A discusión

WEAKENING THE STRONG CONVEXITY OF PREFERENCES*

Christophe Muller

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Correspondence: Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, Campus de Sant Vicent del Raspeig, Apartado de Correos 99, 03080 Alicante, Spain. E-mail: cmuller@merlin.fae.ua.es.

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ABSTRACT

In general models, the strong quasi-concavity of the objective function, sufficient for theoretical properties of demands in consumer theory, is often arbitrary. Then, weaker global concavity conditions that preserve such properties are desirable for such models. We propose a new global concavity condition that implies, for models with several nonlinear constraints: the local uniqueness and the smoothness of the decision functions, and the negativity of the generalised substitution matrix. This condition can be used to specify more general and more flexible economic models.

Keywords: Programming Models, Consumer Economics: Theory, Household Production.

1. Introduction

The global uniqueness, the smoothness of the demands and the negative semi-definiteness of the substitution matrix are among the main theoretical restrictions in consumer theory. The strong quasi-concavity of the utility function (SQC) is a sufficient global condition for these restrictions[1]. It is a cornerstone of consumer and general equilibrium theories. As a matter of fact, global concavity assumptions in economics or in other domains are often responsible for many crucial properties of the used models. They play an essential role in that they largely express the structure of the model, without which a direct empirical approach excluding any formal theory could as well be pursued. Global conditions are important because, as opposed to local conditions, one can impose or check them a priori excluding any knowledge of the optimal decisions.

Although theoretical properties can also be developed for demand correspondences, the uniqueness of the demands considerably simplifies the analysis[2]. It enables one to separate considerations related to the inaccuracy of choices, from the study of decision changes with characteristics of agents and environment. In consumer theory, the uniqueness of the demands is global. The same legitimate

[1] Arrow and Enthoven (1961), Debreu (1972, 1976) discuss these quasi-concavity assumptions.

[2] Ellis (1976).

desire of focusing on the law of decisions is valid for general models. However, global uniqueness is not necessarily appropriate for a decision problem with several constraints. Consider for example the consumption of a person who can obtain his consumption from two distinct domestic technologies. Assume that only the production frontier is observed. Assume also that one technology is enjoyable but has low productivity, while the other technology has opposite characteristics. Then, if the arguments of the person's utility are the penibility of domestic work and consumption, she may be indifferent between two solutions corresponding each to one of these technologies. One does not wish to artificially eliminate this reasonable situation by imposing a unique global solution. Then, what is required is that the decisions are *locally* unique. The smoothness of the decision functions is also important since it allows an easy study of the comparative statics and other variational properties of decisions. All this explains why SQC or similar assumptions are fundamental in consumer economics.

The negativity of the substitution matrix is another a major theoretical restriction. In economic theory, many authors stress the importance of the negativity and symmetry restrictions of the Slutsky matrix in consumer theory[3]. Other au-

[3]e.g. Barten (1977), Afriat (1983), Varian (1984), Takayama (1985), Beavis and Dobbs (1990), El-Hodiri (1991). Shapiro and Braithwait (1979) begin their article with a quotation of Samuelson (1961): "The assumption that [the Slutsky matrix is] ...

thors use the negativity of the Slutsky matrix, or similar properties, to derive sufficient conditions for the law of aggregate demand, which supports the existence of the competitive equilibrium of the whole economy[4]. In the study of price dynamics in general equilibrium, negativity restrictions or related conditions ensure globally stable equilibria[5]. In applied work, these restrictions are used to incorporate theoretical results in estimated models[6].

Unfortunately, the strong quasi-concavity of the utility function that delivers all these restrictions has few theoretical or empirical bases. The quasi-concavity of the utility is related to preference by individual of 'mixtures' of commodities to unbalanced consumption structures. However, various authors are dissatisfied with the hypothesis of strict convexity of preferences (equivalent to the strict quasi-concavity of the utility function, itself very close to SQC)[7]. Also, experimen-

symmetrical and negative semi-definite completely exhausts the empirical implications of utility analysis. All other demand restrictions can be derived as theorems from this single assumption".

[4]e.g. Hildenbrand (1983), Grandmont (1987), Quah (1997).

[5]Khilstrom, Mas-Colell, Sonnenschein (1976).

[6]See Samuelson (1947), Kalman and Intriligator (1973), Chichilnisky and Kalman (1978), Deaton and Muellbauer (1980), Varian (1984), Chung (1994). Kodde and Palm (1987) discuss a parametric test of the negativity of the substitution matrix. In the context of cost function estimation, Gallant and Golub (1984), Diewert and Wales (1987), Koebel et al. (2003) propose methods for imposing curvature conditions on specific flexible functional forms. The latter ones insist on the importance of imposing concavity *globally*, consistently with economic theory. Imposition or verification of the negativity in applied demand systems is common practice.

[7]e.g. Kirman (1982). Other 'technical' conditions on preferences have been attacked as altering the empirical content of models (Ghirardato and Marinacci, 2001).

tal evidence contradicts the convexity of preferences[8]. In fact, the convexity of consumer preferences is intuitive only when comparing standard average baskets with extreme consumption choices concentrated only in few commodities. When comparing two rather balanced commodity baskets, the intuition is somewhat lost and SQC looks rather arbitrary.

