A discusión

EGALITARIAN RULES IN CLAIMS PROBLEMS WITH INDIVISIBLE GOODS*

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ABSTRACT

In this work we deal with rationing problems. In particular with claims problems with indivisible goods, that is, problems in which a certain amount of indivisible units (of an homogeneous good), has to be distributed among a group of agents, when this amount is not enough to satisfy agents' demands. We define discrete rules to solve those problems that involve notions of fairness similar to those supporting the constrained-equal awards and the constrained-equal losses rules in the continuous case. Axiomatic characterizations of those solutions are provided.

JEL classification numbers: D63.

Key words: indivisible goods, claims problems, equal awards solution, equal losses solution.

1 Introduction

A claims problem represents a situation in which a quantity of a certain commodity has to be distributed among some agents and the available resources fall short of total demand. The canonical example of this kind of problems is that in which a firm goes to bankruptcy, that is, the amount it owes to its creditors is greater than the firm's worth. In this problem, in general, a judge has to liquidate the firm and decide how to distribute the amount gotten in the liquidation among the creditors. In this example, the good to be distributed is perfectly divisible. Nonetheless, there are many claims situations involving the distribution of a commodity coming in indivisible units.

Consider the following examples: In order to carry out the administrative tasks at the university departments, a University contracts a certain number of secretaries. On the one hand, this number depends on the financial capabilities of the University. On the other hand, each department demands, depending on its volume, a certain number of secretaries. It happens that the total number of secretaries the departments demand is larger than the available amount. How many secretaries correspond to each department? Another example is the case of renting cars firms: some months the demands of new cars from these firms to the cars manufacturer is so high that the production in that month is not enough to satisfy the whole demand. The manufacturer must decide how to distribute the available cars among the renting firms. Consider also the distribution of radio frequencies among the different broadcasting corporations, whenever there is no an auction mechanism. If the amount of frequencies requested by the firms is too large, the Government should decide how many frequencies are allotted to each corporation.

There is also a particular type of indivisible goods claims problems in which each claimant demands, at most, one unit of the good. This is the case of the waiting lists at hospitals: a certain number of patients demand an operation or transplant each, but it is not possible to carry all the operations out, or the number of available organs is not enough.

In claims problems, the agents are referred to as the creditors or *claimants*, the amount to be distributed is called the *estate* and the creditors' demands are called the *claims*. A *rule* is a function that distributes the estate among the creditors according to their claims, that is, a method to solve claims problems. The first and fundamental approach to this formulation is O'Neill (1982).

Traditionally, the axiomatic literature related with claims problems has focused on situations in which the estate is either an amount of money (and then infinitely divisible) or an indivisible good that can be valuated in monetary terms. In the axiomatic method characterizations of the proposed rules by using some intuitive properties are obtained. The reader is referred to the surveys by Moulin (2001) and Thomson (2003). Two of the most well-known rules in the divisible case are the *constrained-equal awards* and *constrained-equal losses* rules, that correspond, respectively, to the idea of equally divide among the agents absolute awards and absolute losses. This two distinct concepts of equality have a long history and have been advocated by many authors, including Maimonides (12 century) [Aumann and Maschler (1985)]. Characterizations of these rules appear in Moulin (1985), Chun (1988), Young (1988), Dagan (1996), Herrero & Villar (2001) (2002), and Yeh (2004).

Many indivisible claims problems are solved by using priority methods (see Moulin (2000)). The use of priority orderings has also been proposed as a way of obtaining no anonymous solutions in the continuous case. The claimants arrive one at a time and they are fulfilled until the good is run out. Obviously the final allocation depends on the arrival order. A way of recovering anonymity consists of taking the average over all possible arrival orders of the claimants. This procedure gives rise to the so called *random arrival* solution. In the divisible goods case, the amounts recommended by the random arrival solution can, in fact, be allotted to the claimants. In the indivisible goods case, nonetheless, this solution only can be interpreted as an "ex ante" solution, in expected terms, since the final realization is just one among the different possible orderings. Pure priority methods, without randomization, are normally used in the allocation of tickets, elective surgery when there is a waiting list, or the allocation of organs in transplant problems (see Young (1994)).

Moulin (2000) analyzes the family of rules fulfilling three procedural axioms: *consistency* (with respect to variations of the set of agents), *composition up* and *composition down* (with respect to variations of the available resources). In the continuous case, many rules satisfy these properties, including the constrained equal awards and constrained equal losses. Nonetheless, in the case where the commodity comes in indivisible units, the three axioms characterize the family of *priority rules,* where individual demands are met lexicographically according to an exogeneous ordering of the agents. That is, the combination of the three axioms leave no room for any degree of compromise.

Nonetheless, there are natural mechanisms that allocate an approximate egalitarian division of the commodity (or, alternatively, are approximately egalitarian in losses). Moulin himself (2000) mentions some of such a mechanisms: we allocate one unit of the good to each claimant up to the moment in which the smallest claimant is fully satisfied; then we continue allocating one unit of the good to each of the remaining claimants up to the moment in which the second smallest agent is satisfied, an so on. Thus, at some moment, some units are left, but we cannot allocate one unit to each of the remaining agents. What we do then is to use a priority ordering of the agents to allocate the remaining units.

As Moulin points out, previous mechanisms satisfies *consistency* and *composition down,* but fails to satisfy *composition up.*

Previous family of rules (parameterized by the ordering of the set of agents) reflect a notion of fairness similar to the ideas supporting, in the continuous case, the constrained-equal awards. Similarly, another family reflecting the ideas of the constrained-equal losses rules can be defined.

Even though the rules in those families fail to satisfy one of the procedural axioms - composition up or down-, we may ask if they still fulfill some alternative properties satisfied by the continuous egalitarian rules. Thus, we focus in this paper in the axiomatic analysis of approximately egalitarian rules for the discrete claims model.

It happens that the idea of duality, and many of the properties used in the continuos case can be translated into this setting, and many results are recovered, so that we are able to obtain characterization results for our discrete egalitarian rules very much related to characterization results for the egalitarian continuous rules.

The rest of the paper is structured as follows: In Section 2 we set up the claims problems with indivisible goods and the notion of solution or rule for the continuous case and the discrete one. In Section 3 we introduce priority orders and we use them to construct egalitarian solutions in the discrete case. Section 4 is devoted to the properties our rules fulfil. In Section 5 we present our characterization results. Section 6, with final comment and remarks, concludes.

