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WP-AD 2003-07

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Editor: Instituto Valenciano de Investigaciones Económicas, S.A.
Primera Edición Febrero 2003
Depósito Legal: V-898-2003

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* Chattopadhyay acknowledged financial support from the Spanish government (CICyT project BEC2001-0980) and the Ivie. A preliminary and incomplete version of this paper appeared as the first chapter in Moro's doctoral dissertation at Universidad de Alicante. We thank J. Caballe, I. Collado, I. Ortuño, S. Ortigueira and C. Urrutia for their comments on the dissertation. Any remaining errors are our own.

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ABSTRACT

We study an overlapping generations economy with altruistic agents in which the productivity of a child's labour endowment depends on an idiosyncratic shock and on the resources spent by her parent in education her. The parent cannot borrow but can leave a nonnegative bequest which earns a deterministic return in the capital market; this possibility mitigates the liquidity constraint faced by an agent when deciding on the level of education for her child. The shock is assumed to follow a Markov process thus allowing for serial correlation in abilities. A stationary equilibrium of the model is a situation in which the endogenously determined aggregate amounts of capital and efficiency units of labour remain constant, so factor prices are constant, the choices made by agents can be summarized via an invariant distribution, and factor supplies are determined by the mean, taken with respect to the invariant distribution, of the agents' decision rules, and all the markets clear. We develop and discuss conditions under which a stationary equilibrium exists.

Keywords: Stationary Equilibrium, Idiosyncratic Shocks, Altruism, Two Sector Model, Human Capital

1. Introduction

This paper develops a theoretical framework that permits the simultaneous analysis of the distribution of earnings, the distribution of income, and the distribution of bequeathed wealth in general equilibrium. Empirical analysis reveals that each of these distributions has characteristic features that simpler models fail to capture. We propose the study of stationary equilibrium in such a model as the solution concept and provide conditions for the existence of such an equilibrium.

We use an overlapping generations model in which agents live for three periods and a child is born to each middle-aged agent. An agent gets utility from her consumption when middle-aged and when old; in addition, she cares about the utility from consumption that her descendents obtain. This provides us with a model with altruistic agents where, in effect, the family is the decision making unit; as a consequence, in some respects the behaviour of the model is similar to that of a model with an infinite lifetime. Heterogeneity of behaviour is induced by the fact that each agent is assigned an ability level which is random. We allow an agent's ability to depend on her parent's ability so that factors which may influence the family environment can be accommodated; formally, an agent's ability is the realization of a time homogeneous Markov process. The parent can invest in human capital, that is buy education, to affect the productivity of the single unit of labour that the child is endowed with. Furthermore, the parent can decide to leave a bequest which the child receives when she is middle-aged. Both the decisions, however, must be made when the child is young while ability is realized when the child is middle aged. Clearly, both activities affect the child's income and, in turn, the consumption possibilities of all the descendents. The fact that the parent has two options enriches the decision problem faced by the parent. The only sources of income in the model are wage income and the bequest that a child receives from her parent; consumption of the agent when old is determined by what she saved explicitly for retirement out of the income she earned when she was middle-aged. We impose the restriction that the bequest be nonnegative.³ The bequest and any saving for retirement are invested in the capital market where they receive a deterministic return. When we specify a sequence of factor prices, we obtain the family's decision rule as the solution to a recursive dynamic programme.

In order to infer the behaviour of aggregates from that of the individual, we posit that uncertainty is idiosyncratic and that there is a large number of families. So every family uses the same decision rule but different families can be in different "states", where we choose to use the level of education, the level of bequest, and

³This is standard in the literature (see, e.g., Laitner (1992)) and the following argument shows why it is reasonable. Since the only source of income for the agent when old is what she saved for retirement, she is unable to repay a loan taken out in middle age with her own funds when old. So to take out a loan the agent must borrow against her child's future income. This leads to a potential moral hazard problem where the parent borrows ostensibly to educate her child but in fact increases her own consumption. The analysis of such a model, though interesting, is not the purpose of this paper. Hence, we assume—as an institutional datum—that parents cannot force their children to repay their debts. It follows that the parent must finance the child's education with her own funds and cannot borrow implying that the bequest is nonnegative.

realized ability as a description of the state. This fact lets us induce stochastic dynamics using an initial distribution of the state variables and the decision rule to generate a sequence of distribution functions for all the variables of interest. That in turn lets us induce mean values, taken using the distribution over the state vector that prevails in the corresponding period, for physical capital that all the agents taken together wish to hold, the quantity of efficiency units of labour that is available, and the level of demand for the consumption good. Since uncertainty is idiosyncratic, the mean values are not random variables.

The model presented so far is one of a class of models with incomplete markets which have in common the following feature: There is a large number of agents who face unisurable idiosyncratic risks and a constraint on indebtedness.⁴ But the description so far is that of a partial equilibrium; after all, the decision rule adopted leads to mean values as indicated above and the mean values need not be compatible with the sequence of factor prices that the agents assume will prevail. A general equilibrium requires that factor markets and the market for the consumption good also clear at each date. We focus attention on a special kind of general equilibrium, namely, *stationary equilibrium* which is the stochastic analogue of a steady state and requires that factor prices be constant across periods. When factor prices are constant, it is natural to describe the behaviour of families in a manner which is independent of time and an invariant distribution of the state variables is the tool which does exactly that. So for any time invariant specification of factor prices we can obtain the mean values of demand for the three goods using some invariant distribution. A general equilibrium is obtained when the stationary factor prices are such that all the markets clear.

The equilibrium notion that we work with appears to be deceptively simple since factor prices and the values of aggregate variables remain constant over time. However, the interest in the concept derives from the fact that there is movement within the invariant distribution as the economic wealth of families rises and falls even as the distribution remains the same over time. In a fundamental way, the model allows for two means of intergenerational transfers, namely, via human capital accumulation, through expenditure on education, which is a private activity since the technology is available to every agent, and physical capital accumulation, through bequeathed wealth, which is a market based activity. In fact, the agent's decision problem can be viewed as a dynamic portfolio allocation problem in which one of the assets pays a stochastic and nonlinear return while the other pays a deterministic and linear return, where, and this is an essential feature of our analysis, we allow the different endogenous variables to take corner values as the family's income varies. In addition we have the fact that the bequest acts to mitigate the constraint on borrowing to finance a child's education. These facts taken together allow the model to generate rich dynamics. Finally, since we consider a general equilibrium, the associated feedback effects are necessarily present and force us to provide a consistent

⁴Aiyagari (1994) provides a useful primer to such models and their importance for research in macroeconomics.

description of the interaction between agents.

The model that we have specified has a lot in common with that in Loury (1981) where families live in isolation from each other, labour is the only factor of production, and parents invest in their children by providing them with education before the realization of the child's ability. We enrich the framework of analysis by allowing for greater interaction between agents within the same family, via the bequest, and across families, via the markets for factors and the produced good. Also, we allow for serial correlation in abilities. We believe that by augmenting the model we obtain a more powerful tool for studying and explaining phenomena related to the distributions of income, earnings, etc. The body of work done by Laitner considers many of the elements that we add to Loury's model but does not allow for investment in human capital. Laitner (1992) proves existence of a stationary equilibrium and is of particular relevance; in that paper agents live for many periods, a labour supply function which is increasing in an i.i.d. random variable is exogenously given, altruism is two sided, there is exogenous technological progress, and the utility function is isoelastic. The notion of stationary equilibrium has appeared in other contexts, e.g., Bewley (1986) proves existence of stationary monetary equilibrium in an exchange economy.

To shed light on our existence result, let us first consider a model with an infinite lifetime and partial equilibrium. In the deterministic version of the model with a single asset, the steady state (with a time invariant asset position) is pinned down by the requirement that $\beta R = 1$ where β is the discount factor and R is the gross rate of return on the asset. When there is endowment uncertainty but the agent can save in a risk free asset subject to a nonnegativity restriction, the model is called an *income fluctuation problem* and has been analyzed in some detail; from the work of Schechtman and Escudero (1977) it is known that asset accumulation is bounded when $\beta R < 1$ and the utility function satisfies an additional condition.⁵

Consider now the deterministic version of our model with an overlapping generations structure and altruistic agents. In this case $\beta R < 1$ is compatible with a steady state in which no bequest is given. In fact, to see more clearly the nature of the problem, let us consider a variant of our model in which the agent does not care about consumption when old and $\beta R < 1$. Under the circumstances prescribed, the demand for physical capital must be zero since no bequest is given and the agent does not wish to save for old age. So a steady state exists only if $\beta R = 1$ and that too only if the technology by which education is converted into productivity and the technology for producing the consumption good are compatible. This is indicative of the fact that, in the deterministic case, existence of a steady state is easier to prove if the agent cares enough about consumption when old since in that case demand for physical capital is high relative to the level of labour supply at all factor prices satisfying $\beta R \leq 1$.

When there is uncertainty, stationarity of the equilibrium does not require any

⁵Chamberlain and Wilson (2000) show that asset accumulation must diverge if $\beta R > 1$, and they provide conditions under which the same is true even when $\beta R = 1$.

agent to maintain the level of income of her family at a constant value across generations; instead, we require that aggregate behaviour exhibit constancy in time captured by an invariant distribution. This is in contrast to the situation in the deterministic case and gives greater leeway. To show that a stationary equilibrium exists we have to prove the existence of invariant distributions given factor prices and we have to show that the set of such distributions is nice and that the set exhibits continuity as we change factor prices. Since our problem has the flavour of an income fluctuation problem, we are able to prove existence of an invariant distribution for factor prices which obey the restriction $\beta R < 1$ for physical capital accumulation and an analogous condition for investment in human capital. To do so we impose a number of standard conditions on the primitives of the model which are satisfied if the technology by which skills are produced is bounded, continuous and concave, and the technology for producing the consumption good exhibits constant returns to scale and its intensive form is concave. In addition, we need a modification of the condition identified by Schechtman and Escudero (1977) to ensure that invariant distributions exist. We do not impose Inada type boundary conditions. We proceed by using Walras Law and the fact that the technology for producing the consumption good exhibits constant returns to scale to reduce the existence problem to a one dimensional fixed point problem. The result is clinched by ensuring that mean demand for physical capital relative to mean labour supply behaves nicely for factor prices such that βR is close to one, i.e., a boundary behaviour condition; this is similar to the situation in the deterministic case where one needs to ensure that demand for physical capital is positive at all factor prices such that $\beta R \leq 1$. We show that, under a mild condition on the behaviour, around zero, of the function by which human capital is produced from education, the boundary condition is satisfied and a stationary equilibrium exists if the region (near zero) in which the intensive form of the technology for producing the consumption good has marginal returns to capital higher than $1/\beta$ is relatively small, or if mean demand for physical capital diverges to infinity as R approaches $1/\beta$.

It is possible to show that at every stationary equilibrium of our model there is a set of agents whose state receives positive measure under the invariant distribution for whom the nonnegativity constraint on bequests binds. We do not develop these results since the method of proof is standard (see Laitner (1992)). An analysis of the welfare properties of the model and how these results change when a social security system is introduced is desirable and left for future research. We make one final observation: Since our focus is on existence, we have not considered whether the equilibrium set can be characterized more sharply. Following Brock and Mirman (1972), it is common (Loury (1981) and Laitner (1992)) to provide conditions under which the invariant distribution is unique and globally attracting when shocks are i.i.d. Unfortunately, in our Markovian framework, such a result is much more difficult to obtain; we indicate why this is so in Section 6 where we discuss our result in greater detail. Fortunately, existence of a stationary equilibrium can be established even if the invariant distribution is not unique and/or not attracting.

The rest of the paper is organized as follows: The model is specified in Section 2, the individual's decision problem is specified and analysed in Section 3, results on the properties of the set of invariant distributions is the subject matter of Section 4, aggregate variables are defined in Section 5, the definition of stationary equilibrium, the existence theorem, and a discussion are provided in Section 6, and all proofs are grouped together in Section 7. We use a number of results available in the monograph by Stokey and Lucas (1989); however, we provide complete proofs of results that are not stated in the form in which we need them.

2. Model

Consider a world in which at each discrete date three goods are available: a consumption good, a capital good, and labour. There are a large number of families and each generation within a family, to be called an "agent", lives for three periods. In the first period of an agent's life, the agent's parent spends resources in educating the agent, in the second period of life the agent works and consumes and saves out of income, and in the third period the agent consumes out of saving and leaves a bequest. Income consists of wage income, earned by supplying inelastically one unit of labour where the wage depends upon the agent's education and innate ability, and the bequest left by the parent. Agents are competitive and take the return on capital and the wage to be given. Innate ability is assumed to follow a time homogeneous Markov process and the decision on how much to spend on education and how much to leave as a bequest must be made before the realization of ability. Bequests are restricted to be nonnegative. Agents are assumed to be identical in all respects including the uncertainty that they face; they differ by the realization of their own ability and the education and bequest that they receive. Agents are altruistic and take into account the utility obtained by their descendents. We close the model by assuming that there is a technology which permits aggregate capital and labour to be converted into units of the consumption good.

More formally, $t = 0, 1, 2, \dots$ denotes dates. $i \in [0, 1]$ identifies a family. An agent is denoted (i, t) , where i is the family and $t - 1$ is the date of birth. Innate ability, $\alpha_{i,t}$, is a random variable taking values in \mathcal{A} and is realized at date t . We assume that $\alpha_{i,t}$ is generated by a Markov process on the state space $\mathcal{A} \subset [0, 1]$ with transition function P . So $P(a, A)$ denotes the probability that an agent of family i born at date $t - 1$ has ability $\alpha_{i,t}$ in the set A given that the parent had ability $\alpha_{i,t-1} = a$. Let \mathcal{B}^n denote the Borel σ -algebra of R^n and, for X a \mathcal{B}^n -measurable subset of R^n , let \mathcal{B}_X denote the restriction of \mathcal{B}^n to X . We assume

Assumption A.1: (i) $\mathcal{A} \in \mathcal{B}_{[0,1]}$ and \mathcal{A} is compact. (ii) $P : \mathcal{A} \times \mathcal{B}_{\mathcal{A}} \rightarrow [0, 1]$ is a Markov transition function so (a) for all $a \in \mathcal{A}$, $P(a, \cdot)$ is a probability measure on the measurable space $(\mathcal{A}, \mathcal{B}_{\mathcal{A}})$, and (b) for all $A \in \mathcal{B}_{\mathcal{A}}$, $P(\cdot, A)$ is a measurable function.⁶ (iii) P has the Feller Property so for $f : \mathcal{A} \rightarrow R$ a continuous function,

⁶So for all $Y \in \mathcal{B}_{[0,1]}$, $P_{\mathcal{A}}^{-1}(Y) \in \mathcal{B}_{\mathcal{A}}$ where $P_{\mathcal{A}}^{-1}(Y) := \{x \in \mathcal{A} : P(x, A) \in Y\}$.