Even if we admitted SQC for the consumer case, this would be less tolerable for other models. Thus, the presence of heterogenous arguments in the objective function may allow for different 'life styles' or strategies, which may imply non-convexities in preferences. For example, this is the case for the fertility choice between having a large family with limited human capital, or a small family with educated and healthy members. Collective settings for aggregate household decisions may also contradict the convexity of household preferences[9]. In trade theory, in resource economics or in macroeconomics, a country objective function is not necessarily quasi-concave[10]. Finally, in some models the decisions are the characteristics of contracts or preferences may directly incorporate some constraints[11]. There is no reason why the objective function should be quasi-concave in these cases. Clearly, it is desirable to dispose of alternative conditions to the SQC.

[8] Tversky and Kahneman (1991).
[9] Chiappori (1988), Browning and Chiappori (1998).
[10] Dawid and Kopel (1997).
[11] e.g. Stiglitz and Weiss (1992), Heim and Meyer (2004).

In this paper, we provide a new global generalised concavity condition adapted to models with several constraints, possibly non-linear. These models are used in several economic fields. The New Household Economics[12] and agricultural household models[13] involve production and budget constraints. Models with non-linear budget constraints arising from quality effects[14], non-linear taxation[15], labour supply with work costs[16], productive consumption[17], nonlinear wage schedules[18], rationing[19], and non-linear pricing by firms with monopoly power, are also characterised by non-linear constraints. Finally, international trade theory, the study of first-best and second-best optima[20], collective household models[21], or other types of bargaining or incentive models, may include several non-linear constraints for agents' optimal choices.

In general settings, no *global* generalised concavity condition is known that would be as weak as possible for the negativity of the substitution matrix and

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[12] Becker (1965), Lancaster (1966).

[13] Sen (1966), Barnum and Squire (1980), Pitt and Rosenzweig (1985), Singh, Squire and Strauss (1986), Benjamin (1992).

[14] Houthakker (1952), Edlefsen (1981, 1983).

[15] Hausman (1985), Weymark (1987).

[16] Heim and Meyer (2004)

[17] Suen and Hung Mo (1994).

[18] Blomquist (1989).

[19] Madden (1991).

[20] Ben-Israel, Ben-Tal, Charnes (1977), Dixit (1985).

[21] Chiappori (1988, 1992), Browning and Chiappori (1998).

the smoothness and local uniqueness of decisions[22]. Is there such a condition and what are its properties? The aim of this paper is to answer these questions so as to improve the specification of general economic models. In Section 2, we present the general optimisation problem. In Section 3, we recall the consequences of SQC in consumer theory and we analyse a new global generalised concavity condition for optimisation programmes with several constraints. In Section 4, we study the properties of the decision functions under this condition. Finally, we conclude in Section 5. The proofs are given in the appendix.

2. The Optimisation Problem

General behavioural models with several constraints can be represented by the following programme:

$$\max_x \; U(x,\theta) \quad \textit{subject to}: \; g(x,\theta) \leq 0_q, \tag{2.1}$$

where U is the objective function, which is often assumed to be strictly quasi-concave (or strictly concave, e.g. in Varian, 1984). $x \in R^n$ is the n-dimensional

[22] Silberberg (1974), Hatta (1980), Caputo (1999) and Drandakis (2000) study problems with several constraints by using dual methods, although they do not deal with global concavity conditions.

vector of decision functions, $\theta \in R^p$ is the vector of parameters that may be prices and incomes as well as any characteristics of the environment or of the agent. We allow for parameters common to the objective and the constraints[23]. However, these parameters will be omitted for the presentation when they are not necessary. g is a q-dimensional vector of constraint functions. The decisions may be of any type, including possibly negative values, as for variables such as netputs or net trading positions. Positive decisions can be accounted for in the constraints. The set of choices, X, defined by the constraints, is often assumed to be convex. Appendix 1 contains the definitions of the generalised concavity notions that we use in this article, with their properties that are employed.

To be able to use the first-order Kuhn-Tucker conditions (KTC) as necessary for the existence of a solution, one must assume a constraint qualification condition. We follow the common practice of assuming that the gradient vectors of the components of g are linearly independent. Despite their intrinsic interest, changes in regime may correspond to discrete discontinuity jumps of decisions,

[23] Often, exogenous variables or random effects influence preferences as well as constraints. This is useful for applied agricultural household models (Singh, Squire and Strauss, 1986, Pitt and Rosenzweig, 1985), for evolutionary economics (Lesourne, 1993, Young, 1993) and for models in which preferences depend on random states of Nature that may also affect constraints (Viscusi and Evans, 1990). Finally, for Pareto optima, bargaining and incentives models, objective and constraints that all include utility functions, may incorporate the same common characteristics of preferences.

which would justify not paying much attention to negligible marginal substitution effects. In these situations, the negativity property and the smoothness of decisions lose most of their appeal as theoretical restrictions. At solutions where the strict complementarity slackness fails, comparative statics may be problematic because the set of binding constraints may change as the parameter changes, destroying the differentiability of the solutions. To avoid these problems, practitioners generally assume that non-negativity constraints would not bind, but all other constraints always bind. Moreover, researchers are often concerned only with the solutions of one specific regime of interest (one set of binding constraints). This leads us to focus on the following Lagrange conditions, which are the KTC associated with such a specific regime.

$$U_x - g_x \lambda = 0_n$$
$$g(x, \theta) = 0_q \tag{2.2}$$

where 0_q is the q-dimensional vector null and λ is the q-dimensional vector of the Lagrange multipliers. The Lagrange function associated with the problem is $L = U - \lambda' g$. We now discuss global concavity conditions for behavioural models, first by examining the link of global conditions and local properties of decisions.

