A discusión

AUTOREGRESSIVE CONDITIONAL VOLATILITY, SKEWNESS AND KURTOSIS*

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ABSTRACT

This paper proposes a GARCH-type model allowing for time-varying volatility, skewness and kurtosis. The model is estimated assuming a Gram-Charlier series expansion of the normal density function for the error term, which is easier to estimate than the non-central t distribution proposed by Harvey and Siddique (1999). Moreover, this approach accounts for time-varying skewness and kurtosis while the approach by Harvey and Siddique (1999) only accounts for nonnormal skewness. We apply this method to daily returns of a variety of stock indices and exchange rates. Our results indicate a significant presence of conditional skewness and kurtosis. It is also found that specifications allowing for time-varying skewness and kurtosis outperform specifications with constant third and fourth moments.

Keywords: Conditional volatility, skewness and kurtosis; Gram-Charlier series expansion; Stock indices.

JEL Classification: G12, G13, C13, C14

1. Introduction

There have been many papers studying the departures from normality of asset return distributions. It is well known that stock return distributions exhibit negative skewness and excess kurtosis (see, for example, Harvey and Siddique, 1999; Peiró, 1999; and Premaratne and Bera, 2001). Specifically, excess kurtosis (the fourth moment of the distribution) makes extreme observations more likely than in the normal case, which means that the market gives higher probability to extreme observations than in normal distribution. However, the presence of negative skewness (the third moment of the distribution) has the effect of accentuating the left-hand side of the distribution. That is, the market gives higher probability to decreases than increases in asset pricing.

These issues have been widely analyzed in option pricing literature. For example, as explained by Das and Sundaram (1999), the well known volatility smile and smirk effects are closely related to the presence of excess kurtosis and negative skewness in the underlying asset returns distribution.

The generalized autoregressive conditional heteroscedasticity (GARCH) models, introduced by Engle (1982) and Bollerslev (1986), allow for time-varying volatility but neither time-varying skewness nor time-varying kurtosis. Harvey and Siddique (1999) present a way to estimate jointly the time-varying conditional variance and skewness under a non-central t distribution for the error term in the mean equation. Their methodology is applied to several series of stock index returns, and it is found that autoregressive conditional skewness is significant and that the inclusion of skewness affects the persistence in variance. It is important to point out that the paper by Harvey and Siddique (1999) allows for time-varying skewness but still assumes constant kurtosis.

Premaratne and Bera (2001) have suggested capturing asymmetry and excess kurtosis with the Pearson type IV distribution, which has three parameters that can be interpreted as volatility, skewness and kurtosis. This is an approximation to the non-central t distribution proposed by Pearson and Merrington (1958). However, these authors use time-varying conditional mean and variance, but maintain constant skewness and kurtosis over time. Similarly, Jondeau and Rockinger (2000) employ a conditional generalized Student-t distribution to capture conditional skewness and kurtosis by imposing a time-varying structure for the two parameters which control the probability mass in the assumed distribution[1]. However, these parameters do not follow a GARCH structure for either skewness or kurtosis.

The purpose of this research is to extend the work by Harvey and Siddique (1999) assuming a distribution for the error term in the mean equation that accounts for nonnormal skewness and kurtosis. In particular, we jointly estimate time-varying volatility, skewness and kurtosis using a Gram-Charlier series expansion of the normal density function, along the lines suggested by Gallant and Tauchen (1989).

It is also worth noting that, apart from the fact that our approach accounts for time-varying kurtosis while the one by Harvey and Siddique (1999) does not, our likelihood function, based on a series expansion of the normal density function, is easier to estimate than the likelihood function based on the non-central t distribution employed by them.

The joint estimation of time-varying volatility, skewness and kurtosis can be useful in testing option pricing models that explicitly introduce the third and fourth moments of the underlying asset return distribution along the lines suggested by Heston (1993), Bates (1996), and Heston and Nandi (2000). It may also be useful in analyzing the information content of option-implied coefficients of skewness and kurtosis, extending the papers by Day and Lewis (1992), Lamoureux and Lastrapes (1993) and Amin and Ng (1997), among others.

The method proposed in this paper is applied to two different data sets, specifically several daily return series for both exchange rates and stock indices. Our results indicate significant presence of conditional skewness and kurtosis. It is also found that specifications allowing for both time-varying skewness and kurtosis outperform specifications under both constant third and fourth moments.

The rest of the paper is organized as follows. In Section 2 we present our GARCH-type model for estimating time-varying variance, skewness and kurtosis jointly. Section 3 presents the data and the empirical results regarding the estimation of the model. Section 4 compares the models allowing for time-varying skewness and kurtosis and the standard models with constant third and fourth moments. Section 5 concludes with a summary and discussion.

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[1] This generalized Student-t distribution is based on Hansen´s (1994) work.

2. A model for conditional volatility, skewness and kurtosis

In this section we extend the model for conditional variance and skewness proposed by Harvey and Siddique (1999), to account for conditional kurtosis along the lines discussed in the introduction.

Given a series of asset prices $\{S_0, S_1, ..., S_T\}$, we define continuously compounded returns for period t as $r_t = 100 \times \left[\ln\left(S_t / S_{t-1}\right)\right]$, $t = 1, 2, ..., T$. Specifically, we present an asset return model containing either the GARCH(1,1) or NAGARCH (1,1) structure for conditional variance[2] and also a GARCH (1,1) structure for both conditional skewness and kurtosis. Under the NAGARCH (GARCH) specification for conditional variance, the model is denoted as NAGARCHSK (GARCHSK). The NAGARCHSK model is the following:

$$r_t = E_{t-1}\left(r_t\right) + \varepsilon_t; \quad \varepsilon_t \sim \left(0, \sigma_\varepsilon^2\right)$$

$$\varepsilon_t = h^{1/2}\eta_t; \quad \eta_t \sim \left(0,1\right); \quad \varepsilon_t \mid I_{t-1} \sim \left(0, h_t\right)$$

$$h_t = \beta_0 + \beta_1\left(\varepsilon_{t-1} + \beta_3 h_{t-1}^{1/2}\right)^2 + \beta_2 h_{t-1} \tag{1}$$

$$s_t = \gamma_0 + \gamma_1\eta_{t-1}^3 + \gamma_2 s_{t-1}$$

$$k_t = \delta_0 + \delta_1\eta_{t-1}^4 + \delta_2 k_{t-1}$$

where $E_{t-1}\left(\cdot\right)$ denotes the conditional expectation on an information set till period $t-1$ denoted as I_{t-1}. We state that $E_{t-1}\left(\eta_t\right) = 0$, $E_{t-1}\left(\eta_t^2\right) = 1$, $E_{t-1}\left(\eta_t^3\right) = s_t$ and $E_{t-1}\left(\eta_t^4\right) = k_t$ where both s_t and k_t are driven by a GARCH (1,1) structure. Hence, s_t and k_t represent respectively the skewness and kurtosis corresponding to the conditional distribution of the standardized residual $\eta_t = \varepsilon_t h_t^{-1/2}$. Notice that (1) nests the GARCHSK model for $\beta_3 = 0$.