$\int_{\mathcal{A}} f(y)P(x, dy)$ is continuous in x .

In our context it is quite natural to assume that the set of abilities is bounded. The rest of the assumptions are standard technical requirements. A special case of interest in which A.1 holds is when $\alpha_{i,t}$ is distributed independently and identically in time. A.1 allows the support to be a finite set, an interval, or the union of the two.

Agent (i, t) , born at date $t - 1$, receives education $e_{i,t}$ at date $t - 1$ and her ability, $\alpha_{i,t}$, is realized at date t . She supplies one unit of labour inelastically at date t , when she is middle aged, and the random productivity of that unit of labour is determined by a function ψ which combines units of innate ability with units of education into efficiency units of labour. We assume

Assumption A.2: (i) $\psi : \mathcal{A} \times R_+ \rightarrow R_+$ is continuous in α and e . (ii) ψ is concave in e for all $\alpha \in \mathcal{A}$ and $\inf_{\alpha \in \mathcal{A}} \lim_{e \rightarrow \infty} \psi(\alpha, e) > 0$. (iii) Either (a) $\psi(\alpha, 0) = 0$ for all $\alpha \in \mathcal{A}$ or (b) $\inf_{\alpha \in \mathcal{A}} \psi(\alpha, 0) > 0$. (iv) ψ is continuously differentiable in e on R_{++} and $\psi_2(\alpha, e)$ is measurable in $\alpha \in \mathcal{A}$ for all $e \geq 0$.⁷

A.2 (i) and the first part of both A.2 (ii) and A.2 (iv) are standard assumptions requiring continuity of ψ , and that it be concave and continuously differentiable in e . We have imposed very little structure on the dependence of ψ on α beyond continuity. We remark that since $\psi(\alpha, \cdot)$ is defined on R_+ and takes nonnegative values, the fact that $\psi(\alpha, \cdot)$ is concave (in e) implies that it is also weakly increasing (in e). The second part of A.2 (ii) is a nondegeneracy assumption which ensures that for every ability level it is possible to make one's labour endowment productive by acquiring enough education (the limit is well defined if $\psi(\alpha, \cdot)$ is concave, i.e., under the first part of A.2 (ii)); also, the two parts of A.2 (ii) taken together imply that $\psi(\alpha, e) > 0$ for all $(\alpha, e) \in \mathcal{A} \times R_{++}$. A.2 (iii) imposes a form of equity since it requires that with no education, either (a) no one is productive or (b) everyone is productive (though, possibly, with different productivities). A.2 (iii b) allows for the possibility that, even with no education, every agent is able to produce a positive amount, a condition that we find to be acceptable since every agent is assumed to be endowed with a fixed number of labour hours; our existence result will be stated for this case and we will indicate how the conditions need to be changed if A.2 (iii a) is assumed instead.⁸ The second half of A.2 (iv) is a technical requirement which lets us work with the first order conditions to characterize the solution to the optimization problem faced by a family; also, A.2 (iii) lets us simplify the first order conditions (see Lemma 7).

⁷For a real valued function $f : A \rightarrow R$ where $A \subset R^n$, $f_i(x)$ denotes the partial derivative with respect to the i -th coordinate evaluated at $x \in \text{int } A$.

⁸Loury (1981) imposes the conditions that $\psi(0, 0) = 0$ and ψ is strictly increasing in both variables on R_+^2 so that $\psi(\alpha, 0) > 0$ for $\alpha > 0$, a condition intermediate between the two options that we specify in A.2 (iii). Brock and Mirman (1972) impose A.2 (iii a) on the production function. Both of those papers impose additional regularity assumptions and consider the i.i.d. case and require the support of the shock to be a compact interval. They need stronger assumptions since they prove uniqueness and global stability of the invariant distribution.

Since, under A.2 (ii), $\psi(\alpha, \cdot)$ is concave, $\lim_{e \rightarrow 0^+} \psi_2(\alpha, e)$ and $\lim_{e \rightarrow \infty} \psi_2(\alpha, e)$ can be defined (though they can take the value ∞). We define

$$\begin{aligned}\overline{\psi_2^0} &:= \sup_{\alpha \in \mathcal{A}} \lim_{e \rightarrow 0^+} \psi_2(\alpha, e), \\ \underline{\psi_2^0} &:= \inf_{\alpha \in \mathcal{A}} \lim_{e \rightarrow 0^+} \psi_2(\alpha, e), \\ \overline{\psi_2^\infty} &:= \sup_{\alpha \in \mathcal{A}} \lim_{e \rightarrow \infty} \psi_2(\alpha, e), \\ \underline{\psi_2^\infty} &:= \inf_{\alpha \in \mathcal{A}} \lim_{e \rightarrow \infty} \psi_2(\alpha, e).\end{aligned}$$

Furthermore, under A.2 (ii) ψ is uniformly bounded if and only if $\overline{\psi_2^\infty} = 0$.

The agent (i, t) 's wage income is $w \cdot \psi(\alpha_{i,t}, e_{i,t})$ and she receives a bequest $R \cdot b_{i,t}$ at date t , where both w and R are independent of time. The *income* that (i, t) has at date t , $y_{i,t}$, is given by $w \cdot \psi(\alpha_{i,t}, e_{i,t}) + R \cdot b_{i,t}$. This specification of income lets us capture the fact that a positive bequest by the parent does not increase the child's income when the return on saving is zero; of course, under a monotonicity assumption on the payoff the parent will never leave a positive bequest if $R = 0$. The agent (i, t) allocates her income to consumption at date t , denoted $c_{i,t}$, and to educate her child, $e_{i,t+1}$, and saves the rest by renting it as capital. In the last period of her life she receives $R \cdot (y_{i,t} - c_{i,t} - e_{i,t+1})$ as the gross return on her saving and she uses it to consume in old age, denoted $R \cdot s_{i,t}$, and leave a bequest denoted $R \cdot b_{i,t+1}$.

The values of $e_{i,t}$ and $b_{i,t}$ are determined by $(i, t - 1)$'s parent without knowledge about $\alpha_{i,t}$, but $y_{i,t}$, $c_{i,t}$, and $s_{i,t}$, can depend on $\alpha_{i,t}$. The variables c , s , e , and b are required to be nonnegative; the restriction is natural for the first three variables (since the variable s can be identified with consumption when old) while, in the case of bequest, it reflects an institutional arrangement or social code which prevents a child from being held responsible for a debt contracted by her parent.

It is best to think of the family as a unit that extends not only into the infinite future but also into the infinite past, and to imagine that we conduct our analysis by picking a particular date, 0, considering the actions taken by agents upto date -1 as given, and analysing the problem from the perspective of an agent who has to act at date 0, namely an agent born at date -1 . Formally, we shall treat the vector $(e_{i,0}, b_{i,0}, \alpha_{i,0})$ as the initial condition for agent $(i, 0)$.

The utility that accrues to agent $(i, t - 1)$ from her own consumption is given by $u(c_{i,t}, R \cdot s_{i,t})$.

Assumption A.3: (i) $u : R_+^2 \rightarrow R$ is bounded and continuous. (ii) u is strictly increasing. (iii) u is strictly concave. (iv) u is continuously differentiable on R_{++}^2 and, for $i = 1, 2$, $\lim_{x \rightarrow 0^+, y \rightarrow 0^+} u_i(x, y)$ exists.

We define $\bar{u}_i := \lim_{x \rightarrow 0^+, y \rightarrow 0^+} u_i(x, y)$, for $i = 1, 2$, where the limiting value is allowed to be ∞ .

Families are bound together by altruism modelled here as a discounted value of

the utility from consumption obtained by an agent's descendants.

Assumption A.4: $\beta \in (0, 1)$.

A.3 is a standard assumption on the one period payoff function. A.4 is also standard. Notice that we do not allow $\beta = 0$; otherwise there is no motive for intergenerational interaction which causes the economy to collapse to one in which only physical capital is accumulated to provide for consumption in old age.

As to the production side of the economy, we assume that there is a technology which converts units of capital and efficiency units of labour into a produced good and depreciated capital. The technology is specified via a neoclassical production function exhibiting diminishing marginal returns, and constant returns to scale.

Assumption 5: *Let $F : R_+^2 \rightarrow R_+$ be homogeneous of degree one, strictly quasi-concave, strictly increasing, and continuously differentiable on R_{++}^2 . Let $\delta \in (0, 1]$. The input vector (K, L) produces $F(K, L)$ units of the produced good and $(1 - \delta) \cdot K$ units of depreciated capital.*

Since F is differentiable and homogeneous of degree one, the functions F_1 and F_2 are homogeneous of degree zero, and $F(K, L) = K \cdot F_1(K, L) + L \cdot F_2(K, L)$.

One final element of the model remains to be specified. We wish to consider a situation in which, even though each individual family faces risk which is uncorrelated across families, in the aggregate the risk is “washed out”. So we need a law of large numbers for a continuum of independent random variables. The problems that are posed in obtaining such a result are well known as are various solutions;⁹ we have nothing to add to the discussion but we shall follow the established tradition of “ignoring” the problem.¹⁰ Hence, we do not formally state an assumption specifying the nature of the interaction of the shocks faced by the different families; in Section 5 we specify the aggregation procedure that we follow and indicate what this means in terms of the underlying stochastic structure of the model.

This completes the description of the model. We have not yet given a definition of equilibrium; we provide it in Section 6 since the notion of equilibrium that we use requires additional concepts and notation which we introduce in later sections.

Remark 1: We close this section with an observation. A slight change in our specification lets us obtain a two sector model with infinitely lived agents subject to idiosyncratic shocks. This is accomplished by changing A.3 so that only consumption when middle aged gives utility, and by interpreting the bequest as an intertemporal transfer of wealth; the nonnegativity constraint is quite natural in this context. Our specification, in which the agent supplies one unit of labour inelastically in each period and can make an investment in human capital in order to make her labour productive in the next period, corresponds to a situation in which human capital

⁹See Feldman and Gilles (1985) for a lucid discussion and a proposed solution based on a countable set of agents which is, unfortunately, difficult to implement.

¹⁰Equivalently, we can assume directly that the relevant expected values of various random variables are measurable and constant on a common measure space (see, e.g., Bewley (1986)).

depreciates completely each period and seems out of place in a context where the agent lives forever; one might prefer to change the model a bit and posit instead that the education induces a change, relative to the current period, in the number of efficiency units of labour supplied by the agent in the following period.

3. The Optimization Problem

We proceed to formally specify the decision problem faced by a typical family. Set $\Omega := \mathcal{A} \times \mathcal{A} \times \cdots$ with typical element ω , $\mathcal{B}^0 := \{\emptyset, \Omega\}$, and \mathcal{B}^t the σ -algebra generated by $\mathcal{B}_{\mathcal{A}^t} \times \mathcal{A} \times \mathcal{A} \times \cdots$ for $t = 1, 2, \dots$. Set \mathcal{B}^∞ to be the σ -algebra generated by $\cup_{t=0}^\infty \mathcal{B}^t$. The measurable space $(\Omega, \mathcal{B}^\infty)$ with the filtration $\{\mathcal{B}^t\}_{t=0}^\infty$ is used to model the decision problem faced by a family. Given $a \in \mathcal{A}$, let P_a denote a probability measure on $(\Omega, \mathcal{B}^\infty)$ induced by the transition P and initial state a .

Our interest is in a stationary situation because of which we shall restrict attention to the case in which the rate of return on capital, denoted R , and the wage rate, denoted w , do not depend on time. Let $R \in \mathcal{R}$ and $w \in \mathcal{W}$ where, for the moment, $\mathcal{R} = \mathcal{W} := R_+$. We assume that the price of the consumption good is one at every date; this is a normalization and without loss of generality.

Set $y(e, b, \alpha, R, w) := w \cdot \psi(\alpha, e) + R \cdot b$, the *income function*. Under our assumptions on ψ , R , and w , the induced value of y is always nonnegative.

We noted earlier that the vector $(e_{i,0}, b_{i,0}, \alpha_{i,0})$ can be considered as an initial condition for agent $(i, 0)$; this induces an initial income level $y_{i,0} := y(e_{i,0}, b_{i,0}, \alpha_{i,0}, R, w)$.

Given values for $(y_{i,0}, R, w) \in R_+ \times \mathcal{R} \times \mathcal{W}$, we can specify the set of *plans* feasible for the family as

$$\begin{aligned} \Gamma(y_{i,0}, R, w) := \{ & z_i = (c_i, s_i, e_i, b_i) : \\ & (i) (c_{i,t}, s_{i,t}, e_{i,t+1}, b_{i,t+1}) : \Omega \rightarrow R_+^4, t = 0, 1, 2, \dots \\ & (ii) (c_{i,t}, s_{i,t}, e_{i,t+1}, b_{i,t+1}) \text{ is } \mathcal{B}^t \text{-measurable,} \\ & (iii) c_{i,t}(\omega) + s_{i,t}(\omega) + e_{i,t+1}(\omega) + b_{i,t+1}(\omega) \leq y(e_{i,t}(\omega), b_{i,t}(\omega), \alpha_{i,t}(\omega), R, w) \} \end{aligned}$$

where we ignore the qualification ‘‘almost surely.’’ Under A.1 (i)-(ii) and A.2 (i), the set of feasible plans is nonempty and well defined. For notational convenience we shall write $z_{i,t} := (c_{i,t}, s_{i,t}, e_{i,t+1}, b_{i,t+1})$ without subscripts when necessary. The payoff to agent $(i, 0)$ from a feasible plan $z_i \in \Gamma(y_{i,0}, R, w)$ is

$$\mathbb{E} \sum_{t=0}^\infty \beta^t \cdot u(c_{i,t}(\omega), R \cdot s_{i,t}(\omega))$$

where the expectation is taken with respect to the measure $P_{\alpha_{i,0}}$. Under A.1 (i)-(ii), A.3 (i) and A.4, the payoff is well defined and finite. Hence, we can define the *value function* $V : R_+ \times \mathcal{A} \times \mathcal{R} \times \mathcal{W} \rightarrow R$ as

$$V(y_{i,0}, \alpha_{i,0}, R, w) := \sup_{z_i \in \Gamma(y_{i,0}, R, w)} \mathbb{E} \sum_{t=0}^\infty \beta^t \cdot u(c_{i,t}(\omega), R \cdot s_{i,t}(\omega)).$$

We allow V to depend on $\alpha_{i,0}$ since the expectation is taken with respect to the measure $P_{\alpha_{i,0}}$ induced by the initial state making the realization of α at date 0 informative. We allow V to depend on R since consumption when old depends not only on income but also on the return on saving, R . We allow V to depend on w since wage income depends on w .

It should be clear that the agent's identity plays no role in the analysis beyond specifying $y_{i,0}$ and $\alpha_{i,0}$. Consequently, from here onwards we drop the subscripts. Also, set $S := R_+ \times \mathcal{A} \times \mathcal{R} \times \mathcal{W}$ so that $V : S \rightarrow R$.

