A discusión

SOLVING THE NEOCLASSICAL GROWTH MODEL WITH QUASI-GEOMETRIC DISCOUNTING: NON-LINEAR EULER-EQUATION MODELS*

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ABSTRACT

The neoclassical growth model with quasi-geometric discounting is shown by Krusell and Smith (2000) to have multiple solutions. As a result, value-iterative methods fail to converge. The set of equilibria is however reduced if we restrict our attention to the interior (satisfying the Euler equation) solution. We study the performance of the grid-based and the simulation-based Euler-equation methods in the given context. We find that both methods converge to an interior solution in a wide range of parameter values, not only in the "test" model with the closed-form solution but also in more general settings, including those with uncertainty.

JEL Classification: C73, D90, E21

Keywords: quasi-geometric (hyperbolic) discounting, time-inconsistency, neoclassical growth model, numerical methods

1 Introduction

In the recent literature, much attention has been paid to studying the implications of models with quasi-geometric (quasi-hyperbolic) discounting, e.g., Laibson (1997), Laibson, Repetto and Tobacman (1998), Barro (1999), Harris and Laibson (2001), Krusell and Smith (2000, 2003), Krusell, Kuruşçu and Smith (2002). Under such a discounting, the short-run discount factor (applied between today and tomorrow) differs from the long-run discount factor (applied between tomorrow and the day after tomorrow, and onwards). This property leads to a dynamic game played between multiple selves with conflicting (time-inconsistent) preferences.

Krusell and Smith (2000) incorporate quasi-geometric discounting into the deterministic version of the standard neoclassical growth model with one agent. There are no markets; rather, the agent operates her own technology. Under the assumptions of the logarithmic utility function, the Cobb-Douglas production function and full depreciation of capital, the model allows for a closed-form solution. However, numerical algorithms iterating on value function fail to converge to such a solution. Krusell and Smith (2000) explain the failure of the value-iterative approach by the fact that, in addition to its closed-form solution, the model has multiple step-function equilibria.

In a related paper, Krusell et al. (2002) emphasize that the closed-form solution has one property, which distinguishes it from all other solutions, namely, that it is the only solution satisfying the Euler equation. That is, the interior (differentiable) solution to the model is unique. This example indicates that, the value-function and the Euler-equation characterizations of the optimal choice of a quasi-geometric consumer are not equivalent. In this paper, we therefore address the following question: Will numerical algorithms iterating on the Euler equation be successful in arriving at the interior solution, as opposed to those iterating on value function?

We first investigate the performance of the algorithm that solves the Euler equation on a grid of prespecified points, in the context of the deterministic neoclassical growth model with the closed-form solution. We find that the algorithm converges to the closed-form solution in a wide range of parameter values, provided that the grid is not very fine and that the decision rules are updated slowly enough. However, if the difference between the short-run and the long-run discount factors is very large, we observe a lack of convergence. As a further step, we apply the grid algorithm to a more general variant of the model, the one with a Constant Relative Risk Aversion (CRRA) period utility function and a partial depreciation of capital. The solutions delivered by our grid method proved to be identical to those found by the perturbation method developed in Krusell et al. (2002).

We finally employ numerical methods to solve for equilibria in the stochastic version of the neoclassical growth model. To our knowledge, this version of the model has not been studied in the literature yet. The two methods we use here are the stochastic extension of the grid algorithm and the simulation-based variant of the Parameterized Expectation Algorithm (PEA) proposed by den Haan and Marcet (1990). Once again we observe that if the short-run discount factor is not very different from the long-run one, and if the algorithms' parameters (the number of grid points and the updating parameter) are appropriately chosen, both algorithms converge to the interior solution. Our numerical results indicate that the interior solution is unique not only in the model with the closed-form solution but also in more general settings, including those with uncertainty.

The rest of the paper is organized as follows: Section 2 describes the problem of the quasi-geometric consumer, its recursive formulation and its Euler equation. Section 3 discusses the model with the closed-form solution. Section 4 presents the numerical results and finally, Section 5 concludes.

2 The model

We consider a neoclassical economy populated by one quasi-geometric agent.[1] Time is discrete and infinite, $t \in \{0, 1, 2, ...\}$. On each date t, the agent solves the following utility maximization problem:

$$\max_{\{c_\tau, k_{\tau+1}\}_{\tau=t}^{\infty}} \left\{ u(c_t) + E_t \sum_{\tau=t}^{\infty} \beta\delta^{\tau+1-t} u(c_{\tau+1}) \right\} \tag{1}$$

$$\text{s.t.} \qquad c_\tau + k_{\tau+1} = (1-d)\, k_\tau + \theta_\tau f(k_\tau), \tag{2}$$

where initial condition (k_t, θ_t) is given. Here, c_τ is consumption, k_τ is capital, θ_τ is the technology shock, u is the period utility function, f is the production function, E_t is the operator of the conditional expectation, $d \in (0, 1]$ is the depreciation rate of capital, and $\beta > 0$ and $\delta \in (0, 1)$ are the discounting parameters. We assume that u and f are strictly increasing, strictly concave, continuously differentiable and satisfy the Inada conditions and that the random variable $\ln \theta_{t+1}$ follows $AR(1)$ process, $\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$ with $\rho \in [0, 1)$ and $\varepsilon_{t+1} \sim N(0, v^2)$.