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[2] Due to the well known leverage effect, we have chosen the NAGARCH (1,1) specification for the variance equation proposed by Engle and Ng (1993).

Using a Gram-Charlier (GC) series expansion of the normal density function and truncating at the fourth moment[3], we obtain the following density function for the standardized residuals η_t conditional on the information available in $t-1$:

$$\begin{aligned} g\left(\eta_t \mid I_{t-1}\right) &= \phi(\eta_t)\left[1+\frac{s_t}{3!}\left(\eta_t^3-3\eta_t\right)+\frac{k_t-3}{4!}\left(\eta_t^4-6\eta_t^2+3\right)\right] \\ &= \phi(\eta_t)\psi(\eta_t) \end{aligned} \tag{2}$$

where $\phi(\cdot)$ denotes the probability density function, henceforth pdf, corresponding to the standard normal distribution and $\Psi(\cdot)$ is the polynomial part of fourth order corresponding to the expression between brackets in (2). Note that the pdf defined in (2) is not really a density function because for some parameter values in (1) the density $g(\cdot)$ might be negative due to the component $\Psi(\cdot)$. Similarly, the integral of $g(\cdot)$ on $\mathbb{R}$ is not equal to one. We propose a true pdf, denoted as $f(\cdot)$, by transforming the density $g(\cdot)$ according to the method in Gallant and Tauchen (1989). Specifically, in order to obtain a well defined density everywhere we square the polynomial part $\Psi(\cdot)$, and to insure that the density integrates to one, we divide by the integral of $g(\cdot)$ over $\mathbb{R}$. The resulting pdf[4] written in abbreviated form is[5]:

$$f\left(\eta_t \mid I_{t-1}\right) = \phi(\eta_t)\psi^2(\eta_t)/\Gamma_t \tag{3}$$

where

$$\Gamma_t = 1+\frac{s_t^2}{3!}+\frac{\left(k_t-3\right)^2}{4!}.$$

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[3] See Jarrow and Rudd (1982) and also Corrado and Su (1996).

[4] See the appendix for proof that this nonnegative function is really a density function that integrates to one.

[5] An alternative approach under the Gram-Charlier framework is proposed by Jondeau and Rockinger (2001) who also show how constraints on the parameters defining skewness and kurtosis may be implemented to insure that the expansion defines a density. However, their approach does not seem to be feasible in both skewness and kurtosis within the conditional case.

Therefore, after omitting unessential constants, the logarithm of the likelihood function for one observation corresponding to the conditional distribution $\varepsilon_t = h_t^{1/2}\eta_t$, whose pdf is $h_t^{-1/2} f\left(\eta_t \middle| I_{t-1}\right)$, is given by

$$l_t = -\frac{1}{2}\ln h_t - \frac{1}{2}\eta_t^2 + \ln\left(\psi^2\left(\eta_t\right)\right) - \ln\left(\Gamma_t\right). \tag{4}$$

As pointed out before, this likelihood function is clearly easier to estimate than the one based on a non-central t proposed by Harvey and Siddique (1999). In fact, the likelihood function in (4) is the same as in the standard normal case plus two adjustment terms accounting for time-varying skewness and kurtosis. Moreover, it is worth noting that the density function based on a Gram-Charlier series expansion in equation (3) nests the normal density function (when s_t = 0 and k_t = 3), while the noncentral t does not. Therefore, the restrictions imposed by the normal density function with respect to the more general density based on a Gram-Charlier series expansion can be easily tested.

3. Empirical results

3.1. Data and preliminary findings

Our methodology is applied to two different data sets. The first one includes daily returns of five exchange rates series: British Pound/USD (GBP/USD), Japanese Yen/USD (JPY/USD), German Mark/USD (GEM/USD) and Swiss Franc/USD (CHF/USD). The second data set includes five stock indices: S&P500 and NASDAQ100 (U.S.), DAX30 (Germany), IBEX35 (Spain) and the emerging market index MEXBOL (Mexico).

Our data set includes daily closing prices from January 2, 1990 to May 3, 2002 for the five exchange rate series, and from January 2, 1990 to July 17, 2003 for all stock index series except for MEXBOL, which includes data from January 2, 1995 to July 17, 2003. These closing prices are employed to calculate the corresponding continuously compounded daily returns. Table 1 presents some descriptive statistics. Note that all series show leptokurtosis and there is also evidence of negative skewness except for GBP/USD and MEXBOL. It is also worth noting that the Mexican emerging market returns (MEXBOL) show the highest values of the unconditional standard deviation, skewness and kurtosis.

Table 1: Descriptive Statistics for Daily Returns

PANEL A: EXCHANGE RATES				
STATISTIC	GBP/USD	JPY/USD	DEM/USD	CHF/USD
Sample size	3126	3126	3126	3126
Mean	0.0030	-0.0045	0.0072	0.0003
Median	0.0000	0.0120	0.0207	0.0217
Maximum	3.2860	3.3004	3.1203	3.0747
Minimum	-2.8506	-5.7093	-2.9497	-3.7243
Stand. Dev.	0.5731	0.7192	0.6621	0.7197
Skewness	0.2334	-0.5794	-0.0594	-0.2000
Kurtosis	5.7502	7.3298	4.6546	4.5432
Jarque-Bera (p-value)	1013.565 (0.0000)	2616.775 (0.0000)	358.4119 (0.0000)	331.0593 (0.000)

