SPATIAL STABILITY*

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ABSTRACT

We consider a continuous spatial economy consisting of pure exchange local economies. Agents are allowed to change their location over time as a response to spatial utility differentials. These spatial adjustments toward higher utility neighborhoods lead the spatial economy to converge to a spatially uniform allocation of resources, provided that the matrix of price effects is quasi-negative definite. Furthermore our model provides a real time interpretation of the tâtonnement story. Also, spatial fluctuations are shown to be damped at different rates according to their spatial scale.

JEL Classification: *D51, C60, R23*

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1 Introduction

The distribution of economic activities over the geographical space results from the interaction between *dispersion and agglomeration forces*. Dispersion forces create incentives for agents to relocate toward less populated locations, thus contributing to *spatial convergence*. On the other hand, agglomeration forces create incentives for agents to relocate toward more populated locations, thus contributing to *spatial divergence*.

In much of the existing literature, geography is summarized by two or a finite number of countries. Lately, the need to rely on a continuous spatial approach has been reemphasized by Krugman (1996), Fujita *et al.* (1999), and Quah (1996, 2002). When using a continuous approach to describe an economy, variables of the model are spatial distributions over the physical geography. In this context, dynamics is concerned with the evolution of these spatial distributions over time. Of course, like in discrete models, the continuous spatial approach allows to address *stability issues* by determining conditions under which spatial convergence or divergence may happen. In addition it allows to describe the spatial dimension of the convergence process or to assess the spatial extent of agglomerations when they emerge. See Krugman (1996) and Fujita *et al.* (1999) for spatial agglomerations driven by increasing returns, or the spatial growth model of Quah (2002) for spatial clusters driven by spatial externalities.

In models expressed in a continuous spatial setting such as described above, the local market structure plays a central role on the convergence/divergence issue.

On the one hand, in economic geography models of international trade, imperfect competition and increasing returns generate *centripetal forces* leading to the emergence of agglomerations (i.e. concentrations of economic activities in some locations rather than in some others), see, for instance, Krugman (1996), Fujita *et al.* (1999), or Mossay (2003). These models are an extension of Krugman (1991)'s two-country model to the case of a continuous spatial economy. In these models, the continuous spatial approach is crucial in that it determines spatial features of the economy such as the *preferred wavelength* (e.g. Krugman (1996), Fujita *et al.* (1999)), or the *critical wavelength* (e.g. Mossay (2003)) which give an indication on the size and number of emerging agglomerations.

On the other hand, perfect competition at the local level, such as in the examples provided by Sonnenchein (1981, 1982), generates *centrifugal forces* leading to spatial convergence (i.e. the dispersion of economic activities across locations). Sonnenchein shows the global convergence of the economy toward a uniform long-run equilibrium. For that aim, Sonnenschein adopted a structure of perfectly competitive local markets along the lines of Rosen (1974) and developed a general equilibrium model with a continuum of commodities, consumers and mobile firms. Locally, the price of the local good is determined by local supply and local demand. Because the number of firms need not be the same across space, prices and therefore short-run profits may vary across locations, which create an incentive for firms to change their location over time. Adjustments of firms are assumed to be *spatially local* (neighborhood feasible) and *temporally myopic* (expectations are static). These spatial adjustments of firms toward higher profit neighborhoods, lead the economy to converge, thus permitting prices and the distribution of firms to become uniform across locations.

In this paper we study further the implication of perfect competition at the local level in a spatial economy. Our aim is to determine the conditions under which spatial convergence occurs when local markets are pure exchange economies. We thus extend the result obtained in the case of the very specific examples of Sonnenchein (1981, 1982), to a general class of economies. Here, *spatial utility differentials provide adequate migration incentives* for leading the spatial economy to converge to a spatially uniform allocation of resources.

Our model is built with a continuum of commodities and mobile consumers. Consumers exchange their endowment for current consumption in the local market where they are. Further they are allowed to change their location over time. We make the same assumptions concerning the spatial adjustments as the ones made in Sonnenschein (1981, 1982). Namely, *agents move locally in the geographical space and have static expectations when dealing with migration decisions.*

We show that, provided that the matrix of price effects is quasi-negative definite, *local spatial convergence holds for a spatial economy whose local markets consist of pure exchange economies.* Such a sufficient condition also ensures the stability of the tâtonnement process. Therefore the spatial convergence issue is to be related to the Walrasian stability of the underlying spaceless economy. However, while the tâtonnement dynamics proceeds in fictive time, see Hahn (1982), the spatial adjustments of agents across locations, take place in real time. This means that the dynamics of our model provides a *real time interpretation of the tâtonnement story.* Also, we provide a spatial analysis of the convergence process by showing that *spatial fluctuations are damped at a rate inversely quadratic with their spatial scale.*

Even though it might seem anachronistic to devote attention to competitive approaches to geographical economics problems, we do think that perfectly competitive forces deserve to be understood at least as well as agglomerations mechanisms. After all, perfectly competitive forces constitute the most important dispersion force that, together with agglomeration forces, shape a spatial economy.