The period utilities in (1) are weighted by 1, $\beta\delta$, $\beta\delta^2$, $\beta\delta^3$, ... Krusell and Smith (2000) call such discounting quasi-geometric because with the exception of the current period t, the weights decline geometrically over time.

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[1] Krusell et al. (2001) interpret such an economy as the planner's one.

The standard case of geometric discounting corresponds to $\beta = 1$. If $\beta > 1$ ($\beta < 1$), then the short-run discount factor, $\beta\delta$, is higher (lower) than the long-run one, δ, so that the agent is short-run patient (impatient). The case of $\beta < 1$ is also referred to in the literature as quasi-hyperbolic discounting, (see, e.g., Laibson, 1997, Harris and Laibson, 2000). The assumption of quasi-geometric discounting leads to time-inconsistency in preferences in the sense that the relative value of consumption in any two adjacent periods t and $t+1$ depends on the date on which the evaluation is performed. We assume that the agent is fully aware of her preference inconsistency and also, that she cannot commit herself to fulfilling her plans.

We restrict our attention to the recursive first-order Markov equilibrium. We assume that the agent chooses the next period's capital stock k_{t+1} according to a time-invariant policy function, $k_{t+1} = g(k_t, \theta_t)$. We denote by $W(k_t, \theta_t)$ the optimal value of the expected discounted utility of the agent whose current state is k_t and θ_t, and who makes her decisions according to the policy function g. The recursive formulation of the problem (1), (2) is as follows:

$$W(k, \theta) = \max_{k'} \left\{ u\left((1-d)k + \theta f(k) - k'\right) + \beta\delta E\left[V(k', \theta') \mid \theta\right] \right\}, \qquad (3)$$

where $V(k', \theta')$ satisfies the recursive functional equation

$$V(k', \theta') = u\left((1-d)k' + \theta' f(k') - g(k', \theta')\right) + \delta E\left[V\left(g(k', \theta'), \theta''\right) \mid \theta'\right], \tag{4}$$

and k, θ are given. Here, one and two primes are used to denote values of the variables one and two periods from the current date, respectively.

If an interior solution of the problem (3), (4) exists, then it satisfies the Euler equation,

$$u'(c_t) = \delta E_t\left\{u'(c_{t+1})\left(\beta\left(1-d+\theta_{t+1} f'(k_{t+1})\right) + (1-\beta)\frac{\partial g(k_{t+1}, \theta_{t+1})}{\partial k_{t+1}}\right)\right\}. \tag{5}$$

A distinctive feature of the Euler equation (5), compared to the standard one, is the appearance of the last term on the right-hand side: it contains the derivative of the unknown decision rule, $\frac{\partial g(k_{t+1}, \theta_{t+1})}{\partial k_{t+1}}$. The deterministic steady state in such a model satisfies

$$1 = \delta\left(\beta\left(1-d+f'\left(\overline{k}\right)\right) + (1-\beta)\frac{\partial g\left(\overline{k}, 1\right)}{\partial k}\right), \tag{6}$$

where $\overline{k}$ denotes the steady state level of capital. In a standard model ($\beta = 1$), equation (6) delivers $\overline{k}$ straightforwardly. With quasi-geometric discounting ($\beta \neq 1$), however, matters are more complicated. Here, we have only one equation but two unknowns: the steady state level of the function g (since $\overline{k} = g\left(\overline{k}\right)$ by definition) and its first derivative, $\frac{\partial g\left(\overline{k}, 1\right)}{\partial k}$, at this point. The consequence of this fact is that we cannot compute the steady state without solving for the function g.