PANEL B: STOCK INDICES					
STATISTIC	S&P500	NASDAQ	DAX30	IBEX35	MEXBOL
Sample size	3415	3416	3407	3390	2137
Mean	0.0294	0.0383	0.0178	0.0246	0.0511
Median	0.0315	0.1217	0.0641	0.0508	0.0099
Maximum	5.5732	13.2546	7.5527	6.8372	12.1536
Minimum	-7.1127	-10.1684	-8.8747	-8.8758	-14.3139
Stand. Dev.	1.0611	1.6117	1.5056	1.3876	1.8086
Skewness	-0.0995	-0.0099	-0.1944	-0.1854	0.0712
Kurtosis	6.5658	8.3740	6.3210	5.9169	8.6060
Jarque-Bera (p-value)	1814.880 (0.0000)	4110.566 (0.0000)	1587.134 (0.0000)	1221.204 (0.0000)	2800.124 (0.0000)

Before we estimate our NAGARCHSK model, we analyze the dynamic structure in the mean equation of (1). Specifically, the ARMA structure that maximizes the Schwarz Information Criterion (SIC) is selected. All the parameters implied in every model below are estimated by maximum likelihood assuming that the Gram-Charlier series expansion distribution given by (3) holds for the error term, and using Bollerslev and Wooldridge (1992) robust standard errors[6]. If we define the SIC as $ln(L_{ML}) - (q/2)ln(T)$, where q is the number of estimated parameters, T is the number of observations, and L_{ML} is the value of the log likelihood function using the q estimated parameters, then the selected model is the one with the highest SIC. According to SIC, MA(1) and AR(1) models without constant term yield very similar results[7]. However, the AR(1) has the advantage of being consistent

[6] All maximum likelihood estimations in this paper are carried out using the CML subroutine of GAUSS.

[7] The constant term was never significant in each candidate model for the conditional mean.

with the nonsynchronous contracts of individual stocks which constitute the indices. Definitively, the dynamic conditional mean structure for every estimation is represented by an AR(1) model with no constant term.

Table 2 presents the Ljung-Box statistics of order 20, denoted as LB(20), for ε_t^2, ε_t^3 and ε_t^4, where ε_t is the error term from the AR(1) model with no constant term. The statistic for all moments is quite large (p-value = 0.000 in all cases). In other words, the significant serial correlation for ε_t^2, ε_t^3 and ε_t^4 indicate time-varying volatility, skewness and kurtosis. This empirical evidence would justify the implementation of the NAGARCHSK (GARCHSK) model defined in (1) with time-varying volatility, skewness and kurtosis.

Table 2: Ljung-Box Statistics with Order 20 of Residuals From Ar(1) Model

The table presents the Ljung-Box statistic (asymptotic p-value in parenthesis) with order 20, i.e. LB(20), of ε_t^2, ε_t^3 and ε_t^4, where ε_t is the error term from an AR(1) model for daily returns.

SERIES	LB(20) - ε_t^2	LB(20) - ε_t^3	LB(20) - ε_t^4
GBP/USD	825.43 (0.000)	134.37 (0.000)	332.34 (0.000)
JPY/USD	567.01 (0.000)	208.55 (0.000)	196.37 (0.000)
DEM/USD	407.25 (0.000)	70.501 (0.000)	187.38 (0.000)
CHF/USD	317.69 (0.000)	133.75 (0.000)	365.89 (0.000)
S&P500	131.81 (0.000)	120.91 (0.000)	139.79 (0.000)
NASDAQ	3152.1 (0.000)	252.04 (0.000)	315.26 (0.000)
DAX30	2919.1 (0.000)	72.889 (0.000)	489.37 (0.000)
IBEX35	1719.1 (0.000)	131.16 (0.000)	271.49 (0.000)
MEXBOL	488.67 (0.000)	238.18 (0.000)	283.82 (0.000)

3.2. *Model estimation with time-varying volatility, skewness and kurtosis*

Before presenting the estimation results obtained with both the exchange rates and the stock indices series, we summarize the four nested models estimated as follows:

Mean: $$r_t = \alpha_1 r_{t-1} + \varepsilon_t \tag{5-a}$$

Variance (GARCH): $$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} \tag{5-b}$$

Variance (NAGARCH): $$h_t = \beta_0 + \beta_1 \left(\varepsilon_{t-1} + \beta_3 h_{t-1}^{1/2} \right)^2 + \beta_2 h_{t-1} \tag{5-c}$$

Skewness: $$s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1} \tag{5-d}$$

Kurtosis: $$k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1} . \tag{5-e}$$

So, first we estimate the two standard models for the conditional variance: the GARCH (1,1) model (equations (5-a) and (5-b)), and the NAGARCH (1,1) model (equations (5-a) and (5-c)), where a normal distribution is assumed for the unconditional standardized error η_t. Second, we estimate the generalizations of both the standard GARCH and NAGARCH models, with time-varying skewness and kurtosis, named GARCHSK (equations (5-a), (5-b), (5-d) and (5-e)) and NAGARCHSK (equations (5-a), (5-c), (5-d) and (5-e)), assuming in both cases the distribution based on the Gram-Charlier series expansion given by equation (3).

It should be noted that, given that the likelihood function is highly nonlinear, special care must be taken in selecting the starting values of the parameters. As usual in these cases, given that the four models are nested, the estimation is performed following several stages and using the parameters estimated from the simpler models as starting values for more complex ones.

The results for the exchange rate series are presented in Tables 3 and 4 for the GARCH and GARCHSK models respectively. It is found that for all exchange rates series the coefficient for asymmetric variance, that is β_3, is not significant. It confirms that the leverage effect, commonly observed in other financial series, is not observed in the case of exchange rates. Therefore, for the exchange rate series only the results for symmetric variance models are presented.

Table 3: GARCH models – exchange rates

The reported coefficients shown in each row of the table are ML estimates of the standard GARCH model:

$$r_t = \alpha_1 r_{t-1} + \varepsilon_t$$
$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1}$$

for percentage daily returns of British Pound/American Dollar (GBP/USD), Japanese Yen/US Dollar (JPY/USD), German Mark/US Dollar (DEM/USD) and Swiss Franc/US Dollar (CHF/USD) exchange rates, from January 1990 to March 2002. $h_t = var(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ follows a $N(0,h_t)$ distribution. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	GBP/USD	JPY/USD	DEM/USD	CHF/USD
Mean equation	α_1	0.0432 (0.0263)	0.0175 (0.3826)	0.0364 (0.0573)	0.0304 (0.1154)
Variance equation	β_0	0.0031 (0.0459)	0.0086 (0.0645)	0.0051 (0.0663)	0.0111 (0.0715)
	β_1	0.0435 (0.0000)	0.0428 0.0011)	0.0378 (0.0000)	0.0336 (0.0003)
	β_2	0.9468 (0.0000)	0.9402 0.0000)	0.9502 (0.0000)	0.94445 (0.0000)
Log-Likelihood	-	409.3328	-352.5956	-149.3089	-451.7276
SIC	-	393.2391	-368.6843	-165.4027	-467.8213

As expected, the results for all exchange rate series indicate a significant presence of conditional variance. Volatility is found to be persistent since the coefficient of lagged volatility is positive and significant, indicating that high conditional variance is followed by high conditional variance.