So as to investigate the spatio-temporal dynamics of the model, we perform a standard linear stability analysis around a spatially uniform steady state by using the *method of normal modes.* Like in most partial differential equation problems, only linear studies turn out to be tractable in economics. See Quah (2002)'s spatial growth model, or the economic geography models by Krugman (1996), Fujita *et al.* (1999), or Mossay (2003). *The normal mode stability analysis consists in studying how small initial periodic perturbations - called normal modes - evolve over time.* The reason for studying the behavior of periodic perturbations is that they constitute the Fourier components of any arbitrary spatial perturbation. A general perturbation may then be viewed as an appropriate linear combination of these normal modes. The amplification factor corresponding to a normal mode is defined as the growth rate of that normal mode. As it is done in practice, the amplification factor of every normal mode is computed. *If all* *normal modes have a negative amplification factor, meaning that the amplitude of each mode decreases over time, then spatial convergence takes place.* On the other hand, *if some normal modes have a positive amplification factor, meaning that the amplitude of these modes increases over time, then spatial divergence occurs.* For further details concerning the normal mode stability analysis, see a general reference in hydrodynamic stability, e.g. Drazin and Reid (1991).

This paper is organized as follows. In section 2, we describe the continuous spatial environment. We define the short-run equilibrium keeping the distribution of agents fixed in section 3. In section 4 we explain how agents myopically adjust their location over time. Then we define the long-run equilibrium in section 5, and the analysis of spatial convergence is performed in section 6. We conclude thereafter.

2 The Spatial Environment

In this section, the continuous spatial environment is described. We consider a spatial economy ($\mathbb{E}$) with a continuum of locations $s \in \mathbb{C} = [0, 2\Pi]$. Time is denoted by t. There are H types of agents denoted by h, who are distributed along the circle $\mathbb{C}$. The density of type-h agents in location s, at time t, is denoted by $A^h(s,t)$.[1] In each location there are L local goods denoted by l. Each type-h agent is continuously endowed with a constant bundle $\mathbf{E}^h \in \mathbb{R}_+^L$ over time. The instantaneous utility function for type-h

[1] More precisely, there are $A^h(s,t)ds$ type-h agents in region $[s, s+ds]$ at time t.

agent is represented by $U^h(.)$.

In the model below, we will follow Sonnenschein's approach (1981, 1982) by disregarding issues related to intertemporal trading (e.g. borrowing and lending). This approach is also widely used in the economic geography literature, see Krugman (1991) or Ottaviano (1999).

Assumption 1 *There is no intertemporal trade.*

Equivalently, this means that agents spend all their current income on current consumption. The central point of our approach is the spatial separation of markets.

Assumption 2 *Agents can trade with one another at time t if and only if they are in the same location at time t.*

This is the form of spatial separation of markets as introduced by Townsend (1990). Furthermore, like in Sonnenschein (1981, 1982), we consider a perfectly competitive local market structure.

Assumption 3 *Local markets are pure exchange economies.*

In each location s, there is thus a competitive market for the L traded local commodities, at any time t. The role of the local market structure is central. Here, by focusing on local pure exchange economies, we consider the simplest local market structure. The main reason is that it will allow us to identify the role of *local perfect competition* on spatial convergence.

So as to avoid existence and uniqueness difficulties concerning the consumption problem, we will assume that there is a unique bundle which maximizes the consumer's utility. Furthermore we will assume that the individual demand function exists and is continuously differentiable.

Assumption 4 $U^h(.)$ *is twice continuously differentiable, strictly increasing, and strictly concave.*

3 Short-Run Equilibrium

In this section we define what we mean by a short-run equilibrium of the spatial economy $(\mathbb{E})$. The short-run terminology refers to the fact that the distributions of agents are supposed to be fixed. Therefore, local prices are assumed to form instantaneously, and clear all the local markets as in the Walrasian tradition.

Consider a spatial economy $(\mathbb{E})$ where the distributions of agents are $\left\{A^h(s,t)\right\}$, $h = 1, ..., H$. Type-h agents at location s at time t, consider the price vector $\mathbf{P}(s,t)$ prevailing in local market s at time t as given, and solve their consumption program

$$\max_{\mathbf{X}^h(s,t)\in\mathbb{R}^L} U^h(\mathbf{X}^h(s,t))$$

$$\text{st. } \mathbf{P}(s,t)\cdot(\mathbf{X}^h(s,t)-\mathbf{E}^h) = 0 \tag{1}$$

The demand function of type-h agents at location s at time t is expressed by

$$\mathbf{Z}^h(s,t) = \mathbf{X}^h(\mathbf{P}(s,t)) \tag{2}$$

The maximum utility $\mathcal{U}^h$ available in location s at time t corresponds to the indirect utility level Λ^h in s at t

$$\mathcal{U}^h(s,t) = \Lambda^h(\mathbf{P}(s,t)) = U^h(\mathbf{X}^h(\mathbf{P}(s,t))) \tag{3}$$

The market clearing condition for local market s at time t is

$$\sum_{h=1}^{H} A^h(s,t)\mathbf{Z}^h(s,t) = \sum_{h=1}^{H} A^h(s,t)\mathbf{E}^h \tag{4}$$

Definition 1 (Short-Run Equilibrium) *A short-run equilibrium of the spatial economy ($\mathbb{E}$) at time t, is defined, taking the distributions of agents* $\left\{A^h(s,t)\right\}$, $h = 1, ..., H$, *as given, by consumption distributions* $\mathbf{Z}^h(s,t)$, $h = 1, ..., H$, *and a price distribution* $\mathbf{P}(s,t)$ *satisfying the consumption problem (2) and the market clearing condition (4) in all locations* $s \in \mathbb{C}$.

Consider at time t some spatial distributions of agents $\left\{A^h(s,t)\right\}$, $h = 1, ..., H$. As an immediate consequence of Assumption 4, given these spatial distributions of agents, there exists a short-run equilibrium of the spatial economy ($\mathbb{E}$).

