

A discusión

EQUILIBRIUM SELECTION IN THE NASH DEMAND GAME AN EVOLUTIONARY APPROACH*

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ABSTRACT

Equilibrium selection in the Nash demand game is investigated in a learning context with persistent randomness. I adopt a matching framework similar to Kandori, Mailath and Rob (1993) and assume that individuals belong to populations of different sizes. Despite the myopic behavior of individuals, the selected division of the surplus that will be observed most of the time coincides with the Nash bargaining solution. Depending on the matching scenario, either the symmetric or the generalized Nash bargaining solution is selected. In the latter case, the power is larger for the short-side of the market.

JEL Classification: C63, C78, D83.

Key words: bargaining, best response, convention, learning, stochastic stability.

1 Introduction

Splitting the pie can be considered one of the most basic economic activities. For example, in simple transactions the pie symbolizes the potential gains from trade that have to be divided between buyer and seller. Whenever the assumption of perfect competition is relaxed, bargaining is one of the main issues in the analysis of markets. Rubinstein and Wolinsky (1984) and Osborne and Rubinstein (1990) among others have made an explicit use of bargaining models to provide foundations of price formation in decentralized markets.

To illustrate consider a market with just two individuals. One individual (the seller) possesses one good that has zero value for herself but a value equal to one for the other individual (the buyer). In this market there are potential gains from trade that can be exploited if buyer and seller exchange the good. The size of the pie to be divided between buyer and seller is equal to the difference between the value assigned to the good by the buyer and the value assigned to the good by the seller, which is one in this particular case. A simple way to divide the gains from trade (or pie) is by making simultaneous demands, so that trade takes place if and only if the sum of demands is smaller than or equal to the size of the pie. This is just the Nash demand game due to Nash (1953). One basic concern about this game that is more general to bargaining problems is the existence of multiple equilibria. In the Nash demand game any pie split constitutes a Nash equilibrium (N.E. hereafter) of the game.

Nash (1953) addressed the question of equilibrium selection in the Nash demand game by introducing uncertainty in the payoff function. He obtained that the Nash bargaining solution is the unique limiting outcome of the N.E. as uncertainty vanishes.¹ Some related papers are Carlsson (1991) and Muthoo (1996). In Carlsson (1991) the Nash demand game is played by individuals that make errors when choosing their actions. As the probability of error goes to zero the Pareto dominant equilibrium converges to the generalized Nash bargaining solution. The power is a function of the proportion of the unclaimed surplus that goes to each of the players. In Muthoo (1996) the Nash demand game is interpreted as a two-stage bargaining game in which the players make commitments in the first stage that can be modified in the second stage at some cost. As the cost of revoking the commitment made in the first stage goes to infinite only one division of the cake is selected. This division is the generalized Nash bargaining solution with the power as a function of the cost of revoking the commitment.

¹See Binmore and Dasgupta (1987), p.64, for a detailed description.

Evolutionary game theory provides an alternative approach for equilibrium selection. Under this approach selected equilibria are the outcome of the learning process followed by the agents of one or more populations who meet repeatedly to play the game and update their actions according to their previous experience. Young (1993b) provides a nice application to the Nash demand game. In Young (1993b) this game is played repeatedly by the members of two populations. In particular, in each round an individual from each of the populations is randomly chosen to play the game. In order to update their demands agents observe a sample drawn from a given recent history of play, using then the best response to the empirical distribution of the opponents' population play in the sample drawn. Rarely do agents experiment choosing each of the available demands with positive probability. As in the papers mentioned above, under some conditions, the selected equilibrium corresponds to the generalized Nash bargaining solution. The higher power is attained by the agents that observe a larger fraction of the history.

In this paper I also follow an evolutionary approach to analyze equilibrium selection. In the evolutionary bargaining model introduced in this paper the Nash demand game is played repeatedly by the members of two populations of different sizes. In contrast with Young (1993b), I adopt a matching framework in the spirit of Kandori, Mailath and Rob (1993).² In particular, I consider two matching schemes. In the first matching scheme each matching period is divided in m sub-periods. Thus each individual is matched up to m times in a round. Under the second matching scheme each player is matched against each of the individuals in the other population. Therefore, individuals from population A are matched b times, while individuals from population B are matched a times.

Individuals update their actions choosing the best reply to the observed demands taken by their opponents and from time to time experiment. I work under one period memory and assume that agents behavior evolves with inertia. In other words, agents update their actions looking back only at the previous round and, as in Kandori and Rob (1995), the opportunity to adjust arrives independently -across players and time- with a strictly positive probability.