Given an initial condition $(y, \alpha, R, w) \in S$, a plan is *optimal* if it is in $\Gamma(y, R, w)$ and achieves the value $V(y, \alpha, R, w)$.

Our first result follows from a dynamic programming argument.

Theorem 1: *Assume A.1 (i)-(iii), A.2 (i)-(ii), A.3 (i)-(iii), and A.4. The value function $V : S \rightarrow R$ is continuous, bounded, strictly increasing and strictly concave in y . The policy function $\gamma : S \rightarrow R_+^4$ defined as*

$$\gamma(y, \alpha, R, w) := \operatorname{argmax}_{(c,s,e',b') \in \widehat{\Gamma}(y,R,w)} \{ u(c, R \cdot s) + \beta \cdot \int V(y(e', b', \alpha', R, w), \alpha', R, w) P(\alpha, d\alpha') \}$$

is well defined and continuous, and for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$, $\gamma(0, \alpha, R, w) = (0, 0, 0, 0)$, and $\sum_{i=1}^4 \gamma_i(y, \alpha, R, w) = y$.

One expects the decision to educate versus leaving a bequest to depend on the relative returns to the two activities. In particular, we would like to specify conditions such that there is no expenditure on education, and conditions such that there is, necessarily, positive expenditure on education (respectively, a bequest is made) since these properties will be used to show that mean demand has the right kind of boundary behaviour. We would also like to show that when the marginal return to the two activities are not high relative to the discount factor, the agent will not accumulate assets in order to make his income indefinitely large; this latter property will allow us to bound the processes for the endogenous variables thus permitting us to prove existence of invariant distributions of these variables. Since the optimization problem that we solve has some features of an income fluctuation problem, we know, from Schechtman and Escudero (1977), that additional conditions must be imposed to guarantee that the income process is bounded since the assumptions that we have specified so far are *not* sufficient.

We would like to use the characterization of the agent's decision rule via first order conditions. Unfortunately, we run into the possibility that the objective function is not differentiable. However, under the assumptions made in Theorem 1, V is a concave function of y so it is almost everywhere differentiable in y . Furthermore, concavity of V implies that $\lim_{\epsilon \rightarrow 0^+} (V(y + \epsilon, \alpha, R, w) - V(y, \alpha, R, w))/\epsilon$ exists for all $y \geq 0$; set $V^+(y, \alpha, R, w)$ to be the value of the limit. Similarly, for $y > 0$, set $V^-(y, \alpha, R, w) := \lim_{\epsilon \rightarrow 0^+} (V(y, \alpha, R, w) - V(y - \epsilon, \alpha, R, w))/\epsilon$. So the left and right hand derivatives of V with respect to y are well defined at all $y > 0$ and the right hand derivative also exists at $y = 0$.¹¹ This fact allows us to work with appropriate generalizations of first order conditions.

Let us fix some notation. Given $(y, \alpha, R, w) \in S$, set $\gamma_i := \gamma_i(y, \alpha, R, w)$, $i = 1, 2, 3, 4$. Now set $\bar{u}(y, R) := \max\{u_1(\gamma_1, R \cdot \gamma_2), R \cdot u_2(\gamma_1, R \cdot \gamma_2)\}$. $\underline{\psi}_2^0$, $\underline{\psi}_2^0$, $\overline{\psi}_2^\infty$, and

¹¹Lemma 7 (a)-(b) provides conditions under which V is differentiable.

ψ_2^∞ have already been defined.

Theorem 2: Assume A.1 (i)-(iii), A.2 (i)-(iv), A.3 (i)-(iv), and A.4. Let $y > 0$.

- (a) If $w \cdot \psi_2^0 < R$ then $\gamma_3 = 0$. (b) If $w \cdot \psi_2^0 > R$ and $\bar{u}(y, R) < \beta \cdot w \cdot \int \bar{u}(w \cdot \psi(\alpha', 0), R) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha')$ then $\gamma_3 > 0$.
(c) If $w \cdot \psi_2^0 < R$ and $\bar{u}(y, R) < \beta \cdot R \cdot \int \bar{u}(w \cdot \psi(\alpha', 0), R) P(\alpha, d\alpha')$ then $\gamma_4 > 0$.
(d) Let $R \in [0, 1/\beta)$, let there exist some $\tilde{e}(w) \in R_{++}$ such that $w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w)) \in [0, 1/\beta)$, and, for some $\bar{y} \in R_+$ and for all $(\alpha, y) \in \mathcal{A} \times [\bar{y}, \infty)$, let V satisfy

$$V^-(y, \alpha, R, w) > \beta \cdot \max\{R, w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w))\} \int V^-(y, \alpha', R, w) P(\alpha, d\alpha').$$

Then either $\gamma_3 < \tilde{e}(w)$ and $\gamma_4 = 0$, or $w \cdot \psi(\alpha', \gamma_3) + R \cdot \gamma_4 \leq y$ for all $(\alpha', y) \in \mathcal{A} \times [\bar{y}, \infty)$. (e) If $R > 1/\beta$ or $w \cdot \psi_2^\infty > 1/\beta$ then the stochastic process for income diverges P_α almost surely.

Theorem 2 provides us with the conditions, obtained by studying the first order necessary conditions for the optimization problem, that guarantee that the decision rule satisfies certain desirable properties. In part (a) the condition is easily interpreted—leaving a bequest is always more productive at the margin—hence no resources will be spent on education. Similarly, in part (b) a little bit of education is, in all states, more productive than leaving a bequest; hence, if we can ensure that the agent prefers to not consume her entire income when middle aged, which is guaranteed by the second condition that is imposed, then she will in fact educate her child. In an entirely analogous manner we find in part (c) that if a little bit of education is never more productive (at the origin and hence, by concavity of ψ , everywhere) than leaving a bequest then she will in fact leave a bequest (once again, if she does not consume her entire income when middle aged). Theorem 2 (d) shows that if we restrict attention to (R, w) such that $R < 1/\beta$ and $w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w)) < 1/\beta$ for some $\tilde{e}(w) > 0$, and the value function satisfies an additional condition which we discuss below, then income levels above \bar{y} must be transitory; as we show later, this implies that we can restrict attention to $\gamma_i(y, \alpha, R, w)$, $i = 3, 4$, in a compact set. Theorem 2 (e) shows that if the marginal return to either one of the two assets is high relative to the agent's rate of time preference then asset demand necessarily diverges.

Note that a slight strengthening of A.2 (iv) guarantees that if $\max\{R, w \cdot \overline{\psi_2^\infty}\} < 1/\beta$ then there exists $\tilde{e}(w) > 0$ such that $w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w)) < 1/\beta$, a side condition that is imposed in Theorem 2 (d). The result does not cover the case in which $\max\{R, w \cdot \psi_2^\infty\} \leq 1/\beta$ and $\max\{R, w \cdot \overline{\psi_2^\infty}\} \geq 1/\beta$, a case which requires finer tools.¹² This will not be important for us since we strive to provide conditions under which a stationary equilibrium exists with $\max\{R, w \cdot \overline{\psi_2^\infty}\} < 1/\beta$.¹³

¹²In this case the detailed analysis in Chamberlain and Wilson (2000) needs to be used.

¹³Chamberlain and Wilson (2000) show that if $R > 1/\beta$ then asset demand diverges with probability one. They also show that if income is sufficiently stochastic then asset demand diverges with probability one even when $R = 1/\beta$; Aiyagari (1995) uses this result together with the requirement that in equilibrium the market for capital clears at a finite value to argue that, in any stationary equilibrium, $R < 1/\beta$. Huggett (1997) reaches a similar conclusion by integrating an Euler equation using an invariant measure to show that $R \geq 1/\beta$ can be ruled out.

As regards the condition in Theorem 2 (d), the additional assumption that is imposed on the value function is strong precisely because it limits the extent to which, for large values of y , V can vary as α varies. As the larger of R and $w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w))$ approaches $1/\beta$ from below, the derivative of V (which can be shown to exist for large y , see Lemma 7 (a)-(b)) is required to be uniformly close to zero for all y sufficiently large. This translates into a requirement on u which induces a policy function γ such that marginal utility at the optimal solution for very high levels of income is insensitive to changes in the state. We do not discuss further the interpretation of the assumption; the last section in Schechtman and Escudero (1977) provides a succinct discussion of the problem, how the specific assumption avoids the problem, and what properties on the primitives of the model, beyond boundedness and concavity of the utility function when it is defined on one variable (instead of two the way we have it) guarantee that it is satisfied. It is easy to check that for a large class of u satisfying A.3, the function u_1 (or u_2) exhibits the property required to ensure that the derivative of V behaves nicely.¹⁴

Let us also point out that, for factor prices such that $R \leq 1$ and $w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w)) \leq 1$ for some $\tilde{e}(w) > 0$, by following the argument in Theorem 3.3 in Schechtman and Escudero (1977), it should be possible to bound the income process without the additional condition imposed on the value function in Theorem 2 (d). Such an argument is available in Bewley (1986) for an exchange economy with many commodities, infinitely lived agents who discount the future, and constant prices. We do not pursue such a line of argument since we need to impose the additional condition on V for $R \in (1, 1/\beta)$ anyway.

4. Invariant Measures

The decision rule that we have obtained above induces stochastic dynamics and we would like to know when these dynamics exhibit regularities in the long run; that is, we would like to specify conditions under which invariant distributions exist for the endogenous variables, and we would like to show that these invariant distributions form a nice set and change in a nice way as factor prices vary. That is our objective in this section.

We specify the agent's *state* by the triple (e, b, α) where the first two coordinates are endogenous and the last one is exogenous. Let x denote an agent's state; so $x \in X := R_+ \times R_+ \times \mathcal{A}$. Let us define $\hat{\gamma} : X \times \mathcal{R} \times \mathcal{W} \rightarrow R_+^4$ by the rule $\hat{\gamma}(x, R, w) := \gamma(y(e, b, \alpha, R, w), \alpha, R, w)$. Under the assumptions of Theorem 2, $\hat{\gamma}$ inherits the properties of γ since the income function y is continuous and differentiable.

Given $(R, w) \in \mathcal{R} \times \mathcal{W}$, the Markov process for α together with the function $\hat{\gamma}$, which captures both the income function y and the policy function γ , induce stochastic dynamics for the variables e and b . These dynamics can be captured by defining, for each $x \in X$, a function $Q(x, A_{12} \times A_3; R, w)$ on the measurable space

¹⁴Huggett (1993) works with a CES specification and Markovian uncertainty in endowments in a partial equilibrium framework and shows that, under a parameter restriction, the analogue of the condition on V holds. Also Laitner (1992) works with a CES specification.

(X, \mathcal{B}_X) by the rule

$$\begin{aligned} Q(x, A_{12} \times A_3; R, w) &= P(\alpha, A_3) && \text{if } (\hat{\gamma}_3(x, R, w), \hat{\gamma}_4(x, R, w)) \in A_{12} \\ Q(x, A_{12} \times A_3; R, w) &= 0 && \text{if } (\hat{\gamma}_3(x, R, w), \hat{\gamma}_4(x, R, w)) \notin A_{12} \end{aligned}$$

for all $x \in X$, $A_{12} \times A_3 \in \mathcal{B}_X$.

We can show that $Q(\cdot, \cdot; R, w)$ is a Markov transition function (see Lemma 8). Assuming that to be true for the moment, we can introduce the notion of an invariant measure. A measure ν on (X, \mathcal{B}_X) is *invariant* for $Q(\cdot, \cdot; R, w)$ if

$$\nu(A) = \int Q(x, A; R, w)\nu(dx) \quad \text{for all } A \in \mathcal{B}_X.$$

We denote the set of invariant measures of $Q(\cdot, \cdot; R, w)$ by $\mathcal{N}(R, w)$.

We need a notion of continuity when working with measures. We shall use the *topology of weak convergence of measures* which says that for $\mu_i \in \mathcal{M}(X)$, $i = 0, 1, 2, \dots$, where $\mathcal{M}(X)$ denotes the set of probability measures on X for $X \subset \mathbb{R}^n$, the sequence of measures $\{\mu_n\}_{n \geq 1}$ converges to μ_0 , denoted $\mu_n \xrightarrow{w} \mu_0$, if and only if for every bounded and continuous function $f : X \rightarrow \mathbb{R}$, $\int f(x)\mu_n(dx) \rightarrow \int f(x)\mu_0(dx)$. From here onwards, convergence of measures will always be in the weak topology.

Theorem 3: *Assume A.1 (i)-(iii), A.2 (i)-(iv), A.3 (i)-(iv), and A.4, and consider $(R, w) \in [0, \bar{R}] \times [0, \bar{W}] \subset [0, 1/\beta] \times [0, \infty)$. Assume that $\bar{W} \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}) < 1/\beta$ for some $\tilde{e} \in \mathbb{R}_{++}$, that for some $\bar{y} \in \mathbb{R}_+$ and for all $(\alpha, y) \in \mathcal{A} \times [\bar{y}, \infty)$, and for all $(R, w) \in [0, \bar{R}] \times [0, \bar{W}]$, V satisfies*

$$V^-(y, \alpha, R, w) > \beta \cdot \max\{\bar{R}, \bar{W} \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} \int V^-(y, \alpha', R, w)P(\alpha, d\alpha').$$

(a) $\mathcal{N}(R, w)$ is nonempty, closed, and convex.

(b) If $(R_n, w_n) \rightarrow (R_0, w_0)$ and if $\nu_n \in \mathcal{N}(R_n, w_n)$, then there exists $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ and a subsequence $\nu_{n(j)}$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$.

The theorem tells us that, when factor prices are in an appropriate set, invariant measures exist, they form a closed and convex set, and that the set of invariant measures varies continuously when factor prices change. The continuity property will be used to show that the integral over the set of states of any continuous function also exhibits a form of continuity in factor prices. Theorem 3 obtains when, in addition to the standard assumptions used so far, factor prices induce returns that stay away from $1/\beta$, and a uniform version of the Schechtman and Escudero (1977) condition regarding the behaviour of the derivative of the value function for sufficiently large values of income as the state changes is assumed to hold, where the uniformity is with respect to factor prices.

5. Aggregate Variables

So far we have studied the behaviour of a typical family. We wish to infer the behaviour of economic aggregates at a given date from the behaviour of a typical family. We do so by using a standard “trick” which has two parts to it. The fact that we have a large number of families which are subject to shocks that are independent and identically distributed across families indicates that we should be able to aggregate across families to obtain mean values that are, with probability

one, constants. However, in order to not have to differentiate among families, i.e., so as to treat them “equitably,” we model the fact that there are a large number of them by assuming that the set of families forms a continuum.¹⁵ The fact that we have a continuum of families complicates the situation and requires that we *assume* that aggregation does in fact lead to constant mean values and that assumption constitutes the first part of the trick. The second part of the trick involves using the invariant distributions whose existence and properties were discussed in the previous section. These distributions characterize the long run statistical behaviour of a representative family but, since families are identical, we interpret the invariant distributions as cross sections across families. Hence, by integrating the decision rules followed by a typical family using the invariant distributions, we are able to generate the behaviour of economic aggregates.