Moreover, it is found that for the GBP/USD, DEM/USD and CHF/USD exchange rate series, days with high skewness are followed by days with high skewness, since the coefficient for lagged skewness (γ_2) is positive and significant but its magnitude is lower than in the variance case. Also, shocks to skewness are significant though they are less relevant than its persistence. However, there seems to be no conditional skewness evidence for the JPY/USD series since neither γ_1 nor γ_2 is significant in this case.

As in the skewness, the results for the kurtosis equation indicate that days with high kurtosis are followed by days with high kurtosis since the coefficient for lagged kurtosis (δ_2) is positive and significant, and its magnitude is greater than that of skewness but still lower than that of variance. As before, shocks to kurtosis are significant, except for the JPY/USD series.

Table 4: GARCHSK Models – Exchange Rates

The reported coefficients shown in each row of the table are ML estimates of the GARCHSK model:

$$
\begin{aligned}
r_t &= \alpha_1 r_{t-1} + \varepsilon_t \\
h_t &= \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} \\
s_t &= \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1} \\
k_t &= \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}
\end{aligned}
$$

for percentage daily returns of British Pound/US Dollar (GBP/USD), Japanese Yen/US Dollar (JPY/USD), German Mark/US Dollar (DEM/USD) and Swiss Franc/US Dollar (CHF/USD) exchange rates, from January 1990 to March 2002. $h_t = var(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $s_t = skewness(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $k_t = kurtosis(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $\eta_t = \varepsilon_t h_t^{-1/2}$, and $\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ follows the distribution based on a Gram-Charlier series expansion. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	GBP/USD	JPY/USD	DEM/USD	CHF/USD
Mean equation	α_1	0.0219 (0.2537)	-0.0030 (0.8670)	0.0249 (0.3804)	0.0015 (0.9322)
Variance equation	β_0	0.0015 (0.0783)	0.0061 (0.0378)	0.0022 (0.0159)	0.0075 (0.0007)
	β_1	0.0366 (0.0000)	0.0309 (0.0021)	0.0236 (0.0000)	0.0217 (0.0000)
	β_2	0.9550 (0.0000)	0.9537 (0.0000)	0.9690 (0.0000)	0.9611 (0.0000)
Skewness equation	γ_0	0.0053 (0.5379)	-0.0494 (0.0482)	-0.0270 (0.0398)	-0.0242 (0.0989)
	γ_1	0.0093 (0.0004)	0.0018 (0.4190)	0.0175 (0.0054)	0.0054 (0.0688)
	γ_2	0.6180 (0.0000)	0.3414 (0.2097)	0.4421 (0.0000)	0.6468 (0.0002)
Kurtosis equation	δ_0	1.3023 (0.0000)	1.2365 (0.0038)	1.9649 (0.0000)	0.5500 (0.0000)
	δ_1	0.0028 (0.0000)	0.0014 (0.1102)	0.01356 (0.0000)	0.0060 (0.0000)
	δ_2	0.6229 (0.0000)	0.6464 (0.0000)	0.4045 (0.0002)	0.8303 (0.0000)
Log-Likelihood	-	472.3652	-237.6668	-117.5896	-420.9973
SIC	-	432.1309	-277.9012	-157.8240	-461.2317

Finally, it is worth noting that the value of the SIC, shown at the bottom of Tables 3 and 4, rises monotonically in all cases when we move from the simpler models to the more complicated ones, with the GARCHSK model showing the highest value. Therefore, for the four exchange rates series analyzed, the GARCHSK specification seems to be the most appropriate one according to the SIC criterion.

The results for the five stock indices are presented in Tables 5, 6, 7 and 8 for GARCH, NAGARCH, GARCHSK and NAGARCHSK models respectively. As expected, the results shown in Table 5 (GARCH models) indicate significant presence of conditional variance, with the two American indices, that is S&P500 and NASDAQ100, showing the highest degree of persistence. However, Table 6 (NAGARCH models) shows that contrary to the exchange rate case, the coefficient β_3 is negative and significant which states the presence of the leverage effect commonly observed in the stock exchange markets.

Table 5: GARCH Models - Stock Indices

The reported coefficients shown in each row of the table are ML estimates of the standard GARCH model:

$$r_t = \alpha_1 r_{t-1} + \varepsilon_t$$
$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1}$$

for percentage daily returns of S&P500, NASDAQ100, DAX30, IBEX35 stock indices, from January 1990 to July 2003, and MEXBOL from January 1995 to July 2003. $h_t = var(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ follows a $N(0,h_t)$ distribution. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	S&P500	NASDAQ	DAX30	IBEX35	MEXBOL
Mean equation	α_1	0.03394 (0.0544)	0.1266 (0.0000)	0.0179 (0.3133)	0.0943 (0.0000)	0.1564 (0.0000)
Variance equation	β_0	0.0055 (0.0414)	0.0149 (0.0155)	0.0317 (0.0092)	0.05741 (0.0026)	0.0827 (0.0958)
	β_1	0.0587 (0.0000)	0.0948 (0.0000)	0.09394 (0.0000)	0.1035 (0.0000)	0.1194 (0.0098)
	β_2	0.9379 (0.0000)	0.9009 (0.0000)	0.8918 (0.0000)	0.8666 (0.0000)	0.8591 (0.0000)
Log-Likelihood	-	-1459.6826	-2424.1550	-2525.9824	-2441.0090	-2095.6885
SIC	-	-1475.9532	-2440.4262	-2542.2484	-2457.2650	-2111.0210

As regards the skewness equation (see Tables 7 and 8), as before, significant presence of conditional skewness is found, with at least one of the coefficients associated with shocks to skewness (γ_1) and to lagged skewness (γ_2) being significant, except for the S&P500 stock index under the NAGARCHSK specification.