Results are sensitive to the matching scheme adopted. In the long run (as time goes to infinite and in the absence of experimentation) efficiency is only reached under the first matching scheme for m finite. When m is infinite the process converges with

²I have just been aware about a paper by Binmore, Samuelson and Young (2003) in which equilibrium selection in bargaining is also analyzed adopting an evolutionary perspective. They concentrate on best reply rules and consider populations of equal size.

probability one to a particular division of the cake. In the limit as m goes to infinite or under the second matching scheme, however, individuals may end up trapped in a mixed equilibrium. In such a case agreement is not guaranteed.

In the ultra long run we are interested in the states that are robust to experimentation, that is to say, the states that are in the support of the limiting invariant distribution as the rate of experimentation goes to zero. Such states are known as stochastically stable in the literature. In this case, again the results are sensitive to the matching scheme. As the grid of feasible demands becomes finer the symmetric Nash bargaining solution is the only stochastically stable state of the process under the first matching scheme when m is finite. In the other two cases (m infinite or second matching scheme) the selected division of the surplus corresponds to the generalized Nash bargaining solution. The power attained by the agents of each population is a function of the population sizes. The larger the population the lower its power which lends a competitive flavor to this solution.

The paper is organized as follows. Next section contains a description of the model and section 3 presents the main results. The paper is closed with some final remarks. The proofs not included in the main text are in the appendix at the end of the paper.

2 The Model

In the Nash demand game two individuals, say i and j , have to make simultaneous demands over a surplus. Let us normalize the size of the surplus to one and denote such a pair of demands by x_i and x_j respectively. If $x_i + x_j \leq 1$ each individual gets the amount asked and a zero payoff otherwise. In this simple game any pair of demands of the type $(x, 1 - x)$, with $x \in [0, 1]$, constitute a Nash equilibrium (and also a subgame perfect equilibrium).

In the evolutionary model of this paper this one-shot game is repeatedly played by the individuals of two populations: A and B of size a and b respectively. Population A is assumed to be larger than population B , i.e., $a > b$. A discretized version of the Nash demand game is considered, such that the set of actions available to each player is given by the grid $\Omega = \{\delta, 2\delta, \dots, 1 - \delta\}$.

In each round we can distinguish two stages. First, the matching stage when individuals are randomly matched in pairs. Second, the decision stage in which paired individuals have to simultaneously choose a demand from Ω . In this model there are no matching frictions so that the maximum number of matches takes place in the first stage ($2b$ individuals are matched). The two matching schemes analyzed in the

paper will be introduced in a while.

Given an arbitrary pair of demands x and y , we say that they are compatible if $x + y = 1$, strictly compatible if $x + y < 1$, and incompatible if $x + y > 1$. Let us now describe the payoffs. If individuals i and j , asking for x_i and y_j respectively, are matched then they get their demands if and only if x_i and y_j are compatible or strictly compatible, and they get nothing otherwise. The members of population A that are not matched get a zero payoff. Let $u_i(x)$ represent the utility function of agent $i \in A$ and $v_j(y)$ the utility function of individual $j \in B$. Utility functions are assumed to be concave and strictly increasing in x . They are also assumed to be identical among the individuals that belong to the same population. That is to say,

$$u_i(x) = u_j(x), \forall i, j \in A \quad \& \quad v_k(y) = v_l(y), \forall k, l \in B$$

The state of the system in each round t is given by the set of demands made by the individuals of both populations in such a round. Specifically, $s(t)$ denotes the state of the system at period t and is given by the pair of vectors $(x(t), y(t)) \in \Omega^{a+b}$, such that $x_i(t)$ and $y_j(t)$ are the amounts asked by players i and j from populations A and B respectively. If individual k is not matched at round t then $x_k(t)$ should be interpreted as the demand that k would have asked for in case of having been matched. Since such a demand is not realized it is not observed by the rest of individuals.

After all matches have taken place, each individual receives the opportunity of revising her action according to an independent Bernoulli trial, with probability $1 - \lambda_A$ ($1 - \lambda_B$) if she belongs to population A (B). The parameter λ_A (λ_B) can be interpreted as the “inertia” and may reflect informational restrictions. In most cases, agents have to collect information if they want to adjust their actions. In addition of being costly, this information may be available only at certain dates.

When the opportunity to revise the action arrives to an individual then she chooses the (myopic) best reply to the demands taken by the individuals in the opponent population in the last round. The players that update their actions under this rule act myopically in the sense that the distribution of the opponent actions is taken as stationary without anticipating their possible changes.

Associated to this learning model there is a transition matrix $P := (P_{(s'/s)})_{s, s' \in S}$, where S is the space of states as defined above. Given a state s , this matrix tells us how likely the different states are in the following period, i.e., the probability of transiting from s to any other state s' in one period.

The model is complemented with a mutation process. In particular, each player of population A (B) is allowed to experiment occasionally with probability ε_A (ε_B),

in such a case the player chooses each of the demands with equal probability. Experimentation is assumed to be state-independent.