3. Global Concavity Conditions

Theoretical restrictions for the decisions similar to those obtained with SQC in demand theory can be obtained from the sufficient second-order conditions (SSOC) of the optimisation programme, for example in Blackorby and Diewert (1979). However, without a global concavity condition this approach involves several shortcomings. Firstly, the derived decision functions may not satisfy desirable global properties. For example, flexible functional forms used in consumer analysis have been criticised on the grounds that they did not easily allow the global imposition of the convexity of preferences (Diewert and Wales, 1987). Secondly, the consistency of the local duality structures presupposes some global concavity properties (Blackorby and Diewert, 1979). Thus, for the consumer problem the expenditure function must be concave in prices over its domain or the direct utility function must be quasi-concave over its domain. Without these global concavity conditions there is no correspondence of the respective second-order approximations of the expenditure function and of the direct utility function. Meanwhile, the global conditions alleviate difficulties that may arise for the coincidence of the domains of the local utility function and of the other local dual representations of preferences. Therefore, even if local approximations are useful, they do not allow for a precise control of global properties of objective and constraints and of the consistency of the dual. Thirdly, global concavity conditions are used to incorporate decision models in general equilibria frameworks describing the economy by a unique and stable solution. On the whole, one needs global generalised concavity conditions on the optimisation problem, even for obtaining desirable local properties of decision functions. This has been a fertile approach in the economic literature, notably to obtain local uniqueness and differential properties of decisions[24]. Besides, local uniqueness, smoothness and semi-definite negativeness of decisions are global properties when they must be satisfied for the whole domain.

In the consumer problem, the only constraint (to simplify the exposition we ignore any positivity constraint) is the linear budget constraint and x is the vector of consumption. The utility function U is generally assumed to be of type C^2, strictly increasing in the consumption of every commodity and strictly (or strongly) quasi-concave. Under these assumptions, the vector of demands is derived from the Lagrange first-order conditions of the optimisation programme.

[24]e.g. Debreu (1972), Arrow and Enthoven (1961), Laroque (1981), Smale (1982), Dana (1999), Shannon and Zame (2002).

From the budget constraint, one can derive the adding-up and homogeneity restrictions that are somewhat specific to the consumer problem. The other theoretical restrictions characterise the Slutsky matrix S and are discussed in Afriat (1983). The symmetry property (S is a symmetric matrix) results from the separation structure of the optimality problem. The negativity property results from the assumption of strong quasi-concavity of U (Diewert, Avriel and Zang, 1981), which implies the sufficient second-order conditions (SSOC): S is orthogonal to the price vector and is negative definite in the hyperplane orthogonal to the price vector. However, the SSOC would not satisfy us because we search for a global condition.

For general models, SQC may be weakened. Weakening the conditions for properties of optimal solutions is important in mathematical programming[25] and in economics[26] since it limits arbitrary restrictions. We search for a generalised *global* concavity condition that is as weak as possible and implies the local uniqueness and smoothness of decision functions and the negativity of the generalised substitution matrix.

The necessary (respectively sufficient) local second-order conditions correspond

[25] Avriel (1976), Hiriart-Urruty (1996), Auslender (2000).
[26] Debreu (1983).

to the local negative semi-definiteness (respectively the local negative definiteness) of the Hessian matrix of the Lagrange function with respect to decisions at the optimum (respectively, at the optimum for directions in the tangent space to the constraints)[27]. However, we are not interested by local concavity conditions but by global ones. Moreover, because each constraint function and objective function is separately specified in economic models and sometimes separately estimated from different datasets, we look for a condition that can be explicitly expressed in terms of these functions, rather than in terms of the Lagrange function. Next, we recall the definition of the strong quasi-concavity.

Definition 3.1. *Let U be a directionally differentiable real function defined over a convex subset X of R^n. U is called* ***Strongly Quasi-Concave*** *over X if and only if*

$$[x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}v \in X, D_v U(x^0) = 0] \Rightarrow$$

$$[\exists \varepsilon > 0, \exists \alpha > 0, \varepsilon < \bar{t}, \forall t \in [0, \varepsilon],\ U(x^0 + tv) < U(x^0) - \alpha t^2]\,,$$

where D_v denotes the directional derivative operator in direction v.

Equivalent notions have been used[28]. If U is twice differentiable, an equivalent definition is the following (Diewert, Avriel and Zang, 1981).

[27] See Afriat (1971) for a discussion.

[28] Dhrymes (1967), Barten, Lempers and Kloek (1969), Newman (1969), Ginsberg (1973), McFadden (1978). See also Barten and Böhm (1982) for a discussion of the

Definition 3.2. *Let U be a twice differentiable real function defined over a convex subset X of R^n. U is* ***Strongly Quasi-Concave*** *(SQC) if* and *only if* $\forall(x,y) \in X^2$, *such that* $x \neq y$,

$$(\nabla U(x)'(y-x)=0) \Rightarrow (y-x)'\nabla^2 U(x)(y-x) < 0.$$

We now introduce a new notion of generalised concavity by changing the premises of this definition.

Definition 3.3. *The twice differentiable objective function U subject to a differentiable vector of constraint functions g, is called*

Constraint – Strongly Quasi – Concave *(CSQC[29]) with respect to g if and only if U is twice differentiable on X convex subset of R^N, and*

$$\begin{aligned} \forall(x,y) &\in X^2 \ \textit{such that}\ x \neq y \text{ and } g(x) = 0_q, \\ (\nabla g(x)(y-x) &= 0 \ \textit{and}\ \nabla U(x)'(y-x) = 0) \Longrightarrow \\ (y-x)'\,\nabla^2 U(x)\,(y-x) &< 0. \end{aligned} \tag{3.1}$$

The CSQC must be checked for all points x satisfying the constraints, in

properties of strongly quasi-concave utility functions in consumer theory. Crouzeix, Ferland and Zalinescu (1997) extend this notion to sets and relate it to inf-compactness. Quasi-concave functions in a non-differentiable context are discussed in Calzi (1992).