But first we look at the production side of the economy to see if we can restrict factor prices, which should be possible since, with a neoclassical technology, factor demands are determined by equating marginal productivity to factor prices. So, if factors are employed in the quantities K and L , then $R(k) = F_1(k, 1) + 1 - \delta$ and $w(k) = F_2(k, 1)$, where we set $k := K/L$. Clearly, since F is increasing in K , $R(k) \geq 1 - \delta$. Furthermore, as we remarked after Theorem 2, if $R > 1/\beta$ then asset demand diverges; hence, we restrict attention to those pairs (K, L) for which $k \geq k_\beta$ where we define k_β by $F_1(k_\beta, 1) := 1/\beta + \delta - 1$ if $F_1(0, 1) \geq 1/\beta + \delta - 1$; if $F_1(0, 1) < 1/\beta + \delta - 1$ then we set $k_\beta := 0$. Denote $w_\beta := w(k_\beta)$.

Our notion of equilibrium requires that we be able to define aggregate variables by using invariant distributions. To guarantee the existence of invariant distributions and so as to be able to use their continuity properties, we restrict attention to factor prices for which Theorem 3 applies. We proceed to a formal description of the relevant economic aggregates.

Consider $(R, w) \in [1 - \delta, \bar{R}] \times [w_\beta, \bar{W}]$, where $\max\{\bar{R}, \bar{W} \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} < 1/\beta$ for some $\tilde{e} > 0$. Under the conditions of Theorem 3, $\mathcal{N}(R, w) \neq \emptyset$. For $\nu \in \mathcal{N}(R, w)$, define ν_α , the marginal measure on $(\mathcal{A}, \mathcal{B}_\mathcal{A})$, by the rule

$$\tilde{\nu}_\alpha(A) := \int \tilde{\nu}(de, db, A) \quad \text{for all } A \in \mathcal{B}_\mathcal{A}.$$

Mean labour supply, mean asset demand, and mean demand for the produced good can now be defined as

$$\begin{aligned} \mathcal{L}(R, w, \nu) &:= \int \int \psi(\alpha', \hat{\gamma}_3(x, R, w)) \nu_\alpha(d\alpha') \nu(dx), \\ \mathcal{K}(R, w, \nu) &:= \int [\hat{\gamma}_2(x, R, w) + \hat{\gamma}_4(x, R, w)] \nu(dx) \\ \xi(R, w, \nu) &:= \int [\hat{\gamma}_1(x, R, w) + R \cdot \hat{\gamma}_2(x, R, w) + \hat{\gamma}_3(x, R, w)] \nu(dx) \end{aligned}$$

where $\nu \in \mathcal{N}(R, w)$. To obtain mean labour supply we take as given the level of education and integrate, over ability levels, the quantity of efficiency units of labour that is produced and then we integrate over the state vector to capture the effect of the state on the level of education that is chosen. Notice that mean asset

¹⁵If we consider a countable set of families then we need a measure on that set; since any such measure cannot be uniform, it cannot be equitable.

demand includes the bequest and the amount saved for consumption in old age. Mean demand for the produced good has three components: consumption in the two periods of life and education of the child.

Since the policy function satisfies the budget restriction we have

$$\sum_{i=1}^4 \hat{\gamma}_i(e, b, \alpha, R, w) \leq w \cdot \psi(\alpha, e) + R \cdot b$$

where the constraint holds with equality. Given factor prices and an invariant measure $\nu \in \mathcal{N}(R, w)$, by integrating both sides using the product of the measures ν_α and ν , we get¹⁶

$$\int [\sum_{i=1}^4 \hat{\gamma}_i(x, R, w)] \nu(dx) = \int \int [w \cdot \psi(\alpha', e) + R \cdot b] \nu_\alpha(d\alpha') \nu(dx).$$

By substituting the definitions of mean demand for factors and the produced good, and manipulating the expression we obtain the relevant version of Walras Law

$$\xi(R, w, \nu) = w \cdot \mathcal{L}(R, w, \nu) + (R - 1) \cdot \mathcal{K}(R, w, \nu).$$

The expression depends on the invariant measure that is selected but its form does not depend on the specific measure chosen.

6. Equilibrium

We are in a position to define our notion of equilibrium. It is standard in that it requires that agents optimize taking prices as given and that all markets clear where, for the produced good, one has to allow for depreciation.

Definition 1: *A stationary competitive equilibrium is a pair $(K^*, L^*) \in \mathbb{R}_+^2$ such that with $R^* := F_1(K^*/L^*, 1) + 1 - \delta$ and $w^* := F_2(K^*/L^*, 1)$,*

- (i) $\xi(R^*, w^*, \nu^*) = F(K^*, L^*) - \delta \cdot K^*$,
- (ii) $\mathcal{L}(R^*, w^*, \nu^*) = L^*$,
- (iii) $\mathcal{K}(R^*, w^*, \nu^*) = K^*$,
- (iv) $\nu^* \in \mathcal{N}(R^*, w^*)$,

and the functions ξ , \mathcal{L} and \mathcal{K} are induced by the policy function $\hat{\gamma}(\cdot, R^, w^*)$.*

We can now turn to the main result of the paper. We provide sufficient conditions for the existence of a stationary equilibrium. We frame the existence problem in terms of the capital-labour ratio. We start by specifying an appropriately chosen interval with the property that if k belongs to the interval then an invariant distribution exists. We proceed to show that, for every value of k in such an interval, the ratio of the values of the functions \mathcal{K} and \mathcal{L} as we vary over all $\nu \in \mathcal{N}(R(k), w(k))$ is a closed and convex set, and that, as k varies, this set exhibits a form of continuity. Clearly, if at the lower extreme value of k the ratio of factor supplies exceeds k while it falls short of k at the other extreme, then we have a fixed point k^* together with an invariant measure $\nu^* \in \mathcal{N}(R(k^*), w(k^*))$. By considering the value $\mathcal{L}(R(k^*), w(k^*), \nu^*)$ as the value of L^* , and setting $K^* = k^* \cdot L^*$, we have an equilibrium since, using Walras Law, it can be shown that the market for the produced

¹⁶Heuristic justification of the procedure is provided by the observation that e and b are determined without knowledge of α and that ν is an invariant measure.

good clears if the factor markets clear. The conditions labelled (a) and (b) which appear in the statement of Theorem 4 imply the boundary behaviour required of the induced value of the capital-labour ratio. After stating the result we discuss it; in particular, we discuss conditions on the primitives of the model which let us verify assumptions (a) and (b).

Theorem 4: *Assume A.1 (i)-(iii), A.2 (i)-(ii), (iii b), (iv), A.3 (i)-(iv), A.4 and A.5. Consider $k \in [\underline{k}, \bar{k}]$ where $\underline{k} > k_\beta$. Assume that $w(\bar{k}) \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}) < 1/\beta$ for some $\tilde{e} \in R_{++}$, that for some $\bar{y} \in R_+$ and for all $(\alpha, y) \in \mathcal{A} \times [\bar{y}, \infty)$, and for all $(R, w) \in [1 - \delta, R(\underline{k})] \times [w_\beta, w(\bar{k})]$, V satisfies*

$$V^-(y, \alpha, R, w) > \beta \cdot \max\{R(\underline{k}), w(\bar{k}) \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} \int V^-(y, \alpha', R, w) P(\alpha, d\alpha').$$
If (a) $w(\bar{k}) + R(\bar{k}) \cdot \frac{\bar{y}}{\mathcal{L}(R(\bar{k}), w(\bar{k}), \bar{v})} \leq \bar{k}$ for some $\bar{v} \in \mathcal{N}(R(\bar{k}), w(\bar{k}))$, and (b) $\xi(R(\underline{k}), w(\underline{k}), \underline{v}) \geq \mathcal{L}(R(\underline{k}), w(\underline{k}), \underline{v}) \cdot [F(\underline{k}, 1) - \delta \cdot \underline{k}]$ for some $\underline{v} \in \mathcal{N}(R(\underline{k}), w(\underline{k}))$, then a stationary equilibrium exists.

We note for future reference that the proof of Theorem 4 shows that condition (b) in the statement above is equivalent to the requirement

$$\mathcal{K}(R(\underline{k}), w(\underline{k}), \underline{v}) \geq \mathcal{L}(R(\underline{k}), w(\underline{k}), \underline{v}) \cdot \underline{k} \quad \text{for some } \underline{v} \in \mathcal{N}(R(\underline{k}), w(\underline{k})).$$

Theorem 4 is proved for the case in which A.2 (iii b) holds so that mean labour supply is always strictly positive. As we indicated earlier, in our framework where education is used to augment labour productivity, A.2 (iii b) is a reasonable assumption. Even so, below we indicate how an analogous result can be proved when A.2 (iii a) holds instead.¹⁷

Since Theorem 4 also applies to a deterministic economy, that is where we begin our discussion. In that case, the invariance requirement takes a particularly simple form, namely, income does not change. This lets us solve the model easily since the first order conditions become very simple to work with. For k such that $R(k) < 1/\beta$, the individual does not leave a bequest. The level of education is determined completely by the wage rate, and the amount of labour is determined only by the level of education, so $\hat{L}(k) := \psi(\hat{e}(w(k)))$. It follows that, for $k > k_\beta$, the agent's income is easy to determine, $\hat{y}(k) = w(k) \cdot \hat{L}(k)$ since there is no bequest. Again, since no bequest is given and the level of education is directly determined from the wage rate, saving as a function of k can be obtained, $\hat{s}(k)$. An equilibrium without bequest is a value k^* such that $k^* = \hat{s}(k^*)/\hat{L}(k^*)$ and $k^* \geq k_\beta$ where the latter is a necessary condition for "stationary" equilibrium just as in the case under uncertainty. Under standard conditions, the values $\hat{s}(k)/\hat{L}(k)$ induce a continuous function for $k \geq 0$ and, if the boundary conditions in Theorem 4 are satisfied, then an equilibrium with $b = 0$ exists. Another candidate for a steady state is the value k_β which induces $\hat{L}(k_\beta)$. In this case, for every value of $b \geq 0$, a value of $\hat{y}(k_\beta, b)$ is induced as is a value for $\hat{s}(k_\beta, b)$. We have a steady state if b is such that $\hat{y}(k_\beta, b) \geq 0$,

¹⁷The only role played by A.2 (iii b) in the existence proof is to ensure that mean labour supply is strictly positive, a property which can be guaranteed under much weaker assumptions, e.g., by assuming that $\psi(\alpha', 0) > 0$ for all $\alpha' \in A$, where $A \in \mathcal{B}_A$ is such that $P(\alpha, A) > 0$ for all $\alpha \in \mathcal{A}$, which is a weaker version of the assumption made in Loury (1981) indicated in footnote 8 in this paper.

$\widehat{s}(k_\beta, b) \geq 0$, and $\widehat{s}(k_\beta, b) + b = \widehat{K}$, where $\widehat{K} := k_\beta \cdot \widehat{L}(k_\beta)$ is the capital stock which supports k_β as a steady state, so that the market for capital clears. Evidently, if $\widehat{s}(k_\beta)/\widehat{L}(k_\beta) \geq k_\beta$, then there can be no steady state with $b > 0$; if, on the other hand, $\widehat{s}(k_\beta)/\widehat{L}(k_\beta) < k_\beta$ then a steady state with $b > 0$ might exist and whether or not it does depends crucially on the sign of the quantity $w(k_\beta) - \widehat{e}(k_\beta)/\widehat{L}(k_\beta)$.

In parametric examples one can show that, under reasonable restrictions on the values of the parameters, an equilibrium exists. For example, if we specify $u(c, R \cdot s) = c^\gamma \cdot (R \cdot s)^{1-\gamma}$, $\psi(e) = e^\alpha$,¹⁸ and $F(k, 1) = k^\eta$, where $\gamma \in (0, 1)$, $\alpha \in (0, 1)$, and $\eta \in (0, 1)$, then it is easy to check that (imposing the condition $b = 0$ which must hold for $k > k_\beta$),

$$\widehat{L}(k) = [\alpha \cdot \beta \cdot (1 - \eta) \cdot k^\eta]^{\alpha/(1-\alpha)}$$

$$\widehat{s}(k) = (1 - \gamma) \cdot (1 - \alpha \cdot \beta) \cdot (\alpha \cdot \beta)^{\alpha/(1-\alpha)} \cdot [(1 - \eta) \cdot k^\eta]^{1/(1-\alpha)},$$

so that both are increasing functions and can even be convex functions depending on the relative values of α and η , and the fixed point map $\widehat{s}(k)/\widehat{L}(k)$ takes the form $(1 - \gamma) \cdot (1 - \alpha \cdot \beta) \cdot (1 - \eta) \cdot F(k, 1)$, a concave function. At the unique fixed point, $R(k^*) = 1 - \delta + \eta/[(1 - \eta) \cdot (1 - \gamma) \cdot (1 - \alpha \cdot \beta)]$ which could be greater than $1/\beta$ for some parameter values. However, if β is not too high, so the agent does not care that much about her descendents, and γ is not too high, so she wishes to consume in old age thus stimulating the demand for the capital asset, and α and η are not too high, so that decreasing marginal returns to both physical capital and education set in fairly rapidly, then a steady state exists. It turns out that k_β is a steady state with $b > 0$ if and only if at the unique fixed point of the map $\widehat{s}(k)/\widehat{L}(k)$, $R(k^*) < 1/\beta$, i.e., when there is no steady state without bequests. Furthermore, k_β is a steady state even if $\gamma = 1$ so that the individual does not save for retirement. As we remarked earlier, in all the results a key role is played by the quantity $w(k) - \widehat{e}(k)/\widehat{L}(k)$ which, in the Cobb-Douglas specification, is always positive because of which all the nonnegativity restrictions are satisfied and steady states exist. However, one can construct examples where the condition fails at or around k_β and existence of steady states fails to hold; this is possible because k_β and $w(k)$ are determined by the technology in the final good sector while $\widehat{e}(k)$ and $\widehat{L}(k)$ are determined by the technology in the human capital sector.

The deterministic case shows us quite clearly that stationary equilibrium may fail to exist precisely because boundary behaviour needs to be “nice” at the value k_β instead of at zero.