Table 6: NAGARCH Models – Stock Indices

The reported coefficients shown in each row of the table are ML estimates of the NAGARCH model:

$$r_t = \alpha_1 r_{t-1} + \varepsilon_t$$
$$h_t = \beta_0 + \beta_1(\varepsilon_{t-1} + \beta_3 h_{t-1}^{1/2})^2 + \beta_2 h_{t-1}$$

for percentage daily returns of S&P500, NASDAQ100, DAX30, IBEX35 stock indices, from January 1990 to July 2003, and MEXBOL from January 1995 to July 2003. $h_t = var(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ follows a $N(0,h_t)$ distribution. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	S&P500	NASDAQ	DAX30	IBEX35	MEXBOL
Mean equation	α_1	0.0461 (0.0098)	0.1387 (0.0098)	0.0200 (0.2602)	0.0956 (0.0000)	0.1665 (0.0000)
Variance equation	β_0	0.0126 (0.0028)	0.0270 (0.0055)	0.0332 (0.0010)	0.0560 (0.0009)	0.0852 (0.0142)
	β_1	0.0607 (0.0000)	0.1086 (0.0000)	0.0758 (0.0000)	0.0865 (0.0000)	0.0961 (0.0004)
	β_2	0.8776 (0.0000)	0.8605 (0.0000)	0.8855 (0.0000)	0.8609 (0.0000)	0.8169 (0.0000)
	β_3	-0.9588 (0.0000)	-0.4828 (0.0000)	-0.5678 (0.0000)	-0.5326 (0.0000)	-0.8349 (0.0000)
Log-Likelihood	-	-1401.8598	-2385.3512	-2496.0414	-2413.6763	-2050.0510
SIC	-	-1422.1982	-2405.6903	-2516.3739	-2433.9963	-2069.2165

Similar results are obtained for the kurtosis equation under both GARCHSK and NAGARCSK specifications. The coefficient associated with shocks to kurtosis (δ_1) is significant in all cases, except for NASDAQ100 with the GARCHSK model and to some extent for IBEX35 with the NAGARCH model. Moreover, the coefficient associated with lagged kurtosis (δ_2) is significant in all cases except S&P500 under both specifications. Nevertheless, there is significant presence of conditional kurtosis for all stock indices, with both specifications, since at least one of the coefficients associated with shocks to kurtosis or to lagged kurtosis is found to be significant.

As obtained with the exchange rate series, the value of the SIC rises monotonically for all stock index series analyzed when we move from the simpler models to the more complicated ones, with the NAGARCHSK model showing the highest value. This seems to be the most appropriate specification.

Table 7: GARCHSK Models – Stock Indices

The reported coefficients shown in each row of the table are ML estimates of the GARCHSK model:

$$
\begin{aligned}
r_t &= \alpha_1 r_{t-1} + \varepsilon_t \\
h_t &= \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} \\
s_t &= \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1} \\
k_t &= \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}
\end{aligned}
$$

for percentage daily returns of S&P500, NASDAQ100, DAX30, IBEX35 stock indices, from January 1990 to July 2003, and MEXBOL from January 1995 to July 2003. h_t = var(r_t | r_{t-1}, r_{t-2}, …), s_t = skewness(r_t | r_{t-1}, r_{t-2}, …), k_t = kurtosis(r_t | r_{t-1}, r_{t-2}, …), $\eta_t = \varepsilon_t h_t^{-1/2}$, and ε_t | ε_{t-1}, ε_{t-2}, … follows the distribution based on a Gram-Charlier series expansion. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	S&P500	NASDAQ	DAX30	IBEX35	MEXBOL
Mean equation	α_1	0.0211 (0.2285)	0.1229 (0.0000)	0.0080 (0.6557)	0.0949 (0.0000)	0.1775 (0.0000)
Variance equation	β_0	0.0023 (0.1117)	0.0098 (0.0202)	0.0261 (0.0119)	0.0417 (0.0042)	0.1228 (0.0028)
	β_1	0.0387 (0.0000)	0.0822 (0.0000)	0.0851 (0.0000)	0.0843 (0.0000)	0.1663 (0.0000)
	β_2	0.9586 (0.0000)	0.9149 (0.0000)	0.9021 (0.0000)	0.8928 (0.0000)	0.8023 (0.0000)
Skewness equation	γ_0	-0.0458 (0.0518)	-0.0886 (0.0106)	-0.0245 (0.2911)	-0.0446 (0.0161)	0.0228 (0.3101)
	γ_1	0.0085 (0.0139)	0.0078 (0.0032)	0.0048 (0.2006)	0.0189 (0.0000)	0.0125 (0.0136)
	γ_2	0.0227 (0.9187)	0.2174 (0.4136)	0.6781 (0.0168)	0.1352 (0.0852)	0.2969 (0.3112)
Kurtosis equation	δ_0	3.0471 (0.0000)	1.4576 (0.0175)	0.4866 (0.0016)	0.2526 (0.0026)	0.3302 (0.0254)
	δ_1	0.0055 (0.0019)	0.0007 (0.6228)	0.0010 (0.0229)	0.0004 (0.0129)	0.0010 (0.3634)
	δ_2	0.0882 (0.5715)	0.5518 (0.0034)	0.8493 (0.0000)	0.9208 (0.0000)	0.9018 (0.0000)
Log-Likelihood	-	-1404.5752	-2375.0218	-2484.1335	-2414.6928	-2056.0966
SIC	-	-1445.2519	-2415.7000	-2525.7985	-2455.3328	-2094.4277

Table 8: NAGARCHSK Models – Stock Indices

The reported coefficients shown in each row of the table are ML estimates of the NAGARCHSK model:

$$\begin{aligned} r_t &= \alpha_1 r_{t-1} + \varepsilon_t \\ h_t &= \beta_0 + \beta_1(\varepsilon_{t-1} + \beta_3 h_{t-1}^{1/2})^2 + \beta_2 h_{t-1} \\ s_t &= \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1} \\ k_t &= \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1} \end{aligned}$$

for percentage daily returns of S&P500, NASDAQ100, DAX30, IBEX35 stock indices, from January 1990 to July 2003, and MEXBOL from January 1995 to July 2003. $h_t = var(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $s_t = skewness(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $k_t = kurtosis(r_t \mid r_{t-1}, r_{t-2}, \ldots)$, $\eta_t = \varepsilon_t h_t^{-1/2}$, and $\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ follows the distribution based on a Gram-Charlier series expansion. All models have been estimated by ML using the Berndt-Hall-Hall-Hausman algorithm (quasi-maximum likelihood p-values in parenthesis).