2 Preliminaries

Let $\mathbb{N}$ be the set of all potential agents. We denote by $\mathcal{N}$ the family of all finite subsets of $\mathbb{N}$. A claims problem is a tern (N, E, c), where $N \in \mathcal{N}$, $n = |N|$, is a set of agents or claimants, $c \in \mathbb{Z}_+^n$ is the vector of claims or demands[1] (c_i denotes the ith agent's claim), and $E \in \mathbb{Z}_{++}$ represents the estate or amount to be distributed among the agents. The fact that the estate is not enough to satisfy the demands means that $\sum_{i=1}^n c_i > E$. $\mathbb{B}_Z^N$ denotes the set of all claims problems with the fixed set of claimants N, and $\mathbb{B}_Z$ represents the set of all possible problems with variable population.

$$\mathbb{B}_Z^N = \left\{ (N, E, c) \in \mathbb{Z}_{++} \times \mathbb{Z}_+^n : \sum_{i=1}^n c_i > E \right\}$$

[1] $\mathbb{Z}$ represents the set of integer numbers.

$$\mathbb{B}_Z = \bigcup_{N \in \mathcal{N}} \mathbb{B}^N$$

For a given problem (N, E, c) we denote by C and L the total claim and loss respectively:

$$C = \sum_{i=1}^{n} c_i, \qquad L = C - E$$

A rule is a way to distribute the estate among the agents according to their demands.

Definition 2.1 *A* ***continuous rule*** *is a function φ that associates with every $(N, E, c) \in \mathbb{B}_Z$ a unique allocation $\varphi(N, E, c) \in \mathbb{R}^n_+$ such that*

(a) $0 \leq \varphi_i(N, E, c) \leq c_i$, $\forall i \in N$

(b) $\sum_{i \in N} \varphi_i(N, E, c) = E$

The first condition says that nobody can get neither more than she asks for nor a negative outcome; and the second one sets that the estate is entirely distributed.

Three of the most traditional rules in the divisible good case are the proportional, constrained equal awards and constrained equal losses rules. The proportional solution distributes the estate among the claimants proportionally to their demands.

Definition 2.2 *For all $(N, E, c) \in \mathbb{B}_Z$ we define the* ***continuous proportional rule*** *as*

$$p_i(N, E, c) = \lambda c_i,$$

where λ is such that $\sum\limits_{i \in N} p_i(N, E, c) = E$.

The underlying idea of the constrained-equal awards rule is to treat the claimants equally, independently of the differences in claims. All agents receive the same amount provided that this is not higher than her claim.

Definition 2.3 *For all $(N, E, c) \in \mathbb{B}_Z$ we define the* ***continuous constrained equal awards rule*** *as*

$$cea_i(N, E, c) = \max\{c_i, \lambda\}$$

where λ is such that $\sum\limits_{i \in N} cea_i(N, E, c) = E$.

Analogously to the constrained-equal awards rule, the constrained-equal losses rule treats claimants equally with respect to their losses, independently on the differences in claims. Each agent should loss the same amount as the rest of the agents provided that this amount were smaller than her claim.

Definition 2.4 *For all* $(N, E, c) \in \mathbb{B}_Z$ *we define the* ***continuous constrained equal losses rule*** *as*

$$cel_i(N, E, c) = \min\{0, c_i - \lambda\}$$

where λ *is such that* $\sum\limits_{i \in N} cel_i(N, E, c) = E$.

Similarly to the definition of continuous rules, we can define solutions for claims problem with indivisible goods. The difference is that now we impose the allocations to be integer numbers.[2]

Definition 2.5 *A* ***discrete rule*** *is a function* Φ *that associates with every* $(N, E, c) \in \mathbb{B}_Z$ *a unique allocation* $\Phi(N, E, c) \in \mathbb{Z}^n_+$ *such that*

(a) $0 \leq \Phi_i(N, E, c) \leq c_i, \forall i \in N$

(b) $\sum_{i \in N} \Phi_i(N, E, c) = E$

3 σ-Discrete Egalitarian Rules

Let σ be a linear order (a complete, transitive and asymmetric binary relation) on the set of potential agents $\mathbb{N}$. We say that an agent i has priority or is preferred to another agent j whenever $i\sigma j$. We denote by $-\sigma$ the opposite order ($i(-\sigma)j \Leftrightarrow j\sigma i$). Let Ω denote the set of all possible linear orderings on $\mathbb{N}$.

We face the problem of allocating a certain amount of an indivisible good among a set of agents, so that the final allocation is as egalitarian as possible. Consider the following example: $N = \{x, y, z\}$, $E = 9$ and $c = (2, 6, 6)$. We start by given one unit of the good to each of the agents. Still no one is fully satisfied, and 6 units are left. We then give an additional unit to each of the agents. Now, agent x is satisfied, and still 3 units are left. We allot then one unit to each of the remaining claimants, y and z, and still 1 unit is left, but it happens that neither x nor y are satisfied. The problem is now how to allocate the remaining unit. Clearly, two options are open, either to give it to y or to z. The decision

[2] We use small letters for denoting the continuous rules and capital ones for the discrete case.

of allocating this extra unit to one of the remaining agents can be made at random or by using some idea of priority among them. Consider then a linear order $\sigma \in \Omega$. We can allocate the remaining unit according to σ, so that, if $y\sigma z$, then it is agent y the one enjoying the extra unit, that is, the allocation would be $(2, 4, 3)$. Otherwise, it is agent z, and then the allocation would be $(2, 3, 4)$.

Previous procedure indicates that, for each linear order $\sigma \in \Omega$, we have a particular discrete rule. Since the rationale of all those rules is similar to that of the constrained equal awards rule in the continuous case, we will call them all *discrete constrained equal awards rules.* It is easy to see that the allocations recommended by those rules can be obtained according to the following procedure: We may assume that the allocation process takes place in two stages. Let $(N, E, c) \in \mathbb{B}_Z$, in the first one each agent $i \in N$ receives $\lfloor cea_i(N, E, c) \rfloor$ units,[3] that is, the whole part of her corresponding allocation, under the continuous constrained equal awards rule, if the estate were completely divisible. If in this stage some units are still left, ($E' = E - \sum_{i \in N} \lfloor cea_i(N, E, c) \rfloor > 0$) we go to the second step.[4]

Now, we can distinguish two kinds of claimants: those who have already received an integer amount according to *cea*, i.e., $cea_i(N, E, c) \in \mathbb{Z}_+$ (and then $\lfloor cea_i(N, E, c) \rfloor = cea_i(N, E, c)$); and those agents whose allocation is not an integer number. Let us denote by $Q(cea; N, E, c)$ this last group of agents: $Q(cea; N, E, c) = \{j \in N : cea_j(N, E, c) \notin \mathbb{Z}_+\}$. Let $q = |Q(cea; N, E, c)|$.

