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Abstract

In this paper we develop a differentiable approach to deal with incentives in a, possibly small, subset of a general domain of preferences in economies with one public and one private good. We show that, for two agents, there is no social rule which is efficient, nondictatorial and strategy-proof. For the case of more agents the same result occurs when nondictatorship is replaced by Individual Rationality or by Envy-Freeness. *Journal of Economic Literature.*

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Classification Numbers: D61, D78, H41.

1 Introduction

Consider an economy with one public good, one private good and with a finite number of individuals each with private information about his preferences. The assumption that these preferences are representable by an utility function which is linear in the private good - i.e. a Transferrable Utility (TU) domain- is made in a large number of papers concerning incentives in public good economies (e.g. Clarke [5], Groves [8], Groves and Loeb [9], Green and Laffont [11], Laffont and Maskin [15], Walker [27], Hurwicz and Walker [13], and Beviá and Corchón [3]).

One of the reasons for the popularity of this class of preferences is that they are very handy: The efficient quantity of the public good is determined independently of the private good allocated to individuals -thus, it makes sense to speak of the optimal level of the public good independently of how it is financed- and the distribution of the private good is independent of efficiency. Unfortunately, once we move away from a TU domain all these properties are lost: The efficient quantity of the public good depends on the consumption of the private good and therefore how the public good is financed matters for efficiency.

In this paper we consider the incentives problem in a domain in which, possibly, efficiency and distribution of the private good can not be separated. Our main concern is to verify the existence of allocation mechanisms that satisfy certain desirable properties such as efficiency, nondictorship, envy-freeness or individual rationality, in which each agent's space of preferences depends on a parameter that belongs to an arbitrary open subset of the real line and, for each agent, truthfully reporting his true preferences is a dominant strategy, i.e. strategy-proofness.

Having lost the insights obtained in the case where utility is linear in the private good we approach this problem with the help of differential calculus so we assume that all the relevant functions are smooth. This assumption is just a little bit stronger than continuity because continuous functions can be approximated by differentiable functions (see Bartle [1, p. 172]), and continuity seems like a sensible requirement in mechanism design (see e.g., Peleg [20] or Postlewaite and Wettstein [21]). Differentiability has been used extensively in the literature dealing with incentives in a TU domain.

Our first result is that with two agents, efficiency, non-dictatorship and strategy-proofness are incompatible in any domain that includes an open set in which preferences are representable by an utility function that is additively separable between the public and the private good, in which the subutility that refers to the public good takes certain form and in which the marginal rate of substitution between the public and the private good is a strictly monotonic function of the type of the agent, i.e. the so called single-crossing property of the indifference curves (Proposition 1).

When the number of agents is arbitrary, we have to strengthen the requirement of non-dictatorship. Our next result says that with more than two agents, efficiency, strategy-proofness and envy-freeness (see Foley [7]) are all incompatible requirements in *any* domain that includes an open set of preferences satisfying the single crossing property (Proposition 2). When envy-freeness is substituted by individual rationality we need a requirement of regularity -that implies the single-crossing property holds locally- but we only need to assume that the outcome is efficient for a specific profile of preferences. With these assumptions in hand we show that there is no strategy-proof and individually rational mechanism defined in *any* open set (Proposition 3).[1]

We stress that the domains of preferences used in this paper can include those representable by quasi-linear utility functions. Therefore, our results are also applicable to TU domains.

We now review other contributions to incentive compatibility in NTU domains with public goods. Ledyard and Roberts [17] showed that with two agents, there is no incentive compatible individually rational and efficient mechanism in a domain that include certain utility functions. Saijo [22] obtained a similar result strengthening the assumption of the domain but imposing a weaker individual rationality assumption and considering n person economies. Serizawa [25, pp. 503-8], building on Moulin [19], showed that there is no incentive compatible mechanism defined in the domain of all continuous, strictly increasing and quasi-concave utility functions that is Individually Rational, Non-Exploitative, Non-Bossy[2] and Efficient. Finally

[1] These results are similar to those in TU domains but the intuition behind them is different: In TU domains, the usual route ([13, p. 694] and [27, pp.1527-8]) is to show that Groves mechanisms are the only strategy-proof mechanisms ([10] and [12]) but they seldom balance the budget. The proof by Beviá and Corchón [3] hinges on the fact that the efficient quantity of the public good is independent of the consumption of the private good. These routes can not be taken in our case.

[2] In a Non-Bossy mechanism, no agent can affect the bundle consumed by any other agent without affecting the bundle consumed by her. Notice that, in our framework, either budget balance plus two agents or envy-free implies non-bossy.

Schummer [24] has shown results similar to ours if preferences are linear in both goods. He also has shown how to extend these results to domains that strictly include linear preferences in the case of two agents ([24, Corollary 2, p. 715]). Our results differ from those in two accounts: Assumptions -we only need a small domain but at the same time our assumptions on the form of utility functions are very weak- and methods -our approach is based on differential calculus that provides a general approach to deal with incentives in general NTU domains.

The closest paper to ours is by Satterthwaite and Sonnenschein [23]. They prove that any Non-Bossy, strategy-proof mechanism defined in an open set in which second order conditions of payoff maximization hold with strict inequality (Regularity) and satisfying certain technical properties must yield serial dictatorship (p. 590, Theorem 1). This result is neither implied, nor it implies our results: On the one hand the Non-Bossy assumption is implied by our assumptions in Propositions 1 (two agents plus budget balance) and 2 (envy-freeness) and their set up is more general than our's. On the other hand, our assumptions on the domain are generally weaker than those by SS, for instance our single-crossing assumption is implied by their Regularity condition (see our comments after Proposition 3), we dispose completely of the additional technical conditions that are difficult to interpret and obtain sharper results -Dictatorship, lack of Envy-Freeness or Individual Rationality- instead of Serial Dictatorship.