	Parameter	**S&P500**	**NASDAQ**	**DAX30**	**IBEX35**	**MEXBOL**
Mean equation	α_1	0.0358 (0.0466)	0.1255 (0.0000)	0.0152 (0.4009)	0.1024 (0.0000)	0.1742 (0.0000)
Variance equation	β_0	0.0083 (0.0006)	0.01841 (0.0038)	0.0278 (0.0005)	0.04460 (0.0004)	0.1000 (0.0001)
	β_1	0.0416 (0.0000)	0.0986 (0.0000)	0.0696 (0.0000)	0.0729 (0.0000)	0.1202 (0.0000)
	β_2	0.9099 (0.0373)	0.8801 (0.0000)	0.8961 (0.0000)	0.8800 (0.0000)	0.7834 (0.0000)
	β_3	-1.0116 (0.0000)	-0.4351 (0.0000)	-0.5597 (0.0000)	-0.5795 (0.0003)	-0.7703 (0.0000)
Skewness equation	γ_0	-0.0451 (0.0373)	-0.0618 (0.0005)	-0.0261 (0.2285)	-0.0204 (0.1174)	0.0525 (0.0782)
	γ_1	0.0091 (0.1034)	0.0103 (0.0025)	0.0050 (0.1883)	0.0045 (0.1423)	0.0180 (0.0045)
	γ_2	0.0552 (0.7418)	0.4572 (0.0000)	0.6573 (0.0124)	0.5325 (0.0022)	0.1922 (0.5459)
Kurtosis equation	δ_0	3.1652 (0.0000)	1.6929 (0.0003)	0.4536 (0.0016)	0.2012 (0.0858)	1.9901 (0.0011)
	δ_1	0.0150 (0.0000)	0.0053 (0.0025)	0.0009 (0.0161)	0.0004 (0.0749)	0.0055 (0.0004)
	δ_2	0.0293 (0.6645)	0.4684 (0.0014)	0.8581 (0.0000)	0.9365 (0.0000)	0.4017 (0.0271)
Log-Likelihood	-	-1371.4169	-2351.1665	-2461.0251	-2382.5437	-2016.8569
SIC	-	-1416.1613	-2395.9126	-2505.7566	-2427.2477	-2059.0212

4. Comparison of models

One way to start comparing the models is to compute a likelihood ratio test. It is easy to see that the density function based on a Gram-Charlier series expansion in equation (2) nests the normal density function when $s_t = 0$ and $k_t = 3$ (alternatively when $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\delta_1 = 3$ and $\delta_2 = \delta_3 = 0$). Therefore, the restrictions imposed by the normal density function with respect to the more general density based on a Gram-Charlier series expansion can be tested by means of a likelihood ratio test. The results are contained in Table 9. The value of the LR statistic is quite large in all cases, indicating the rejection of the null hypothesis that the true density is the restricted one, i.e. the normal density function.

Table 9: Likelihood Ratio Tests

The table shows the values of the maximized log-likelihood function (logL) when the distribution for the error term is assumed to be normal (standard GARCH or NAGARCH specification) and when it is assumed to be a Gram-Charlier series expansion of the normal density (GARCHSK or NAGARCHSK specification), the likelihood ratio (LR) and asymptotic p-values, for the series employed in the paper.

PANEL A: EXCHANGE RATES

STATISTIC	GBP/USD	JPY/USD	DEM/USD	CHF/USD
LogL(GARCH)	409.3	-352.6	-149.3	-451.7
LogL(GARCHSK)	472.4	-237.7	-117.6	-421.0
LR **(p-value)**	126.1 (0.00)	229.9 (0.00)	63.4 (0.00)	61.5 (0.00)

PANEL B: STOCK INDICES

STATISTIC	S&P500	NASDAQ100	DAX30	IBEX35	MEXBOL
LogL(NAGARCH)	-1401.9	-2385.4	-2496.0	-2413.7	-2050.1
LogL(NAGARCHSK)	-1371.4	-2351.2	-2461.0	-2382.5	-2016.9
LR **(p-value)**	60.9 (0.00)	68.4 (0.00)	70.0 (0.00)	62.3 (0.00)	72.8 (0.00)

A second way consists of comparing the properties of the conditional variances obtained in each model. Figure 1 shows the behavior of conditional variance for one of the exchange rate series -GBP/USD- with both GARCH and GARCHSK models, and for one of the stock index series -S&P500- with both NAGARCH and NAGARCHSK specifications. It is clear that conditional variances obtained with models accounting for time-varying skewness and kurtosis are smoother than those obtained with standard GARCH or NAGARCH models. This is confirmed by the results in Table 10, which shows some descriptive statistics for these conditional variances. In fact, conditional variances obtained with GARCHSK or NAGARCHSK models show less standard deviation, skewness and kurtosis than those obtained under the standard models. This fact was observed by Harvey and Siddique (1999) with their time-varying skewness but a constant kurtosis specification.

Table 10: Descriptive Statistics for Conditional Variances

The table shows the main descriptive statistics for the conditional variances obtained from GARCH and GARCHSK models for GBP/USD series, and from NAGARCH and NAGARCHSK models for S&P500 series.