In a short-run equilibrium, such as defined above, consumption levels of agents may vary across locations, depending on the agent distributions $\left\{A^h(s,t)\right\}$, $h = 1, ..., H$. There is thus an incentive for agents to change their location when their indirect utility is not equalized through space. Before dealing with this in the section below, we define the spaceless economy (Σ) underlying the spatial economy ($\mathbb{E}$).

Definition 2 (The Spaceless Underlying Economy) *The spaceless economy* (Σ) *underlying the spatial economy* $(\mathbb{E})$ *is a pure-exchange economy. There are* H *types of agents denoted by* h *and* L *goods denoted by* l*. Like in the local markets of the spatial economy* $(\mathbb{E})$*, type-h agents are endowed with the bundle* $\mathbf{E}^h$ *and have utility function* U^h*, and Assumption 4 holds. In this spaceless economy* (Σ)*, type-h agents are in number* $B^h \geq 0$*. They choose a consumption bundle* $\mathbf{X}^h$ *so as to maximize* $U^h(\mathbf{X}^h)$ *while being constrained by* $\mathbf{P} \cdot (\mathbf{X}^h - \mathbf{E}^h) = 0$*. Equilibrium prices* $\mathbf{P}^*$ *are such that*

$$\sum_{h=1}^{H} B^h \mathbf{X}^h(\mathbf{P}^*) = \sum_{h=1}^{H} B^h \mathbf{E}^h \tag{5}$$

Major properties of the spatial economy $(\mathbb{E})$ will require restrictions on the corresponding underlying economy (Σ). We will therefore frequently refer to (Σ) in the following sections.

4 Spatial Adjustment

We now introduce spatial dynamics by specifying a local process of adjustment of agents through the geographical space. Following Sonnenschein (1981, 1982), spatial adjustments are assumed to be *spatially local* and *temporally myopic*. This means that agents are only required to know prices locally. We make this clear in the two following assumptions.

Assumption 5 *Migration is local.*

Migration is viewed here as a local process, in the sense that no instantaneous migration can take place between two locations separated by a finite distance. Nevertheless, the continuous flows involved allow finite and even large scale migration over time. Local migration behavior is consistent with empirical findings according to which the intensity of migration flows declines with the increasing distance between origin and destination, see Ravenstein (1885), Shaw (1975), or Wheeler *et al.* (1998). By making the local migration assumption we tend to focus on what we believe to be the most important part of migration flows.

Eventhough long range and international migrations play an important role in explaining the growth of large cities, the local aspect of migration is found back in many migratory patterns in world history. A first illustratory case is the westward movement of Anglo-Saxons and other Europeans in the United States during the 19th century. They gradually moved away from the East coast toward the West coast. *This movement was a slow and continuous westward process.*[2] Another case where the local aspect is found, is the rural migratory patterns in France during the 18th and 19th centuries, see Ariès (1979). *Peasants migrated over short distances* and gave rise to numerous small-size cities (called "bourgs" in French) spread all over France.

Assumption 6 *Agents have static expectations.*

[2] A continuous spatial approach was proposed by Hotelling (1921) to describe these migration flows. Migration was assimilated to a pure diffusion process. However, no microeconomic foundation was proposed by Hotelling.

This assumption means that when making their migration choice at time t, agents do believe that the spatial distribution of prices $\mathbf{P}(s,t)$ prevailing at time t, will prevail forever.

Assumptions 5 and 6 concerning the migration decision process have the following implication. Consider agents located in s^* at time t^*. As migration is local, they will move in the surrounding of s^*, basing their migration choice by regarding $\mathcal{U}^h(s,t)$ as equal to $U^h(\mathbf{X}^h(\mathbf{P}(s,t^*),\mathbf{E}^h)$ for s in the neighborhood of s^*, and $t > t^*$. As in Sonnenchein (1981, 1982), agents adjust their location by choosing the optimal velocity in order to maximize the difference between the rate of change of utility and quadratic migration costs,

$$\max_{V^h(s^*,t^*)} V^h(s^*,t^*)\partial_s\mathcal{U}^h(s^*,t^*) - \frac{\left[V^h(s^*,t^*)\right]^2}{2k^h} \tag{6}$$

where V^h denotes the velocity of type-h agents and k^h is inversely related to the intensity of their migration cost.

As a result of (6), type-h agents at location s at t move over the distance $V^h(s,t)dt$ during time span dt with

$$V^h(s,t) = k^h\partial_s\mathcal{U}^h(s,t) \tag{7}$$

where k^h reflects the adjustment speed to spatial utility differentials for type-h agents.

This means that migration of agents is modelled as a gradient adjustment process. At each point in time, type-h agents at location s move in the direction of the highest indirect utility $\mathcal{U}^h(s,t)$.

So far, we have described how agents change their location over time. We still need to say how these spatial adjustments affect the distributions of agents A^h. This will allow us to describe how the distribution of agents A^h evolves over time. It turns out that once the distribution of adjustment $V^h(s,t)$ is known at time t, we can deduce how the distribution A^h changes over time.

Consider a region $\Gamma = [s, s+ds]$. Its overall type-h population, at time t, is $A^h(s,t)ds$. The conservation law of population h asserts that the rate of increase of population h in region Γ is equal to the flow of population h into Γ through its borders s and $s+ds$; see Figure 1.

$$\partial_t \left[A^h(s,t)ds\right] = -(\Phi^h(s+ds,t) - \Phi^h(s,t)) \tag{8}$$

where $\Phi^h(s,t)$ represents the flow of type-h agents through location s at time t. It corresponds to $A^h(s,t)V^h(s,t)$.