Several questions remain open. For instance the study of the performance of mechanisms when the equilibrium concept is Bayesian Equilibrium. Also, in a TU domain with more than two agents, there are strategy-proof, non-dictatorial and efficient mechanisms in restricted domains (see Groves and Loeb [9] and Tian [26]). We do not know if a similar result may occur in other subdomains of a NTU domain. All these are left for future research.

Two sections follow. Section 2 presents the model and some definitions. Section 3 presents our main results.

2 The Model

There are I agents. Let $y \in \mathbb{R}_+$ be the amount of the public good and $x_i \in \mathbb{R}_+$ the consumption of the private good (money). Let ω be the aggregate endowment of the private good. We will consider two cases: one in which each agent i privately owns an initial endowment of the private good

$\omega_i \in \mathbb{R}_{++}$ and so $\omega = \sum_i \omega_i$, and the case in which ω is publicly owned by all agents. An *allocation* is a vector $a = (y, x_1, ..., x_I)$.

We assume that all the relevant functions are twice continuously differentiable.

The preferences of agent i are representable by a concave utility function $U_i(y, x_i, \theta_i)$ with

$$\frac{\partial U_i(y, x_i, \theta_i)}{\partial y} > 0 \text{ and } \frac{\partial U_i(y, x_i, \theta_i)}{\partial x_i} > 0$$

and where θ_i, the type of agent i, is drawn from an interval $\Theta_i = (\underline{\theta}_i, \overline{\theta}_i) \subset \mathbb{R}$. In Propositions 1 and 2 we will assume that for all i and θ_i

$$\frac{\frac{\partial U_i(y,x_i,\theta_i)}{\partial y}}{\frac{\partial U_i(y,x_i,\theta_i)}{\partial x_i}}$$

is strictly monotonic in θ_i, i.e. the so-called Spence-Mirrless *single-crossing* property.[3] This property guarantees that for each agent and for any pair of preferences, her associated indifference curves cross just once (in Proposition 3 the single-crossing property will be derived from another assumption).

In Proposition 1 we consider preferences that are representable by a utility function of the form $U_i(a, \theta_i) = H(y) + v_i(x_i, \theta_i)$, where $H(\cdot)$ and $v_i(\cdot)$ are concave and strictly increasing, and $\frac{\partial v_i(\cdot,\theta_i)}{\partial x_i}$ is strictly monotonic in θ_i, for every i. These preferences are also representable by the utility function $U_i(a, \theta_i) = F_i(\theta_i) f(y) + g_i(x_i, \theta_i)$ with $F_i(\cdot) > 0$. Two examples of this class of utility functions are Cobb-Douglas, $U_i(a, \theta_i) = y^{\alpha_i(\theta_i)} x_i^{\beta_i(\theta_i)}$, or quasi-linear, $U_i(a, \theta_i) = \theta_i f(y) + x_i$ with $\theta_i > 0$.[4]

A vector $\theta = (\theta_1, \theta_2, ..., \theta_I)$ will be called an *economy.* The space of all possible economies is denoted by $\Theta = \prod_{i=1}^{I} \Theta_i$. The cost function, denoted by $c(\cdot)$, is also twice continuously differentiable with $c'(y) > 0$, and $c''(y) \geq 0$, for every y. The set of feasible allocations is

$$A \equiv \{(y, x_1, ..., x_I) : y \in \mathbb{R}_+, x_i \in \mathbb{R}_+ \text{ for all } i, \sum_{i=1}^{I} x_i + c(y) \leq \omega\}.$$

An allocation $(y, x_1, ..., x_I)$ is *Individually Rational* if $U_i(y, x_i, \theta_i) \geq U_i(0, \omega_i, \theta_i)$ for all i.

[3] This property plays an important role in other parts of economics, e.g. signalling.
[4] Not every quasi-linear utility function can be written in the way we assumed above.

A Social Choice Function (SCF) is a function $f : \Theta \to A$. We will write $f(\cdot) = (y(\cdot), x_1(\cdot), ..., x_I(\cdot))$. A SCF $f(\cdot)$ is *Budget Balanced* (BB) if,

$$\sum_{i=1}^{I} x_i(\theta) + c(y(\theta)) = \omega, \ \forall \theta \in \Theta.$$

A SCF $f(\cdot)$ is *Pareto Efficient* (PE) if there is no economy $\theta = (\theta_1, ..., \theta_I)$ and feasible allocation $a \in A$ such that $U_i(a, \theta_i) \geq U_i(f(\theta), \theta_i)$ for every i, and $U_i(a, \theta_i) > U_i(f(\theta), \theta_i)$ for some i. If an allocation is PE and interior (see Campbell and Truchon [4] and Conley and Diamantaras [6] for non-interior PE allocations) it must satisfy the Lindahl-Bowen-Samuelson condition

$$c'(y) = \sum_{i=1}^{I} \frac{\frac{\partial U_i(y, x_i, \theta_i)}{\partial y}}{\frac{\partial U_i(y, x_i, \theta_i)}{\partial x_i}},$$

which for $U_i(a, \theta_i) = H(y) + v_i(x_i, \theta_i)$ reads

$$c'(y) = \frac{\partial H(y)}{\partial y} \sum_{i=1}^{I} \frac{1}{\frac{\partial v_i(x_i, \theta_i)}{\partial x_i}}.$$