[29] *To simplify the presentation, we use CSQC both for 'Constraint-Strong Quasi-Concave' and 'Constraint-Strong Quasi-Concavity'.*

particular for x non-optimal. This is a crucial requirement, since, at the stage of model specification, the actual optimum may be unknown. The CSQC differs from the SSOC by several elements. First, it is global. Second, an orthogonality condition involving the gradient ∇U intervenes in the premises of the CSQC and not in that of the SSOC. Third, the negativity condition in the conclusion of the CSQC is for $\nabla^2 U$, whereas it is for $\nabla^2 L$ in the SSOC. Also, the CSQC differs from the SQC by the presence in the premises of the choice set and orthogonality conditions with respect to the constraint gradients.

The fact that the constraints intervene in the CSQC is more natural than it may seem at first sight, because both objective and constraints characterise the optimisation problem and they should therefore be considered together. Novshek (1980) states that "*the second-order conditions impose constraints on the curvature of level sets for f relative to the curvature of level sets for g [here f describes the objective and g describes the constraints]. The absolute properties of [$f_{ij}(x)$] (positive definite, negative definite, corresponding to a saddle point, etc.) are unimportant. The properties of [$f_{ij}(x)$] relative to [$g^i_j(x)$] are important.*" This quote is consistent with the definition of the CSQC in which the curvature properties of U are considered relatively to g [30].

[30] See also Ben-Tal (1980) for second order conditions involving Hessian functions and

The tangent subspace at x to the constraints and to the indifference hypersurfaces (i.e. the subspace orthogonal to the gradients of functions $g^i (i = 1, ..., q)$ and U) generates a differential manifold, as x varies along the frontier of the constraints. This manifold is generally nonlinear and is neither a hyperplane as in consumer theory with one linear constraint, nor a sub-vector space as when considering only local conditions with several constraints. It is exclusively along this manifold that the negativity of $\nabla^2 U$ is imposed by the CSQC.

The sole consideration of the frontier of the constraints in the definition of the CSQC is motivated by the search for as weak a condition as possible. Indeed, since the objective function is generally specified as increasing in its arguments, it is generally useless to incorporate restrictions occurring at points that are never reached at the equilibrium because they are not at the frontier.

Some functions U and g may yield an empty set of directions corresponding to the premises of the definition of the CSQC. When that is the case, essentially no arbitrary restriction of generalised concavity is imposed on the optimisation problem. However, this seems unlikely to happen in models of interest. In order to characterise the CSQC by the shape of the graph of the objective function, we need an additional definition.

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curvatures.

Definition 3.4. *f : X convex set $\subset R \to R$ attains a* ***strong local maximum*** *(SLM) at $t_0 \in X$ if and only if $\exists\, \alpha > 0, \exists\, \varepsilon > 0, \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap X$, $f(t) \leq f(t_0) - \alpha(t - t_0)^2$.*

Intuitively, a function attains a SLM when its curvature to the origin at the maximum is at least as strong as that of a quadratic function. Definition 3.1 shows that a strongly quasi-concave function U is such that for directions v orthogonal to ∇U in x^0, $h(t) \equiv U(x^0 + t\,v)$ attains a SLM at $t = 0$. We now show that the CSQC can also be characterised in terms of SLM in some directions. By considering particular values of α, one can extend this characterisation to define properties of 'α-constraint-quasiconcavity', similarly to the α-quasiconvexity in Crouzeix, Ferland and Zalinescu (1997).

Proposition 3.5. *Let U be a twice differentiable function over X. Then,*

U CSQC with respect to g over X if and only if

$(x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}\,v \in X, \forall i = 1, \ldots, q, g^i(x^0) = 0$ and $\nabla g^i(x^0)'v = 0$ and $\nabla U(x^0)'v = 0) \Rightarrow h(t) \equiv U(x^0 + t\,v)$ attains a SLM at $t = 0$.

It is well known that the convexity of a function $f(x)$ is equivalent to the convexity of its epigraph, $E_p(f)$, the set of couples (x, y) where $y \geq f(x)$. The following proposition characterises the epigraph in the case of CSQC.

Proposition 3.6. *Let the function U, from X convex set of R^n to R, be CSQC with respect to the vector of constraints g. We consider the graph of U:*

$$\partial E_p(U) = \{(x_1,\ldots,x_n,U(x_1,\ldots,x_n)) \,|\, (x_1,\ldots,x_n) \in X\},$$

and we define a "g–U-admissible" direction at x, as $d \in R^n$ such that $\nabla g^i(x)'d = 0$, for all i and $\nabla U(x)'d = 0$ (i.e. a direction of the domain generated by the tangent space to the constraints and to the indifference hypersurfaces). Then,

(a*) the frontier of the epigraph of U is strictly below all its tangent hyperplanes, in any g–U-admissible direction at the frontier of constraints;*

(b) the curvature (to the origin) of the frontier of the epigraph of U in any g-U-admissible direction at the frontier of constraints is strictly positive;

(c) the dimension of the subspace spanned by the ∇g^i $(i = 1,\ldots,q)$, which is q because of the constraint qualification condition, is greater than the number of positive or null eigenvalues of $\nabla^2 U$ at the frontier of constraints.

