\right) \right]}{2 \left[ (z-1)^\sigma T + (z-1)T^\sigma (\frac{1}{\lambda}-1)^\sigma \right]^2}$$

which is also strictly positive for any $z > 1$ given that the affine function $(-1 + z\sigma +$

$\mu(1 + (z-2)\sigma))$ has value $(\sigma - 1)(1 - \mu) > 0$ in $z = 1^+$ and its slope is $\sigma(\mu + 1) > 0$. $\blacksquare$