Let $\sigma \in \Omega$. In the second stage we distribute the E' remaining units among some agents in $Q(cea; N, E, c)$ according to the order σ; we give one and only one unit to each of the E' claimants with the highest priority in $Q(cea; N, E, c)$. Let $Q^\sigma(cea; N, E, c)$ be the ordered set $Q(cea; N, E, c)$ with the restriction of σ. We denote by $Q^\sigma_a(cea; N, E, c)$ the set of the a first agents in $Q^\sigma(cea; N, E, c)$.

Thus, we may formally define, for each $\sigma \in \Omega$, the *σ-discrete constrained equal awards rule* as follows:

Definition 3.1 *Let $\sigma \in \Omega$. Then, for all $(N, E, c) \in \mathbb{B}_Z$ we define the **σ-discrete constrained equal awards rule** as*

$$CEA^\sigma_i(e) = \begin{cases} \lfloor cea_i(N, E, c) \rfloor + 1 & \text{if } i \in Q^\sigma_{E'}(cea; N, E, c) \\ \lfloor cea_i(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor cea_i(N, E, c) \rfloor > 0$.

[3] For any $x \in \mathbb{R}_+$, $\lfloor x \rfloor$ denotes the largest integer number s.t. $\lfloor x \rfloor \leq x$.

[4] If no unit remains it is due to the fact that the allocation under the continuous rule is an integer share by now.

Similarly, we may wish to solve allocations problems of this type so that agents' losses are as equal as possible. Consider the following example: $N = \{x, y, z\}$, $c = (2, 4, 6)$ and $E = 5$. We now propose the following mechanism. Since the total demand is 12, and there are only 5 units to share, agents should lose 7 units in aggregate. We start by given all agents their full demand, and then we subtract one unit to each agent. Still the allocation is not feasible. Thus, we take out an additional unit to each agent, so that they have $(0, 2, 4)$. Still, an additional unit should be subtracted. Since agent x has an allotment of zero units, we cannot take out any additional unit from x. Consequently, this unit should be subtracted either from y or from z. As before, we may choose between y and z at random or either by using some priority order. Keeping in mind that the priority order should favor those agents with higher priority, if $y\sigma z$, then we take out the unit from z, otherwise, it is taken out from y.

Again, for each ordering of the set of potential agents we have a different discrete rule, all of them sharing the rationale of the *constrained equal losses rule.* As before, for a given order $\sigma \in \Omega$, the allocation recommended by previous procedure can be obtained in two steps. In the first one each agent $i \in N$ receives $\lfloor cel_i(N, E, c) \rfloor$ units. Now, if still some units remain ($E' = E - \sum_{i \in N} \lfloor cel_i(N, E, c) \rfloor > 0$) we go to the second step. Again, we divide the set of claimants into two groups: those who have already obtained a whole allocation and those who do not. $Q(cel; N, E, c) = \{j \in N : cel_j(N, E, c) \notin \mathbb{Z}_+\}$ denotes this last subset of agents. In the second stage we distribute the E' remaining units among some agents in $Q(cel; N, E, c)$ according to the order σ; we give one and only one unit to each of the E' claimants with the highest priority in $Q(cel; N, E, c)$.

As before, we may formally define, for each $\sigma \in \Omega$, the *σ-discrete constrained equal losses rule* as follows:

Definition 3.2 *Let $\sigma \in \Omega$. Then, for all $(N, E, c) \in \mathbb{B}_Z$ we define the **σ-discrete constrained equal losses rule** as*

$$CEL_i^\sigma(e) = \begin{cases} \lfloor cel_i(N, E, c) \rfloor + 1 & \text{if } i \in Q_{E'}^\sigma(cel; N, E, c) \\ \lfloor cel_i(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor cel_i(N, E, c) \rfloor > 0$.

Previously we stated the formal relationship between the continuos constrained equal awards and constrained equal losses rules with our discrete versions. Consider now a particular problem (N, E, c), involving the set of agents N. Given an order $\sigma \in \Omega$, let us call σ_N the restriction of σ to N. If $n = |N|$, there are only $n!$ different orders σ_N on N. And thus, under the perspective of the discrete egalitarian rules described above, there are only $n!$ alternative recommendations to distribute E among the N claimants in the problem. Assume that, as in the *random arrival rule,* the order σ_N corresponds to the arrival order of the agents, and that all orderings are equally likely. Now, for each arrival order, we apply the discrete CEA^{σ_N} (or equivalently, CEL^{σ_N}). Then, it turns out that the average of the allocations obtained when the arrival orders change are simply the allocations recommended by the continuous *cea* and *cel* rules. This result is stated in the following Proposition.

Proposition 3.1 *Let $(N, E, c) \in \mathbb{B}_Z$. For any order σ_N on the set of agents N, let $CEA^{\sigma_N}(N, E, c)$ and $CEL^{\sigma_N}(N, E, c)$ stand for the allocations recommended by the (σ_N)-discrete constrained-equal awards and (σ_N)-discrete constrained equal losses rules respectively. It holds that:*

(a) $cea(N, E, c) = \frac{1}{n!} \sum_{\sigma_N \in \Omega} CEA^{\sigma_N}(N, E, c)$

(b) $cel(N, E, c) = \frac{1}{n!} \sum_{\sigma_N \in \Omega} CEL^{\sigma_N}(N, E, c)$

Proof.