Let Δ be the $I-1$ unit simplex and $int(\Delta)$ be the interior of the $I-1$ unit simplex. Under our assumptions, a SCF $f(\cdot)$ is PE if there exist I functions $\alpha_1(\cdot), ..., \alpha_I(\cdot) \in \Delta$ such that, for every economy $\theta = (\theta_1, ..., \theta_I)$,

$$\begin{aligned} f(\theta) \ \in \ & \arg \max_{y, x_1, \ldots, x_I} \sum_{i=1}^{I} \alpha_i(\theta)[U_i(y, x_i, \theta_i)] \\ & \text{subject to} \sum_{i=1}^{I} x_i + c(y) = \omega. \end{aligned}$$

A Pareto efficient SCF is *non-dictatorial* if $\alpha_i(\cdot) \neq 1$, for any i and *non-exclusive* if in the above maximization $\alpha_1(\cdot), ..., \alpha_I(\cdot) \in int(\Delta)$. With $I = 2$ non-exclusive and non-dictatorial are equivalent. A SCF is *Envy-Free* if for each economy it selects an allocation in which no agent prefers the bundle consumed by any other agent (see Foley [7]). In our case it is equivalent to require that the consumption of all agents is the same. Envy-Free is an appropriate requirement for economies in which the property of the private good is common.[5] Finally, a SCF is *Individually Rational (IR)* if for each economy it selects individually rational allocations. This is an appropriate requirement for economies in which the property of the private good is private and agents have the option of not participating in the mechanism.

The previous concepts are illustrated by the example below which serves to appreciate the differences between a TU and a NTU domain.

Example 1 *Let* $I = 2$, $u_i = y + F(\theta_i)x_i^\beta$, $0 < \beta < 1$, $F(\theta_i) > 0$, *and* $c(y) = y$. *Assume*

$$\omega > 2(\frac{\beta F(\theta_1)F(\theta_2)}{F(\theta_1) + F(\theta_2)})^{\frac{1}{1-\beta}}.$$

The Lindahl-Bowen-Samuelson equation reads

$$x_2(\theta) = [\frac{[\beta F(\theta_1) - x_1(\theta)^{1-\beta}]F(\theta_2)}{F(\theta_1)}]^{\frac{1}{1-\beta}}.$$

Thus, in this case PE does not determine the level of the public good, but a locus of points (x_1, x_2). *If an allocation is Envy-Free and PE, we have that*

$$\begin{aligned} x_1(\theta) &= x_2(\theta) = (\frac{\beta F(\theta_1)F(\theta_2)}{F(\theta_1) + F(\theta_2)})^{\frac{1}{1-\beta}}, \\ y(\theta) &= \omega - 2(\frac{\beta F(\theta_1)F(\theta_2)}{F(\theta_1) + F(\theta_2)})^{\frac{1}{1-\beta}}. \end{aligned}$$

Let S_i denote the agent i's set of possible actions or messages.

A *mechanism* $\Gamma = (S_1, ..., S_I, g(\cdot))$ is a collection of I sets $(S_1, ..., S_I)$ and an outcome function $g : S_1 \times ... \times S_I \to A$.

A *strategy* for player i is a function $s_i : \Theta_i \to S_i$, that gives the action chosen by i for each type θ_i.

We also adopt the notational convention of writing

$$\begin{aligned} \theta_{-i} &= (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_I), \\ \theta &= (\theta_i, \theta_{-i}), \\ s_{-i}(\cdot) &= (s_1(\cdot), ..., s_{i-1}(\cdot), s_{i+1}(\cdot), ..., s_I(\cdot)), \\ s &= (s_i, s_{-i}), \\ S_{-i} &= S_1 \times ...S_{i-1} \times S_{i+1} \times ...S_I, \\ S &= S_i \times S_{-i}. \end{aligned}$$

[5] An alternative motivation for this property is that any anonymous and coalition-proof mechanism yields no-envy allocations, see Moulin [19, p.310].

The strategy profile $s^*(\cdot) = (s_1^*(\cdot), ..., s_I^*(\cdot))$ is a *dominant strategy equilibrium* of mechanism $\Gamma = (S_1, ..., S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$U_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq U_i(g(s_i', s_{-i}), \theta_i),$$

for all $s_i' \in S_i$ and all $s_{-i} \in S_{-i}$.

The mechanism $\Gamma = (S_1, ..., S_I, g(\cdot))$ *implements the SCF f in dominant strategies* if every dominant strategy equilibrium $s^*(\cdot) = (s_1^*(\cdot), ..., s_I^*(\cdot))$ of mechanism Γ is such that $g(s^*(\theta)) = f(\theta)$, for each economy $\theta \in \Theta$.

A *direct mechanism* is a mechanism in which $S_i = \Theta_i$ for all i and the outcome function is the SCF to be implemented, i.e. $g(\cdot) = f(\cdot)$.

The SCF $f(\cdot)$ is *dominant strategies incentive compatible* (DSIC) (or strategy-proof) if $s_i^*(\theta_i) = \theta_i$ is a dominant strategy equilibrium of the direct mechanism $\Gamma = (\Theta_1, ..., \Theta_I, f(\cdot))$ for all $\theta_i \in \Theta_i$ and $i = 1, ..., I$. By the Revelation Principle if a SCF is implementable in dominant strategies, it is DSIC.