We can approximate (8) for small time variations, and get that the net increase of population h in region Γ during time span $[t, t+dt]$ is due to type h-agents flows through borders s and $s+ds$

$$\begin{aligned} \frac{A^h(s,t+dt)ds - A^h(s,t)ds}{dt} &= -\left[\Phi^h(s+ds,t) - \Phi^h(s,t)\right] \\ = \quad &\left[V^h(s,t)A^h(s,t) - V^h(s+ds,t)A^h(s+ds,t)\right] \end{aligned}$$

which can be rewritten as

$$\frac{A^h(s,t+dt) - A^h(s,t)}{dt} = \frac{V^h(s,t)A^h(s,t) - V^h(s+ds,t)A^h(s+ds,t)}{ds}$$

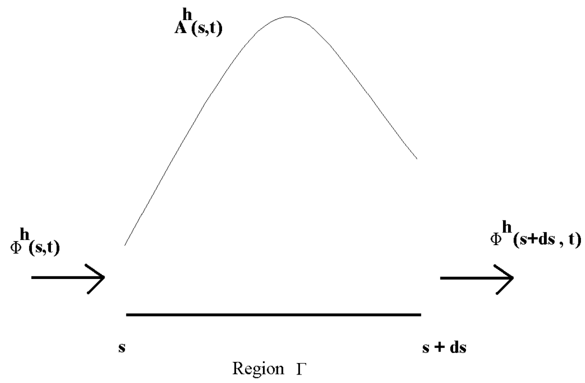

Figure 1: Conservation Law for Population h

and then as

$$\partial_t A^h(s,t) = -\partial_s \left[V^h(s,t) A^h(s,t) \right]$$

which leads to

$$\partial_t A^h(s,t) + \partial_s \left[A^h(s,t) V^h(s,t) \right] = 0 \tag{9}$$

This is the evolution law for the agent distribution A^h where V^h is given by (7).[3]

The dynamics of the spatial economy ($\mathbb{E}$) over space and time is governed by equations (2), (4), (7) and (9). We are interested in whether the spatial economy ($\mathbb{E}$) converges to a long-run equilibrium.

[3] An equivalent formulation to (9) is $D_t A^h = -A \partial_s V^h$, where D_t denotes the material derivative ($D_t \equiv \partial_t + V^h \partial_s$), see a reference on Fluid Dynamics (e.g. Batchelor (2000)).

5 Long-Run Equilibrium

In this section we define what we mean by long-run equilibrium.

Definition 3 (Long-Run Equilibrium) *A long-run equilibrium of the spatial economy* ($\mathbb{E}$) *is defined as a stationary state* $\{A^h(s),\ \mathbf{Z}^h(s),\ V^h(s),\ \mathbf{P}(s)\}$, $h = 1, ..., H$ *of the system of equations (2), (4), (7), and (9).*

Here the particular long-run equilibria of ($\mathbb{E}$) under interest are uniform long-run equilibria. In a uniform long-run equilibrium, the net flow of agents through any location is zero, and all local markets are time-invariant, and similar in all respects.

Definition 4 (Uniform Long-Run Equilibrium) *A uniform long-run equilibrium of the spatial economy* ($\mathbb{E}$) *is defined as a uniform stationary state* $\left\{A^h,\ \mathbf{Z}^h,\ V^h,\ \mathbf{P}\right\}$, $h = 1, ..., H$ *of the system of equations (2), (4), (7), and (9).*

Consider a uniform long-run equilibrium where the number of type-h agents is $\overline{A}^h$ in each location. The corresponding local prices, consumption and adjustments are

$$\mathbf{Z}^h = \overline{\mathbf{Z}}^h; \mathbf{P} = \overline{\mathbf{P}}; V^h = 0; A^h = \overline{A}^h\ , \qquad h = 1, ..., H \tag{10}$$

where $\overline{\mathbf{Z}}^h$ is defined by $\mathbf{X}^h(\overline{\mathbf{P}})$ and $\overline{P}$ is such that $\sum_h \overline{A}^h \mathbf{X}^h(\overline{\mathbf{P}}) = \sum_h \overline{A}^h \mathbf{E}^h$.

In general, there may be many price systems $\overline{\mathbf{P}}$ satisfying (10), or equivalently being an equilibrium price of the underlying economy (Σ) for $B^h = \overline{A}^h$.

6 Behavior around the Long-Run Equilibrium

The purpose of this section is to study whether the spatial economy $(\mathbb{E})$ evolves back to a uniform long-run equilibrium when initially deviated from it. This will be done in several steps. First, we will consider a spatial economy $(\mathbb{E})$ which is initially at a uniform long-run equilibrium such as given by (10). The economy is then deviated from this equilibrium. The deviations under study are supposed to be small. So as to measure how far away the spatial economy is from its initial equilibrium, it is useful to decompose the variables into their uniform long-run equilibrium value and their corresponding deviation. The equations describing the time evolution of these deviations are naturally called *perturbation equations.* Because initial perturbations are assumed to be small, we can describe their early evolution by linearizing the equations governing them. One then obtains *linearized perturbation equations.* The next step is to characterize the early evolution of these perturbations. This will be done by decomposing the perturbation into its Fourier components called *normal modes.* The procedure then consists in determining whether each of these normal modes dies away or increases in amplitude over time. This stability analysis is the *method of normal modes.* If all normal modes die away, the uniform long-run equilibrium will be said stable, and spatial convergence occurs. However, if at least one normal mode increases in amplitude over time, then the long-run equilibrium will be said unstable, and spatial divergence occurs.