3 The model with a closed-form solution

In this section, we consider a variant of the model that allows for a closed-form interior solution. Let us assume that the period utility function is logarithmic, $u(c) = \ln(c)$, that the production function is Cobb-Douglas, $f(k) = k^{\alpha}$ with $\alpha \in (0,1)$ and that capital depreciates fully in each period, $d = 1$. As such, the value function V and the policy function $k' = g(k,\theta)$, solving (3), (4), are given by

$$V(k,\theta) = \frac{1}{1-\delta}\left(\ln\frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} + \frac{\delta\alpha}{1-\delta\alpha}\ln\frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}\right) + \frac{\alpha}{1-\delta\alpha}\ln k + \frac{1}{(1-\delta\rho)(1-\delta\alpha)}\ln\theta, \tag{7}$$

$$k' = \frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}\theta k^{\alpha}. \tag{8}$$

3.1 Multiplicity of discontinuous solutions

Krusell and Smith (2000) study the deterministic version of the model ($\theta_t = 1$ for all t) and find that numerical algorithms iterating on value function fail to converge to the closed-form solution. They explain the failure of the value-iterative approach by the fact that the model has multiple solutions. "The multiplicity takes two forms. First, there is a continuum of stationary points for the consumer's asset holdings. Second, for each stationary point, there is a continuum of paths leading into this stationary point" (Krusell and Smith, 2000, p. 17). The interval of possible stationary points (steady states) for the capital stock is given by

$$\overline{k} \in \left((f')^{-1} \left(\frac{1}{\beta\delta} \right), \ (f')^{-1} \left(\frac{1 - \delta (1 - \beta)}{\beta\delta} \right) \right). \tag{9}$$

The paths leading to each steady state are discontinuous (they take the form of step functions).

3.2 Uniqueness of an interior solution

Krusell et al. (2002) argue that the closed-form solution is a unique interior solution to the model. This fact can be shown by computing the limit of the solution to the finite-horizon model "by hand", as is done in the standard geometric-discounting case (see, Manuelli and Sargent, 1987). The iterations on value function "by hand" lead to a sequence of value functions, which converges to value function (7).

Why is it that the discontinuous solutions described in Krusell and Smith (2000) do not affect the convergence when iterations on value function are performed "by hand"? It is because such solutions are ruled out by the assumption that the equilibrium is interior (i.e., satisfies the Euler equation).

It is therefore of interest to investigate whether numerical algorithms iterating on the Euler equation converge to interior solutions.

4 Computing an interior equilibrium

In this section, we investigate the performance of two non-linear Euler-equation methods in the context of the model with quasi-geometric discounting. We begin by describing the methods used. We then present the numerical results obtained for the model with the closed-form solution, and we finally discuss the results obtained for more general variants of the model and compare them to those presented in Krusell et al. (2002).

4.1 Description of the Euler-equation methods used

Our first method solves the Euler equation on a grid of prespecified points, and it is applied to both the deterministic and stochastic versions of the model. The second method is the simulation-based version of the Parameterized Expectation Algorithm (PEA) by den Haan and Marcet (1990), and it is applied exclusively to the stochastic version of the model.

For all numerical experiments, we assume the Constant Relative Risk Aversion (CRRA) utility function, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, where $\sigma > 0$, and the Cobb-Douglas production function, $f(k) = k^{\alpha}$, and we fix $\delta = 0.95$, $\alpha = 0.36$.

In the stochastic case, we parameterize the process for shock by $\rho = 0.95$ and $v = 0.01$. If $\sigma = 1$ and $d = 1$, we obtain the model with the closed-form solution.

4.1.1 Grid algorithm

We restrict the domain of the capital stock to the interval $[k_{\min},\ k_{\max}] = \left[0.25\overline{k}^{\,*},\ 4\overline{k}^{\,*}\right]$, where $\overline{k}^{\,*}$ is the steady state value of capital in the model with standard geometric discounting. We consider an equally-spaced grid of N points. To evaluate the policy function outside the grid, we use the cubic polynomial interpolation. To solve the stochastic version of the model, we approximate the autoregressive process for the logarithms of shocks by a Markov chain with seven states, $\Theta \equiv \left\{0, \pm\frac{5v}{3}, \pm\frac{5v}{2}, \pm 5v\right\}$, as in Tauchen (1986). For each state $\ln\theta \in \Theta$, we parametrize the next period's capital stock as a function of the current capital stock.

By substituting consumption from the Euler equation (5) in budget constraint (2), we obtain

$$\begin{aligned}&\widetilde{g}\left(k,\theta\right) \equiv k' = \left(1-d\right)k + \theta k^{\alpha} - \\ &-\left\{\delta \sum_{\ln\theta' \in \Theta}\left[\frac{\beta\left(1-d+\theta'\alpha g\left(k,\theta\right)^{\alpha-1}\right) + \left(1-\beta\right)\frac{\partial g\left(g\left(k,\theta\right),\theta'\right)}{\partial g\left(k,\theta\right)}}{\left(\left(1-d\right)g\left(k,\theta\right) + \theta' g\left(k,\theta\right)^{\alpha} - g\left(g\left(k,\theta\right),\theta'\right)\right)^{\sigma}}\right]\pi\left(\theta' \mid \theta\right)\right\}^{-1/\sigma},\end{aligned}$$

where $\pi\left(\theta' \mid \theta\right)$ is the probability of θ' conditional on θ.