Let us turn to the general case and discuss the conditions on the primitives of the model which guarantee that the sufficient conditions labelled (a) and (b) in Theorem 4 hold. Under A.2 and A.5 both $F(k, 1)$ and ψ are concave functions; if, in addition, ψ is also a bounded function, the usual case in such models, then \bar{k} is unrestricted and can be arbitrarily large so that the fact that $F(k, 1)$ is concave, that $R(k) < 1/\beta$, and that, by A.2 (iii b), $\min_{\alpha \in \mathcal{A}} \psi(\alpha, 0) > 0$, ensure that (a) holds for some \bar{k} sufficiently large (even though $w(k)$ may be an unbounded function). Of course, (a) might hold in other cases too. As for (b), it requires that demand

¹⁸In the example ψ satisfies A.2 (iii a).

for the produced good be nonnegligible; alternatively, that mean asset demand is nonnegligible. (b) will hold if we assume that $\gamma_3 = 0$, we choose $\min_{\alpha \in \mathcal{A}} \psi(\alpha, 0)$ to be arbitrarily small, and we assume that $F(k, 1)$ is such that k_β is sufficiently small (this last property corresponds to η close to zero in the parametric case where $F(k, 1) = k^\eta$). To see that (b) will hold under the above specification, notice that $\gamma_3 = 0$ implies that $\mathcal{L}(R(\underline{k}), w(\underline{k}), \underline{\nu}) = \min_{\alpha \in \mathcal{A}} \psi(\alpha, 0)$, its lowest possible value, so that by choosing \underline{k} to be sufficiently close to k_β , where the latter is close to zero by the choice of F , we can ensure that $\mathcal{L}(R(\underline{k}), w(\underline{k}), \underline{\nu}) \cdot [F(\underline{k}, 1) - \delta \cdot \underline{k}]$ is sufficiently small (alternatively, that $\mathcal{L}(R(\underline{k}), w(\underline{k}), \underline{\nu}) \cdot \underline{k}$ is sufficiently small) for all $\underline{\nu} \in \mathcal{N}(R(\underline{k}), w(\underline{k}))$, while since $\gamma_3 = 0$, any income must be spent on consumption when middle aged, consumption when old, or leaving a bequest where the first two generate demand for the produced good while the last two generate demand for the capital asset. In either case the right hand side of the inequality in (b) is made arbitrarily small while the conditions are sufficient to guarantee that the left hand side of the inequality is strictly positive. Alternatively, (b) holds if $\gamma_3 = 0$ and if mean asset demand can be shown to diverge to infinity as R approaches $1/\beta$. Such a result is reported in Laitner (1992) for a CES utility function; Aiyagari (1994, 1995) refers to a paper by Bewley where a general result is proved.¹⁹ Theorem 2 (a) tells us that $w(\underline{k}) \cdot \overline{\psi}_2^0 < R(\underline{k})$ is a sufficient condition to get $\gamma_3 = 0$; Theorem 2 (c) can be used to show that, under the same condition, $\mathcal{K}(R(\underline{k}), w(\underline{k}), \underline{\nu}) > 0$ (under A.2 (iii b), the side condition on marginal utility in the statement of Theorem 2 (c) is automatically satisfied case for some set of $\alpha \in \mathcal{A}$ which receives positive measure under the invariant distribution).

Is it possible to assert that existence follows if we maintain all the conditions in Theorem 4 except for (b) which we replace by the statement that mean asset demand diverges as we get closer to k_β ? The answer is not evident since both education and bequests play the role of assets through which intertemporal transfers are achieved. But the fact that the return to education is given by a concave function while bequests earn a linear return indicates that, if ψ is a bounded function then, beyond a cut-off value for income, the demand for the physical asset should dominate and the mean demand for physical capital should diverge faster as total mean demand for assets diverges implying that arbitrarily high levels of income are achieved. But proving such a result would require much more detailed information about the cross partial derivatives of the value function (or equivalent lattice theoretic arguments); the standard conditions that one imposes do not appear to tell us much.

Evidently, if $k_\beta > 0$ then any stationary equilibrium identified in the theorem is interior in the sense that the stock of physical capital is positive. However, the average level of bequests need not be positive; a very strong sufficient condition under which it is positive is to require that \underline{k} be sufficiently close to k_β , so that $\beta \cdot R(\underline{k})$ is close to one, in addition to requiring that $w(\underline{k}) \cdot \overline{\psi}_2^0 < R(\underline{k})$ and that $\min_{\alpha \in \mathcal{A}} \psi(\alpha, 0)$ is arbitrarily small, since in that case the result follows from Theorem

¹⁹Professor Bewley has indicated to us that he never finished the paper though he seems to remember that he did have a proof of the result.

2 (c). A sufficient condition for the mean level of education to be positive at some stationary equilibrium is to require that for some \tilde{k} and all $\nu \in \mathcal{N}(R(\tilde{k}), w(\tilde{k}))$, $\mathcal{K}(R(\tilde{k}), w(\tilde{k}), \nu) \geq \tilde{k} \cdot \mathcal{L}(R(\tilde{k}), w(\tilde{k}), \nu)$ and \tilde{k} is such that at the induced factor prices, the conditions in Theorem 2 (b) hold so that γ_3 is necessarily positive (always since, under A.2 (iii b), y is always strictly positive).

What can we say about existence when we assume A.2 (iii a) instead of A.2 (iii b)? As we remark in Section 7 after the proof of Lemma 11, the principal problem that we need to overcome is the fact that measures which have support on the set $\{(0, 0)\} \times \mathcal{A}$ can now be invariant measures. Eliminating such measures leaves us with a set which is not closed. So we need to rework Theorem 3 to prove existence of a closed and convex set of invariant measures which does not include the kind of measure indicated above. This is a difficult exercise and its contribution to improving our understanding of the problem is not obvious; furthermore, as we indicate in Section 7, the conditions required for constructing such a set for k near \underline{k} can interfere with being able to meet the boundary condition (b) in Theorem 4. However, let us suppose for the moment that it is possible to accomplish the modification to Theorem 3. In that case the proof of Theorem 4 can proceed as before since, on some compact subset of the set $(\underline{k}, \bar{k}]$, we can invoke the assumptions under which Theorem 2 (b) guarantees that γ_3 is positive, so that there is a nonempty compact set on which mean labour supply is uniformly positive. The boundary conditions can continue to be the same (modulo the additional difficulty in being able to verify them that we just indicated) and some fairly straight forward changes to the conditions spelled out in Theorem 4 will lead to an existence theorem.

We end this section by indicating why we have not tried to obtain uniqueness and stability results for the invariant distributions. With the assumptions that we have specified, we cannot ensure that the decision rules are monotone. However, in the i.i.d. case, with a few additional assumptions, we can show that the income process is monotone. Under A.2 (iii b), by adapting the argument in Hopenhayn and Prescott (1992) for the Brock-Mirman model, it should be possible to prove uniqueness and stability. Under A.2 (iii a), which is the original Brock-Mirman assumption, the argument in Section 7 alluded to in the previous paragraph shows that for k near k_β there is a conflict between being able to carry out the Brock-Mirman construction (of eliminating the possibility that a set of the form $[0, \epsilon)$ is in the support of some invariant measure) and being able to satisfy the boundary condition (b) in Theorem 4. Since we would rather ensure existence, we choose not to opt for unicity when A.2 (iii a) is imposed. We do not know what happens in the Markov case in so far as uniqueness and stability are concerned.

7. Proofs

We prove each of the theorems in turn with a few preliminary results preceding each proof. We shall refer to several results available in the monograph by Stokey and Lucas (1989) (henceforth SL). We provide complete proofs of some results which are not stated in SL.

Theorem 1 is proved after proving Lemma 1 through Lemma 5.

Recall that $y(e, b, \alpha, R, w) := w \cdot \psi(\alpha, e) + R \cdot b$. For $y \in R_+$, the one period feasibility correspondence is given by

$$\widehat{\Gamma}(y, R, w) := \{(c, s, e', b') \in R_+^4 : c + s + e' + b' \leq y\}.$$

When necessary, we use z to denote elements of $\widehat{\Gamma}(y)$.

Lemma 1: *For all $y \geq 0$, $\widehat{\Gamma}(y, R, w)$ is nonempty, convex, and compact. At every $y \geq 0$, $\widehat{\Gamma}$ is an upper and lower hemicontinuous correspondence.*

Proof: Since $y \geq 0$, the vector $(0, 0, 0, 0) \in \widehat{\Gamma}(y, R, w)$ showing nonemptiness. Since $\widehat{\Gamma}(y, R, w)$ is defined via the intersection of sets specified by weak linear inequalities, it is convex and closed; since the normal vectors of the linear inequalities generating $\widehat{\Gamma}(y, R, w)$ are the unit vectors and the vector $(1, 1, 1, 1)$, $\widehat{\Gamma}(y, R, w)$ is compact.

We turn to the continuity properties of $\widehat{\Gamma}$. Since $\widehat{\Gamma}$ is independent of (R, w) it suffices to prove continuity in y . Let $y_n \rightarrow y_0$ and consider $\bar{y} := \sup_n \{y_n\}$. Evidently, $\widehat{\Gamma}(y_n, R, w) \subset \widehat{\Gamma}(\bar{y}, R, w)$. Since $\bar{y} \geq 0$, the result in the previous paragraph shows that $\widehat{\Gamma}(\bar{y}, R, w)$ is a compact set. As $\widehat{\Gamma}$ has closed graph, it follows from SL Theorem 3.4 that it is upper hemicontinuous.

To check for lower hemicontinuity, let $y_n \rightarrow y_0$ and let $z_0 \in \widehat{\Gamma}(y_0, R, w)$. If $y_0 = 0$ then, necessarily, $z = (0, 0, 0, 0)$ and the result follows since $(0, 0, 0, 0) \in \widehat{\Gamma}(y, R, w)$ for all $y \geq 0$. If instead $y_0 > 0$ then set $e'_n := e'_0 \cdot (y_n/y_0)$ and analogously for the other variables. Since $y_n \geq 0$, and $y_n \rightarrow y_0$, the resulting sequence has the property that $z_n \rightarrow z_0$ and $z_n \in \widehat{\Gamma}(y_n, R, w)$ for all n . ■

Recall that $S := R_+ \times \mathcal{A} \times \mathcal{R} \times \mathcal{W}$. For a function $f : S \rightarrow R$, consider the optimization problem, parameterized by (y, R, w) ,

$$\max_{(c, s, e', b') \in \widehat{\Gamma}(y, R, w)} \{ u(c, R \cdot s) + \beta \cdot \int f(y(e', b', \alpha', R, w), \alpha', R, w) P(\alpha, d\alpha') \}.$$

Denote the solution to the optimization problem with $Tf : S \rightarrow R$.

Lemma 2: *Consider the optimization problem with $f : S \rightarrow R$ a bounded and continuous function. Let A.1 (i)-(iii), A.2 (i), and A.3 (i) hold. (a) Then the induced function Tf is well defined, bounded and continuous. (b) If, in addition, A.2 (ii) and A.3 (iii) hold, and f is concave in y and also weakly increasing in y , then Tf is concave in y . (c) If, in addition, A.3 (ii) holds and f is weakly increasing in y , then Tf is strictly increasing in y .*

Proof: (a) Consider $(y, \alpha, R, w) \in S$. By Lemma 1, $\widehat{\Gamma}(y, R, w)$ is nonempty, convex, compact valued, and continuous. Consider $(y_n, \alpha_n, R_n, w_n) \rightarrow (y_0, \alpha_0, R_0, w_0)$, where $(y_0, \alpha_0, R_0, w_0) \in S$, and $\{z_n\}$ a sequence in $\widehat{\Gamma}(y_n, R_n, w_n)$ with $z_n \rightarrow z_0$. By Lemma 1, $\widehat{\Gamma}$ is upper hemicontinuous so $z_0 \in \widehat{\Gamma}(y_0, R_0, w_0)$. Let $z_i = (c_i, s_i, e'_i, b'_i)$. Since u is continuous under A.3 (i), $u(c_n, R_n \cdot s_n) \rightarrow u(c_0, R_0 \cdot s_0)$. Since, under A.2 (i), $y(e, b, \alpha, R, w)$ is a continuous function, for each $\alpha' \in \mathcal{A}$, $y(e'_n, b'_n, \alpha', R_n, w_n) \rightarrow y(e'_0, b'_0, \alpha', R_0, w_0)$. Under A.1 (i)-(ii), $\int f(y, \alpha', R, w) P(\alpha, d\alpha')$ is well defined and, since f is continuous, \mathcal{A} is compact under A.1 (i), y is continuous under A.2 (i), and the transition function P satisfies the Feller Property under A.1 (iii), by SL Lemma 9.5

$$\int f(y(e'_n, b'_n, \alpha', R_n, w_n), \alpha', R_n, w_n) P(\alpha_n, d\alpha')$$

$$\rightarrow \int f(y(e'_0, b'_0, \alpha', R_0, w_0), \alpha', R_0, w_0) P(\alpha_0, d\alpha').$$

It follows that the maximand is a continuous function of (y, α, R, w, z) , where $z \in \widehat{\Gamma}(y, R, w)$. By Berge's Theorem of the Maximum (SL Theorem 3.6) a maximizer exists for each $(y, \alpha, R, w) \in S$, and the function Tf can be defined pointwise and is continuous. Tf is bounded since under A.3 (i) u is bounded and by hypothesis f is bounded.

(b) We show that Tf is concave in y . For $i = 1, 2$, let $y_i \geq 0$ and let $Tf(y_i, \alpha, R, w)$, the maximal value, be achieved at $z_i \in \widehat{\Gamma}(y_i, R, w)$. For $\lambda \in (0, 1)$, let $y^\lambda := \lambda \cdot y_1 + (1 - \lambda) \cdot y_2$ and $z^\lambda := \lambda \cdot z_1 + (1 - \lambda) \cdot z_2$. Trivially, $z^\lambda \in \widehat{\Gamma}(y^\lambda)$.

By SL Lemma 9.5, if f is concave in y for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$, then $\int f(y, \alpha', R, w) \cdot P(\alpha, d\alpha')$ is concave in y for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$. It follows that

$$\begin{aligned} & \int [\lambda \cdot f(y(e'_1, b'_1, \alpha', R, w), \alpha', R, w) + (1 - \lambda) \cdot f(y(e'_2, b'_2, \alpha', R, w), \alpha', R, w)] P(\alpha, d\alpha') \\ & \leq \int f(\lambda \cdot y(e'_1, b'_1, \alpha', R, w) + (1 - \lambda) \cdot y(e'_2, b'_2, \alpha', R, w), \alpha', R, w) P(\alpha, d\alpha') \\ & \leq \int f(y(\lambda \cdot e'_1 + (1 - \lambda) \cdot e'_2, \lambda \cdot b'_1 + (1 - \lambda) \cdot b'_2, \alpha', R, w), \alpha', R, w) P(\alpha, d\alpha'), \end{aligned}$$

where the second inequality follows from the fact that f is weakly increasing in y together with the fact that

$$\begin{aligned} & \lambda \cdot y(e'_1, b'_1, \alpha', R, w) + (1 - \lambda) \cdot y(e'_2, b'_2, \alpha', R, w) \\ & \leq y(\lambda \cdot e'_1 + (1 - \lambda) \cdot e'_2, \lambda \cdot b'_1 + (1 - \lambda) \cdot b'_2, \alpha', R, w) \end{aligned}$$

where we use A.2 (ii), the fact that ψ is concave in e . Furthermore, since u is strictly concave

$$\lambda \cdot u(c_1, R \cdot s_1) + (1 - \lambda) \cdot u(c_2, R \cdot s_2) \leq u(c_1^\lambda, R \cdot s^\lambda)$$

with a strict inequality if $(c_1, s_1) \neq (c_2, s_2)$.