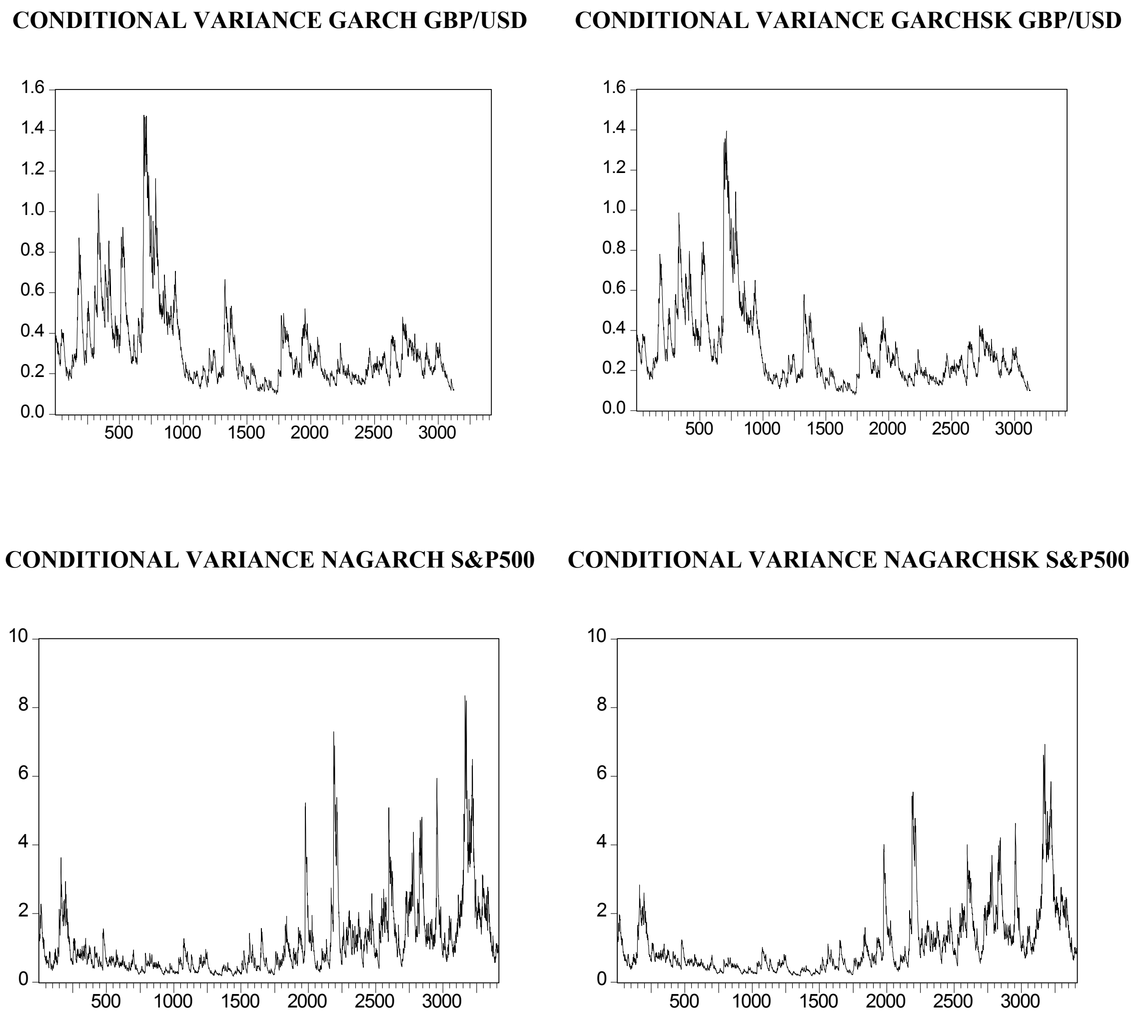
	GBP/USD		**S&P500**	
STATISTIC	h_t - GARCH	h_t – GARCHSK	h_t - NAGARCH	h_t - NAGARCHSK
Sample size	3124	3124	3413	3413
Mean	0.3264	0.3026	1.1394	1.0928
Median	0.2647	0.2432	0.7692	0.7513
Maximum	1.4762	1.3944	8.3534	6.9340
Minimum	0.0988	0.0776	0.1731	0.1771
Stand. Dev.	0.2034	0.1980	1.0575	0.9533
Skewness	2.2384	2.1624	2.5160	2.2077
Kurtosis	9.4659	8.9007	11.1431	8.9475
Jarque-Bera (p-value)	8050.721 (0.0000)	6966.893 (0.0000)	13030.79 (0.0000)	7802.598 (0.0000)

The in-sample predictive ability of the different models is compared by means of two metrics. The variable predicted is the squared forecast error (ε_t^2) and the predictors are the conditional variances (h_t) from, respectively, the standard GARCH or NAGARCH models and GARCHSK or NAGARCHSK models. The two metrics are:

Median absolute error: $MAE = med(|\varepsilon_t^2 - h_t|)$

Median percentage absolute error: $MPAE = med\left(\frac{|\varepsilon_t^2 - h_t|}{\varepsilon_t^2}\right)$

Figure 1: Estimated Conditional Variances With Nagarch and Nagarchsk Models



The metrics are based on the median since it is more robust than the mean. The results are shown in Table 11. Models accounting for time-varying skewness and kurtosis in general outperform standard GARCH or NAGARCH models. They are the best performing models under the two metrics with all the exchange rate and stock index series except for NASDAQ100 and IBEX35 under the median absolute error but not under the median percentage absolute error.

Furthermore, it is worth noting that the series that performs best, based on these metrics, is the MEXBOL stock index, which is the series with the highest values of unconditional standard deviation, skewness and kurtosis (see Table 1). This result could suggest the potential application of our methodology to financial series from emerging economies, characterized by higher risk and more pronounced departures from normality.

Table 11: In-Sample Predictive Power

The variable predicted is the squared forecast error (ε_t^2) and the predictors are the conditional variances (h_t) from, respectively, the standard GARCH or NAGARCH models and GARCHSK or NAGARCHSK models. Two metrics are chosen to compare the predictive power ability of these models:

1. Median absolute error $MAE = med(| \varepsilon_t^2 - h_t |)$
2. Median percentage absolute error $MPAE = med\left(\frac{| \varepsilon_t^2 - h_t |}{\varepsilon_t^2} \right)$

The metrics are based on the median given the high dispersion of the error series.

SERIES		MAE	MPAE
GBP/USD	G	0.2030	1.9227
	GSK	0.1874	1.6567
JPY/USD	G	0.3369	2.2226
	GSK	0.3165	2.0134
DEM/USD	G	0.3058	1.7982
	GSK	0.2895	1.6028
CHF/USD	G	0.3749	1.8096
	GSK	0.3635	1.6788
S&P500	NG	0.5884	1.7690
	NGSK	0.5723	1.7670
NASDAQ	NG	0.9061	1.3801
	NGSK	0.9209	1.3075
DAX30	NG	1.0225	1.5102
	NGSK	1.0207	1.5071
IBEX35	NG	1.0081	1.4610
	NGSK	1.0109	1.4349
MEXBOL	NG	1.6743	1.6508
	NGSK	1.6308	1.5531

Finally, Table 12 shows some descriptive statistics for 30-day simple moving average measures of both skewness and kurtosis and also, the statistics corresponding to the conditional skewness and kurtosis under the GARCHSK/NAGARCHSK models, specifically the GBP/USD and S&P 500 series. The main feature of Table 12 is that conditional measures of skewness and kurtosis show less standard deviation than those obtained with the 30-day simple moving average. Also, at least for the S&P 500 series, conditional measures seem to provide more pronounced (negative) skewness and higher leptokurtosis (see the median statistics). Meanwhile, for the GBP/USD series though leptokurtosis is higher for the conditional case, a positive skewness is higher under the moving average case (see median statistics).