We implement the following iterative procedure: Fix some policy function on the grid, $g(k,\theta)$, and use it to re-calculate $\widetilde{g}(k,\theta)$ in each point of the grid. Compute the policy function for the next iteration by using updating, $\eta\widetilde{g}(k,\theta)+(1-\eta)g(k,\theta)$, where $\eta\in(0,1]$. Iterate until $\widetilde{g}(k,\theta)=g(k,\theta)$ with a given precision.

4.1.2 Parameterized expectations algorithm

We re-write the Euler equation (5) in terms of the policy function for consumption, $c(k,\theta)$, and approximate the conditional expectation as

$$E_t\left\{c_{t+1}^{-\sigma}\left(1-d+\alpha\theta_{t+1}k_{t+1}^{\alpha-1}-(1-\beta)\frac{\partial c(k_{t+1},\theta_{t+1})}{\partial k_{t+1}}\right)\right\}$$
$$\simeq\exp(\xi_0+\xi_1\log\theta_t+\xi_2\log k_t),$$

where $\xi=(\xi_0,\xi_1,\xi_2)$ is a vector of coefficients to be found. Under the above approximation, we have

$$c_t=\{\delta\exp(\xi_0+\xi_1\log\theta_t+\xi_2\log k_t)\}^{-1/\sigma},\qquad\frac{\partial c(k_{t+1},\theta_{t+1})}{\partial k_{t+1}}=-\frac{\xi_2}{\sigma}\frac{c_{t+1}}{k_{t+1}}.$$

We draw and fix a random series for technology $\{\theta_t\}_{t=1}^{T}$ and perform the following steps: Calculate recursively the series $\{c_t(\xi),k_{t+1}(\xi)\}_{t=1}^{T}$ by using the assumed policy function for consumption. Run a non-linear least-squares regression of the variable within the expectation on the approximating function in order to re-estimate the vector of coefficients, $\widetilde{\xi}$. Compute the coefficients for the next iteration by using the updating, $\eta\widetilde{\xi} + (1-\eta)\,\xi$, where $\eta \in (0,1]$. Iterate until $\widetilde{\xi} = \xi$ with a given precision.

4.2 Numerical results for the model with the closed-form solution

We find that whether the algorithm converges to the closed-form solution or not depends on specific values of the model's and the algorithms' parameters. As far as the grid algorithm is concerned, the convergence depends crucially on the number of grid points for capital, N, and the value of β. We shall also mention that in order to ensure convergence, the policy function should be updated much more slowly than in the usual geometric discounting case, e.g., $\eta = 0.01$.

In the deterministic case, for example, if $N = 100$, the algorithm converges to the closed-form solution under $\beta \in [0.4, 1.6]$. When the grid is refined, the range of values of β leading to convergence narrows down: if $N = 300$, the algorithm converges under $\beta \in [0.8, 1.2]$; if $N = 1000$, the convergence range is $\beta \in [0.95, 1.05]$ and, finally, if $N = 10000$, the algorithm diverges even under $\beta \in [0.99, 1.01]$.[2] In the first panel of Table 1, we

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[2] The fact that the accuracy of approximation can affect the convergence is also observed by Krusell and Smith (2000) for value-iterative methods: "The algorithm may converge if g is approximated with very low accuracy (with few grid points, or with an inflexible

Table 1. The steady-state value of capital in the deterministic model.[a]

β	0.8	0.9	1	1.1	1.2
The model with the closed form solution ($d = 1, \sigma = 1$).					
Exact solution	.147426	.167507	.187032	.205955	.224254
Approximation	.147405	.167492	.187025	.205955	.224254
The model with $d = 0.1$.					
$\sigma = 0.5$	1.9986	2.8734 (2.87)[b]	3.8219	4.8013	5.7838
$\sigma = 1$	2.3900	3.0902 (3.09)[b]	3.8219	4.5690	5.3205
$\sigma = 2$	2.6960	3.2536 (3.25)[b]	3.8219	4.3943	4.9667
$\sigma = 3$	2.8373	3.3282 (3.33)[b]	3.8219	4.3149	4.8049
$\sigma = 4$	2.9226	3.3729 (3.37)[b]	3.8219	4.2672	4.7076
$\sigma = 5$	2.9810	3.4035 (3.40)[b]	3.8219	4.2345	4.6411
$\sigma = 6$	3.0240	3.4260 (3.43)[b]	3.8219	4.2106	4.5922
$\sigma = 7$	3.0574	3.4435 (3.44)[b]	3.8219	4.1919	4.5543

Notes: [a] Parameter values: $\alpha = 0.36$, $\delta = 0.95$, $N = 100$.