Consider now the following example of a well-known mechanism.

Example 2 *(A voluntary contribution game).*[6] *Let $s_i = x_i$. Assume that $U_i(y, x_i, \theta_i) = y + v_i(s_i, \theta_i)$ and that the public good is produced under constant returns to scale, $y = k\sum_{i=1}^{I}(\omega_i - s_i)$. Payoff functions are*

$$k\sum_{i=1}^{I}(\omega_i - s_i) + v_i(s_i, \theta_i), \; i = 1, ..., I,$$

Assuming interiority, the first order condition of payoff maximization reads

$$k = \frac{\partial v_i(s_i, \theta_i)}{\partial s_i}.$$

Notice that: 1) The contribution game has a dominant strategy. 2) If $\frac{\partial v_i(\omega_i, \theta_i)}{\partial s_i} < k$, agent i makes a positive contribution in equilibrium and 3) the level of the public good is smaller than any level compatible with Pareto efficiency, i.e. agents free-ride (the first two assertions follow directly from first order condition of payoff maximization, a proof of the last assertion can be obtained from the authors upon request).[7] *We remark that in the standard quasi-linear*

[6] An overview of these games is provided by Bergstrom, Blume and Varian [2].

[7] These characteristics were first noticed by Keser [14] for a special case of the utility functions assumed in this example.

framework 1) and 2) are not true: In particular, only one agent makes a positive contribution (see e.g. Mas-Colell, Whinston and Green [18, pp. 361-2]). Thus, the preferences in this example are suited to capture situations where free riding does not take the extreme form that it does in a TU domain.[8]

3 Results

In this section we gather our main findings. Let us begin by proving three lemmas which characterize the properties of strategy-proof, non-exclusive and efficient SCF. In these lemmas, we assume that a SCF is defined in an open set Θ in which utility functions can be written as $H(y) + v_i(x_i, \theta_i)$, where $H(\cdot)$ and $v_i(\cdot)$ are concave and strictly increasing, the consumption of both the private and the public good is positive for each agent and $\frac{\partial v_i(\cdot,\theta_i)}{\partial x_i}$ is strictly monotonic in θ_i for every i, i.e. the single crossing assumption.

Lemma 1: *If a non-exclusive SCF $f(\cdot) = (y(\cdot), x_1(\cdot), ..., x_I(\cdot))$ is PE and DSIC, then there exist I functions $\alpha_1(\cdot), ..., \alpha_I(\cdot) \in int(\Delta)$ such that,*

$$\alpha_i(\theta) \sum_{j=1}^{I} \frac{\partial x_j(\theta)}{\partial \theta_i} = \frac{\partial x_i(\theta)}{\partial \theta_i}, \text{ for every } i = 1, ..., I \text{ and every } \theta \in \Theta. \quad (1)$$

Proof. If a non-exclusive SCF $f(\cdot) = (y(\cdot), x_1(\cdot), ..., x_I(\cdot))$ is PE, from the first order conditions of the maximization of $\sum_{i=1}^{I} \alpha_i(\theta)[H(y)+v_i(x_i, \theta_i)]$ over $\sum_{i=1}^{I} x_i + c(y) = \omega$, we have that for every $\theta \in \Theta$,

$$\alpha_i(\theta) \frac{\partial v_i(x_i(\theta), \theta_i)}{\partial x_i} = \frac{1}{c'(y(\theta))} \frac{\partial H(y)}{\partial y}, \text{ for every } i = 1, ..., I, \quad (2)$$

and

$$\sum_{i=1}^{I} x_i(\theta) + c(y(\theta)) = \omega. \quad (3)$$

[8] We may wonder if the voluntary contribution mechanism (VCM) is not Pareto dominated for other DSIC mechanism in a non-negligible subdomain of the domain considered in this example. Without further assumptions this conjecture is not true: Consider the confiscatory mechanism in which, no matter what, $y = w$. If the number of agents is large enough the confiscatory mechanism Pareto dominates the VCM because the loss of utility caused by zero consumption of money is compensated by the amount of the public good received in the confiscatory mechanism. Therefore, the question is: Are there non constant DSIC mechanisms that Pareto dominate VCM?

If $f(\cdot) = (y(\cdot), x_1(\cdot), ..., x_I(\cdot))$ is also DSIC, then, for every $\theta \in \Theta$ and i,

$$\frac{\partial H(y)}{\partial y}\frac{\partial y(\theta)}{\partial \theta_i} + \frac{\partial v_i(x_i(\theta), \theta_i)}{\partial x_i}\frac{\partial x_i(\theta)}{\partial \theta_i} = 0. \tag{4}$$

Now, by differentiating condition (3) with respect to θ_i we have that, for every $\theta \in \Theta$ and for every i,

$$\sum_{j=1}^{I} \frac{\partial x_j(\theta)}{\partial \theta_i} + c'(y(\theta))\frac{\partial y(\theta)}{\partial \theta_i} = 0. \tag{5}$$

From conditions (2), (4), and (5) the result follows. ■

Next two Lemmas deal with the two person case. In order to simplify notation, we will write $\alpha_1(\cdot) = \alpha(\cdot)$ and $\alpha_2(\cdot) = 1 - \alpha(\cdot)$.