We can normalize the price of good L by fixing $P_L = 1$. We denote $(P_1, ..., P_{L-1})$, $(X_1^h, ..., X_{L-1}^h)$, $(Z_1^h, ..., Z_{L-1}^h)$, and $(E_1^h, ..., E_{L-1}^h)$ respectively by $\mathbf{p}$, $\mathbf{x}^h$, $\mathbf{z}^h$ and $\mathbf{e}^h$.

6.1 Perturbation Linearized Equations

In order to perform the linearization of equations (2), (4), (7) and (9), we decompose the variables $\mathbf{z}^h(s,t)$, $A^h(s,t)$, $V^h(s,t)$, and $\mathbf{p}(s,t)$ into their uniform steady state value and their corresponding deviation. We denote these deviations by $\mathbf{w}^h, a^h, \mathbf{q}, v^h,$ and write

$$\begin{aligned} \mathbf{z}^h(s,t) &= \overline{\mathbf{z}}^h + \mathbf{w}^h(s,t); A^h(s,t) = \overline{A}^h + a^h(s,t); \\ \mathbf{p}(s,t) &= \overline{\mathbf{p}} + \mathbf{q}(s,t); V^h(s,t) = 0 + v^h(s,t) \end{aligned} \tag{11}$$

The perturbation equations are then obtained by substitution of (11) into equations (2), (4), (7), and (9). Neglecting second-order terms such as $a^h\mathbf{w}^h$ leads to the following perturbation linearized equations (see Appendix A)

$$\partial_t a^h(s,t) + \overline{A}^h \partial_s v^h(s,t) = 0 \tag{12}$$

$$\sum_{h=1}^{H} \overline{A}^h \mathbf{w}^h(s,t) + \sum_{h=1}^{H} (\overline{\mathbf{z}}^h - \mathbf{e}^h) a^h(s,t) = 0 \tag{13}$$

$$\mathbf{w}^h(s,t) = \frac{\partial \mathbf{x}^h}{\partial \mathbf{p}}(\overline{\mathbf{p}})\ \mathbf{q}(s,t) \tag{14}$$

$$v^h(s,t) = k^h \partial_s \left[\frac{\partial \Lambda^h}{\partial \mathbf{p}}(\overline{\mathbf{p}}) \cdot \mathbf{q}(s,t) \right] \tag{15}$$

6.2 Normal Mode Analysis

The main idea in what follows is to study how periodic spatial perturbations evolve over time. For the sake of simplicity, you may think of periodic perturbations as being like $\sin(\omega s)$. High (low) values of ω correspond to high (low) frequency pertubations. The reason for studying the behavior of periodic perturbations is that they constitute the Fourier components of any arbitrary spatial perturbation. In addition, because equations have been linearized, we can combine linearly the temporal evolutions of these periodic perturbations so as to deduce the evolution of any arbitrary perturbation. However, according to the normal mode analysis,[4] we need to consider more general perturbations in order to deal with our problem.

Definition 5 (Spatial Mode) *A spatial mode is determined by its frequency ω, and is defined as $\exp[I\omega s]$, with $I^2 = -1$.*

As suggested in the case of sinusoidal functions, high (low) frequency modes correspond to small (large) spatial scales. Our analysis consists in determining the evolution of each of the normal modes. A priori, there should be no reason for normal modes to have the same time behavior.

Definition 6 (Amplification Factor) *The amplification factor corresponding to normal mode ω will be denoted $\beta(\omega)$.*

A negative (positive) real part of $\beta(\omega)$ means that normal mode ω is damped (growing) over time. Provided $\beta(\omega)$ has a negative real part whatever ω, the spatial economy

[4] For further details concerning the normal mode stability analysis, see a general reference in hydrodynamic stability literature, e.g. Drazin and Reid (1991).

will converge to the uniform long-run equilibrium. We can now explicitly define the concept of convergence in the context of our continuous spatial economy.

Definition 7 (Spatial Local Stability) *A uniform long-run equilibrium such as defined by (10) is locally stable for the spatial economy ($\mathbb{E}$) provided that the amplification factor $\beta(\omega)$ has a negative real part whatever the normal mode ω.*

As the perturbation equations have been linearized, we look for solutions where all perturbations are proportional to $\exp[\beta(\omega)t+I\omega s]$, where $\beta(\omega)$ denotes the amplification factor corresponding to normal mode ω

$$\begin{bmatrix} a^h(s,t) \\ \mathbf{q}(s,t) \\ \mathbf{z}^h(s,t) \\ v^h(s,t) \end{bmatrix} = \exp[\beta(\omega)t + I\omega s] \begin{bmatrix} a^h \\ \mathbf{q} \\ \mathbf{z}^h \\ v^h \end{bmatrix} \tag{16}$$

where for instance, a^h is the constant amplitude of the density perturbation associated with type-h population.

In what follows, we determine the relationship $\beta(\omega)$, i.e. the amplification factor of each normal mode ω so that (16) is a non-trivial solution of equations (12), (13), (14) and (15).

6.3 Spatial Convergence

The following definitions and lemmas will be of use in Proposition 1.