Since the maximal value of the problem with $y = y^\lambda$ is at least as large as the value obtained by choosing z^λ , the strict concavity of u , assumed in A.3 (iii), the concavity of f in y and the fact that f is weakly increasing in y , hypothesised above, imply that Tf is concave in y .

(c) We show that Tf is strictly increasing in y . Consider y_i where $y_1 < y_2$. Let $Tf(y_i, \alpha, R, w)$, the maximal value, be achieved at $z_i \in \widehat{\Gamma}(y_i, R, w)$. For $\epsilon > 0$ sufficiently small, $z_1 + \epsilon \cdot (1, 0, 0, 0) \in \widehat{\Gamma}(y_2, R, w)$. Moreover, by A.3 (ii), u is strictly increasing in the first coordinate, so the value of the maximand is strictly larger at $z_1 + \epsilon \cdot (1, 0, 0, 0)$ than at z_1 at which it achieves the value $Tf(y_1, \alpha, R, w)$. It follows that $Tf(y_1, \alpha, R, w) < Tf(y_2, \alpha, R, w)$ as required. \blacksquare

We have shown that, given $f \in C(S)$, where $C(S)$ is the space of bounded and continuous functions with domain S equipped with the *sup* norm, one can think of the maximized value of the problem parametrized by the vector (y, α, R, w) as the image of an operator T where $T : C(S) \rightarrow C(S)$. Furthermore, we have provided conditions under which T preserves concavity and monotonicity of f in y .

Lemma 3: (a) Under A.1 (i)-(iii), A.2 (i), and A.3 (i), there exists a unique $v^* : S \rightarrow R$ which is continuous and bounded and solves the functional equation

$$\begin{aligned} f(y, \alpha, R, w) = Tf(y, \alpha, R, w) := & \max_{(c, s, e', b') \in \widehat{\Gamma}(y, R, w)} \{ u(c, R \cdot s) \\ & + \beta \cdot \int f(y(e', b', \alpha', R, w), \alpha', R, w) P(\alpha, d\alpha') \}. \end{aligned}$$

(b) If A.2 (ii) and A.3 (iii) are also satisfied then v^* is concave in y . (c) If A.3 (ii) is also satisfied then v^* is strictly increasing in y .

Proof: (a) For $f \in C(S)$, by Lemma 2 (a), $Tf \in C(S)$ also.

We show that the operator T is monotone. Consider $f_1 \in C(S)$ and $f_2 \in C(S)$ satisfying $f_1(y, \alpha, R, w) \leq f_2(y, \alpha, R, w)$ for all $(y, \alpha, R, w) \in S$. Fix $(y, \alpha, R, w) \in S$; this also fixes $\widehat{\Gamma}(y, R, w)$. For every $z = (c, s, e', b') \in \widehat{\Gamma}(y, R, w)$, and for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$,

$$\int f_1(y(e', b', \alpha', R, w), \alpha', R, w)P(\alpha, d\alpha') \leq \int f_2(y(e', b', \alpha', R, w), \alpha', R, w)P(\alpha, d\alpha').$$

It follows easily that $Tf_1(y, \alpha, R, w) \leq Tf_2(y, \alpha, R, w)$ for all $(y, \alpha, R, w) \in S$.

We show that the operator discounts at β which, by A.4, is in the set $(0, 1)$. For $a \in R_+$, and $f \in C(S)$, define $(f + a) \in C(S)$ by the rule $(f + a)(y, \alpha, R, w) := f(y, \alpha, R, w) + a$. It follows that

$$\begin{aligned} \int (f + a)(y(e', b', \alpha', R, w), \alpha', R, w)P(\alpha, d\alpha') \\ = \int f(y(e', b', \alpha', R, w), \alpha', R, w)P(\alpha, d\alpha') + a. \end{aligned}$$

As a consequence, $T(f + a)(y, \alpha, R, w) = Tf(y, \alpha, R, w) + \beta \cdot a$, thus verifying the requirement.

It follows that T satisfies Blackwell's sufficient conditions for being a contraction of modulus less than one (SL Theorem 3.3). Hence, by Banach's Fixed Point Theorem (SL Theorem 3.2), T has a unique fixed point in the space $C(S)$. Denote the fixed point with v^* .

(b) By Lemma 2 (b) and (c), if $f \in C(S)$ and f is concave in y and f is weakly increasing in y then Tf is concave and weakly increasing in y . Since the subspace of functions which are concave and weakly increasing in y is closed it follows that v^* , whose existence has already been shown, must be in the subspace. Hence v^* is concave and weakly increasing in y .

(c) By (b) above, v^* is weakly increasing in y . By Lemma 2 (c), if $f \in C(S)$ is weakly increasing in y , then $Tf \in C(S)$ is strictly increasing in y . It follows that v^* must exhibit the property. \blacksquare

We define the *policy correspondence* for our problem. $\gamma : S \rightarrow R_+^4$ is obtained as

$$\begin{aligned} \gamma(y, \alpha, R, w) := \{(\tilde{c}, \tilde{s}, \tilde{e}', \tilde{b}') \in \widehat{\Gamma}(y, R, w) : (\tilde{c}, \tilde{s}, \tilde{e}', \tilde{b}') := \\ \operatorname{argmax}_{(c, s, e', b') \in \widehat{\Gamma}(y, R, w)} \{u(c, R \cdot s) + \beta \cdot \int v^*(y(e', b', \alpha', R, w), \alpha', R, w)P(\alpha, d\alpha')\}\}. \end{aligned}$$

Lemma 4: Assume A.1 (i)-(iii), A.2 (i), A.3 (i), and A.4. The policy correspondence γ is well defined, upper hemicontinuous, and admits a measurable selection.

Proof: That the policy correspondence is well defined, upper hemicontinuous, and compact valued, follows from an application of Berge's Theorem (SL Theorem 3.6) to the optimization problem specified above since, under the stated assumptions, by Lemma 3 (a), v^* is continuous in the choice variables as is u while, by Lemma 1, $\widehat{\Gamma}(y, R, w)$ is nonempty, continuous, and compact valued for each $(y, \alpha, R, w) \in S$.

Since S is a Borel set and γ is nonempty, upper hemicontinuous, and compact valued, by the Measurable Selection Theorem (SL Theorem 7.6), γ admits a mea-

surable selection. ■

We can now relate the function v^* to the value function V .

Lemma 5: *Assume A.1 (i)-(iii), A.2 (i), A.3 (i), and A.4. Then, given any initial condition $(y, \alpha, R, w) \in S$, there exist measurable selections from the policy correspondence γ , and any plan generated by such measurable selections is optimal; hence, v^* satisfies $v^* = V$ so that the value function V is bounded and continuous. Furthermore, any plan which achieves $V(y, \alpha, R, w)$ is P_α -almost everywhere a plan generated by measurable selections from γ .*

Proof: Under the stated assumptions, Lemma 3 (a) shows that v^* is continuous and bounded. Since v^* is continuous, it is measurable. Consider any initial condition (y, α, R, w) and a plan $\bar{z} \in \Gamma(y, R, w)$. Following the argument in SL Theorem 9.2, such a feasible plan can be evaluated using v^* since we have shown that v^* is measurable and satisfies the functional equation; furthermore, since v^* is bounded and, under A.4, $\beta \in (0, 1)$, $v^*(y, \alpha, R, w)$ is an upper bound for the payoff from such a plan. But Lemma 4 shows that the policy correspondence γ admits a measurable selection. It can be verified that a plan generated from the initial state (y, α, R, w) by using measurable selections from γ attains the upper bound. Hence $v^* = V$.

Furthermore, under A.3 (i), u is bounded. Since $v^* = V$ and v^* is measurable, V is measurable. Following the argument in SL Theorem 9.4 we can show that for any initial state (y, α, R, w) and a plan $\bar{z} \in \Gamma(y, R, w)$ which achieves $V(y, \alpha, R, w)$, there exists a plan generated by γ that also achieves $V(y, \alpha, R, w)$ and that agrees P_α -almost everywhere with the plan \bar{z} . ■

Proof of Theorem 1: All the properties of V except for strict concavity in y follow from Lemma 3 and Lemma 5. For strict concavity, consider $y_1 < y_2$. By Lemma 5, $V(y_i, \alpha, R, w)$ is achieved at a plan generated by a measurable selection from γ denoted $\{(c_{t,i}(\omega), s_{t,i}(\omega), e_{t+1,i}(\omega), b_{t+1,i}(\omega))\}$. Since V is strictly increasing in y , $(c_{t,1}, s_{t,1}) \neq (c_{t,2}, s_{t,2})$ on a set of P_α positive measure for some t . Since u is strictly concave,

$$\begin{aligned} & V(\lambda \cdot y_1 + (1 - \lambda) \cdot y_2, \alpha, R, w) \\ & \geq \mathbf{E} \sum_{t=0}^{\infty} \beta^t \cdot u(\lambda \cdot c_{t,1}(\omega) + (1 - \lambda) \cdot c_{t,2}(\omega), R \cdot [\lambda \cdot s_{t,1}(\omega) + (1 - \lambda) \cdot s_{t,2}(\omega)]) \\ & > \mathbf{E} \sum_{t=0}^{\infty} \beta^t \cdot \lambda \cdot u(c_{t,1}(\omega), R \cdot s_{t,1}(\omega)) + \mathbf{E} \sum_{t=0}^{\infty} \beta^t \cdot (1 - \lambda) \cdot u(c_{t,2}(\omega), R \cdot s_{t,2}(\omega)) \\ & = \lambda \cdot V(y_1, \alpha, R, w) + (1 - \lambda) \cdot V(y_2, \alpha, R, w) \end{aligned}$$

thus proving that V is strictly concave in y .

The rest of the proof of Theorem 1 is in Lemma 6 which follows. ■

Lemma 6: *Assume A.1 (i)-(iii), A.2 (i)-(ii), A.3 (i)-(iii), and A.4. Then, the policy correspondence γ is a well defined continuous function, $\sum_{i=1}^4 \gamma_i(y, \alpha, R, w) = y$ for all $(y, \alpha, R, w) \in S$, and $\gamma(0, \alpha, R, w) = (0, 0, 0, 0)$ for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$.*

Proof: From Lemma 4 we know that γ is well defined and upper hemicontinuous. By Lemma 5, $v^* = V$ and by Theorem 1, V is strictly concave. It follows that γ is obtained as the solution of an optimization problem in which a strictly concave function is maximized on a convex set so that the solution set consists of a single

point. This shows that γ is a well defined continuous function.

Under A.3 (ii), u is strictly increasing. This ensures that the budget restriction holds with equality.

The last property stated follows trivially from the definition of $\widehat{\Gamma}(y, R, w)$. \blacksquare

We prove Theorem 2 after proving Lemma 7 which analyzes the first order necessary conditions for the optimization problem which leads to γ . In Lemma 7, we allow the various derivatives to take the value “ $+\infty$ ” so long as they are well defined as limits as is the case under A.2 and A.3.

Let us fix some notation first. Given $(y, \alpha, R, w) \in S$, set $\gamma_i := \gamma_i(y, \alpha, R, w)$, $i = 1, 2, 3, 4$, and if, in addition, $\alpha' \in \mathcal{A}$ is also specified, then set $y' := w \cdot \psi(\alpha', \gamma_3) + R \cdot \gamma_4$, and set $\gamma'_i := \gamma_i(y', \alpha', R, w)$. Now set $\bar{u}(y, R) := \max\{u_1(\gamma_1, R \cdot \gamma_2), R \cdot u_2(\gamma_1, R \cdot \gamma_2)\}$. $\bar{u}(y', R)$ can now be defined in the obvious way. We have $\bar{u}(0, R) = \max\{\bar{u}_1, R \cdot \bar{u}_2\}$.

Lemma 7: *Assume A.1 (i)-(iii), A.2 (i)-(iv), A.3 (i)-(iv), and A.4. Then, the policy correspondence γ has the following properties for $(y, \alpha, R, w) \in S$:*

(a) *If $\gamma_i > 0$, $i = 1, 2$, then $V(y, \alpha, R, w)$ is differentiable in y with derivative $V'(y, \alpha, R, w) = \bar{u}(y, R)$.*

(b) *For all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$, there exists $\tilde{y}(\alpha, R, w)$ such that for $y > \tilde{y}(\alpha, R, w)$, $\gamma_i > 0$ for some $i = 1, 2$.*

(c) *For $y > 0$, if $\gamma_3 = \gamma_4 = 0$ then*

$$\bar{u}(y, R) \geq \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha')$$

$$\bar{u}(y, R) \geq \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha').$$

If, in addition, $\gamma'_1 = \gamma'_2 = 0$ for all (y', α', R, w) that are induced, or $y' = 0$, then

$$\bar{u}(y, R) \geq \beta \cdot w \cdot \bar{u}(0, R) \cdot \int \psi_2(\alpha', 0) P(\alpha, d\alpha')$$

$$\bar{u}(y, R) \geq \beta \cdot R \cdot \bar{u}(0, R).$$

(d) *If $\gamma_3 > 0$ then, $w > 0$, $y' > 0$ for all $\alpha' \in \mathcal{A}$, and*

$$\beta \cdot w \int V^-(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') = V^-(y, \alpha, R, w)$$

$$\geq V^+(y, \alpha, R, w) = \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha')$$

while if $\gamma_4 > 0$ then, $R > 0$, $y' > 0$ for all $\alpha' \in \mathcal{A}$, and

$$\beta \cdot R \int V^-(y', \alpha', R, w) P(\alpha, d\alpha') = V^-(y, \alpha, R, w)$$

$$\geq V^+(y, \alpha, R, w) = \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha')$$

and if in addition $\gamma_1 > 0$, or $\gamma_2 > 0$ then the conditions hold with equality.

(e) *If $\gamma_3 > 0$ and $\gamma_4 = 0$ then*

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', \gamma_3) - R] P(\alpha, d\alpha') \geq 0,$$

while if $\gamma_3 = 0$ and $\gamma_4 > 0$ then

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', 0) - R] P(\alpha, d\alpha') \leq 0.$$

Proof: As noted in the proof of Lemma 6, γ is obtained as the solution of an optimization problem in which the objective function is strictly concave and increasing, and the feasible set is convex. So consider $(y, \alpha, R, w) \in \mathcal{S}$ and note first that y' is a function of (γ, α', R, w) so, under A.2 (ii)-(iii), since ψ , w , R , and γ_i take nonnegative values, $y' > 0$ for some α' if and only if it is positive for all $\alpha' \in \mathcal{A}$ which we shall write as if $y' > 0$ for (some, hence, for) all $\alpha' \in \mathcal{A}$.