(a) *Let $(N, E, c) \in \mathbb{B}_Z$ and $i \in N$. If the agent i is such that he receives a whole amount under the cea, the statement is trivial, since $\frac{1}{n!} \sum_{\sigma_N \in \Omega} CEA_i^{\sigma_N}(N, E, c) = \frac{1}{n!} \sum_{\sigma_N \in \Omega} cea_i(N, E, c) = cea_i(N, E, c)$. Let us suppose then that this is not the case and $i \in Q(cea; e)$.*

$$\frac{1}{n!} \sum_{\sigma_N \in \Omega} CEA_i^{\sigma_N}(N, E, c) = \lfloor cea_i(N, E, c) \rfloor + \frac{1}{n!} \sum_{\substack{\sigma_N \\ i \in Q^{\sigma_N}(cea; N, E, c)}} 1$$

Note that the amount $A = \sum_{\substack{\sigma_N \\ i \in Q^{\sigma_N}(cea; N, E, c)}} 1$ *is the same for all the agents. Then, adding the above expression across the demanders we can obtain the value of A in this way*

$$\begin{aligned} E &= \sum_{k \in N} \lfloor cea_i(N, E, c) \rfloor + \frac{1}{n!} \sum_{k \in N} \sum_{\substack{\sigma_N \\ k \in Q^{\sigma_N}(cea; N, E, c)}} 1 = \\ &= E - E' + \frac{1}{n!} \sum_{k \in Q(cea; N, E, c)} \sum_{\substack{\sigma_N \\ k \in Q^{\sigma_N}(cea; N, E, c)}} 1 = \\ &= E - E' + \frac{1}{n!} qA, \end{aligned}$$

where $E' = E - \sum_{i \in N} \lfloor cea_i(N, E, c) \rfloor$. Hence $A = \frac{E' n!}{q}$ and then

$$\frac{1}{n!} \sum_{\sigma_N \in \Omega} CEA_i^{\sigma_N}(N, E, c) = \lfloor cea_i(N, E, c) \rfloor + \frac{1}{n!} A = cea_i(N, E, c)$$

(b) The case of the constrained equal losses rule goes in a similar way.

■

Similar to the continuous case, two rules are dual rules if one of them allocates awards in the same way the other one allocates losses.

Definition 3.3 *Two discrete rules* Φ *and* Φ^* *are* ***dual rules*** *if for all* $(N, E, c) \in \mathbb{B}_Z$

$$\Phi^*(N, E, c) = c - \Phi(N, L, c)$$

In the continuos case, *cea* and *cel* are dual rules. Here we also obtain duality between discrete CEA and CEL rules, but with respect to opposite orderings.

Proposition 3.2 *Let* $\sigma \in \Omega$, *and let* $-\sigma$ *be the opposite ordering. Then,* CEA^σ *and* $CEL^{-\sigma}$ *are dual rules.*

Proof. *Let* $Q(cea; N, E, c)$ *and* $Q(cel; N, L, c)$ *be the sets of agents whose allocation via cea and cel are integer numbers.*[5] *It is easy to check that, since cea and cel are dual rules,* $Q(cea; N, E, c) = Q(cel; N, E, c) =: Q$, *and then* $\lfloor cea_i(N, E, c) \rfloor = c_i - \lfloor cel_i(N, L, c) \rfloor$ *for all* $i \notin Q$ *and* $\lfloor cea_i(N, E, c) \rfloor = c_i - \lfloor cel_i(N, L, c) \rfloor - 1$ *for all* $i \in Q$.

If we denote by $E' = E - \sum_{i \in N} \lfloor cea_i(N, E, c) \rfloor$ *and* $L' = L - \sum_{i \in N} \lfloor cel_i(N, L, c) \rfloor$ *the remaining units, we have the following relation*

$$\begin{aligned} E' &= E - \textstyle\sum_{i \in N} \lfloor cea_i(N, E, c) \rfloor = E - \sum_{i \in Q} \lfloor cea_i(N, E, c) \rfloor - \sum_{i \notin Q} \lfloor cea_i(N, E, c) \rfloor = \\ &= E - \textstyle\sum_{i \in Q} (c_i - \lfloor cel_i(N, L, c) \rfloor - 1) - \sum_{i \notin Q} (c_i - \lfloor cel_i(N, L, c) \rfloor) = \\ &= q - (C - E - \textstyle\sum_{i \in N} \lfloor cel_i(N, L, c) \rfloor) = \\ &= q - L' \end{aligned}$$

On the other hand $Q^{-\sigma}_{L'} = Q^\sigma \setminus Q^\sigma_{q-L'} = Q^\sigma \setminus Q^\sigma_{E'}$. *Therefore,* $k \in Q^\sigma_{E'}$ *if and only if* $k \notin Q^{-\sigma}_{L'}$.

If $k \notin Q$ *then* $CEA^\sigma_k(N, E, c) = cea_k(N, E, c) = c_k - cel_k(N, L, c) = CEL^{-\sigma}_k(N, L, c)$. *If* $k \in Q$, $CEA^\sigma_k(N, E, c) = \lfloor cea_k(N, E, c) \rfloor + 1$ *iff* $k \in Q^\sigma_{E'}$, *but we have shown above that this happens if and only if* $k \notin Q^{-\sigma}_{L'}$, *iff* $CEL^{-\sigma}_k(N, L, c) = \lfloor cel_k(N, L, c) \rfloor$. *Therefore* $CEA^\sigma_k(N, E, c) = \lfloor cea_k(N, E, c) \rfloor + 1 = c_i - \lfloor cel_k(N, L, c) \rfloor - 1 + 1 = c_i - CEL^{-\sigma}_k(N, L, c) = \lfloor cel_k(N, L, c) \rfloor$. ■

[5] See Section 3.2 for notation

4 Properties

Here we look for properties our rules may fulfil. Some of the following properties have been studied in the continuous case, and their rationale and appealingness are preserved in the discrete case. In some other cases, we have to adapt the fairness principle at hand so that it becomes meaningful in the discrete case.

The most common and appealing requirement in the continuous case is a property of impartiality. In one if its forms, the so called *equal treatment of equals,* it says that in any problem, if two claimants have identical claims, then they should receive the same amount.[6] Unfortunately, no discrete rule could fulfill this property. It is enough to consider the case of two agents with identical claims, and $E = 1$. Instead of this condition we consider a weak version which sets that if in a problem two players have the same claims, then their allocations differ, at most, in one unit.[7]

Definition 4.1 *Φ satisfies* ***weak equal treatment*** *if for all $(N, E, c) \in \mathbb{B}_Z$ and all $i, j \in N$; if $c_i = c_j$, then $|\Phi_i(N, E, c) - \Phi_j(N, E, c)| \leq 1$.*

We introduce now a stronger version of the above definition that also makes use of the priority order σ. We say that a rule satisfies σ-weak equal treatment if whenever two claimants with the same claim are not allotted the same amount, the agent who receives the extra unit is the one with the highest priority according to σ.