Lemma 2. *Let $I = 2$. Suppose that $f(\cdot)$ is a non-dictatorial SCF that is PE and DSIC with associated function $\alpha(\cdot) \in int(\Delta)$. Then, the solution of the PDE (1) can be written as a $\mathbb{C}^1$* function of $\alpha(\cdot)$.

Proof. If $f(\cdot) = (y(\cdot), x_1(\cdot), x_2(\cdot))$ is a non-dictatorial SCF that is both PE and DSIC, then, by Lemma 1, the functions $x_1(\cdot)$ and $x_2(\cdot)$ have to satisfy the system of partial differential equations (1) for some function $\alpha(\cdot) \in int(\Delta)$, i.e. for every $\theta \in \Theta$:

$$\frac{\partial x_1(\theta)}{\partial \theta_1} = \frac{\alpha(\theta)}{1-\alpha(\theta)}\frac{\partial x_2(\theta)}{\partial \theta_1}, \tag{6}$$

$$\frac{\partial x_1(\theta)}{\partial \theta_2} = \frac{\alpha(\theta)}{1-\alpha(\theta)}\frac{\partial x_2(\theta)}{\partial \theta_2}. \tag{7}$$

If there exist functions $x_1(\cdot)$ and $x_2(\cdot)$ that solve the above equations, by differentiating with respect to θ_2 in (6) and with respect to θ_1 in (7), these functions have to satisfy

$$\begin{aligned}
\frac{\partial^2 x_1(\theta)}{\partial \theta_1 \theta_2} &= \frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial \theta_2}\frac{\partial x_2(\theta)}{\partial \theta_1} + \frac{\alpha(\theta)}{1-\alpha(\theta)}\frac{\partial^2 x_2(\theta)}{\partial \theta_1 \theta_2} \\
\frac{\partial^2 x_1(\theta)}{\partial \theta_2 \theta_1} &= \frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial \theta_1}\frac{\partial x_2(\theta)}{\partial \theta_2} + \frac{\alpha(\theta)}{1-\alpha(\theta)}\frac{\partial^2 x_2(\theta)}{\partial \theta_2 \theta_1}.
\end{aligned}$$

Since, for any pair of twice continuously differentiable functions $x_1(\cdot)$ and $x_2(\cdot)$, $\frac{\partial^2 x_1(\theta)}{\partial\theta_1\theta_2} = \frac{\partial^2 x_1(\theta)}{\partial\theta_2\theta_1}$ and $\frac{\partial^2 x_2(\theta)}{\partial\theta_1\theta_2} = \frac{\partial^2 x_2(\theta)}{\partial\theta_2\theta_1}$, we have that

$$\frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial\theta_2}\frac{\partial x_2(\theta)}{\partial\theta_1} - \frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial\theta_1}\frac{\partial x_2(\theta)}{\partial\theta_2} = 0, \tag{8}$$

which is a linear partial differential equation in x_2.

Also, write equations (6) and (7), as follows:

$$\frac{\partial x_2(\theta)}{\partial\theta_1} = \frac{1-\alpha(\theta)}{\alpha(\theta)}\frac{\partial x_1(\theta)}{\partial\theta_1}, \tag{9}$$

$$\frac{\partial x_2(\theta)}{\partial\theta_2} = \frac{1-\alpha(\theta)}{\alpha(\theta)}\frac{\partial x_1(\theta)}{\partial\theta_2}. \tag{10}$$

Then, by differentiating with respect to θ_2 in (9) and with respect to θ_1 in (10), and applying an analogous reasoning of symmetry of the second cross derivatives of $x_1(\cdot)$ and $x_2(\cdot)$, we get

$$\frac{\partial[\frac{1-\alpha(\theta)}{\alpha(\theta)}]}{\partial\theta_2}\frac{\partial x_1(\theta)}{\partial\theta_1} - \frac{\partial[\frac{1-\alpha(\theta)}{\alpha(\theta)}]}{\partial\theta_1}\frac{\partial x_1(\theta)}{\partial\theta_2} = 0, \tag{11}$$

which is a linear partial differential equation in x_1.

The solution to Equations (8) and (11) is found in Zachmanouglou and Thoe [28, pp. 62, Example 2.2 and Problem 2.3] and is given by

$$\begin{aligned} x_1(\theta) &= h_1(\alpha(\theta)) \\ x_2(\theta) &= h_2(\alpha(\theta)). \end{aligned}$$

Where $h_i(\cdot)$ is an arbitrary $\mathcal{C}^1$ function of a single variable and $\alpha(\theta) = c$ is the general solution of the ordinary differential equation associated with both Equation (8) and Equation (11).[9,10]

By BB we can also write

$$y(\theta) = g(\alpha(\theta)) = c^{-1}(\omega - h_1(\alpha(\theta)) - h_2(\alpha(\theta))).$$

[9] The bridge between the notation in Zachmanouglou and Thoe [28] and ours is: $x = \theta_1$, $y = \theta_2$, $a(x, y) = \frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial\theta_2}$, $b(x, y) = -\frac{\partial[\frac{\alpha(\theta)}{1-\alpha(\theta)}]}{\partial\theta_1}$, $z = x_1$ in (11) and $z = x_2$ in (8).

[10] By making some algebra, the ordinary differential equation can be written as $\frac{\partial\alpha(\theta)}{\partial\theta_1}d\theta_1 + \frac{\partial\alpha(\theta)}{\partial\theta_2}d\theta_2 = 0$ and trivially $\alpha(\theta) = c$ is its general solution.