A straight forward variational argument, together with the differentiability of $\psi(\alpha, \cdot)$ and u , A.2 (iv) and A.3 (iv), lead to the following first order necessary conditions for the optimization problem:

$$V^+(y, \alpha, R, w) = \max\{u_1(\gamma_1, R \cdot \gamma_2), R \cdot u_2(\gamma_1, R \cdot \gamma_2),$$

$$\beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) \cdot P(\alpha, d\alpha'), \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha')\}$$

if $y \geq 0$, where the corresponding first order conditions must hold with equality if $\gamma_i > 0$, $i = 1, 2, 3, 4$, and

$$u_1(\gamma_1, R \cdot \gamma_2) = V^-(y, \alpha, R, w) \quad \text{if } y > 0 \text{ and } \gamma_1 > 0,$$

$$R \cdot u_2(\gamma_1, R \cdot \gamma_2) = V^-(y, \alpha, R, w) \quad \text{if } y > 0 \text{ and } \gamma_2 > 0,$$

and if $y' > 0$ for all $\alpha' \in \mathcal{A}$ then

$$\beta \cdot w \int V^-(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') = V^-(y, \alpha, R, w) \quad \text{if } \gamma_3 > 0,$$

$$\beta \cdot R \int V^-(y', \alpha', R, w) P(\alpha, d\alpha') = V^-(y, \alpha, R, w) \quad \text{if } \gamma_4 > 0.$$

The fact that V is concave in y provides us with the condition

$$V^-(y, \alpha, R, w) \geq V^+(y, \alpha, R, w) \quad \text{for } y > 0.$$

(a) Suppose $\gamma_1 > 0$; it follows that $y > 0$. Hence, it must be the case that the following first order condition holds:

$$u_1(\gamma_1, R \cdot \gamma_2) = V^-(y, \alpha, R, w) \geq V^+(y, \alpha, R, w) = u_1(\gamma_1, R \cdot \gamma_2).$$

It follows that $V(y, \alpha, R, w)$ is differentiable in y if $\gamma_1 > 0$. An analogous argument shows that $V(y, \alpha, R, w)$ is differentiable in y if $\gamma_2 > 0$.

We have shown that if either $\gamma_1 > 0$ or $\gamma_2 > 0$ then $V(y, \alpha, R, w)$ is differentiable in y with derivative $V'(y, \alpha, R, w) = \bar{u}(y, R)$.

(b) Since V is a bounded function and V is increasing and concave in y , it follows that, for all $(\alpha, R, w) \in \mathcal{A} \times \mathcal{R} \times \mathcal{W}$, there exists \tilde{y} such that

$$V^+(\tilde{y}, \alpha, R, w) = \bar{u}(0, R).$$

Hence, for $y > \tilde{y}(\alpha, R, w)$, $\gamma_1 > 0$ or $\gamma_2 > 0$ or both.

(c) Suppose that $y > 0$ but that $\gamma_3 = \gamma_4 = 0$.

For γ_3 , the following first order condition must hold

$$V^+(y, \alpha, R, w) \geq \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha').$$

Since $y > 0$ and $\gamma_3 = \gamma_4 = 0$, by Lemma 6, $\gamma_1 > 0$, or $\gamma_2 > 0$, or both. Using the result in (a) above we have $V^-(y, \alpha, R, w) = V^+(y, \alpha, R, w) = \bar{u}(y, R)$. So we can rewrite the first order condition as

$$\bar{u}(y, R) \geq \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha'). \quad (*)$$

In an entirely analagous manner we get the first order condition for γ_4

$$\bar{u}(y, R) \geq \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha').$$

Consider γ'_i . If $\gamma'_1 = \gamma'_2 = 0$ for all (y', α', R, w) that are induced, then, by the first order conditions, $V^+(y', \alpha', R, w) \geq \bar{u}(0, R)$ so that by substituting in (*), we get

$$\bar{u}(y, R) \geq \beta \cdot w \cdot \bar{u}(0, R) \cdot \int \psi_2(\alpha', 0) P(\alpha, d\alpha').$$

The same condition is obtained if $y' = 0$ since in that case, necessarily, $\gamma'_1 = \gamma'_2 = 0$ (see Lemma 6).

For γ_4 we obtain the analogous condition

$$\bar{u}(y, R) \geq \beta \cdot R \cdot \bar{u}(0, R)$$

corresponding to the two cases, $\gamma'_1 = \gamma'_2 = 0$ for all (y', α', R, w) that are induced, or $y' = 0$.

(d) Suppose $\gamma_3 > 0$; it follows that $y > 0$. Hence, it must be the case that

$$\begin{aligned} \beta \cdot w \int V^-(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') &= V^-(y, \alpha, R, w) \\ &\geq V^+(y, \alpha, R, w) = \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') \end{aligned}$$

where the first equality is required to hold if $y' > 0$ for all $\alpha' \in \mathcal{A}$. If $w = 0$ then the last equality implies that $V^+(y, \alpha, R, w) = 0$ which is incompatible with the fact that V is increasing in y . Hence, $w > 0$. But then since $w > 0$ and $\gamma_3 > 0$, by A.2 (ii)-(iii), $y' > 0$ for (some, hence, for) all $\alpha' \in \mathcal{A}$. It follows that the first equality in the condition must hold.

In an analogous manner we can show that if $\gamma_4 > 0$, then $R > 0$, $y' > 0$ for all $\alpha' \in \mathcal{A}$, and the following first order condition holds:

$$\begin{aligned} \beta \cdot R \int V^-(y', \alpha', R, w) P(\alpha, d\alpha') &= V^-(y, \alpha, R, w) \\ &\geq V^+(y, \alpha, R, w) = \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha'). \end{aligned}$$

In either case, if $\gamma_1 > 0$, or $\gamma_2 > 0$ then, by (a) above, $V^-(y, \alpha, R, w) = V^+(y, \alpha, R, w)$ and the conditions hold with equality.

(e) Suppose that $\gamma_3 > 0$ and $\gamma_4 = 0$. By (d) above, $w > 0$, $y' > 0$, and

$$\begin{aligned} \beta \cdot w \int V^-(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') &= V^-(y, \alpha, R, w) \\ &\geq V^+(y, \alpha, R, w) = \beta \cdot w \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha'). \end{aligned}$$

Also, as in (c) above, we obtain a first order condition for γ_4

$$V^+(y, \alpha, R, w) \geq \beta \cdot R \int V^+(y', \alpha', R, w) P(\alpha, d\alpha').$$

The two conditions can be combined to obtain

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', \gamma_3) - R] P(\alpha, d\alpha') \geq 0.$$

Similarly, we obtain

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', 0) - R] P(\alpha, d\alpha') \leq 0$$

as a necessary condition if $\gamma_3 = 0$ and $\gamma_4 > 0$. ■

Proof of Theorem 2: (a) Lemma 7 (d) and (e) show that if $\gamma_3 > 0$ then $w > 0$ and

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', \gamma_3) - R] P(\alpha, d\alpha') \geq 0$$

with equality if $\gamma_4 > 0$ also holds. Since, by Theorem 1, V is strictly increasing and concave in y , $V^+(y', \alpha', R, w) > 0$ so that the condition above implies that for some $A \in \mathcal{B}_{[0,1]}$ such that $P(\alpha, A) > 0$, $w \cdot \psi_2(\alpha', \gamma_3) \geq R$ for all $\alpha' \in A$. Since, by A.2 (ii), $\psi(\alpha, \cdot)$ is concave for every $\alpha \in \mathcal{A}$, and $\underline{\psi}_2^0 := \sup_{\alpha \in \mathcal{A}} \lim_{e \rightarrow 0^+} \psi_2(\alpha, e)$, we have $\underline{\psi}_2^0 \geq \psi_2(\alpha', \gamma_3)$. This leads to $w \cdot \underline{\psi}_2^0 \geq R$ which contradicts the hypothesis that $w \cdot \underline{\psi}_2^0 < R$. Hence, under the stated conditions, $\gamma_3 = 0$.

(b) Lemma 7 (e) shows that if $\gamma_3 = 0$ and $\gamma_4 > 0$, when $y > 0$, then

$$\beta \cdot \int V^+(y', \alpha', R, w) \cdot [w \cdot \psi_2(\alpha', 0) - R] P(\alpha, d\alpha') \leq 0.$$

Using an argument similar to the one used in (a) above, we obtain $w \cdot \underline{\psi}_2^0 \leq R$ as a necessary implication which contradicts the hypothesis that $w \cdot \underline{\psi}_2^0 > R$. Hence, under the stated conditions, $\gamma_3 = 0$ and $\gamma_4 > 0$ cannot hold.

Now consider the case in which $\gamma_3 = \gamma_4 = 0$ and $y > 0$. It follows that $y' = w \cdot \psi(\alpha', 0) \geq 0$. Since

$$\bar{u}(y, R) < \beta \cdot w \cdot \int \bar{u}(w \cdot \psi(\alpha', 0), R) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha')$$

and $\bar{u}(w \cdot \psi(\alpha', 0), R) \leq V^+(y', \alpha', R, w)$, which follows from the first order conditions developed in the proof of Lemma 7 for γ evaluated at the vector (y', α', R, w) induced by the choices γ_i , we have

$$\bar{u}(y, R) < \beta \cdot w \cdot \int V^+(y', \alpha', R, w) \cdot \psi_2(\alpha', 0) P(\alpha, d\alpha')$$

which contradicts one of the necessary conditions obtained in Lemma 7 (c). It follows that $\gamma_3 = \gamma_4 = 0$ and $y > 0$ cannot hold either under the stated conditions.

We have shown that under the stated conditions, if $y > 0$ then $\gamma_3 > 0$ necessarily.

(c) We omit the proof which is very similar to the proof of (b) just given.

(d) Let $(R, w) \in \mathcal{R} \times \mathcal{W}$ be such that for some $\tilde{e}(w) \in R_{++}$, $\max\{R, w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}(w))\} \in [0, 1/\beta)$. Let $\bar{y} \in R_+$ be as specified in the statement of the theorem. Consider $(y, \alpha, R, w) \in \mathcal{S}$ where $y \geq \bar{y}$. Set $y' := w \cdot \psi(\alpha', \gamma_3) + R \cdot \gamma_4$ with $\gamma_i := \gamma_i(y, \alpha, R, w)$.

Suppose that $\gamma_3 \geq \tilde{e}(w)$. It follows from the hypothesis that

$$\begin{aligned} V^-(y', \alpha, R, w) &> \beta \cdot \max\{R, w \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} \cdot \int V^-(y', \alpha', R, w) P(\alpha, d\alpha') \\ &\geq \beta \cdot w \cdot \int V^-(y', \alpha', R, w) \cdot \psi_2(\alpha', \gamma_3) P(\alpha, d\alpha') \\ &= V^-(y, \alpha, R, w), \end{aligned}$$

where we use the Euler equation for γ_3 derived in Lemma 7 (d), and the fact that $\sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}) \geq \psi_2(\alpha', \gamma_3)$ since, by A.2 (ii), $\psi(\alpha, \cdot)$ is concave for every $\alpha \in \mathcal{A}$ and $\gamma_3 \geq \tilde{e}$. The concavity of V in y implies that $y' < y$.

An analogous argument works if $\gamma_4 > 0$ and delivers the same result.

The remaining case in which $0 < \gamma_3 < \tilde{e}(w)$ and $\gamma_4 = 0$ was singled out in the statement of the theorem.

(e) Follows from Corollary 1 to Theorem 2 in Chamberlain and Wilson (2000). ■

We prove Theorem 3 after proving Lemma 8 and Lemma 9.

Recall that we have defined $\hat{\gamma} : X \times \mathcal{R} \times \mathcal{W} \rightarrow R_+^4$ by the rule $\hat{\gamma}(x, R, w) := \gamma(y(e, b, \alpha, R, w), \alpha, R, w)$.

Lemma 8: *Assume A.1 (i)-(iii), A.2 (i)-(ii), A.3 (i) and (iii), and A.4. Given $x \in X$ and $(R, w) \in \mathcal{R} \times \mathcal{W}$, the function $Q(x, A_{12} \times A_3; R, w)$ defined on the measurable space (X, \mathcal{B}_X) induced by the Markov transition P , the income function $y(e, b, \alpha, R, w)$, and the policy function $\gamma(y, \alpha, R, w)$ is a transition function with the Feller property. If $(R_n, w_n) \rightarrow (R_0, w_0)$ then $Q(x, \cdot; R_n, w_n) \xrightarrow{w} Q(x, \cdot; R_0, w_0)$ for all $x \in X$.*

Proof: Under A.2 (i), ψ is continuous in α and in e . It follows that $w \cdot \psi(\alpha, e) + R \cdot b$ is continuous in x . Further, under A.1 (i)-(iii), A.2 (i)-(ii), A.3 (i) and (iii), and A.4, Lemma 6 showed that the function γ is continuous, hence measurable. It follows that $\hat{\gamma}(x, R, w)$ is continuous, hence measurable, in x and in (R, w) . By SL Theorem 9.13, $Q(\cdot, \cdot; R, w)$ is a Markov transition. By SL Theorem 9.14, $Q(\cdot, \cdot; R, w)$ has the Feller property since P has the Feller property, A.1 (iii), and $\hat{\gamma}$ is continuous.