[b] The numbers in parenthesis correspond to the solution reported by Krusell et al. (2002).

compare the exact and approximate solutions for the steady state value of the capital stock under $N = 100$ and $\beta \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$. We observe that the algorithm delivers a relatively high degree of precision, even when the number of nodes is not very large.

In the stochastic case, the performance of the grid algorithm is similar. The range of β, under which the algorithm converges to the closed-form solution for each particular N, is however somewhat larger. For example, if $N = 100$, the convergence is achieved under $\beta \in [0.3, 2.0]$.

Under the PEA, we observe similar regularities. To be more specific, if β is not very different from one and the policy function is updated slowly enough, the algorithm systematically converges to the closed-form solution. For example, if $\eta = 0.01$, we have convergence under $\beta \in [0.4, 1.8]$. By using a lower value of η, we can somewhat increase the interval of convergence, e.g, if $\eta = 0.001$, the algorithm converges under $\beta \in [0.3, 2.0]$.

The results in Krusell and Smith (2003) allow us to gain intuition on why the model's and the algorithms' parameters can affect the convergence in the quasi-geometric discounting case. This paper specifically shows that the multiple discontinuous solutions described in Krusell and Smith (2000) satisfy a difference (non-differentiable) analogue of the Euler equation (5). The

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functional form)".

consequence is that within the multiplicity interval, the numerical methods fail to distinguish the true (closed-form) solution to the Euler equation from a bunch of nearby discontinuous "pseudo solutions".

Let us consider the grid algorithm. If β is not very different from one, then the multiplicity interval is relatively small. If, in addition, the grid is not very fine (i.e., N is small), then there are few nodes in this interval. In such a case, the algorithm finds the closed-form solution. When the value of β deviates significantly from one, the multiplicity interval increases, and so does the number of nodes in this interval. Similarly, the number of nodes in the multiplicity interval increases if the grid is refined. When the number of nodes lying in the multiplicity interval becomes sufficiently large, the algorithm produces a cycling.[3]

Under the PEA, the policy function is computed on the simulated time series of the technology shocks and the capital stocks, i.e., the grid here is endogenous. When β is close to one, the multiplicity interval is small, and most of the simulated capital stocks are outside of this interval. In such a case, we have convergence to the closed-form solution. When β deviates from

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[3] This suggests the following modification of the algorithm. Construct the grid so that all nodes are placed outside the multiplicity interval and compute the decision rules in the multiplicity interval by using interpolation. We find that this method performs very well if β is not very different from one, e.g., $\beta \in [0.4, 1.6]$, however it also fails when β differs from one significantly, so that the multiplicity interval is very large.

one significantly, a high portion of the simulated capital stocks belongs to the multiplicity interval, which leads to the failure of the algorithm.