Condition (b) can be shown to correspond to a strictly positive curvature of the objective function obtained by substituting the constraints in the initial objective, as in Afriat (1971). Condition (c) illustrates that the more constraints the less restrictive is the CSQC. The index '$n -$ (the number of positive eigenvalues of $\nabla^2 U$)' can be used as an index of non-convexity[31]. In a von Neumann-

[31] More general indices of non-convexity are proposed by Thach and Konno (1995).

Morgenstern framework, the CSQC is related to risk aversion in the domains of choices defined by the constraints and by the indifference hypersurfaces. It is worth noting, as the next proposition shows, that CSQC utility functions can be ordinal and therefore that they correspond to relevant restrictions on the representation of preferences.

Proposition 3.7. *The CSQC is an ordinal property of the preferences.*

In the next section, we examine the consequences of the CSQC for the decisions.

4. The Properties of the Decision Functions under CSQC

4.1. The Link of CSQC with SQC and the Second-Order Conditions

We now turn to the link of the CSQC with the SQC and the SSOC, not only because of the intrinsic interest of these conditions, but also because the SSOC is a convenient intermediate to derive some properties of decisions. First, the CSQC and the SQC can be ranked.

Proposition 4.1. *If function U is strongly quasi-concave, then it is CSQC.*

The reciprocal proposition is not true because, even at the optimum, $\nabla U' v = 0$ does not generally imply $\nabla g^{i'} v = 0$ for all i. When there are several constraints, the CSQC is a weaker condition than the strong quasi-concavity, because it is associated with a local curvature that is strictly positive only in a sub-space of dimension $n - q$ or generally $n - q - 1$, and only at the frontier of the constraints, while this curvature must be strictly positive in a whole hyperplane for U strongly quasi-concave. The CSQC is an assumption that does not locally impose any structure on the preferences in a subspace of dimension q at least, and therefore globally in a large domain. Moreover, out of the frontier of the constraints, the CSQC is tantamount to the absence of restrictions[32]. We now turn to the relationship between the CSQC of the Lagrange function and the CSQC of the objective function, as a first step towards the second-order conditions of optimality.

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[32] The case of families of constraint and utility functions with θ varying instead of given a priori, leads to obvious generalisations and stronger global conditions, although still weaker than SQC. These generalisations do not change the core of our results.

Proposition 4.2.

If g is quasi-convex, the CSQC of the utility function implies the CSQC of the Lagrange function associated with the optimisation programme, whether calculated with optimal or non-optimal Lagrange multipliers.

The CSQC of the Lagrange function is important because it characterises the shape of the optimisation problem under CSQC of the objective function. However, the Lagrange function cannot be directly used to impose structural conditions, because of its dependence on a priori unknown Lagrange multipliers. Moreover, applied researchers want to impose restrictions on objective and constraint functions independently, not on a combination of them. The following proposition shows that CSQC ensures that the second-order conditions are satisfied when the choice set is convex[33].

Proposition 4.3. *If g is quasi-convex, then U CSQC implies the sufficient local second order conditions of optimality.*

The contention that g is quasi-convex is little restrictive since it is equivalent to assume that the choice set is convex. By contrast with the necessary second-

[33] This does not imply that any optimal situation must satisfy the CSQC. For example, $(x-2)^4$ does not satisfy the CSQC, but it has a minimum at $x = 2$.

order conditions, the SSOC are not necessary the consequences of any optimisation programme. A condition like CSQC is required to obtain the SSOC. Locally, SSOC implies the negativity of the generalised substitution matrix (Pauwels, 1979). However, without global conditions one would have to know the optimum to be able to a priori check SSOC in a tractable way. We are now ready to examine the properties of the decision functions under CSQC.

4.2. The Properties of the Decision Functions

In consumer theory, the budget set is bounded, therefore compact in finite dimension. This implies that there is always a solution to the maximisation of an upper semi-continuous utility function. In general models, the choice set inside a given regime is defined as $C = \{x \in R^n, g(x, \theta) = 0_q\}$ and is no longer necessarily bounded. Firstly, we consider a non-empty choice set C to avoid absurd situations. Secondly, the problem optimum is given by the KTC and corresponds to a tangential contact point of the constraint frontier with an indifference hypersurface. The CSQC yields strict curvatures that seems to geometrically imply the existence of the optimum. Surprisingly, this is not the case[34] and additional

[34] Indeed, in the following example there is no optimum. Let be a 2-dimensional vector of decisions $x = (x_1, x_2)' \in R^2$, and a constraint vector of one dimension described by the equation: $x_1 + x_2 = 1$. Assume that the objective function U is strictly increasing and differentiable in x_1 and x_2, and CSQC. Because the constraint is a unique line, this

assumptions are necessary to guarantee the existence of the optimum. Since U and g are continuous, if one assumes that the decisions belong to a bounded set Ξ, then the choice set C is compact and is not empty by hypothesis, and an upper semi-continuous utility function has a maximum in this set. Another possibility is to assume that U is coercive, upper semi-continuous on a closed feasible set with at least one point where U is finite.

We now present a characterisation of the CSQC in terms of a bordered Hessian, similarly to consumer theory with strong quasi-concavity (Barten & Böhm, 1982). This is interesting first because this type of characterisation is usual for various concavity conditions, and also because determinants can be easily calculated from data to check these concavity conditions. The non-singularity of another related bounded Hessian will be necessary to the derivation of properties of decisions.