Definition 4.2 *Φ satisfies* ***σ-weak equal treatment*** *if for all $(N, E, c) \in \mathbb{B}_Z$ and all $i, j \in N$; it happens that $c_i = c_j$ implies that $|\Phi_i(N, E, c) - \Phi_j(N, E, c)| \leq 1$, and if $|\Phi_i(N, E, c) - \Phi_j(N, E, c)| = 1$ then $\Phi_i(N, E, c) = \Phi_j(N, E, c) + 1 \Leftrightarrow i\sigma j$*

The next group of properties refers to changes in the estate, when the set of agents and the claims remain fixed.

The first one is a very straightforward property, it says that if the estate increases no agent will receive less than she got initially.

Definition 4.3 *Φ satisfies* ***estate monotonicity*** *if for all (N, E, c), $(N, E', c) \in \mathbb{B}_Z$, if $E' \geq E$ then $\Phi(N, E', c) \geq \Phi(N, E, c)$.*

[6] A continuous rule φ satisfies equal treatment of equals if for all $(N, E, c) \in \mathbb{B}_Z$ and all $i, j \in N$; if $c_i = c_j$, then $\varphi_i(N, E, c) = \varphi_j(N, E, c)$.

[7] This notion was introduced by Balinski & Young (1977), and they referred to that as balancedness.

The following two properties have to do with mistakes in the estimate of the estate, either from below or from above. Basically, they guarantee a sort of invariance in the final allocation, after correcting the mistakes. These two properties are of a procedural nature.

Imagine that when estimating the value of the estate, we were pessimistic, so that the real value is larger than expected. Then two possibilities are open, either to forget about the initial allocation and just solve the new problem, or keep the tentative allocation and then allocate the rest of the estate among the claimants, after reducing their claims by the amount already obtained. The property requires that the final allocation should not depend on this timing.

Definition 4.4 *Φ satisfies* ***composition up (Young, 1988)*** *if for all $(N, E, c) \in \mathbb{B}_Z$ and all $E_1, E_2 \in \mathbb{Z}_{++}$ such that $E_1 + E_2 = E$ it holds that $\Phi(N, E_1, c) + \Phi(N, E_2, c) = \Phi(N, E, c)$.*

Now, suppose that, once the tentative estate has been distributed among the claimants, it happens that it was too optimistic a estimate, so that the real value of the estate is smaller than expected. Then we have two possibilities: the first one is to invalidate the allocation and make a new one with the reduced estate; the second one is to consider a new claims problem in which the claims correspond to the tentative allocation and the new estate is the reduced one. The next property asks these two procedures to result in the same outcome.

Definition 4.5 *Φ satisfies* ***composition down (Moulin, 1987)*** *if for all $(N, E, c) \in \mathbb{B}_Z$ and all $E' \in \mathbb{Z}_+$ with $E' > E$ then $\Phi(N, E, c) = \Phi(N, E, \Phi(N, E', c))$.*

Trivially, both composition up and composition down imply estate monotonicity.

The next group of properties are protective properties in favor of small claimants. They refer to how small a claim should be for its owner to receive the whole claim. One way to decide that threshold in a claims problem is the following: substitute it for the claim of any other agents whose claim is higher, and check whether there would then be enough to compensate everyone. The first property exploits the idea that only claimants responsible for the bankruptcy should be rationed.

Definition 4.6 *Φ satisfies* ***conditional full compensation (Herrero & Villar, 2002)*** *if for all $(N, E, c) \in \mathbb{B}_Z$, if $\sum_{j=1}^{n} \min\{c_i, c_j\} \leq E$ then $\Phi_i(N, E, c) = c_i$.*

The following property proposes an alternative threshold: when an individual's claim is smaller than the equal division of the estate, the individual should be fully compensated.

Definition 4.7 Φ *satisfies* ***exemption (Herrero & Villar, 2001)*** *if for all* $(N, E, c) \in \mathbb{B}_Z$*, if* $c_i \leq \lfloor E/n \rfloor$ *then* $\Phi_i(N, E, c) = c_i$

Note that exemption implies conditional full compensation. Moreover, in the case of two agents they coincide.

Similar to the previous properties, the next ones are protective properties for agents with large enough claims. They refer to cases where agents with small enough claims (below a certain threshold) should receive nothing, favouring large claimants. Different thresholds give rise to different properties.

Definition 4.8 Φ *satisfies* ***conditional null compensation (Herrero & Villar, 2002)*** *if for all* $(N, E, c) \in \mathbb{B}_Z$*, if* $\sum_{j=1}^{n} \min\{c_i, c_j\} \leq E$ *then* $\Phi_i(N, E, c) = 0$.

Definition 4.9 Φ *satisfies* ***exclusion (Herrero & Villar, 2001)*** *if for all* $(N, E, c) \in \mathbb{B}_Z$*, if* $c_i \leq \lfloor L/n \rfloor$ *then* $\Phi_i(N, E, c) = 0$

Obviously, exclusion implies conditional null compensation and in the two-agents case they are equivalent.

Now, we consider properties that refer to changes in the set of agents. The first one, *consistency,* has been studied in a variety of models of distributive justice. Suppose that, after solving the problem (N, E, c), a proper subset of agents $S \subset N$ decides to reallocate the total amount they have received, that is, they face a new claims problem: $(S, \sum_{i \in S} a_i, c_S)$, where $c_S = (c_i)_{i \in S}$ and a is the allocation recommended by the rule to the problem (N, E, c). A rule satisfies consistency if the reallocation is only a restriction to the subset S of the initial allocation.

Definition 4.10 Φ *satisfies* ***consistency (Aumann & Maschler, 1985)*** *if for all* $(N, E, c) \in \mathbb{B}_Z$*, all* $S \subset N$ *it holds that* $\Phi_i(N, E, c) = \Phi_i(S, \sum_{j \in S} \Phi_j(N, E, c), c_S)$ $\forall i \in S$.

The next property is a sort of converse of consistency, but when we only look at subgroups of agents of size 2. If an allocation for a problem is such that for all two-person subgroup, the solution chooses the restriction of the allocation to the subgroup for the associated reduced problem to this subgroup, then that allocation should be the solution outcome for the original problem.

Let us define the set $c.con(E, c; \Phi) = \{x \in \mathbb{Z}_+^n : \sum_{i \in N} x_i = E$ and for all $S \subset N$ such that $|S| = 2, x_S = \Phi(\sum_{i \in S} x_i, c_s)\}$

Definition 4.11 *Φ satisfies* ***converse consistency (Chun, 1999)*** *if for all $(N, E, c) \in \mathbb{B}_Z$, $c.con(E, c; \Phi) \neq \phi$, and if $x \in c.con(E, c; \Phi)$ then $x = \Phi(N, E, c)$.*

As in the continuous case, duality of properties can also be established for discrete solutions.