The inclusion of experimentation defines a new transition matrix, say $P^\varepsilon := (P_{(s'/s)}^\varepsilon)_{s,s' \in S}$. All components of this matrix are strictly positive and therefore it is ergodic. As a consequence the Markov process defined has a unique invariant distribution, $\mu(\varepsilon)$. We are interested in the analysis of the limiting invariant distribution as the rate of experimentation vanishes.

Let us now introduce the two matching schemes considered in this paper.

2.1 Matching Schemes

MATCHING SCHEME 1: m sub-periods

Under this scheme each matching period is divided in m sub-periods so that individuals are matched m times in a round. Since there are not matching frictions the maximal number of matches, mb , takes place in each matching period. Note that under this scenario the probability that an individual i from population A is not matched is $(1 - (b/a))^m > 0$.

MATCHING SCHEME 2: all play with all

The second matching scenario considered is such that in each round all the individuals from each population play with all the individuals in the opposite population. Thus, in each round each individual from population A ends up playing b times and each individual from population B plays a times. The distinguishing feature of this scenario is that now the demands of all individuals (even the ones in the larger population) are observed in each matching period.

3 Results

As usual in this type of literature the analysis is divided into two parts. In the first part I characterize the minimal absorbing sets of the unperturbed process, i.e., the process free from experimentation (long run). In the second part, we allow for experimentation and characterize the set of stochastically stable states or, in other words, the states that are robust to this experimentation (ultra-long run).

3.1 Long run

Let us start with some definitions.

Definition 1. (Minimal Absorbing set, Vega-Redondo (1997)): A set $A \subset S$ is defined to be a limit (or minimal absorbing) set of the dynamics if this set is closed under finite chains of positive-probability transitions. That is: (i) $\forall s \in A, \forall s' \notin A, P_{(s'|s)} = 0$; (ii) $\forall s, s' \in A, \exists m \in \mathbb{N}$ such that $P_{(s'|s)}^m > 0$. Where P is the unperturbed transition matrix, and $P_{(s'|s)}^m$ corresponds to the m -step unperturbed transition matrix P .

Definition 2. (Convention): A state $s = (x, y)$, s.t. $x \in \Omega^a, y \in \Omega^b$, is called “convention” if $x_i = x^*, \forall i \in A$ & $y_j = y^*, \forall j \in B$,³ and it is denoted by (x^*, y^*) . We say that a convention is “efficient” if $x^* + y^* = 1$, the set of efficient conventions is denoted by C^E . A convention is “inefficient” if $x^* + y^* < 1$, this set is denoted by C^I . Finally, a convention is “strongly inefficient” if $x^* + y^* > 1$, this set is denoted by C^{SI} .

Let us denote the set of pure Nash equilibria of the (population) Nash demand game by **NE**. Formally, this set is as follows:

$$\mathbf{NE} = \left\{ (\mathbf{x}, \mathbf{y}) \in \Omega^{a+b} : \begin{array}{l} x_i \in \arg \max_{x \in \Omega} x \hat{F}(1 - x | \mathbf{y}), \forall x_i \in \mathbf{x} \\ y_j \in \arg \max_{y \in \Omega} y \hat{F}(1 - y | \mathbf{x}), \forall y_j \in \mathbf{y} \end{array} \right\}$$

Where $\hat{F}(1 - x | \mathbf{y})$ is the empirical distribution function given the set of demands $\mathbf{y}$, $\hat{F}(1 - y | \mathbf{x})$ is defined analogously. $\hat{F}(1 - x | \mathbf{y})$ should be read as the probability that a demand from vector $\mathbf{y}$ is lower or equal to the amount $1 - x$ or, alternatively, the proportion of demands in vector $\mathbf{y}$ that are smaller than or equal to $1 - x$.

Proposition 1 *Under the matching scheme 1, for any finite m , the minimal absorbing sets of the evolutionary bargaining process are the states s that constitute an efficient convention. That is to say, the states of the type $(x^*, 1 - x^*)$, where $x^* \in \Omega$. Thus, as $t \rightarrow \infty$ an efficient convention is reached with probability one.*

In the limit, as $m \rightarrow \infty$, the minimal absorbing sets are the states $s \in \mathbf{NE}$.

Proof. Let us start considering the case in which m is finite. Note that a state $(\mathbf{x}, \mathbf{y})$ such that x_i is not a best reply to the vector of demands $\mathbf{y}$ for some $x_i \in \mathbf{x}$ or y_j is not a best reply to the vector of demands $\mathbf{x}$ for some y_j cannot be absorbing. This

³Note that this definition of convention is slightly different from the ones offered by Young (1993a) and Josephson and Matros (2000).

is because, with positive probability, the individuals (i or j) that are not choosing a best reply will switch their demands, choosing a best reply.