Proposition 4.4. *Let be*

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is obtained with U strongly quasi-concave. But there exist families of strongly convex indifference curves that satisfy these conditions and are asymptotically tangent to the constraint when x_1 goes to $+\infty$ or x_2 goes to $-\infty$. In that case, there is no optimum since the contact point is at the infinite.

$$H = \begin{bmatrix} U_{xx} & g_x^1 & \cdots & g_x^q \\ g_x^{1\prime} & 0 & & 0 \\ \vdots & & 0 & \\ g_x^{q\prime} & 0 & & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} L_{xx} & g_x^1 & \cdots & g_x^q \\ g_x^{1\prime} & 0 & & 0 \\ \vdots & & 0 & \\ g_x^{q\prime} & 0 & & 0 \end{bmatrix}.$$

(a) If U is CSQC, then H is non-singular at any solution of the KTC.

(b) If U is CSQC and g is quasi-convex, then J is non-singular at any solution of the KTC.

The non-singularity of H characterises the CSQC at the optimum. The non-singularity of J enables us to use the implicit function theorem under the CSQC when g is quasi-convex, and we exploit it in the next proposition.

Proposition 4.5. *Let* $(x_0, \lambda_0, \theta_0)$ *be such that the KTC of Problem 2.1 are satisfied with U CSQC and g quasi-convex. Then,*

(a) $\exists V_0$ *open neighbourhood of* θ_0, $\forall V \subset V_0$, *open and connected neighbourhood of* θ_0, *there is a unique function* h: $V \to R^{2n}$, *such that*

$(x_0, \lambda_0) = h(\theta_0)$ *and* $\forall \theta \in V, \exists (x,\lambda) \in R^{2n}$, $(x, \lambda) = h(\theta)$

and $KTC[h(\theta), \theta] = 0$, *where KTC[.] is the vector of functions corresponding to the equations in the KTC.*

(b) h is of type C^1 in V and its derivative is

$$h'(\theta) = -\left[D_{x,\lambda} KTC[\,h(\theta),\ \theta]\right]^{-1} \circ \left[D_\theta KTC[\,h(\theta),\ \theta]\right].$$

(c) If moreover, U and g are of type C^{p+1} in a neighbourhood of $(x_0,\ \lambda_0,\ \theta_0)$, then h is of type C^p in a neighbourhood of θ_0.

(d) If moreover, U and g are analytic in a neigbourhood of $(x_0,\ \lambda_0,\ \theta_0)$, then h is analytic in a neigbourhood of θ_0.

The first component of h defines the vector decision functions. Proposition 4.5 proves the local uniqueness and smoothness of the decision functions. It also justifies the usual calculus of the derivatives of the decision functions. We now discuss the negativity property.

Proposition 4.6. *Under the CSQC and the convexity of the choice set, the generalised substitution matrix, S, is negative semi-definite and is negative definite in the tangent space to the constraints.*

The negative semi-definiteness of matrix S at the optimum is related to the local stability of the equilibrium that is ensured when matrix L_{xx} is negative definite in the tangent space to the constraints. At the optimum, the CSQC jointly with the quasi-convexity of g is, to our knowledge and for the moment, the weakest available global condition for the negativity of the substitution matrix in a general context.

In practice, the definition of the CSQC suggests to verify the semi-definite negativity of $\nabla^2 U$ only in a limited domain. In particular, the knowledge of all decisions and all multipliers is not needed, and neither are verifications over the whole spaces of decisions, multipliers and directions. Because firstly multipliers need not be considered, and secondly decisions and directions need be checked only in reduced domains, checking globally the CSQC may often be tractable. Proposition 4.4 (a) provides a necessary condition on a bordered determinant that could be used for the test. However, this test would be only valid for the solutions of the KTC, which may be hard to calculate. Checking CSQC can also be directly implemented by using a grid of the domain of decisions, calculating for each knot of the grid all the eigen-values of $\nabla^2 U$ in the $(U - g)-$admissible directions and verifying that they are all negative. Statistical tests are available (Kodde and Palm, 1987) to account for approximations done in the model or with the grid. In some cases, a grid may not be necessary if the functional forms used for U and g lead to simplifications. Then, the CSQC may sometimes be easier to test than the SQC because the domain to explore for this test is much smaller.

5. Conclusion

In general models, the strong quasi-concavity of the objective function, which is sufficient for theoretical properties of demands in consumer theory, is often arbitrary and weaker global concavity conditions are desirable. We propose a new global concavity condition, the 'constraint-strong quasi-concavity (CSQC)' of the objective function that implies, for models with several non-linear constraints, the local uniqueness and the smoothness of the decision functions as well as the negativity of the generalised substitution matrix when used jointly with the convexity of the choice set. CSQC is weaker than the strong quasi-concavity and is parsimonious because it is strictly based on what is required globally for the negativity of the generalised substitution matrix. Indeed, it does not restrict the curvature of the objective function in directions that are not compatible with the constraints or not compatible with the augmentation of the objective level. Moreover, the CSQC may be easier to check numerically for specific models than the strong quasi-concavity. Finally, using the CSQC allows the extension of the set of possible functional forms as compared with the strong quasi-concavity, thereby increasing modelling flexibility.