Definition 8 (Stable Matrix) *A matrix* $\mathbf{M} = (m_{ij})_{i,j=1,...,L-1}$ *is stable if all its eigenvalues have negative real parts, that is when* $\operatorname{Re}\lambda \leq 0$*, for any eigenvalue* λ *of* $\mathbf{M}$*.*

Definition 9 (Quasi-Negative Definite Matrix) *A matrix* $\mathbf{M} = (m_{ij})_{i,j=1,...,L-1}$ *is quasi-negative definite (resp. semidefinite) if* $(\mathbf{M} + \mathbf{M}^T)$ *is negative definite (resp. semidefinite).*

Lemma 1 *Consider matrices* $\mathbf{M} = (m_{ij})_{i,j=1,...,L-1}$ *and* $\mathbf{N} = (n_{ij})_{i=1,...,H}^{j=1,...,L-1}$*. If* $\mathbf{M}$ *is quasi-negative semidefinite, then the matrix product* $\mathbf{NMN}^T$ *is quasi-negative semidefinite.*

Proof. See Appendix B. ■

Lemma 2 *Consider a matrix* $\mathbf{Q} = (q_{ij})_{i,j=1,...,H}$ *and a diagonal matrix* $\mathbf{D} = diag(d_i)_{i=1,...,H}$*, with strictly positive entries* $d_i > 0$*. If* $\mathbf{Q}$ *is quasi-negative semidefinite, then the matrix product* $\mathbf{DQ}$ *is stable.*

Proof. See Appendix C. ■

Here is the spatial convergence result of this paper.

Proposition 1 (Local Spatial Stability) *If the matrix of price effects is quasi-negative definite in the underlying spaceless economy* (Σ) *for* $B^h = \overline{A}^h$*, then a spatial long-run equilibrium such as defined by (10) is locally stable.*

Proof.

Instead of replacing straight the normal representation (16) in the perturbation linearized equations (12), (13), (14) and (15), we first obtain a system of equations involving the variables $\{a^h\}_{h=1,..,H}$ only. Then we will replace a^h by its normal mode representation.

Some manipulations of equations (12), (13), (14) and (15), detailed in Appendix D, lead to the following system of equations

$$\begin{aligned}\partial_t a^h(s,t) &= -k^h\overline{A}^h\lambda^h(\overline{\mathbf{p}})(\overline{\mathbf{z}}^h - \mathbf{e}^h)\cdot \\ &\quad [\sum_{j=1}^{H}\overline{A}^j D_{\mathbf{p}}\mathbf{x}^j(\overline{\mathbf{p}})]^{-1}\sum_{k=1}^{H}(\overline{\mathbf{z}}^k - \mathbf{e}^k)\partial^2_{ss}a^k(s,t) \qquad (17)\end{aligned}$$

where $\lambda^h(\overline{\mathbf{p}})$ is the Lagrange multiplier associated with the consumption program of type-h agents.

Replacing $a^h(s,t)$ by its expression (16) in (17) yields

$$\beta a^h - \sum_{k=1}^{H} c_{hk}a^k = 0 \qquad (18)$$

where we define $c_{hk} = k^h\overline{A}^h\lambda^h(\overline{\mathbf{p}})\omega^2(\overline{\mathbf{z}}^h - \mathbf{e}^h)^T[\sum_j \overline{A}^j D_{\mathbf{p}}\mathbf{x}^j(\overline{\mathbf{p}})]^{-1}(\overline{\mathbf{z}}^k - \mathbf{e}^k)$.

The relation (18) may be rewritten in the following matrix form

$$\beta\mathbf{a} = \mathbf{C}\mathbf{a} \qquad (19)$$

where $\mathbf{a}$ denotes the vector $(a^1, ..., a^H)$ and $\mathbf{C}$ the matrix with entries $(c_{hk})_{h,k=1,...H}$.

By inspection of relation (19), the amplification factor β corresponds to the eigenvalues of matrix $\mathbf{C}$. Thus the uniform long-run equilibrium (10) is stable provided that matrix $\mathbf{C}$ has eigenvalues with negative real parts.

It turns out that matrix $\mathbf{C}$ can be rewritten as

$$\mathbf{C} = \mathbf{DNMN}^T \tag{20}$$

where we define matrices $\mathbf{D}$, $\mathbf{M}$, $\mathbf{N}$ as follows

$$\begin{aligned}
\mathbf{D} &= \omega^2 diag\left[k^1\overline{A}^1\lambda^1(\overline{\mathbf{p}}), ..., k^H\overline{A}^H\lambda^H(\overline{\mathbf{p}})\right] \\
\mathbf{M} &= [\sum_{j=1}^{H}\overline{A}^j D_{\mathbf{p}}\mathbf{x}^j(\overline{\mathbf{p}})]^{-1} \\
\mathbf{N} &= \left[\overline{\mathbf{z}}^1 - \mathbf{e}^1, ..., \overline{\mathbf{z}}^H - \mathbf{e}^H\right]^T
\end{aligned}$$

Since the matrix of price effects $\sum_j \overline{A}^j D_{\mathbf{p}}\mathbf{x}^j(\overline{\mathbf{p}})$ is quasi-negative definite, so is matrix $\mathbf{M}$. By using Lemma 1, the matrix product $\mathbf{NMN}^T$ is quasi-negative semi-definite. Finally, as matrix $\mathbf{D}$ is a diagonal matrix with strictly positive elements, it results from Lemma 2 that matrix $\mathbf{C}$ is stable. As a consequence, the amplification factor $\beta(\omega)$ has a negative real part whatever normal mode ω. And the uniform long-run equilibrium such as defined by (10) is stable. ■

First, Proposition 1 shows that *spatial utility differentials provide adequate migration incentives* for leading the spatial economy to converge to a spatially uniform allocation of resources.