The previous claim implies that if a state is absorbing it must belong to the set $\mathbf{NE}$. It is easy to check that efficient conventions are absorbing. Now it will be shown that the states $s \in \mathbf{NE} \setminus C^E$ cannot be absorbing. Let us consider a state $s \in \mathbf{NE} \setminus C^E$, for such state to be absorbing the smallest demand contained in vector $\mathbf{y}$, denoted by $\underline{y}$, must be compatible with the highest demand in vector $\mathbf{x}$. Thus the expected payoff associated to $\underline{y}$ will be $v(\underline{y})$ for all matching realizations. Since all demands made by the individuals in population B must be a best reply to $\mathbf{x}$, then $v(\underline{y}) = y' \hat{F}(1 - y' | \mathbf{x})$, for some other $y' \in \mathbf{y}$ and such that $y' > \underline{y}$. Since the demands contained in vector $\mathbf{x}$ are heterogeneous and m is finite, there will exist a matching realization with a vector of observed demands, denoted by $\mathbf{x}^o$, such that $\hat{F}(1 - y' | \mathbf{x}^o) \neq \hat{F}(1 - y' | \mathbf{x})$. But then the pair of demands $\underline{y}$ and y' will not be indifferent any more.

Let us now prove convergence towards an efficient convention. Suppose that the initial state $s(0) = (\mathbf{x}(0), \mathbf{y}(0))$ does not belong to the set $\mathbf{NE}$. Two things may happen. First, if the best reply to vector $\mathbf{x}(0)$ is unique then with positive probability all the individuals in population B will switch to such a best reply, denoted by y^* . In the next period, $t = 1$, the whole population A will ask for $1 - y^*$, having then reached convention $(1 - y^*, y^*)$. Consider now the alternative scenario in which there are multiple best replies to vector $\mathbf{x}(0)$. The argument used above to show that heterogeneous states that belong to set $\mathbf{NE}$ are not absorbing can be used now to show that if there are multiple best replies to vector $\mathbf{x}(0)$ these will be abandoned gradually till only one demand prevails in population B , say demand y^* . This is possible because of the inertia and the assumption that m is finite. Once such state is reached, all individuals of population A , not asking for $1 - y^*$, will switch their demands to $1 - y^*$ with positive probability. Thus in a finite number of periods the efficient convention $(1 - y^*, y^*)$ will be reached. As $t \rightarrow \infty$, and starting from any initial state, an efficient convention will be reached with probability 1. This concludes the proof of the first part of the proposition.

It is straightforward to realize that as $m \rightarrow \infty$ all states in $\mathbf{NE}$ are absorbing since in this case all demands from the individuals of population A will be observed with probability one. It is also easy to see that a state $s \notin \mathbf{NE}$ cannot be absorbing for the same reason as when m was finite. It remains proving that the probability of reaching an absorbing state is one as $t \rightarrow \infty$. Consider an initial state $s(0) = (\mathbf{x}(0), \mathbf{y}(0)) \notin \mathbf{NE}$. There are three possible scenarios:

- (i) The case in which there is a unique best reply to either $\mathbf{x}(0)$ or $\mathbf{y}(0)$. In such

a situation an efficient convention can be reached in just two rounds.

(ii) If there were several best replies to $\mathbf{x}(0)$ and either none of them or only one were present in vector $\mathbf{y}(0)$ then, with positive probability all individuals in population B would choose the same best reply to vector $\mathbf{x}(0)$, say y^* , entering the basin of attraction of the efficient convention $(1 - y^*, y^*)$. The case in which there are several best replies to $\mathbf{y}(0)$ and at the most one of them is present in vector $\mathbf{x}(0)$ is analogous.

(iii) Finally, consider the case in which there are several best replies to vector $\mathbf{x}(0)$ and to vector $\mathbf{y}(0)$ and some of them are present in the initial vector of demands $\mathbf{y}(0)$ and $\mathbf{x}(0)$. Starting from such state, with positive probability the individuals from population B not choosing a best reply will choose the same best reply with positive probability and $\mathbf{y}(1) \neq \mathbf{y}(0)$. Vector $\mathbf{y}(1)$ will contain a finite number of different demands, say $k < b$. If all demands in $\mathbf{x}(0)$ are a best reply to $\mathbf{y}(1)$ then we have reached a state in the **NE** set. If not, then $\mathbf{x}(1) \neq \mathbf{x}(0)$ and some of the demands in vector $\mathbf{y}(1)$ will be abandoned such that in the new vector $\mathbf{y}(2)$ there will be $k' < k$ different demands. Since the populations are finite we have that in a finite number of periods the process will have reached a state in **NE**

■

Proposition 2 *Under the matching scheme 2, the minimal absorbing sets of the evolutionary bargaining process are given by the states $s \in \mathbf{NE}$.*

(The proof of this proposition parallels the proof of the second part of proposition 1).