Definition 4.12 *We say that $\mathcal{P}^*$ is dual property of $\mathcal{P}$ if for every rule Φ it is true that Φ satisfies $\mathcal{P}$ if and only if its dual rule Φ^* satisfies $\mathcal{P}^*$*

The following property is straightforward and the proof is very similar to the analogous result in the continuous case (see Herrero & Villar, 2001).

Proposition 4.1 *The following pairs of properties are dual properties:*

(a) σ-weak equal treatment and $(-\sigma)$-weak equal treatment

(b) Composition up and composition down

(c) Conditional full compensation and conditional null compensation

(d) Exemption and exclusion.

Moreover, weak equal treatment, estate monotonicity, consistency and converse consistency are auto-dual properties.

The next four results for the continuous case are also valid (without modifications) in the discrete one.

Theorem 4.1 (Herrero & Villar, 2001) *If a rule Φ is characterized by a set of independent properties $\Pi = \{\mathcal{P}_1, \dots, \mathcal{P}_k\}$, and if for any $\mathcal{P}_i$ there exists a dual property $\mathcal{P}_i^*$, then the dual rule Φ^* is characterized by the corresponding set of dual properties $\Pi^* = \{\mathcal{P}_1^*, \dots, \mathcal{P}_k^*\}$. Moreover, the properties in Π^* are also independent.*

Proposition 4.2 ([Elevator lemma] Thomson, 2000) *If a rule Φ is bilaterally consistent and coincides with a conversely consistent rule Φ' in the two agent case, then it coincides with Φ' in general.*

Proposition 4.3 (Chun, 1999) *Estate monotonicity and consistency together imply converse consistency.*

Proposition 4.4 (Yeh, 2004) *Exemption and consistency togerther imply conditional full compensation.*

5 Characterizations

We are now ready to present some characterizations results. The fist one is a characterization of the σ-discrete constrained-equal awards rule.

Theorem 5.1 *The σ-discrete constrained equal awards rule is the unique rule that satisfies σ-weak equal treatment, conditional full compensation and composition down.*

Proof. *Let $\sigma \in \Omega$. It is easy to check that the σ-discrete constrained equal awards rule satisfies the three properties.*

Let us prove the converse. Let us suppose that there exists a discrete rule Φ satisficing σ-weak equal treatment, conditional full compensation and composition down. Let $N \in \mathcal{N}$ and $c \in \mathbb{Z}^n$ be a vector of demands. We will show that, for any value of the estate E such that $0 < E < C$, $\Phi(N,E,c) = CEA^\sigma(N,E,c)$. Let us define $\delta^t(c) = t.th\max_{j\in N}\{c_j\}$, that is $\delta^1(c)$ is the highest claim, $\delta^2(c)$ is the second highest claim, and so on. $N^t(c) = \{j \in N : c_j = \delta^t(c)\}$ and $n^t(c) = |N^t(c)|$.

Step 1. Let us suppose that E is such that $C - n^1(c)(\delta^1(c) - \delta^2(c)) \leq E < C$. We can distinguish two cases:

- *(a) If $i \notin N^1(c)$, then $\Phi_i(N,E,c) = c_i = CEA_i^\sigma(N,E,c)$ by conditional full compensation.*
- *(b) If $i \in N^1(c)$, by using σ-weak equal treatment, then $\Phi_i(N,E,c) = CEA_i^\sigma(N,E,c)$.*

Note that, if $E = C - n^1(c)(\delta^1(c) - \delta^2(c))$ then

$$\Phi_i(N,E,c) = CEA_i^\sigma(N,E,c) = \begin{cases} c_i & \text{if } i \notin N^1(c) \\ \delta^2(c) & \text{if } i \in N^1(c) \end{cases}$$

Step 2. We define $E^1 = C - n^1(c)(\delta^1(c) - \delta^2(c))$, $c^1 = \Phi(N,E^1,c) = CEA^\sigma(N,E^1,c)$ and $C^1 = \sum_{j\in N} c_j^1$. Let us suppose that E is such that $C^1 - (n^1(c)+n^2(c))(\delta^2(c) - \delta^3(c)) \leq E < C^1$. We can distinguish two cases:

- *(a) If $i \notin N^1(c) \cup N^2(c)$, then $\Phi_i(N,E,c^1) = c_i = CEA_i^\sigma(N,E,c^1)$ by conditional full compensation.*
- *(b) If $i \in N^1(c) \cup N^2(c)$, by using σ-weak equal treatment, then $\Phi_i(N,E,c^1) = CEA_i^\sigma(N,E,c^1)$.*

  Applying composition down

$$\Phi(N,E,c) = \Phi(N,E,c^1) = CEA^\sigma(N,E,c^1) = CEA^\sigma(N,E,c)$$

Note that, if $E = C^1 - (n^1(c)+n^2(c))(\delta^2(c)-\delta^3(c))$ *then* $\Phi_i(N,E,c^1) = CEA_i^\sigma(N,E,c^1) =$
$$\begin{cases} c_i^1 & \text{if } i \notin N^1(c) \cup N^2(c) \\ \delta^3(c) & \text{if } i \in N^1(c) \cup N^2(c) \end{cases}$$
By repeating this procedure we cover all the possible values for E.

■

Herrero & Villar (2002) show that the continuous constrained equal awards rule is the only rule satisfying conditional full compensation and composition down. Here, unlike the continuous case, composition down and conditional full compensation together do not imply σ-weak equal treatment. Since this last property is directly related with the order σ, then a natural question arises: Would it be possible to characterize the family of all the discrete constrained-equal awards rules by using only conditional full compensation and composition down? The answer is not. Example (5.1) shows a rule that satisfies conditional full compensation and composition down that is not a member of the discrete constrained-equal awards family.