Secondly, Proposition 1 relates the convergence of the spatial economy (Σ) to the quasi-negativeness of the matrix of price effects. The sufficient condition we derived, also ensures the stability of the tâtonnement process. This is because quasi-negativeness implies D-stability, see Hahn (1982). Therefore the spatial convergence issue is to be related to the Walrasian stability of the spaceless economy (Σ) underlying the spatial economy ($\mathbb{E}$). Moreover, while the tâtonnement dynamics proceeds in fictive time, the spatial adjustments of agents across locations, take place in real time. This means that the dynamics of our model provides a *real time interpretation of the tâtonnement story.*

Furthermore, we provide a fine description of the spatial convergence process by relating the convergence rate β of a normal mode to its frequency ω. Different spatial scale perturbations are not damped at the same rate. It is thus important to distinguish spatial scales when dealing with spatial convergence. In particular, because of quadratic adjustments costs, $\beta(\omega)$ is quadratic in the frequency ω, see relationships (19) and (20). The implication of this is that *myopic migration leads high frequency* (small spatial scale) *perturbations to converge faster than low frequency* (large spatial scale) *perturbations.*

When perfect competition holds within local markets, as is the case in this work, agents mobility contributes to spatial convergence provided that the matrix of price effects is quasi-negative definite. On the other hand, when monopolistic competition holds in the spatial economy - as is the case in Krugman (1996), Fujita *et al.* (1999), or in Mossay (2003) -, agents mobility contributes to divergence and to the emergence of spatial agglomerations. As Krugman's framework is a benchmark model for continuous spatial economies characterized by *monopolistic competitive markets*, our model seems to be the corresponding ideal benchmark model for continuous spatial economies characterized by *perfectly competitive local markets.* In these two spatial environments, the role of the local structure on the spatial convergence/divergence issue appears as central. Both frameworks constitute thus a step forward in the understanding of how spatial convergence or agglomerations emerge.

7 Conclusion

Our paper reinforces and generalizes the view of Sonnenschein (1981, 1982) to the case of local pure exchange economies: myopic behavior concerning migration is sufficient to ensure spatial convergence in a local perfect competition environment. This is because, under perfect competition, *spatial utility differentials provide adequate migration incentives* for leading the spatial economy to converge to a spatially uniform allocation of resources.

We make clear the fact that the spatial convergence issue is closely related to the quasi-negative definiteness of the matrix of price effects, which also ensures the stability of the tâtonnement process of the underlying spaceless economy. Unlike the tâtonnement dynamics, the spatial adjustments of agents across locations, take place in real time. This means that the dynamics of our model provides a *real time interpretation of the tâtonnement story*. In addition, we have shown that the convergence rate is not the same for all normal modes. High frequency modes (corresponding to small spatial scale modes) are damped faster than low frequency modes (corresponding to large spatial scale modes). This means that because of quadratic adjustments costs, *myopic migration behavior leads small spatial scale shocks to be damped faster than larger spatial* *scale shocks.*

Finally, Sonnenschein (1982) seems to put forward the argument that the conditions leading to spatial convergence should be less restrive when people have rational expectations than when they have static expectations. In order to address this last issue, it is necesary to investigate what happens when consumers have forward-looking abilities and optimize their migration decisions over time.

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Appendix A: Perturbation Linearized Equations

Here we establish the perturbation linearized equations (12), (13), (14), (15). Substituting the decomposition (11) into the evolution equations (2), (4), (7) and (9), yields the following perturbation equations

$$\partial_t a^h(s,t) + \partial_s(v^h(s,t)(\overline{A}^h + a^h(s,t))) = 0$$

$$\sum_{h=1}^{H}(a^h(s,t)(\overline{\mathbf{z}}^h + \mathbf{w}^h(s,t)) + \overline{A}^h \mathbf{w}^h(s,t)) = \sum_{h=1}^{H} a^h(s,t)\mathbf{e}^h$$

$$\overline{\mathbf{z}}^h + \mathbf{w}^h(s,t) = \mathbf{x}^h(\overline{\mathbf{p}} + \mathbf{q}(s,t))$$

$$v^h(s,t) = k^h \partial_s \Lambda^h(\overline{\mathbf{p}} + \mathbf{q}(s,t)) \quad , \qquad h = 1, ..., H$$

Then by neglecting second-order terms, we get the perturbation linearized equations (12), (13), (14), and (15).

Appendix B: Proof of Lemma 1

Let $\mathbf{y}$ and μ be respectively an eigenvector and an eigenvalue of $\mathbf{NMN}^T + \mathbf{NM}^T\mathbf{N}^T$. Because this matrix is symmetric, $\mathbf{y}$ is a real vector and μ a real number. We have

$$(\mathbf{NMN}^T + \mathbf{NM}^T\mathbf{N}^T)\mathbf{y} = \mu\mathbf{y}$$

$$\mathbf{N}(\mathbf{M}+\mathbf{M}^T)\mathbf{N}^T\mathbf{y} = \mu\mathbf{y}$$

Premultiplying this relationship by $\mathbf{y}^T$ leads to

$$\mathbf{y}^T\mathbf{N}(\mathbf{M}+\mathbf{M}^T)\mathbf{N}^T\mathbf{y} = \mu\mathbf{y}^T\mathbf{y}$$