Only under scenario 1 and when m is finite efficiency is reached in the long run. Either if $m \rightarrow \infty$ or if all individuals play with the whole population of opponents the evolutionary process can be trapped in a “mixed equilibrium”⁴ in which agreement is not guaranteed for all pairs of matched individuals. The reason behind such diverging results is that when m is finite the fragility of mixed equilibria (in the normal sense used in game theory) shows up. This is because in such scenario the vector of observable demands of population A may change from period to period even if no individual from this population switches her action. As a consequence, a mixed equilibrium can be abandoned in a finite number of matching periods. This is similar to what happens in Young (1993b)’s model where the long run efficiency is possible

⁴We use the term mixed equilibrium referring to the heterogeneous states (in which individuals choose different demands within a population) that belong to set **NE**.

because of the assumption that individuals can observe only a small fraction of the previous actions taken. If individuals were able to observe the whole previous history of play or, in the framework of this paper, the actions taken by all the opponents “mixed equilibria” would not be abandoned unless we introduce some noise or experimentation. Indeed when we introduce the possibility of experimentation efficiency is guaranteed in the long run.

3.2 Ultra-long run

When experimentation is allowed the perturbed bargaining model has the interesting property that, as the rate of experimentation vanishes, the limiting invariant distribution exists and is unique. Because of the strong ergodicity of the perturbed process, this invariant distribution leads to a very nice interpretation: each component represents the average time spent on each state. The states contained in its support are the stochastically stable states. These are the states that will be observed most of the time in the ultra long run. Because this part is rather technical, the introduction of some concepts becomes necessary.⁵

Let μ_ε denote the unique invariant distribution of the perturbed process P^ε . And, let μ^* denote the limiting distribution as the mutation rate tends to zero, i.e.,

$$\mu^* = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon$$

We say that s is a stochastically stable state if $\mu^*(s) > 0$ (where $\mu^*(s)$ indicates the probability assigned by μ^* to state s), i.e., if s belongs to the support of μ^* , denoted as $\text{supp } \mu^*$.

Let us consider any two states s and s' , an $(s'|s) - \text{path}$ is a sequence of states $\zeta = (s^1, \dots, s^q)$, such that $s^1 = s$ and $s^q = s'$. That is, it is a sequence that begins in s and ends in s' . Given any two arbitrary states, s and s' , the resistance $R(s'|s)$, is the minimum number of experiments involved in any $(s'|s) - \text{path}$. Note that if s is an absorbing state of the process then $R(s'|s) > 0$ for all states $s' \neq s$. Let $r(s'|s)$ be the least resistance over all $(s'|s) - \text{paths}$. This represents the minimum number of mutations required to go from s to s' with positive probability. Therefore $r(s'|s) \leq R(s'|s)$.

Given the set of states S and an arbitrary element of this set s , a $s - \text{tree}$ is a tree on the set S in the normal sense used in game theory (that is, a directed graph

⁵Most of the definitions introduced can be found in Young (1998) -chapter 3 and appendix, and in Fudenberg and Levine (1998) -chapter 5.

that branches out) except that the direction of motion is the reverse of the usual one. The set of s-trees is denoted by T_s . We say that the total resistance of an s-tree is the sum of the least resistances of its edges $s \rightarrow s'$. The stochastic potential of a state s , denoted by $\gamma(s)$, is the least total resistance among all its s-trees, that is,

$$\gamma(s) = \min_{T \in T_s} \sum_{(s', s'') \in T} r(s''|s')$$

The states with minimum stochastic potential are the stochastically stable states we are looking for. Under the evolutionary bargaining model considered in this paper and as the precision of demands goes to zero only one state is selected as stochastically stable. The next two propositions show that the selected state depends on the matching scheme under consideration.

Proposition 3 *Under the matching scenario 1 and finite m , for every precision $\delta > 0$, there is at least one and at the most two stochastically stable states, and as $\delta \rightarrow 0$ they converge to the (symmetric) Nash bargaining solution.*

Proposition 4 *Either under the matching scenario 2 or in the limit, as $m \rightarrow \infty$ (under scenario 1), for every precision $\delta > 0$, there is at least one and at the most two stochastically stable states, and as $\delta \rightarrow 0$ they converge to the generalized Nash bargaining solution with power $\frac{b}{a+b}$ and $\frac{a}{a+b}$ for the individuals of populations A and B respectively.*

(The proofs are contained in the appendix)

Again a finite m in scenario 1 gives a different equilibrium selection from the limit case as $m \rightarrow \infty$ and scenario 2. When m is finite the symmetric Nash bargaining solution arises as the only stochastically stable state. If the individuals of both populations share the same preferences then belonging to population A or to population B does not make a big difference (of course the expected payoff faced by the individuals that belong to the larger population will be smaller since these are not matched with positive probability). However, in the remaining two cases (infinite m or scenario 2) the selected division of the surplus gives an advantage to the smaller population. Thus, an element that played no role in Young (1993b)'s framework, the population size, determines the bargaining power in this model.