Several extensions of this paper are possible. First, 'α-CSQC' notions could be developed to (1) extend the analysis from functions to sets, (2) produce weaker notions of generalised concavity, (3) build the bridge between the CSQC and notions of generalised strict quasiconcavity. Second, the CSQC can be related to the degree of non-convexity in optimisation problems (as defined in Thien Thach and Konno, 1997). Third, in consumer theory, the negativity of the Slutsky matrix is related to revealed preferences axioms (Kihlstrom, Mas-Colell and Sonnenschein, 1976). We conjecture that analog results could be derived by limiting the decisions to the constrained choice set (as in Chavas and Cox, 1993). Such results would express the curvature properties embodied in the CSQC hypothesis. Fourth, since imposing strong concavity globally jeopardises the flexibility of usual flexible functional forms for cost functions and utility functions, one could investigate if, in the presence of several constraints, imposing only the CSQC would permit to preserve the flexibility of such functional forms. Finally, game theory problems and equilibria problems seem likely to be fertile application fields of the CSQC.

Appendices:

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Definitions of the generalised concavity notions

The generalised convexity properties for a function f are given by the corresponding generalised concavity properties of -f.

Let X be a convex set of R^N.

f : X -> R, a differentiable function, is called concave if and only if

$\forall(x, y) \in X^2,\ f(y) - f(x) \leq \nabla f(x)(y - x)$.

f : X -> R, a differentiable function, is called strictly concave if and only if

$\forall(x, y) \in X^2,\ x \neq y, f(y) - f(x) < \nabla f(x)(y - x)$.

f : X -> R, a differentiable function, is called quasi-concave if and only if

$\forall(x, y) \in X^2, (f(x) \leq f(y)) \Rightarrow \nabla f(x)(y - x) \geq 0$.

This is equivalent by contraposition to

$\forall(x, y) \in X^2, \nabla f(x)(y - x) < 0 \Rightarrow (f(x) > f(y))$.

Moreover, if f is quasi-concave and twice differentiable then

$\forall(x, y) \in X^2$, such that x$\neq y, \nabla f(x)(y - x) = 0 \Rightarrow$

$(y - x)' f_{xx}(x)(y - x) \leq 0$.

f : X -> R, is called strictly quasi-concave if and only if

$\forall(x, y) \in X^2, x \neq y, \forall \lambda \in]0, 1[,\ f(y) \leq f(x) \Rightarrow f(x) < f((1 - \lambda)x + \lambda y)$.

Then, f is strictly quasi-concave and, if, moreover, it is twice differentiable, it satisfies the same second-order condition as any quasi-concave function.

f : X -> R, a twice differentiable function, is called strongly quasi-concave if and only if $\forall(x, y) \in X^2$, such that $x \neq y$,

$(\nabla f(x)(y - x) = 0) \Rightarrow (y - x)' f_{xx}(x)(y - x) < 0$.

Then, f is strictly quasi-concave.

f : X -> R, a twice differentiable function, is called constraint-strongly quasi-concave (CSQC) with respect to g, where g : X -> R^m is a differentiable vector function, if and only if

$\forall(x,y) \in X^2$ such that $x \neq y$ and $g(x) = 0_q$, ($\nabla g(x)(y-x) = 0$ and $\nabla f(x)(y-x) = 0) \Rightarrow (y-x)' f_{xx}(x)(y-x) < 0$.

Proofs

Proof of Proposition 3.5:

$\Longrightarrow$] Let U be CSQC over X, let $x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}v \in X$, $\forall i = 1, \ldots q$, $g^i(x^0) = 0_q$, $\nabla g^i(x^0)'v = 0$, $\nabla U(x^0)'v = 0$.

Let $h(t) \equiv U(x^0 + tv)$. We calculate $j(t) \equiv h(t) - h(0) = U(x^0 + tv) - U(x^0)$.

A second order Taylor expansion of $U(x^0 + tv)$ gives

$j(t) = (t^2/2)v'\nabla^2 U(x^0)v + t^2\varepsilon(t)$ because $\nabla U(x^0)v = 0$.

U CSQC implies that for x^0 and v such that $\forall i = 1, \ldots, q$, $g^i(x^0) = 0_q$, $\nabla g^i(x^0)'v = 0$ and $\nabla U(x^0)'v = 0$, we have $v'\nabla^2 U(x^0)v < 0$. Then, since $\nabla^2 U$ is continuous by hypothesis, $j(t) < 0$ when t is small enough. Therefore, $h(t)$ attains a SLM at $t = 0$ for any $0 < \alpha < Min\{-\frac{1}{2}v'\nabla^2 U(x^0)v \,:\, v'v = 1\}$.

$\Longleftarrow$] Let be $x^0 \in X$, $v'v = 1$, $\bar{t} > 0$, $x^0 + \bar{t}v \in X$, $(\forall i = 1, \ldots, q, g^i(x^0) = 0$ and $\nabla g^i(x^0)'v = 0) \Rightarrow h(t) \equiv f(x^0 + t\,v)$ attains a SLM at $t = 0$.

Then, $\exists\, \alpha > 0$,

$$j(t) + \alpha\, t^2 \leq 0. \qquad (1)$$

Besides, a second order Taylor expansion of $U(x^0 + t\,v)$ about x^0 yields

$U(x^0 + t\,v) = U(x^0) + \frac{t^2}{2}\, v'\, \nabla^2 U(x^0)\, v + t^2\, \varepsilon(t)$ where $\varepsilon(t) \to 0$ when $t \to 0$ because $v' \nabla U(x^0) = 0$.

Therefore,

$$j(t) = \frac{t^2}{2}\, v'\, \nabla^2 U(x^0)\, v + t^2\, \varepsilon(t) \qquad (2)$$

Eqs. 1 and 2 imply that $\frac{1}{2}v'\,\nabla^2U(x^0)\,v \leq -\alpha - \varepsilon(t)$, which gives for t small enough $v'\,\nabla^2U(x^0)\,v < 0$, which proves that U is CSQC. QED.