4.3 Comparison to Krusell et al. (2002) and further examples

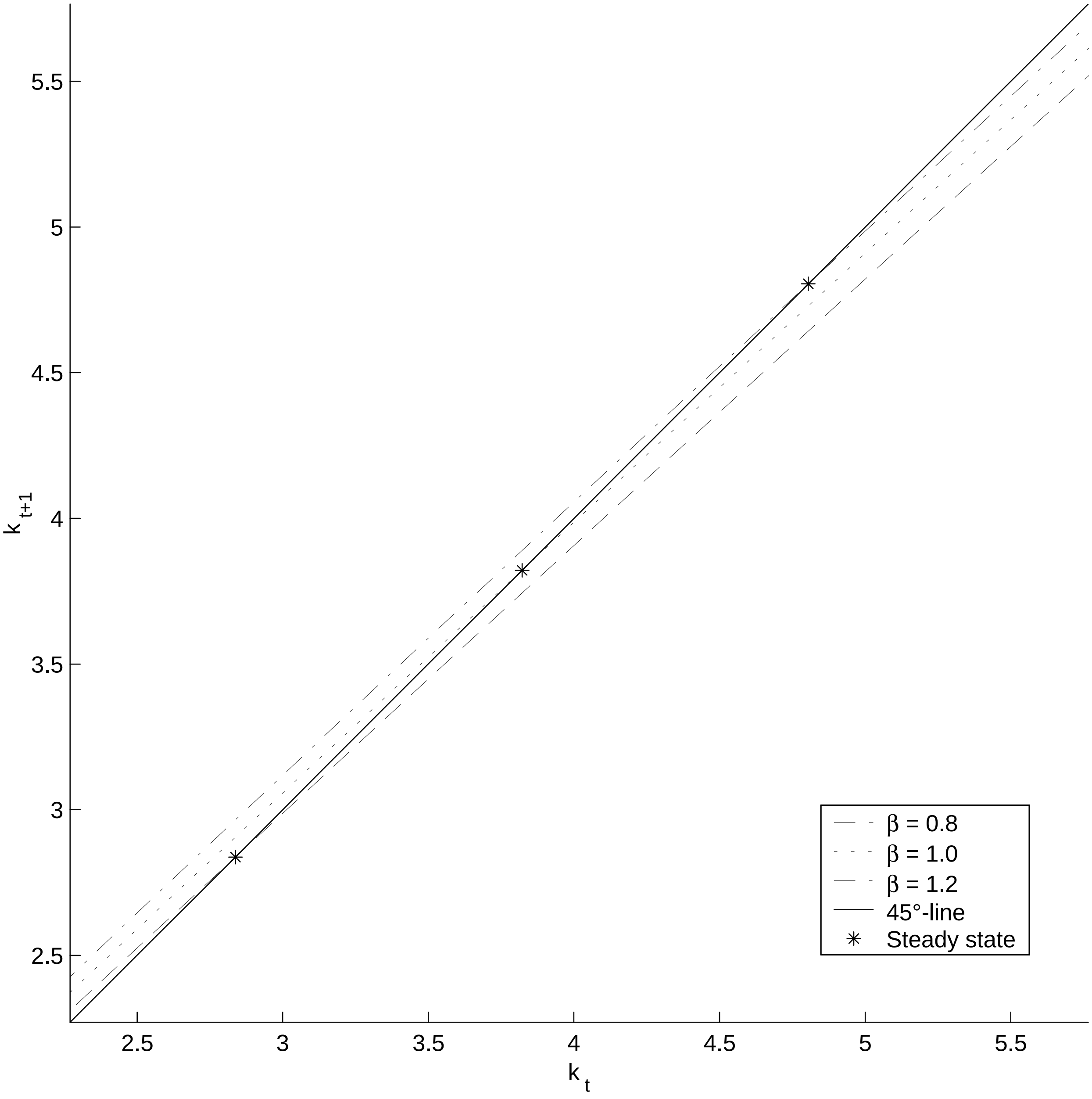
Except for the case when the model allows for the closed-form solution, no proof of existence and uniqueness of an interior solution to the model has been provided in the literature. The most extensive study of these issues is done by Krusell et al. (2002), where a perturbation algorithm is applied to compute a solution to the deterministic model with $\sigma \neq 1$ and $d \neq 1$, and the indications obtained from that study is that the interior equilibrium is unique. It is of interest, therefore, to investigate the performance of our computational methods in the model with $\sigma \neq 1$ and $d \neq 1$ and to determine whether they yield the same equilibria as those found in Krusell et al. (2002).

The convergence properties of our computational methods in the model with $\sigma \neq 1$ and $d \neq 1$ proved to be very similar to those in the model with the closed-form solution. Specifically, both the grid algorithm and the PEA converge to a unique interior solution provided that the value of β is not very different from one and that the policy function is updated slowly enough. To achieve the convergence under the grid algorithm, we should use the grid which is not very fine.

Krusell et al. (2002) compute the solution to the deterministic version of the model under $\beta = 0.9$, $d = 0.1$, and $\sigma \in \{0.5, 1, 2, 3, 4, 5, 6, 7\}$. We consider the same values of the parameters d and σ, and explore several values of β, namely, $\beta \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$. In Table 1, we report the steady state values of the capital stock computed by our grid algorithm for the deterministic model. For the sake of comparison, we also provide the results obtained by Krusell et al. (2002). The main thing to be noted here is that our solutions are identical to those computed by the perturbation method in Krusell et al. (2002). Regarding the properties of the solutions, we can observe the following tendencies: The steady state value of capital increases (decreases) with σ for a given value of β when $\beta < 1$ ($\beta > 1$), and it increases with β for a given value of σ. The latter tendency is illustrated in Figure 1, where we plot the computed decision rules and the corresponding steady states under $\beta \in \{0.8, 1.0, 1.2\}$ and $\sigma = 3$.