Let us devote some lines to the intuition behind such a result. When δ is small we have that the tree with minimum stochastic potential is formed by one step transitions of the type $(x, 1-x) \rightarrow (x+\delta, 1-x-\delta)$ or $(x, 1-x) \rightarrow (x-\delta, 1-x+\delta)$.

The first type of transition is beneficial for individuals of population A , let us call them $A-trans$. The second type is favorable for individuals of population B and will be called $B-trans$. An $A-trans$ requires $\left\lceil a^{\min} * \frac{v'(1-x)}{v(1-x)} \right\rceil$ mutations by individuals of population A , where $a^{\min}$ is the minimum number of individuals of population A that can be observed in a matching period. Under scenario 1 when m is finite $a^{\min} = b$, while $a^{\min} = a$ when $m \rightarrow \infty$ or under scenario 2. On the other hand, a $B-trans$ requires $\left\lceil b * \frac{u'(x)}{u(x)} \right\rceil$ mutations by individuals of population B .

Note that an $A-trans$ ($B-trans$) is cheaper (more expensive) than a $B-trans$ ($A-trans$) as long as $\left\lceil a^{\min} * \frac{v'(1-x)}{v(1-x)} \right\rceil < (>) \left\lceil b * \frac{u'(x)}{u(x)} \right\rceil$. The cost of an $A-trans$ is increasing in x , while the cost of a $B-trans$ is decreasing in x . Therefore, the cost minimizing tree will have $A-trans$ for small values of x and $B-trans$ for large values of x . The transition cost will reach its maximum at the x^* such that $a^{\min} * \frac{v'(1-x^*)}{v(1-x^*)} = b * \frac{u'(x^*)}{u(x^*)}$, where $(x^*, 1-x^*)$ is the stochastically stable division. Thus, when $a^{\min} = b$ the stochastically stable division of the surplus coincides with the symmetric Nash bargaining solution. In the other two cases, in which $a^{\min} = a$, the selected division of the surplus coincides with the generalized Nash bargaining solution, giving a higher power to the smaller population.

4 Final Remarks

This paper has analyzed the problem of equilibrium selection in the Nash demand game using an evolutionary model in which individuals update their actions following the best reply rule. In agreement with the existing literature the Nash bargaining solution arises as our favorite candidate. The distinguishing feature of this paper is that when the generalized Nash bargaining solution is selected (under the second matching scheme or as $m \rightarrow \infty$ in the first matching scheme) it has a competitive flavor, giving a higher power to the individuals in the smaller population.

Under the first matching scheme and for a finite number of rounds per period (finite m), the symmetric Nash bargaining solution is selected despite the differences in the sizes of each population. In all cases, if both populations are identical (in terms of preferences and size) the surplus is divided evenly.

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5 Appendix

Proof. (Proposition 3: Stochastic Stability under BR: scheme 1, finite m) The structure of the proof follows Young (1993b). In particular, the proof is divided in three main steps. The first step is devoted to the computation of the minimum

resistance of moving from any absorbing state to any other absorbing state. The aim of the second step is to find the tree(s) with minimum stochastic potential. In the final step we prove that as the precision of demands vanishes, i.e. $\delta \rightarrow 0$, the stochastically stable conventions converge to the (symmetric) Nash bargaining solution.

Step 1. When m is finite efficient conventions are the only absorbing states. Denote by $r(x'|x)$ the least resistance of moving from convention $(x, 1-x)$ to convention $(x', 1-x')$. Take the efficient convention $(x, 1-x)$ and consider the cheapest transition to enter the basin of attraction of a different absorbing state. This implies computing the minimum number of mutations required to provoke a change in the best reply of the agents one of the populations, A or B . Consider the transition from convention $(x, 1-x)$ to a different convention $(x', 1-x')$. Let $x' > x$. Then,

(i) The minimum number of mutations by players of population A , denoted by i , required to make the members of population B best reply with $1-x'$, is such that,

$$v(1-x') \geq \frac{a^o - i}{a^o} v(1-x) \iff i \geq a^o \frac{v(1-x) - v(1-x')}{v(1-x)}$$

where a^o is the number of observed demands in population A . The cheapest transition will occur when a^o reaches its minimum value, denoted by $a^{\min}$ which is equal to b when m is finite. Further, the minimum i will occur when $x' = x + \delta$. Therefore,

$$i = \left\lceil b \frac{v(1-x) - v(1-x-\delta)}{v(1-x)} \right\rceil$$

where $\lceil x \rceil$ is the minimum integer larger than or equal to x .