Example 5.1 *Let* $(N,E,c) \in \mathbb{B}_Z$ *a claims problem with only two claimants,* $|N| = 2$, *called 1 and 2, and let* $\sigma \in \Omega$. *Consider a rule that, if the claims were different or the estate were an even integer number, coincides with the discrete constrained-equal awards rule for the order* σ. *If the claims were equal, we differentiate two cases: if the estate were* $1, 5, 9, 13, \ldots$ *the rule will also coincide with the discrete constrained-equal awards rule for the order* σ*; otherwise, if the estate were* $3, 7, 11, \ldots$ *it will coincide with the discrete constrained-equal awards but with the opposite order* $(-\sigma)$. *Such a rule satisfies composition down, exemption and consistency (vacuously) but there does not exist an order* σ *such that it satisfies* σ*-weak equal treatment.*

$$\Phi(N,E,c) = \begin{cases} CEA^\sigma(N,E,c) & \text{if } c_1 \neq c_2 \text{ or } E \in \{2k\}_{k\in\mathbb{Z}} \\ CEA^\sigma(N,E,c) & \text{if } c_1 = c_2 \text{ and } E \in \{4k-3\}_{k\in\mathbb{Z}} \\ CEA^{-\sigma}(N,E,c) & \text{if } c_1 = c_2 \text{ and } E \in \{4k-1\}_{k\in\mathbb{Z}} \end{cases}$$

Now we have a characterization of the σ-discrete constrained-equal losses rule.

Theorem 5.2 *The* σ*-discrete constrained equal losses rule is the unique rule that satisfies* σ*-weak equal treatment, conditional null compensation and composition up.*

Proof. *It is sufficient to apply the results shown in Theorems (4.1) and (5.1), and Propositions (3.2) and (4.1).* ■

Analogously to the constrained-equal awards case, Herrero & Villar (2002) characterize the continuous constrained-equal losses rule by using conditional null compensation and composition up but not σ-weak equal treatment. The dual rule of that defined in Example (5.1) satisfies conditional null compensation and composition up but not σ-weak equal treatment. This fact shows that it is not possible to characterize the discrete constrained equal losses family by only using conditional null compensation and composition up.

We present now an alternative characterization of the σ-discrete constrained equal awards rule.

Theorem 5.3 *The σ-discrete constrained equal awards rule is the unique discrete rule that satisfies σ-weak equal treatment, exemption, composition down, and consistency.*

Proof. *It follows inmediately from Theorem (5.1) and Proposition (4.4).* ■

As in the previous results, Herrero & Villar (2001) prove that, in the continuous case, the unique rule that satisfies composition down, exemption and consistency is the constrained equal awards rule. Our result is very similar to theirs adding σ-weak equal treatment. Again, it is not possible to characterize the family of all the discrete constrained-equal awards rules using the three axioms proposed by Herrero & Villar. Example (5.1) also provides the answer.

Here we show the dual result of Theorem (5.1). It consists in a characterization of the σ-discrete constrained equal losses rule.

Theorem 5.4 *The σ-discrete constrained equal losses rule is the unique discrete rule that satisfies σ-weak equal treatment, exclusion, composition up, and consistency.*

Proof. *It is sufficient to apply the results shown in Theorems (4.1) and (5.3), and Propositions (3.2) and (4.1).* ■

The next characterization of the σ-discrete constrained equal awards rule is similar to that presented in Yeh (2004).

Theorem 5.5 *The σ-discrete constrained equal awards is the unique discrete rule that satisfies σ-weak equal treatment, exemption, composition down and converse consistency.*

Proof. *Let $\sigma \in \Omega$. It is easy to check that CEA^{σ} fullfils the four properties, let us see the converse. Suppose that there exists a rule Φ different from CEA^{σ} and which satisfies σ-weak equal treatment, exemption, composition down and converse consistency. Since exemption coincides with conditional full compensation in the two claimants case,* *by Theorem (5.1), $\Phi = CEA^{\sigma}$ in this case. Φ fulfills converse consistency, then, by using the Elevator Lemma, $\Phi = CEA$.* ■

The dual result of the previous theorem provides us a new characterization of the σ-discrete constrained-equal losses rule.

Theorem 5.6 *The σ-discrete constrained equal losses is the unique discrete rule that satisfies σ-weak equal treatment, exclusion, composition up and converse consistency.*

Proof. *It is sufficient to apply the results shown in Theorems (4.1) and (5.6), and Propositions (3.2) and (4.1)* ■

Again, Example (5.1) shows that it is not possible to characterize the whole family of the discrete constrained equal awards rules (discrete constrained equal losses rules) by only using exemption, composition down and converse consistency (exclusion, composition up and converse consistency).

The results in this section can be summarized in Table (1).

Property	CEA^{σ}	CEL^{σ}
σ-weak equal treatment	Y(*)(+)(-)	Y(*)(+)(-)
Estate monotonicity	Y	Y
Composition down	Y(*)(+)(-)	N
Composition up	N	Y(*)(+)(-)
Conditional full compensation	Y(+)	N
Exemption	Y(*)(-)	N
Conditional null compensation	N	Y(+)
Exclusion	N	Y(*)(-)
Consistency	Y(*)	Y(*)
Converse consistency	Y(-)	Y(-)

Table 1: This table summarizes the result in former sections; "Y" means that the rule satisfies that property while "N" that it does not. On the other hand Y(*) (respectively Y(+) and Y(-))means that this property, together with the others marked with (*) ((+),(-)) in the same column, characterize the rule.

5.1 Independence of Properties

The characterizations in Theorems (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6)are tight. We here prove the independence of the properties.

Example 5.2 *A rule, Φ^σ, that satisfies σ-weak equal treatment, conditional full compensation, exemption, consistency, converse consitency and that no satisfies composition down can be described as follows: First, let us suppose that the estate were completely divisible; then start by dividing the estate among the agents with the lowest claims, up to the moment in which those agents are satiated; then, if there is still some estate left, divide it equally among agents with the second lowest claim, and so on. Formally, let $(E, c) \in \mathbb{B}^N$, let $\mu^1(c) = \min_{j \in N}\{c_j\}$, $M^1(c) = \{j \in N : \mu^1(c) = c_j\}$, $m^1(c) = |M^1(c)|$, $\mu^2(c) = \min_{j \in N \setminus M^1(c)}\{c_j\}$, $M^2(c) = \{j \in N : \mu^2(c) = c_j\}$, $m^2(c) = |M^2(c)|$, etc. Then for each $j \in M^k(c)$*

$$\varphi_j(E, c) = \begin{cases} 0 & \text{if } 0 \leq E \leq \sum_{s<k} m^s(c)\mu^s(c) \\ \dfrac{E - \sum_{s<k} m^s(c)\mu^s(c)}{m^k(c)} & \text{if } \sum_{s<k} m^s(c)\mu^s(c) \leq E \leq \sum_{s\leq k} m^s(c)\mu^s(c) \\ c_j & \text{otherwise} \end{cases}$$

We define now the discrete rule

$$\Phi_j^\sigma(N, E, c) = \begin{cases} \lfloor \varphi_j(N, E, c) \rfloor + 1 & \text{if } j \in Q_{E'}^\sigma(\varphi; N, E, c) \\ \lfloor \varphi_j(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor \varphi_i(N, E, c) \rfloor > 0$.