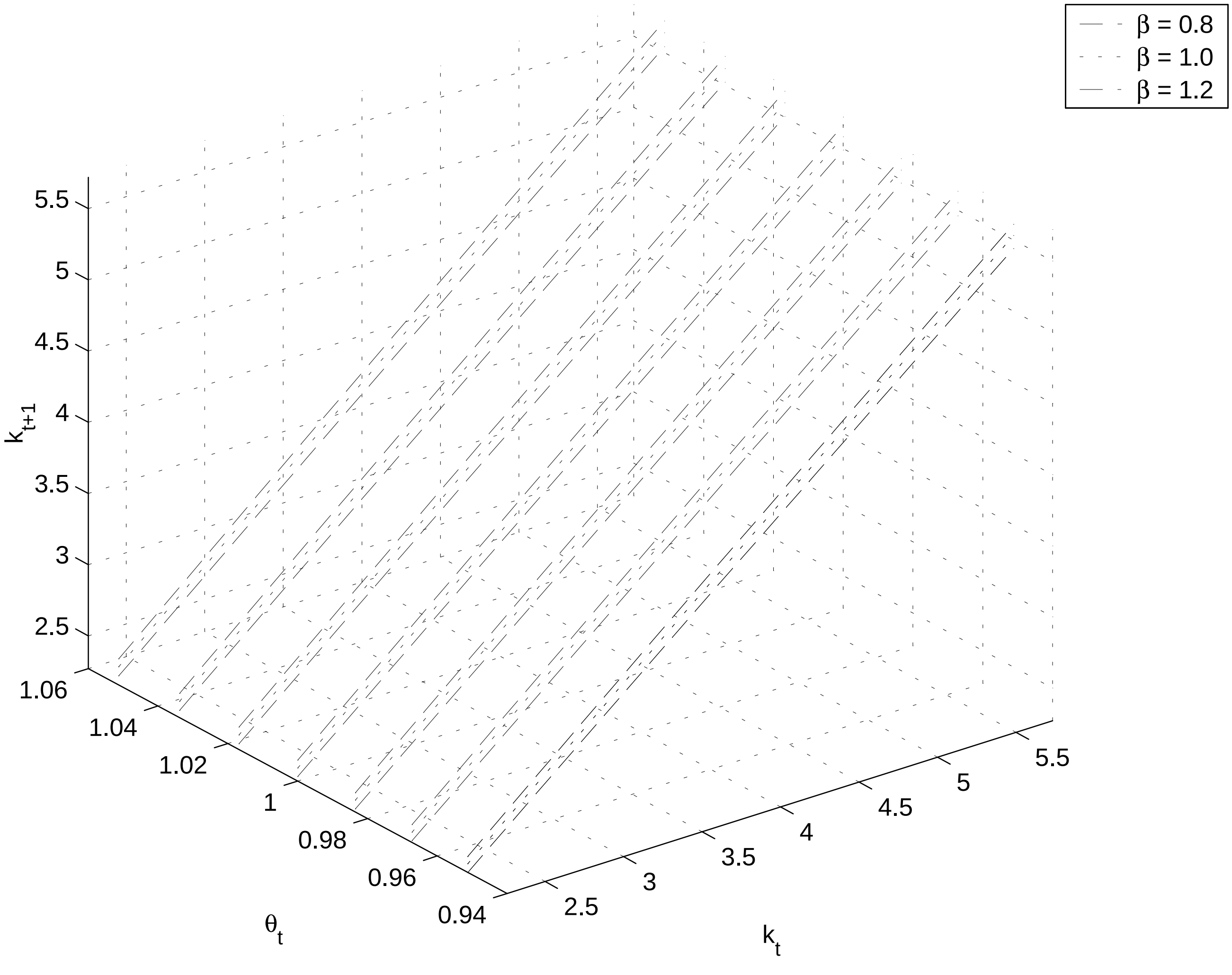
We finally investigate the properties of the solutions to the stochastic version of the neoclassical growth model with $\sigma \neq 1$ and $d \neq 1$. We report the results under $\beta \in \{0.8, 1.0, 1.2\}$ and $\sigma = 3$. In Figure 2, we plot the policy functions computed by the grid algorithm. In Figure 3, we plot the

Figure 1. The grid algorithm: the policy function in the deterministic model.[a]