(ii) The minimum number of mutations by agents of population B required to make the members of population A best reply with x' , denoted by j , is such that,

$$\frac{j}{b} u(x') \geq u(x) \iff j \geq b \frac{u(x)}{u(x')}$$

since the minimum j occurs when $x' = 1 - \delta$,

$$j = \left\lceil b \frac{u(x)}{u(1-\delta)} \right\rceil$$

Applying an analogous reasoning, the minimum number of mutations required to transit towards convention $(x', 1-x')$, with $x' < x$ is given by

$$\min \left\{ \left\lceil b \frac{u(x) - u(x-\delta)}{u(x)} \right\rceil, \left\lceil b \frac{v(1-x)}{v(1-\delta)} \right\rceil \right\}$$

Under the assumption that u and v are concave this expression can be further simplified since in such a case,

$$\frac{v(1-x)}{v(1-\delta)} \geq \frac{u(x) - u(x-\delta)}{u(x)}$$

See Young (1993b) for the proof of this claim. Thus,

$$\min \left\{ \left\lceil b \frac{u(x) - u(x-\delta)}{u(x)} \right\rceil, \left\lceil b \frac{v(1-x)}{v(1-\delta)} \right\rceil \right\} = \left\lceil b \frac{u(x) - u(x-\delta)}{u(x)} \right\rceil$$

The minimum number of mutations required to escape from convention $(x, 1-x)$ is given by $r_\delta(x'|x)$, such that

$$r_\delta(x'|x) = \min \left\{ \left\lceil b \frac{v(1-x) - v(1-x-\delta)}{v(1-x)} \right\rceil, \left\lceil b \frac{u(x) - u(x-\delta)}{u(x)} \right\rceil, \left\lceil b \frac{u(x)}{u(1-\delta)} \right\rceil \right\}$$

Note that $r_\delta(x'|x)$ is either strictly increasing, strictly decreasing or first increasing and then decreasing, reaching at least one and at the most two adjacent maxima in the interval $[\delta, 1-\delta]$

Step 2. Let us now build the tree with minimum stochastic potential. Let x_δ^* be a maximum of $r_\delta(x'|x)$ and build the following tree, $T_{x_\delta^*}$, rooted at this convention, connecting the absorbing states as follows:

- (i) If $(x, 1-x)$ is an efficient convention such that $x < x_\delta^*$ and $\frac{u(x)}{u(1-\delta)} < \frac{v(1-x)-v(1-x-\delta)}{v(1-x)}$, put a direct edge connecting such convention with convention $(\delta, 1-\delta)$, which requires $\left\lceil b \frac{u(x)}{u(1-\delta)} \right\rceil$ mutations.
- (ii) If $(x, 1-x)$ is an efficient convention such that $x < x_\delta^*$ and $\frac{u(x)}{u(1-\delta)} > \frac{v(1-x)-v(1-x-\delta)}{v(1-x)}$, put a direct edge connecting such convention with convention $(x+\delta, 1-x-\delta)$. Such a transition requires $\left\lceil b \frac{v(1-x)-v(1-x-\delta)}{v(1-x)} \right\rceil$ mutations.
- (iii) If $(x, 1-x)$ is an efficient convention such that $x > x_\delta^*$, put a direct edge from this state to convention $(x-\delta, 1-x+\delta)$. This implies a cost of $\left\lceil b \frac{u(x)-u(x-\delta)}{u(x)} \right\rceil$ mutations.

In such a tree efficient conventions $(x, 1-x)$, where $x < x_\delta^*$ are connected either to $(x+\delta, 1-x-\delta)$ or to $(1-\delta, \delta)$, while efficient conventions where $x > x_\delta^*$ are connected to the adjacent efficient convention $(x-\delta, 1-x+\delta)$. This gives raise to a tree in which the root is given by the efficient convention $(x_\delta^*, 1-x_\delta^*)$.

We claim that this tree has minimum stochastic potential. To prove this claim, take another tree with root at state $s = (\hat{x}, 1-\hat{x})$, where $\hat{x}$ does not maximizes

$r_\delta(x'|\hat{x})$. In such a tree, T_s , there must be an outgoing edge from $(x_\delta^*, 1 - x_\delta^*)$ to some other state with resistance of at least $r_\delta(x'|x_\delta^*)$. In addition, for every other node $(x, 1 - x)$ the resistance of the outgoing edge in the new tree is at least equal to $r_\delta(x'|x)$. Therefore, $\gamma(T_s) \geq \gamma(T_{x_\delta^*}) + r_\delta(x'|x_\delta^*) - r_\delta(x'|\hat{x})$. Since $r_\delta(x'|x_\delta^*) - r_\delta(x'|\hat{x}) > 0$, our claim is proven. An absorbing state $s = (x, 1 - x)$ is stochastically stable if and only if it maximizes $r_\delta(x'|x)$ over Ω .