Let us denote $\mathbf{N}^T\mathbf{y}$ by $\mathbf{u}$, we can write

$$\mathbf{u}^T(\mathbf{M}+\mathbf{M}^T)\mathbf{u} = \mu\mathbf{y}^T\mathbf{y}$$

Since $\mathbf{M}$ is quasi-negative semidefinite and $\mathbf{y}^T\mathbf{y} > 0$, we have that $\mu \leq 0$, meaning that $\mathbf{NMN}^T + \mathbf{NM}^T\mathbf{N}^T$ is negative semidefinite, and thus that $\mathbf{NMN}^T$ is quasi-negative semidefinite. ■

Appendix C: Proof of Lemma 2

Let $\mathbf{y}$ and μ be respectively an eigenvector and an eigenvalue of $\mathbf{DQ}$. As $\mathbf{DQ}$ need not be symmetric, $\mathbf{y}$ is in general a complex vector and μ a complex number. We have

$$\mathbf{DQy} = \mu\mathbf{y}$$

Premultiplying this relationship by $\mathbf{D}^{-1}$ leads to

$$\mathbf{Qy} = \mu\mathbf{D}^{-1}\mathbf{y}$$

Premultiplying now by $\overline{\mathbf{y}}^T$, the complex conjugate vector of $\mathbf{y}^T$, leads to

$$\overline{\mathbf{y}}^T\mathbf{Qy} = \mu\overline{\mathbf{y}}^T\mathbf{D}^{-1}\mathbf{y} \tag{21}$$

We now use the fact $\overline{\mathbf{f}}^T \mathbf{C}\mathbf{g}$ is the complex conjugate of $\overline{\mathbf{g}}^T \mathbf{C}^T \mathbf{f}$, where $\overline{\mathbf{f}}$ and $\overline{\mathbf{g}}$ denote respectively the complex conjugate vectors of vectors $\mathbf{f}$ and $\mathbf{g}$. We then get the complex conjugate of (21)

$$\overline{\mathbf{y}}^T \mathbf{Q}^T \mathbf{y} = \overline{\mu}\, \overline{\mathbf{y}}^T \mathbf{D}^{-1} \mathbf{y} \tag{22}$$

where $\overline{\mu}$ denote the complex conjugate of μ.

By adding up relationships (21) and (22), we get

$$\overline{\mathbf{y}}^T (\mathbf{Q} + \mathbf{Q}^T) \mathbf{y} = (\mu + \overline{\mu})\, \overline{\mathbf{y}}^T \mathbf{D}^{-1} \mathbf{y}$$

Since $\mathbf{Q}$ is quasi-negative semidefinite and $\overline{\mathbf{y}}^T \mathbf{D}^{-1} \mathbf{y} > 0$, we have that $(\mu + \overline{\mu}) \leq 0$, meaning that $\mathrm{Re}\,\mu \leq 0$. And thus $\mathbf{DQ}$ is stable. ■

Appendix D: Some Manipulations to get equation (17)

Equation (15) may be rewritten as

$$v^h(s,t) = k^h \sum_{k=1}^{L-1} \frac{\partial q_k}{\partial s}(s,t) \frac{\partial \Lambda^h}{\partial p_k}(\overline{\mathbf{p}})$$

We may then express $\partial_s v^h$ as

$$\begin{aligned} \partial_s v^h(s,t) &= k^h \sum_{k=1}^{L-1} \frac{\partial^2 q_k}{\partial s^2}(s,t) \frac{\partial U^h}{\partial p_k}(\overline{\mathbf{p}}) \\ &= k^h \sum_{k=1}^{L-1} D_{p_k} \Lambda^h(\overline{\mathbf{p}}) (\frac{\partial^2 q_k}{\partial s^2}(s,t)) \\ &= k^h D_{\mathbf{p}} \Lambda^h(\overline{\mathbf{p}}) \cdot \frac{\partial^2}{\partial s^2} \mathbf{q}(s,t) \end{aligned} \tag{23}$$

Substituting (23) into (12) leads to

$$\partial_t a^h(s,t) + k^h \overline{A}^h D_{\mathbf{p}} \Lambda^h(\overline{\mathbf{p}}) \cdot \partial^2_{ss} \mathbf{q}(s,t) = 0 \tag{24}$$

We get the equilibrium price deviation $\mathbf{q}$ by substituting (14) in (13),

$$\mathbf{q}(s,t) = -\sum_{h=1}^{H} [\sum_{j=1}^{H} \overline{A}^j D_{\mathbf{p}} \mathbf{x}^j(\overline{\mathbf{p}})]^{-1} (\overline{\mathbf{z}}^h - \mathbf{e}^h) a^h(s,t) \tag{25}$$

where the underlying economy (Σ) is regular for $B^j = \overline{A}^j$ since, by assumption, the matrix of price effects is quasi-negative definite in the underlying spaceless economy (Σ) for $B^j = \overline{A}^j$.

By substituting (25) into (24), we get

$$\partial_t a^h(s,t) - k^h \overline{A}^h (D_{\mathbf{p}} \Lambda^h(\overline{\mathbf{p}})) \cdot [\sum_{j=1}^{H} \overline{A}^j D_{\mathbf{p}} \mathbf{x}^j(\overline{\mathbf{p}})]^{-1} \sum_{k=1}^{H} (\overline{\mathbf{z}}^k - \mathbf{e}^k) \partial^2_{ss} a^k(s,t) = 0$$

Moreover we use the fact that

$$D_{\mathbf{p}} \Lambda^h(\overline{\mathbf{p}}) = -\lambda^h(\overline{\mathbf{p}})(\overline{\mathbf{z}}^h - \mathbf{e}^h)$$

where $\lambda^h(\overline{\mathbf{p}})$ is the Lagrange multiplier associated with the consumption program of type-h agents.

This leads to equation (17).