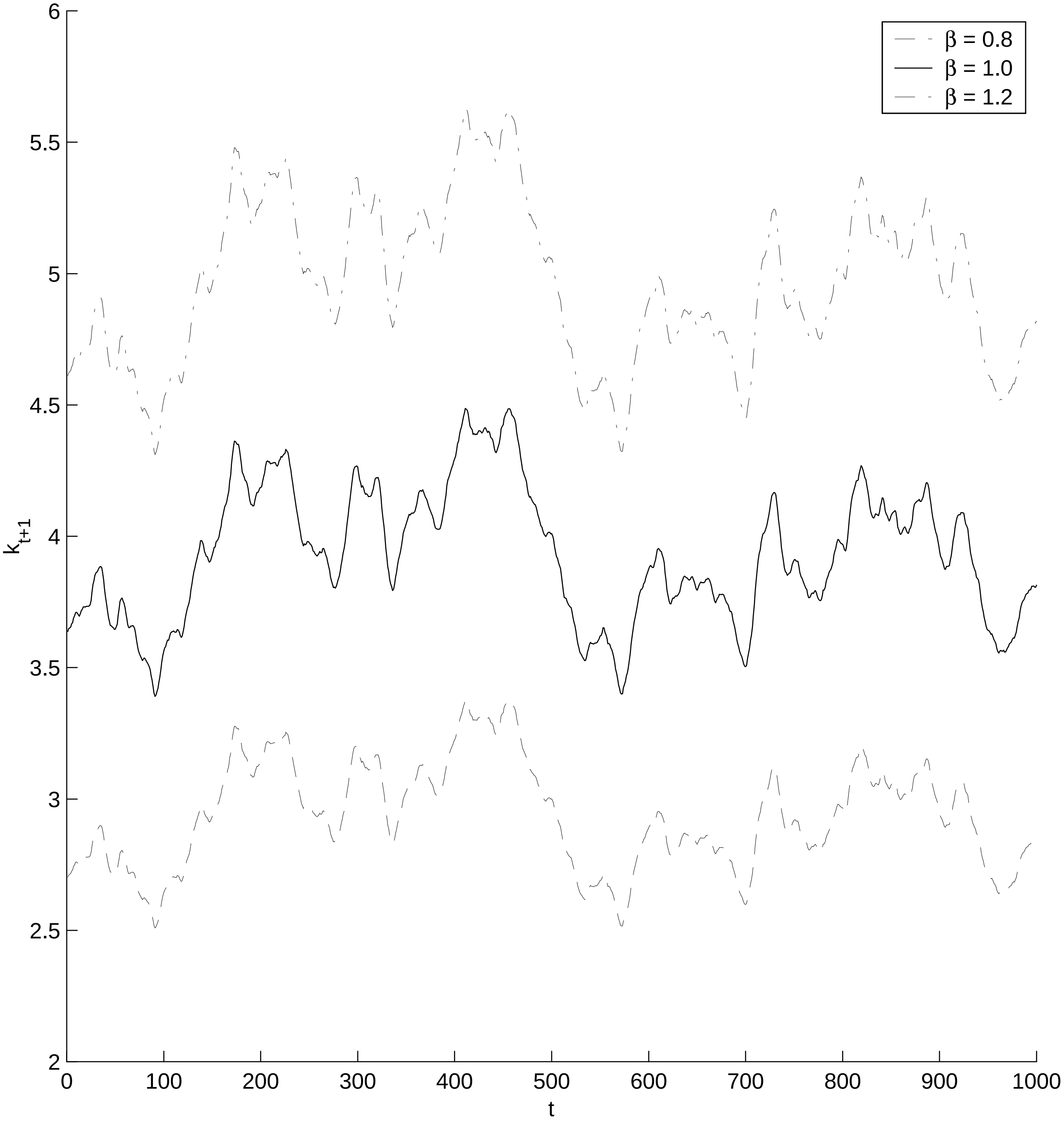
[a] Parameter values: α=0.36, d=0.1, δ=0.95, σ=3, N=100.

Figure 2. The grid algorithm: the policy function in the stochastic model.[a]



[a] Parameter values: α=0.36, d=0.1, δ=0.95, σ=3, ρ=0.95, v=0.01, N=100.

Figure 3. The PEA: the time-series solution in the stochastic model.[a]



[a] Parameter values: α=0.36, d=0.1, δ=0.95, σ=3, ρ=0.95, v=0.01.

time-series solutions for the capital stock obtained by using the PEA (we use an identical sequence of technology shocks in the three simulations). The noteworthy finding in the figures is that the solutions under all three values of β are very similar. The main difference is that an agent with $\beta > 1$ ($\beta < 1$) holds more (less) capital than the one with $\beta = 1$, i.e., the short-run patient (impatient) agent tends to over-save (under-save) relative to the one with $\beta = 1$.

5 Conclusion

This paper studies the possibility of using non-linear Euler-equation methods for computing equilibrium in the neoclassical growth model, where the agent has quasi-geometric discounting. Under the logarithmic utility function, the Cobb-Douglas production function and full depreciation of capital, the model is known to yield a closed-form solution. However, this is not the only solution. As shown in Krusell and Smith (2000), the model also has multiple discontinuous solutions in the form of step-functions. As a consequence of such multiplicity, we observe several features, which are not typical for the standard geometric discounting case. First, the methods considered allow us to find the solution for only a limited range of values of the discounting parameter β, even though the solution exists for any nonnegative value of this parameter. Second, under the grid method, we cannot achieve an arbitrary accuracy by refining the grid, because the method fails to converge when the grid becomes too fine. Finally, to enforce convergence, we have to update the decision rules very slowly (much more slowly than in the usual geometric discounting case).

The conclusions of this paper are therefore twofold. The performance of the traditional Euler-equation methods in the context of a model with quasi-geometric discounting is not entirely satisfactory, and other methods, such as the perturbation method proposed in Krusell et al. (2002), must be developed. Yet, the Euler-equation methods, like those we studied here, can be a simple and useful alternative in many empirical applications, in spite of all their limitations. Indeed, we have been able to find the solution to the model in a wide range of parameter values. This is not only true for the "test" model with the closed-form solution but also for more general settings.

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