All these establish the result. ■

Lemma 3. *Let $I = 2$. Suppose that $f(\cdot)$ is a non-dictatorial, PE and DSIC SCF with an associated function $\alpha \in int(\Delta)$. Then,* $\frac{\partial \alpha(\theta)}{\partial \theta_i} \neq 0$, $i = 1, 2$.

Proof. Suppose that $\frac{\partial \alpha(\theta)}{\partial \theta_i} = 0$, some i, say $i = 1$. By PE (Equation (2) in Lemma 1) and Lemma 2 we have that $\forall \theta \in \Theta$

$$\alpha(\theta)\frac{\partial v_1(h_1(\alpha(\theta)), \theta_1)}{\partial x_1} = \frac{1}{c'(g(\alpha(\theta)))}\frac{\partial H(g(\alpha(\theta)))}{\partial y}. \tag{2'}$$

Differentiating (2') with respect to θ_1 we see that the left hand side is different from zero, because the single-crossing property, i.e.

$$\alpha(\theta)[\frac{\partial^2 v_1(x_1, \theta_1)}{\partial x_1^2}h_1'(\alpha(\theta))\frac{\partial \alpha(\theta)}{\partial \theta_1} + \frac{\partial^2 v_1(x_1, \theta_1)}{\partial x_1 \theta_1}] = \alpha(\theta)\frac{\partial^2 v_1(x_1, \theta_1)}{\partial x_1 \theta_1} \neq 0,$$

$$\text{but since } \frac{\partial \alpha(\theta)}{\partial \theta_i} = 0 \text{ we have that } \frac{\partial(\frac{\frac{\partial H(g(\alpha(\theta)))}{\partial y}}{c'(g(\alpha(\theta)))})}{\partial \theta_1} = 0.$$

Therefore equation (2') does not hold for all θ and we obtain a contradiction. A similar argument can be done with θ_2 and so the proof follows. ■

With these lemmas in hand we prove the following result.

Proposition 1: *When $I = 2$ there is no PE, DSIC and non-dictatorial SCF defined on any open Θ in which utility functions can be written as $H(y) + v_i(x_i, \theta_i)$, where $H(\cdot)$ and $v_i(\cdot)$ are concave and strictly increasing, the single-crossing assumption holds and the consumption of the private and the public good is positive for each agent.*

Proof. Suppose there exists a SFC $f(\cdot) = (y(\cdot), x_1(\cdot), x_2(\cdot))$ that is PE, DSIC and non-dictatorial in some open set Θ with the properties required above. Then, by Lemmas 1 and 2, this SFC has to be of the form:

$$x_1(\theta) = h_1(\alpha(\theta)),\ x_2(\theta) = h_2(\alpha(\theta)),\ y(\theta) = g(\alpha(\theta)).$$

By Lemma 3 $\frac{\partial \alpha(\theta)}{\partial \theta_i} \neq 0$, $i = 1, 2$. Now consider a change in both θ_1 and θ_2 around θ that leave α unchanged, i.e.

$$\frac{d\theta_2}{d\theta_1} = -\frac{\frac{\partial \alpha(\theta)}{\partial \theta_1}}{\frac{\partial \alpha(\theta)}{\partial \theta_2}}.$$

But again (applying the same reasoning as in the proof of Lemma 3) the single-crossing assumption plus the fact that, although both θ_1 and θ_2 have changed α has not changed, imply that Equation (2') can not hold. ■

As pointed out by Groves and Loeb [9], PE, non-dictatorship and DSIC are compatible in small domains when $I > 2$. Thus, in economies with more than two agents we strengthened the requirement of non-dictatorship to that of Envy-Freeness or Individual Rationality.

Proposition 2: *When $\#I > 2$, there is no PE, DSIC and Envy-Free SCF defined on any open set Θ in which the single-crossing assumption holds and the consumption of the private and the public good is positive for each agent.*

Proof. Clearly, if an allocation is Envy-Free it is non-exclusive. If a non-exclusive SCF $f(\cdot) = (y(\cdot), x_1(\cdot), ..., x_I(\cdot))$ is both PE and DSIC then, reasoning like in Lemma 1 we find that for every $\theta \in \Theta$ and $i = 1, ..., I$,

$$\alpha_i(\theta)\frac{\partial U_i(y, x_i, \theta_i)}{\partial y}\sum_{j=1}^{I}\frac{\partial x_j(\theta)}{\partial \theta_i} = \frac{\partial x_i(\theta)}{\partial \theta_i}\sum_{j=1}^{I}\alpha_j(\theta)\frac{\partial U_j(y, x_i, \theta_i)}{\partial y}.$$

As we noticed before, envy-freeness implies that $x_i(\theta) = x(\theta)$, for every i and θ. Therefore, the previous equation becomes

$$\alpha_i(\theta)I\frac{\partial U_i(y, x, \theta_i)}{\partial y}\frac{\partial x(\theta)}{\partial \theta_i} = \frac{\partial x(\theta)}{\partial \theta_i}\sum_{j=1}^{I}\alpha_j(\theta)\frac{\partial U_j(y, x, \theta_i)}{\partial y}, \text{ for every } i \text{ and } \theta.$$