Table 12. Descriptive Statistics for Simple Moving Average and Conditional Coefficients of Skewness And Kurtosis

The table shows the main descriptive statistics for 30-day simple moving average and conditional skewness and kurtosis, for GBP/USD (using the GARCHSK model) and for S&P500 (using the NAGARCHSK model) series.

	GBP/USD Skewness		GBP/USD Kurtosis	
STATISTIC	Mov.-Aver. S_t	Conditional S_t	Mov.-Aver. K_t	Conditional K_t
Sample size	3096	3096	3096	3096
Mean	0.0012	0.0167	3.4223	3.4969
Median	0.0164	0.0134	3.1494	3.4665
Maximum	2.6956	2.8019	12.1532	9.0119
Minimum	-1.8734	-0.9511	1.4685	3.4540
Stand. Dev.	0.5836	0.1074	1.2151	0.1570
	S&P 500 Skewness		**S&P500 Kurtosis**	
STATISTIC	Mov.-Aver S_t	Conditional S_t	Mov.-Aver. K_t	Conditional K_t
Sample size	3385	3385	3385	3385
Mean	-0.0171	-0.0503	3.2323	3.3387
Median	0.0240	-0.0477	2.9155	3.2649
Maximum	1.7763	0.4568	12.2603	34.6040
Minimum	-2.4311	-2.8565	1.7728	3.2608
Stand. Dev.	0.5142	0.0806	1.2030	0.7273

5. Conclusions

It is well known that the generalized autoregressive conditional heteroscedasticity (GARCH) models, introduced by Engle (1982) and Bollerslev (1986) allow for time-varying volatility but neither time-varying skewness nor time-varying kurtosis. However, given the increasing attention that time-varying skewness and kurtosis have attracted in option pricing literature, it may be useful to analyze a model that jointly accounts for conditional second, third and fourth moments.

Harvey and Siddique (1999) present a way of jointly estimating time-varying conditional variance and skewness, assuming a non-central t distribution for the error term in the mean equation. We propose a GARCH-type model allowing for time-varying volatility, skewness and kurtosis. The model is estimated assuming a Gram-Charlier series expansion of the normal density function, along the lines suggested by Gallant and Tauchen (1989), for the error term in the mean equation. This distribution is easier to estimate than the non-central t distribution proposed by Harvey and Siddique (1999). Moreover, our approach accounts for time-varying skewness and kurtosis while the one by Harvey and Siddique (1999) only accounts for time-varying skewness.

Our model is estimated using daily returns of four exchange rate series, five stock indices and the emerging market index MEXBOL (Mexico). Our results indicate significant presence of conditional skewness and kurtosis. Moreover, it is found that specifications allowing for time-varying skewness and kurtosis outperform specifications with constant third and fourth moments.

Finally, it is important to point out two main implications of our GARCHSK and NAGARCHSK model. First, they can be useful in estimating future coefficients of volatility, skewness and kurtosis, which are unknown parameters in option pricing models that account for nonnormal skewness and kurtosis. For example, estimates of volatility, skewness and kurtosis from the NAGARCHSK model, based on historical series of returns, could be compared with option implied coefficients in terms of their out of sample option pricing performance. Second, our models could be useful in testing the information content of option implied coefficients of volatility, skewness and kurtosis. This could be done by including option implied coefficients as exogenous terms in the equations of volatility, skewness and kurtosis, extending the papers by Day and Lewis (1992), Lamoureux and Lastrapes (1993) and Amin and Ng (1997), among others.

APPENDIX

Here we show that the nonnegative function $f(\eta_t | I_{t-1})$ in (3) is really a density function, which is it integrates to one. We can rewrite $\psi(\eta_t)$ in (2) as:

$$\psi(\eta_t) = 1 + \frac{s_t}{\sqrt{3!}} H_3(\eta_t) + \frac{k_t - 3}{\sqrt{4!}} H_4(\eta_t)$$

where $\{H_i(x)\}_{i \in N}$ represents the Hermite polynomials such that $H_0(x) = 1,\ H_1(x) = x$ and for $i \geq 2$ they hold the following recurrence relation:

$$H_i(x) = \left(xH_{i-1}(x) - \sqrt{i-1}H_{i-2}(x)\right) / \sqrt{i}\,.$$

It is verified that $\{H_i(x)\}_{i \in N}$ is an orthonormal basis satisfying that:

$$\int_{-\infty}^{\infty} H_i^2(x)\phi(x)\,dx = 1, \quad \forall i \qquad \text{(A-1)}$$

$$\int_{-\infty}^{\infty} H_i(x) H_j(x)\phi(x)\,dx = 0, \quad \forall i \neq j \qquad \text{(A-2)}$$

where $\phi(\cdot)$ denotes the N(0,1) density function. If we integrate the conditional density function in (3) and given conditions (A-1) and (A-2):

$$(1/\Gamma_t)\int_{-\infty}^{\infty} \phi(\eta_t)\left[1 + \frac{s_t}{\sqrt{3!}} H_3(\eta_t) + \frac{k_t - 3}{\sqrt{4!}} H_4(\eta_t)\right]^2 d\eta_t$$

$$= (1/\Gamma_t)\left[\int_{-\infty}^{\infty} \phi(\eta_t)\,d\eta_t + \frac{s_t^2}{3!}\int_{-\infty}^{\infty} H_3^2(\eta_t)\phi(\eta_t)\,d\eta_t + \frac{(k_t - 3)^2}{4!}\int_{-\infty}^{\infty} H_4^2(\eta_t)\phi(\eta_t)\,d\eta_t\right]$$

$$= (1/\Gamma_t)\left[1 + \frac{s_t^2}{3!} + \frac{(k_t - 3)^2}{4!}\right]$$

$$= 1.$$

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