Example 5.3 *A rule that satisfies σ-weak equal treatment, exemption, composition down, but violates consistency and converse consistency.*

$$\Phi^\sigma(N, E, c) = \begin{cases} \overline{\Phi}^\sigma(N, E, c) & \text{if } |N| = 3 \text{ and for each pair } \{i, j\} \in N,\ c_i \neq c_j \\ CEA^\sigma(E, c) & \text{otherwise} \end{cases}$$

where $\overline{\Phi}^\sigma$ is described as follows: suppose that $c_1 \leq c_2 \leq \ldots \leq c_n$ and consider $\overline{\varphi}$ given by

$$\overline{\varphi}(N, E, c) = \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & \text{if } \frac{E}{3} \leq c_1 \\ \left(c_1, \frac{E}{3} + \frac{1}{3}\left(\frac{E}{3} - c_1\right), \frac{E}{3} + \frac{2}{3}\left(\frac{E}{3} - c_1\right)\right) & \text{if } c_1 < \frac{E}{3} \leq \min\left\{\frac{c_1+3c_2}{4}, \frac{2c_1+3c_3}{5}\right\} \\ (c_1, c_2, E - c_1 - c_2) & \text{otherwise} \end{cases}$$

Then,

$$\overline{\Phi}_j^\sigma(N, E, c) = \begin{cases} \lfloor \overline{\varphi}_j(N, E, c) \rfloor + 1 & \text{if } j \in Q_{E'}^\sigma(\overline{\varphi}; N, E, c) \\ \lfloor \overline{\varphi}_j(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor \overline{\varphi}_i(N, E, c) \rfloor > 0$.

1. *The rule Φ^* is defined by the previous algorithm. Imagine that the set of players is ordered in this way $\{n, n-1, \dots, 1\}$, if we take the problem $E = 10$ and $c = (2, 6, 8)$, $\Phi^*(E, c) = (2, 3, 5)$; while for $S = \{2, 3\}$, $c_S = (6, 8)$, $\Phi^*(3 + 5, c_s) = (4, 4)$.*

Example 5.4 *Similar to the discrete constrained equal awards and losses rules, we can define some kind of proportional rule as follows:*

$$P_j^\sigma(N, E, c) = \begin{cases} \lfloor p_j(N, E, c) \rfloor + 1 & \text{if } j \in Q_{E'}^\sigma(p; N, E, c) \\ \lfloor p_j(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor p_i(N, E, c) \rfloor > 0$, and p denotes the continuous proportional rule. This σ-discrete proportional rule satisfies σ-weak equal treatment, composition down, consistency and converse consistency but it violates exemption and conditional full compensation.

Example 5.5 *$CEA^{-\sigma}$ satisfies exemption, composition down, conditional full compensation, and consistency but not σ-weak equal treatment.*

6 Conclusions

In this work we have considered claims problems with indivisible goods, that is, problems in which the estate, the claims and the allocations are indivisible units of an homogeneous good. If the three procedural properties *consistency, composition up* and *composition down* are requested, only pure priority rules are left, leaving no room for any sort of compromise. Nonetheless, it is possible to construct *approximately egalitarian* rules to solve this types of problems. That is, rules that allocate either awards or losses in a way so that they are as equal as possible among the agents. When indivisible goods are involved, those egalitarian procedures give rise to multiplicity of allocations, unless we state some priority order among the agents. We do so, and use the priority order only to allocate the "extra" units, that is, those units that are left after applying our egalitarian principle as far as possible. This order is exogenously given, and it can be chosen either at random or by using any sort of priority principle. By using this method, we construct discrete rules that very much share the spirit of the constrained equal awards and constrained equal losses rules for the continuous case. A family of rules is obtained in either case, one rule for each linear order on the set of agents. It happens that for every problem, the allocation recommended by the corresponding continuous rule is the average of the allocations recommended for all the discrete rules, for the different orders.

Many of the properties of the continuous case can be extended to the discrete one, as the idea of duality, that also works in this context. Any discrete constrained equal awards rule is the dual rule of the constrained equal losses rule, under opposite orders.

Even though our discrete rules do not satisfy simultaneously the procedural properties of composition up and composition down, they fulfill one of them each, as well as many other properties used in the literature for the continuous case. Thus, we obtained characterization results for our rules by using properties very much related to those used in the continuous case. The main difference now is that pure anonymity principles cannot be fulfilled for our rules, and thus, we consider some related properties, stating that when two agents have identical claims, their awards are equal or else differ in one unit. Furthermore, if two agents with identical claims do not receive the same awards, then the one with the highest award goes first in the priority order. Our characterization results are similar to those in Herrero & Villar (2001, 2002) and Yeh (2003) for the continuous case, by adding the aforementioned relaxation of equal treatment of equals. Unlike the continuous case, now this "impartiality property" is necessary to get a characterization.

Theorem 5.3 is related to Theorem 1, page 175 in Young (1994). Consider the case where each agent demands at most one unit of the indivisible commodity. Then, it happens that consistency implies composition down, and exemption is vacuously fulfilled. Furthermore, for this particular case, the σ-discrete constrained equal awards rule coincides with the priority rule defined by the ordering σ. Therefore, Theorem 5.3 simply says that a rule satisfies consistency and σ-weak equal treatment iff it is the σ-priority rule.

Our discrete extensions are also related to the continuous rules in a different way. We may consider a two step procedure to allocate awards. Given a problem, in the first step, we allocate to every agent the whole part of her allocation under the *cea* (or the *cel*) rule. Then we allocate the remaining units according to the priority order. Previous two-step procedure can be also used in order to define some other types of discrete rules, by considering, at the first step, any continuous rule. Hence, for each continuous rule we get a family of discrete associated rules. Even though theoretically, the application of this procedure gives rise to new discrete rules, not in all cases the rules obtained are natural and well-behaved (see Example 5.4). This happens, in particular with respect to the proportional solution. Thus, further research is needed in order to provide with alternative discrete rules.

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