By PE, we get

$$\alpha_i(\theta)\frac{\partial U_i(y, x, \theta_i)}{\partial x_i}c'(y) = \sum_{j=1}^{I}\alpha_j(\theta)\frac{\partial U_j(y, x, \theta_i)}{\partial y}.$$

Since $\alpha_i(\theta) \neq 0$, the two previous equations imply that

$$I\frac{\partial U_i(y, x, \theta_i)}{\partial y}\frac{\partial x(\theta)}{\partial \theta_i} = \frac{\partial x(\theta)}{\partial \theta_i}\frac{\partial U_i(y, x, \theta_i)}{\partial x_i}c'(y) \text{ for every } i \text{ and } \theta.$$

Suppose that there is an economy θ for which $\frac{\partial x(\theta)}{\partial \theta_i} \neq 0, \forall i = 1, ..., I$. By continuity of $x(\cdot)$ there is a set $\mathcal{V}$, for which $\frac{\partial x(\hat{\theta})}{\partial \theta_i} \neq 0, \forall \hat{\theta} \in \mathcal{V}$. >From the previous equation we obtain that

$$\frac{I}{c'(y)} = \frac{\frac{\partial U_i(y,x,\hat{\theta}_i)}{\partial x_i}}{\frac{\partial U_i(y,x,\hat{\theta}_i)}{\partial y}} = \frac{\frac{\partial U_j(y,x,\hat{\theta}_j)}{\partial x_j}}{\frac{\partial U_j(y,x,\hat{\theta}_j)}{\partial y}}, \text{ for every } i \neq j, \text{ for all } \widehat{\theta} \in \mathcal{V}.$$

Perturbing θ_j and $\theta_k, k, j \neq i$ in such a way that $x(\hat{\theta})$ -and therefore $y(\hat{\theta})-$ remains constant we see that the single-crossing property implies that the previous equation can not hold. Thus $\frac{\partial x(\theta)}{\partial \theta_i} = 0$ some i, and by DSIC

$$\frac{\partial U_i(y, x_i, \theta_i)}{\partial y}\frac{\partial y(\theta)}{\partial \theta_i} + \frac{\partial U_i(y, x_i, \theta_i)}{\partial x_i}\frac{\partial x(\theta)}{\partial \theta_i} = 0,$$

we have that $\frac{\partial y(\theta)}{\partial \theta_i} = 0$ for some i.

Consider now the Lindahl-Bowen-Samuelson condition,

$$c'(y(\theta)) = \sum_{i=1}^{I} \frac{\frac{\partial U_i(y,x_i,\theta_i)}{\partial y}}{\frac{\partial U_i(y,x_i,\theta_i)}{\partial x_i}},$$

and let a small variation in, say θ_i. But since $\frac{\partial x(\theta)}{\partial \theta_i} = \frac{\partial y(\theta)}{\partial \theta_i} = 0$, the single-crossing property is contradicted. ■

Finally we substitute the requirement of f being Envy-Free by the assumption that f is Individually Rational (IR).

We need some extra definitions and notation. A SCF f is *Weakly Regular* (WR) at θ' if there is an agent, say i, for whom to tell the truth implies that second order conditions of payoff maximization hold with strict inequality at θ'. Let the first order condition of IC be written as

$$\frac{\partial U_i(y(\theta), x_i(\theta), \tilde{\theta}_i)}{\partial y}\frac{\partial y(\theta)}{\partial \theta_i} + \frac{\partial U_i(y(\theta), x_i(\theta), \tilde{\theta}_i)}{\partial x_i}\frac{\partial x_i(\theta)}{\partial \theta_i} \equiv \Omega(\theta, \tilde{\theta}_i) = 0 \quad (12)$$

where θ is the economy announced by agents and $\tilde{\theta}_i$ is the true type of i. With this notation in hand, f is WR at θ' iff

$$\exists i, \text{ with } \frac{\partial \Omega(\theta', \tilde{\theta}_i)}{\partial \theta_i} < 0, \text{ when } \tilde{\theta}_i = \theta'_i.$$

The WR condition is a much weaker version of an assumption called *Regularity* (R) by Satterthwaite and Sonnenschein [23, p. 590]. R requires that second order conditions hold strictly for *all* economies and *all* agents. The relationship of R and WR with the single-crossing property is deferred after the proof of our next result.

Finally, let $\hat{\theta}$ be an economy such that $(0, \omega_1, \omega_2, ..., \omega_I)$ is PE and there is, at least one agent with non linear indifference curves.

Proposition 3: *There is no DSIC and IR SCF defined on an open set* Θ *with* $\hat{\theta} \in \Theta$ *that is WR and PE at* $\hat{\theta}$.

Proof. We first see that if $f(\hat{\theta})$ is a PE allocation, it must be that $f(\hat{\theta}) = (0, \omega_1, \omega_2, ..., \omega_I)$. Notice that any other PE allocation is such that either some agents have less utility than in $(0, \omega_1, \omega_2, ..., \omega_I)$, which contradicts that f is IR or such that all agents are indifferent. In the latter case, a convex combination of this allocation and $(0, \omega_1, \omega_2, ..., \omega_I)$ can improve the utility of, at least, one individual, contradicting that $(0, \omega_1, \omega_2, ..., \omega_I)$ is PE.