Proof of Proposition 3.6:

(a) Let d be a $(g-U)-$admissible direction at x at the frontier of the constraints, then from a Taylor expansion of U at x, we have

$U(x+d) = U(x) + \nabla U(x)'\,d + \frac{1}{2}\,d'\,\nabla^2U(x)\,d + \|d\|^2\,\varepsilon(d)$ where $\lim\varepsilon(d) = 0$ when $\|d\| \to 0$. Because U is CSQC we have at the frontier of the constraints $d'\,\nabla^2U(x)\,d < 0$. Choosing α small enough shows that the hypersurface $\partial E_p(U)$ is strictly below all its tangent hyperplanes in $(g-U)-$admissible directions at the frontier of the constraints.

(b) is deduced from the fact that the curvature of the epigraph in a direction d at x can be associated with $-d'\nabla^2U(x)d$ with a positive factor of proportionality to adjust for the local metric of the hypersurface. Under the CSQC, all points of the frontier of the constraints are 'elliptic' for U in any $g-U-$admissible direction, while they may be 'parabolic or hyperbolic' in the whole space.

(c) is a direct consequence of the definition of the CSQC and of the fact that since ∇^2U is symmetric there exists an orthogonal basis of eigenvectors of ∇^2U whose first q vectors generate the subspace spanned by the ∇g^i (theorem of the incomplete base). The orthogonality condition with respect to ∇U generally enables one to add a unity from the number of possible positive or null eigenvalues of ∇^2U, although not for an optimum since in that case the KTC are satisfied and ∇U is a linear combination of the ∇g^i.

Proof of Proposition 3.7:

$V = FoU$ gives $\nabla V = (F'oU).\nabla U$,

and $\nabla^2V = (F'oU)\nabla^2U + (F''oU)\nabla U\nabla U'$.

Then, with $F' > 0$, and moreover $\forall\, i = 1,\ldots,q$, $\nabla g^{i'}v = 0$ and $\nabla U'v = 0$ at the frontier of constraints implies $v'\nabla^2Uv < 0$, we have $v'\nabla^2Vv < 0$. Therefore, V is CSQC. QED.

Proof of Proposition 4.1:

The premises of the definition of the CSQC imply that of the definition of the SQC. QED.

Proof of Proposition 4.2:

We first give the proof for one constraint function g only. U is CSQC and g is quasiconvex. Then, for all x and y such that $x \neq y, g(x) = 0_q, \nabla g(x)(x-y) = 0$ and $\nabla U(x)(x-y) = 0$, we have $(x-y)'$ $\nabla^2 U(x)(x-y) < 0$ and $(x-y)'\nabla^2 g(x)(x-y) \geq 0$. Then, for all vectors $\lambda \geq 0$, for all x and y such that $g(x) = 0_q, \nabla g(x)(x-y) = 0$ and $\nabla U(x)(x-y) = 0$, we have $(x-y)'\nabla^2 L(x,\lambda)(x-y) < 0$ and the Lagrange function is CSQC for any vector of non-negative multipliers. In particular, this result is true for optimal solution and with optimal Kuhn-Tucker multipliers. The extension of the proof to several inequality constraints is straightforward because the Hessian matrix of a linear combination of functions is the linear combination of the Hessian matrices of these functions. QED.

Proof of Proposition 4.3: Consequence of Proposition 4.2, applied at an optimal solution. QED.

Proof of Proposition 4.4:

(a) Assume that H is singular. Then, $\exists\, z \in R^n$, $\exists\, r \in R^q$, such that

$$U_{xx}z + \sum_{i=1}^{q} r_i g_x^i = 0 \tag{3}$$

$$(z', r')' \neq 0_{n+q} \tag{4}$$

$$g_x^{i\prime} z = 0, \forall i = 1, \ldots, q \tag{5}$$

$z = 0$ and $r \neq 0$ is impossible since $\sum_i r_i g_x^i = 0$, is a system of n equations with q unknown variables (r_i, $i = \ldots, q$), which implies $r_i = 0$ for all i because of the hypothesis of constraint qualification (∇g is full rank).

$z \neq 0$ is also impossible because from eqs. 3 and 5 we would obtain

$z'U_{xx}z = 0$ and $g_x^{i\prime}z = 0, \forall i = 1, \ldots, q$, in contradiction to the CSQC for a solution of the KTC, ∇U lies in the vector space generated by ∇g. Therefore, no non-null vector $(z', r')'$ exists such that $(z', r')H = 0$, which implies that H is non singular.

(b) The proof is similar to that of (a), taking advantage of the fact that $g_x^{i\prime}z = 0, \forall i = 1, \ldots, q$ implies $z'g_{xx}^{i\prime}z \geq 0, \forall i = 1, \ldots, q$.

Proof of Proposition 4.5:

The system describing the KTC has $n + q$ equations whose vector function is denoted KTC[.], and $n + q + p$ variables *(x, λ, θ)*. Because ∇U, ∇g and g are of type C^1, KTC[.] is of type C^1. Since R^{n+q+p} is an open set (this is as well the case if the regime of interest is defined by strict inequality for non-binding constraints) and since $|J|$, which is the Jacobian determinant associated with the KTC for their solution in (x, λ), is non-singular at a solution of the KTC when U is CSQC and g is quasiconcave, we can apply the theorem of implicit functions. This generates all the results of the proposition. QED.

Proof of Proposition 4.6: The CSQC implies the SSOC and the SSOC implies S negative semidefinite in the tangent space to the constraints and orthogonal to the constraint gradients (Pauwels, 1979).