Step 3. Finally, it remains to prove the convergence towards the Nash Bargaining solution as $\delta \rightarrow 0$. First note that, even if initially there were two adjacent maxima, say $(x_\delta^*, 1 - x_\delta^*)$ and $(x_\delta^* + \delta, 1 - x_\delta^* - \delta)$, as δ tends to zero, the distance between these two elements will become negligible and in the limit there will be only one maximum, say $(x^*, 1 - x^*)$. For each precision δ , define the function

$$f_\delta(x'|x) = \frac{r_\delta(x'|x)}{\delta b}$$

such that

$$f_\delta(x'|x) = \min \left\{ \left[\frac{(v(1-x) - v(1-x-\delta))/\delta}{v(1-x)} \right], \left[\frac{(u(x) - u(x-\delta))/\delta}{u(x)} \right], \left[\frac{u(x)/\delta}{u(1-\delta)} \right] \right\}$$

Since $f_\delta(x'|x)$ is proportional to $r_\delta(x'|x)$, x maximizes $f_\delta(x'|x)$ if and only if it also maximizes $r_\delta(x'|x)$. Further, note that as $\delta \rightarrow 0$, the last element of $f_\delta(x)$ goes to infinite. Hence, we can restrict our attention to the expression:

$$\min \left\{ \left[\frac{(v(1-x) - v(1-x-\delta))/\delta}{v(1-x)} \right], \left[\frac{(u(x) - u(x-\delta))/\delta}{u(x)} \right] \right\}$$

Under differentiability of u and v (Young (1993b) also considers the more general case of subdifferentiability),

$$\lim_{\delta \rightarrow 0} f_\delta(x'|x) = \min \left\{ \frac{v'(1-x)}{v(1-x)}, \frac{u'(x)}{u(x)} \right\}$$

This function reaches its maximum when

$$\frac{v'(1-x)}{v(1-x)} = \frac{u'(x)}{u(x)}$$

which is a necessary and sufficient condition for x to be the unique maximum function

$$u(x)v(1-x)$$

Therefore, the root of the tree with minimum stochastic potential will be the efficient convention $(x^*, 1 - x^*)$, where this division is the (symmetric) Nash bargaining solution. ■

Proof. (Stochastic Stability under BR: scheme 1 as $m \rightarrow \infty$ and scheme 2). When $m \rightarrow \infty$ under scheme 1 or under scheme 2 the set of stochastically stable states is larger than in the case in which m is finite. Heterogeneous states $s \in \mathbf{NE}$ are also absorbing. Denote by $r(x'|a)$ the least resistance of moving from the absorbing state $a = (\mathbf{x}, \mathbf{y})$ to the efficient convention $(x', 1 - x')$. It is easy to see that $r(\bar{x}|a) = 1$, where $\bar{x} = \max x_i$, such that $x_i \in x$. Note that, by definition, in state a it must be that

$$x_i F(1 - x_i | \mathbf{y}) = x_k F(1 - x_k | \mathbf{y}), \quad \forall i, k \in A$$

where $F(1 - x_i | \mathbf{y})$ is the probability that x_i is compatible with a demand from vector $\mathbf{y}$. Since $\bar{x}$ is the maximum in vector x for the previous equality to hold we need that

$$F(1 - \bar{x} | \mathbf{y}) < F(1 - x_i | \mathbf{y}), \quad \forall i \in A \text{ such that } x_i < \bar{x}$$

just one mutation from a member of population B switching from $y_j > 1 - \bar{x}$ to $1 - \bar{x}$ is sufficient to provoke a change in the best reply of all the agents asking for an amount lower than $\bar{x}$. This proves our initial statement.

The analysis of the cheapest transitions between efficient conventions parallels the case in which m was finite. The only difference is that now the minimum number of demands that can be observed in each matching period from the individuals of the larger population (A), the previous $a^{\min}$, is equal to a rather than b . Thus,

$$r_\delta(x'|x) = \min \left\{ \left\lceil a \frac{v(1-x) - v(1-x-\delta)}{v(1-x)} \right\rceil, \left\lceil b \frac{u(x) - u(x-\delta)}{u(x)} \right\rceil, \left\lceil b \frac{u(x)}{u(1-\delta)} \right\rceil \right\}$$

The tree with minimum stochastic potential is constructed as before. Now, for the absorbing states $s = (\mathbf{x}, \mathbf{y}) \in \mathbf{NE} \setminus C^E$ we have to put a direct edge from s to convention $(\bar{x}, 1 - \bar{x})$, with $\bar{x}$ defined as above. The root, as before, is the efficient convention $(x_\delta^*, 1 - x_\delta^*)$ such that $r_\delta(x'|x)$ is maximized. As $\delta \rightarrow 0$, the division that maximizes the minimum resistance satisfies the following condition:

$$\frac{a}{a+b} \frac{v'(1-x^*)}{v(1-x^*)} = \frac{b}{a+b} \frac{u'(x^*)}{u(x^*)}$$

This division $(x^*, 1 - x^*)$ is just the generalized Nash bargaining solution with power $\frac{b}{a+b}$ for the individuals from population A and $\frac{a}{a+b}$ for the individuals from population B , which proves our claim. ■