Since f is DSIC for all $\theta \in \Theta$, differentiating (12) we have that

$$\frac{\partial \Omega(\theta, \tilde{\theta}_i)}{\partial \theta_i} + \frac{\partial \Omega(\theta, \tilde{\theta}_i)}{\partial \tilde{\theta}_i} = 0 \text{ for } \tilde{\theta}_i = \theta_i.$$

Since f is WR at $\hat{\theta}$ we have that for $\tilde{\theta}_i = \hat{\theta}_i$, $\frac{\partial \Omega(\hat{\theta}, \tilde{\theta}_i)}{\partial \theta_i} < 0$ and thus,

$$\frac{\partial \Omega(\hat{\theta}, \tilde{\theta}_i)}{\partial \tilde{\theta}_i} \equiv \frac{\partial^2 U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial y \tilde{\theta}_i} \frac{\partial y(\hat{\theta})}{\partial \theta_i} + \frac{\partial^2 U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i \tilde{\theta}_i} \frac{\partial x_i(\hat{\theta})}{\partial \theta_i} > 0.$$

If $\frac{\partial x_i(\hat{\theta})}{\partial \theta_i} = 0$, Equation (12) above imply that $\frac{\partial y(\hat{\theta})}{\partial \theta_i} = 0$ and the previous equation can not hold. Thus, $\frac{\partial x_i(\hat{\theta})}{\partial \theta_i} \neq 0$. Therefore, there is a θ_i' in a neighborhood of $\hat{\theta}_i$ such that $x_i(\theta_i', \hat{\theta}_{-i}) > x_i(\hat{\theta}) = \omega_i$. Let,

$$f(\theta_i', \hat{\theta}_{-i}) = (y(\theta_i', \hat{\theta}_{-i}), x_1(\theta_i', \hat{\theta}_{-i}), x_2(\theta_i', \hat{\theta}_{-i}), ..., x_I(\theta_i', \hat{\theta}_{-i})).$$

Since $y(\theta_i', \hat{\theta}_{-i}) \geq 0$ and $x_i(\theta_i', \hat{\theta}_{-i}) > \omega_i$,

$$U_i(y(\theta_i', \hat{\theta}_{-i}), x_i(\theta_i', \hat{\theta}_{-i}), \hat{\theta}_i) > U_i(0, \omega_i, \hat{\theta}_i) = U_i(f(\hat{\theta}), \hat{\theta}_i).$$

Since f is IR, $\forall j \neq i$,

$$U_i(y(\theta_i', \hat{\theta}_{-i}), x_j(\theta_i', \hat{\theta}_{-i}), \hat{\theta}_j) \geq U_j(0, \omega_j, \hat{\theta}_j) = U_j(f(\hat{\theta}), \hat{\theta}_j).$$

But this contradicts that f is PE at $\theta = \hat{\theta}$. $\blacksquare$

Notice that, in contrast with other results in the literature, we only assume that f selects PE allocations at $\theta = \hat{\theta}$. Also, the WR assumption can be disposed with at the cost of considering a special domain of preferences (a proof is available under request).

We now prove that DSIC and WR imply that the single-crossing assumption holds around $\hat{\theta}_i$ for agent i. A similar argument can be used to show that DSIC and R implies the single crossing assumption.

Notice that FOC of DSIC can be written as

$$\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i}(MRS_i(\hat{\theta}, \tilde{\theta}_i)\frac{\partial y(\hat{\theta})}{\partial \theta_i} + \frac{\partial x_i(\hat{\theta})}{\partial \theta_i}) \equiv \Omega(\hat{\theta}, \tilde{\theta}_i) = 0,$$

$$\text{where } MRS_i(\hat{\theta}, \tilde{\theta}_i) \equiv \frac{\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial y}}{\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i}}.$$

Recall from the proof of Proposition 3 that DSIC and WR imply that $\frac{\partial \Omega(\hat{\theta}, \tilde{\theta}_i)}{\partial \tilde{\theta}_i} > 0$ for $\tilde{\theta}_i = \hat{\theta}_i$. Thus,

$$\frac{\partial \Omega(\hat{\theta}, \tilde{\theta}_i)}{\partial \tilde{\theta}_i} = \frac{\partial^2 U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i \tilde{\theta}_i}(MRS_i(\hat{\theta}, \tilde{\theta}_i)\frac{\partial y(\hat{\theta})}{\partial \theta_i} + \frac{\partial x_i(\hat{\theta})}{\partial \theta_i})+$$

$$+\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i}\frac{\partial MRS_i(\hat{\theta}, \tilde{\theta}_i)}{\partial \tilde{\theta}_i}\frac{\partial y(\hat{\theta})}{\partial \theta_i} > 0, \text{ for } \tilde{\theta}_i = \hat{\theta}_i.$$

Notice that $\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i} > 0$ implies that FOC of DSIC are satisfied if and only if

$$MRS_i(\hat{\theta}, \tilde{\theta}_i)\frac{\partial y(\hat{\theta})}{\partial \theta_i} + \frac{\partial x_i(\hat{\theta})}{\partial \theta_i} = 0.$$

Then, from the inequality above,

$$\frac{\partial U_i(y(\hat{\theta}), x_i(\hat{\theta}), \tilde{\theta}_i)}{\partial x_i}\frac{\partial MRS_i(\hat{\theta}, \tilde{\theta}_i)}{\partial \tilde{\theta}_i}\frac{\partial y(\hat{\theta})}{\partial \theta_i} > 0,$$

and this implies $\frac{\partial MRS_i(\hat{\theta}, \tilde{\theta}_i)}{\partial x_i} \neq 0$, i.e. that the single-crossing condition holds around $\hat{\theta}_i$ for agent i.

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