Now, consider $(R_n, w_n) \rightarrow (R_0, w_0)$. Fix $x \in X$ and a closed set $A_{12} \times A_3 \subset X$. Set $\hat{\gamma}_{i,n} := \hat{\gamma}_i(x, R_n, w_n)$, and $\hat{\gamma}_{i,0} := \hat{\gamma}_i(x, R_0, w_0)$. Since $\hat{\gamma}$ is a continuous function, it follows that $\hat{\gamma}_{i,n} \rightarrow \hat{\gamma}_{i,0}$. If $(\hat{\gamma}_{3,0}, \hat{\gamma}_{4,0}) \in A_{12}$ then $Q(x, A; R_0, w_0) = P(\alpha, A_3)$, and either (a) for all n there exists $N(n) > n$ such that $(\hat{\gamma}_{3,N(n)}, \hat{\gamma}_{4,N(n)}) \in A_{12}$, so that for all n there exists $N(n) > n$ such that $Q(x, A; R_{N(n)}, w_{N(n)}) = Q(x, A; R_0, w_0) =$

$P(\alpha, A_3)$ which in turn implies that $\limsup_{n \rightarrow \infty} Q(x, A; R_n, w_n) = Q(x, A; R_0, w_0)$, or (b) for every n , $(\hat{\gamma}_{3,n}, \hat{\gamma}_{4,n}) \notin A_{12}$, so that, for all n , $Q(x, A; R_n, w_n) = 0 \leq Q(x, A; R_0, w_0) = P(\alpha, A_3)$ which in turn implies that $\limsup_{n \rightarrow \infty} Q(x, A; R_n, w_n) \leq Q(x, A; R_0, w_0)$. If instead, $(\hat{\gamma}_{3,0}, \hat{\gamma}_{4,0}) \notin A_{12}$ then $Q(x, A; R_0, w_0) = 0$ and, since A_{12} is closed, there exists N such that for all $n > N$, $(\hat{\gamma}_{3,n}, \hat{\gamma}_{4,n}) \notin A_{12}$ so that $Q(x, A; R_n, w_n) = 0$ for all $n > N$ with the implication that $\limsup_{n \rightarrow \infty} Q(x, A; R_n, w_n) = 0 = Q(x, A; R_0, w_0)$. We have shown that in every case $\limsup_{n \rightarrow \infty} Q(x, A; R_n, w_n) \leq Q(x, A; R_0, w_0)$ for all $A \subset X$ and A closed. SL Theorem 12.3 (b) shows that $Q(x, \cdot; R_n, w_n) \xrightarrow{w} Q(x, \cdot; R_0, w_0)$ for all $x \in X$. \blacksquare

Lemma 9: *Assume A.1 (i)-(iii), A.2 (i)-(ii), A.3 (i) and (iii), and A.4. Consider $(R, w) \in \mathcal{R} \widehat{\times} \mathcal{W}$ where the set of factor prices $\mathcal{R} \widehat{\times} \mathcal{W}$ has the property that $\gamma_i(y, \alpha, R, w)$, $i = 3, 4$, is uniformly bounded where \bar{e} and \bar{b} are the upper bounds. (a) $\mathcal{N}(R, w)$ is nonempty, closed, and convex. (b) If $(R_n, w_n) \rightarrow (R_0, w_0)$ and $\nu_n \in \mathcal{N}(R_n, w_n)$ then there exists a $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ and a subsequence $\nu_{n(j)}$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$.*

Proof: Define $\bar{X} := [0, \bar{e}] \times [0, \bar{b}] \times \mathcal{A}$. Under the stated conditions, $x \in \bar{X} \subset X$ where \bar{X} is a compact set.

(a) By Lemma 8, for every $(R, w) \in \mathcal{R} \widehat{\times} \mathcal{W}$, $Q(\cdot, \cdot; R, w)$ is a Markov transition with the Feller property. Since we have a Markov transition function with the Feller property on a compact state space, by SL Theorem 12.10, for every $(R, w) \in \mathcal{R} \widehat{\times} \mathcal{W}$ there exists an invariant measure proving that $\mathcal{N}(R, w) \neq \emptyset$.

Convexity of $\mathcal{N}(R, w)$ follows directly from the definition of an invariant measure.

To prove closure, fix $(R, w) \in \mathcal{R} \widehat{\times} \mathcal{W}$, and let $\nu_n \xrightarrow{w} \bar{\nu}$, where $\nu_i \in \mathcal{N}(R, w)$ for all $i = 1, 2, \dots$. For $f : \bar{X} \rightarrow R$ a continuous function,

$$(1) \int f(x) \nu_n(dx) \rightarrow \int f(x) \bar{\nu}(dx) \text{ since } \nu_n \xrightarrow{w} \bar{\nu},$$

$$(2) \int (\int f(x') Q(x, dx'; R, w)) \nu_n(dx) \rightarrow \int (\int f(x') Q(x, dx'; R, w)) \bar{\nu}(dx)$$

by (1) using the fact that $\int f(x') Q(x, dx'; R, w)$ is a continuous function on \bar{X} since, by Lemma 8, $Q(\cdot, \cdot; R, w)$ is a Markov transition with the Feller property. Also, since f is measurable and $\nu_n \in \mathcal{N}(R, w)$ for all n ,

$$\int f(x) \nu_n(dx) = \int f(x') (\int Q(x, dx'; R, w) \nu_n(dx))$$

which leads to

$$(3) \int f(x) \nu_n(dx) = \int (\int f(x') Q(x, dx'; R, w)) \nu_n(dx)$$

where we use SL Theorem 8.3 to verify that the order of integration does not matter.

From the triangle inequality and (1)-(3),

$$\int f(x) \bar{\nu}(dx) = \int (\int f(x') Q(x, dx'; R, w)) \bar{\nu}(dx)$$

for all continuous functions $f : \bar{X} \rightarrow R$. Using SL Theorem 8.3,

$$\int f(x) \bar{\nu}(dx) = \int f(x') (\int Q(x, dx'; R, w) \bar{\nu}(dx))$$

for all continuous functions $f : \bar{X} \rightarrow R$. Corollary 2 to SL Theorem 12.6 implies that $\bar{\nu} \in \mathcal{N}(R, w)$ as required.

(b) To prove continuity, consider $(R_n, w_n) \rightarrow (R_0, w_0)$ and $\nu_n \in \mathcal{N}(R_n, w_n)$ for all $n = 1, 2, \dots$. Since \bar{X} is compact, $\mathcal{M}(\bar{X})$ is compact in the weak topology (Parthasarathy Chapter 2, Theorem 6.4) with the implication that there exists $\bar{\nu}$ and a subsequence such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$. For $f : \bar{X} \rightarrow R$ a continuous function

we set $g_i(x) := \int f(x')Q(x, dx'; R_i, w_i)$. As shown in Lemma 8, $Q(x, \cdot; R_n, w_n) \xrightarrow{w} Q(x, \cdot; R_0, w_0)$ for all $x \in \overline{X}$ so that $g_n(x) \rightarrow g_0(x)$ for all $x \in \overline{X}$. By the Dominated Convergence Theorem (SL Theorem 7.10), $\int g_{n(j)}(x)\nu_{n(j)}(dx) \rightarrow \int g_0(x)\nu_{n(j)}(dx)$ where we can take $\|f\|$, the norm of f in the *sup* norm, as the dominating function. We have shown that for $f : \overline{X} \rightarrow R$ a continuous function,

$$\int (\int f(x')Q(x, dx'; R_{n(j)}, w_{n(j)}))\nu_{n(j)}(dx) \rightarrow \int (\int f(x')Q(x, dx'; R_0, w_0))\nu_{n(j)}(dx)$$

or, using the fact that $\nu_{n(j)} \in \mathcal{N}(R_{n(j)}, w_{n(j)})$, that

$$(1) \int f(x)\nu_{n(j)}(dx) \rightarrow \int (\int f(x')Q(x, dx'; R_0, w_0))\nu_{n(j)}(dx).$$

Further, for $f : \overline{X} \rightarrow R$ a continuous function,

$$(2) \int f(x)\nu_{n(j)}(dx) \rightarrow \int f(x)\bar{\nu}(dx) \text{ since } \nu_{n(j)} \xrightarrow{w} \bar{\nu},$$

and

$$(3) \int (\int f(x')Q(x, dx'; R_0, w_0))\nu_{n(j)}(dx) \rightarrow \int (\int f(x')Q(x, dx'; R_0, w_0))\bar{\nu}(dx)$$

by (2) using the fact that $\int f(x')Q(x, dx'; R_0, w_0)$ is a continuous function on \overline{X} since $Q(\cdot, \cdot; R_0, w_0)$ is a Markov transition with the Feller property. From the triangle inequality and (1)-(3),

$$\int f(x)\bar{\nu}(dx) = \int (\int f(x')Q(x, dx'; R_0, w_0))\bar{\nu}(dx)$$

for all continuous functions $f : \overline{X} \rightarrow R$. That $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ now follows as in the earlier proof of the closure of $\mathcal{N}(R, w)$ by using SL Theorem 8.3 and Corollary 2 to SL Theorem 12.6. \blacksquare

Proof of Theorem 3: (a) and (b) Under the stated assumptions, by Theorem 2 (d), either $\gamma_3(y, \alpha, R, w) \leq \tilde{e}$ and $\gamma_4(y, \alpha, R, w) = 0$, or $y \geq \bar{y}$ is not sustainable. Since $\gamma_i \geq 0$ and, by Theorem 1, $\sum_{i=1}^4 \gamma_i(y, \alpha, R, w) = y$ for all $(y, \alpha, R, w) \in S$, it follows that, for the purpose of analyzing the long run dynamics of the system, we can restrict attention to $e \in [0, \bar{e}]$ and $b \in [0, \bar{b}]$ where $\bar{e} := \max\{\tilde{e}, \bar{y}\}$ and $\bar{b} := \bar{y}$.

So, under the stated conditions, there are uniform upper bounds on $\hat{\gamma}_3$ and $\hat{\gamma}_4$. Hence, Lemma 9 (a) shows that for all $(R, w) \in [0, \bar{R}] \times [0, \bar{W}]$, $\mathcal{N}(R, w)$ is nonempty, closed, and convex; that there exists a subsequence and $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$ follows from Lemma 9 (b). \blacksquare

Theorem 4 is proved after proving Lemma 10 and Lemma 11.

In Lemma 10 we put together the properties that mean demand and supply satisfy.

Lemma 10: *Assume A.1 (i)-(iii), A.2 (i)-(iv), A.3 (i)-(iv), A.4, and A.5, and consider factor prices in the set $[1 - \delta, \bar{R}] \times [w_\beta, \bar{W}]$ where $\bar{R} < 1/\beta$ and $\bar{W} \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}) < 1/\beta$ for some $\tilde{e} \in R_{++}$. Assume further that for some $\bar{y} \in R_+$ and for all $(\alpha, y) \in \mathcal{A} \times [\bar{y}, \infty)$, and for all $(R, w) \in [1 - \delta, \underline{R}] \times [w_\beta, \bar{W}]$, V satisfies*

$$V^-(y, \alpha, R, w) > \beta \cdot \max\{\underline{R}, \bar{W} \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} \int V^-(y, \alpha', R, w)P(\alpha, d\alpha').$$

(a) *For all (R, w) , $\mathcal{N}(R, w) \neq \emptyset$ and, for $\nu \in \mathcal{N}(R, w)$, $\mathcal{L}(R, w, \nu)$, $\mathcal{K}(R, w, \nu)$, and $\xi(R, w, \nu)$ are well defined, nonnegative, and uniformly bounded by $\max_{\alpha \in \mathcal{A}} \psi(\alpha, \bar{e})$, \bar{y} and $(1 + 1/\beta) \cdot \bar{y}$ respectively.*

(b) *If, in addition, A.2 (iii a) holds, then there exists $\hat{\nu} \in \mathcal{N}(R, w)$ where $\hat{\nu}$ has support $\{(0, 0)\} \times \mathcal{A}$ so $\mathcal{L}(R, w, \hat{\nu}) = 0$. If, in addition, $w \cdot \overline{\psi}_2^0 < R$, then $\mathcal{L}(R, w, \nu) = 0$ for all $\nu \in \mathcal{N}(R, w)$.*

(c) If, in addition, A.2 (iii a) holds, $\beta \cdot w \cdot \underline{\psi}_2^0 > 1$ also holds, and $\tilde{\nu} \in \mathcal{N}(R, w)$ is such that $\tilde{\nu}(\{(0, 0)\} \times \mathcal{A}) < 1$, then $\mathcal{L}(R, w, \tilde{\nu}) > 0$.

(d) If, in addition, A.2 (iii b) holds, then $\mathcal{L}(R, w, \nu) \geq \min_{\alpha \in \mathcal{A}} \psi(\alpha, 0) > 0$ for all $\nu \in \mathcal{N}(R, w)$. If, in addition, $w \cdot \overline{\psi}_2^0 < R$, then $\max_{\alpha \in \mathcal{A}} \psi(\alpha, 0) \geq \mathcal{L}(R, w, \nu)$ for all $\nu \in \mathcal{N}(R, w)$.

(e) If $\nu_i \in \mathcal{N}(R, w)$ with $\nu_i \xrightarrow{w} \bar{\nu}$ then $\mathcal{L}(R, w, \nu_i) \rightarrow \mathcal{L}(R, w, \bar{\nu})$, and similarly for \mathcal{K} and ξ . If $(R_n, w_n) \rightarrow (R_0, w_0)$ and $\nu_n \in \mathcal{N}(R_n, w_n)$, then there exists a $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ and a subsequence $\nu_{n(j)}$ such that $\mathcal{L}(R_{n(j)}, w_{n(j)}, \nu_{n(j)}) \rightarrow \mathcal{L}(R_0, w_0, \bar{\nu})$, and similarly for \mathcal{K} and ξ .

Proof: Consider factor prices in the set $[1 - \delta, \overline{R}] \times [w_\beta, \overline{W}]$. Under the assumptions of Lemma 10, we can restrict attention to states in the compact set $\overline{X} := \bar{e} \times \bar{b} \times \mathcal{A}$ defined in the proofs of Theorem 3 and Lemma 9. Our analysis involves studying the function $\hat{\gamma}$, which, by Theorem 1, is continuous, on the compact set $\overline{X} \times [1 - \delta, \overline{R}] \times [w_\beta, \overline{W}]$.

(a) By Theorem 3 (a), $\mathcal{N}(R, w) \neq \emptyset$. Since $\hat{\gamma}$ restricted to \overline{X} is integrable, since it is measurable, nonnegative, and bounded above, the functions ξ , \mathcal{L} , and \mathcal{K} are well defined. That they are nonnegative is trivial. Evidently, \mathcal{L} can be uniformly bounded from above by $\max_{\alpha \in \mathcal{A}} \psi(\alpha, \bar{e})$; similarly, use the budget constraint and the fact that \bar{y} is a uniform upper bound on income to conclude that \mathcal{K} can be bounded from above by \bar{y} , and ξ can be bounded from above by $(1 + 1/\beta) \cdot \bar{y}$.

(b) For the first part, Theorem 1 shows that $\hat{\gamma}_3 = \hat{\gamma}_4 = 0$ if $y = 0$. In addition, under A.2 (iii a), $e = b = 0$ implies that $y = 0$. It follows that

$$Q((0, 0, \alpha), A_{12} \times A_3; R, w) = P(\alpha, A_3) \quad \text{if } (0, 0) \in A_{12}.$$

Consider a set A_{12} such that $(0, 0) \in A_{12}$. In such a case, the invariance requirement takes the form

$$\hat{\nu}(A_{12} \times A_3) = \int_{x \in A_{12} \times A_3} Q(x, A_{12} \times A_3; R, w) \hat{\nu}(dx).$$

Since $\hat{\nu}$ has support $\{(0, 0)\} \times \mathcal{A}$, $\hat{\nu}(A_{12} \times A_3) = \hat{\nu}_\alpha(A_3)$ if $(0, 0) \in A_{12}$ and $\hat{\nu}(A_{12} \times A_3) = 0$ if $(0, 0) \notin A_{12}$, where $\hat{\nu}_\alpha$ is the marginal measure induced by $\hat{\nu}$ on α . Hence the invariance requirement becomes

$\hat{\nu}_\alpha(A_3) = \int_{x \in \{(0, 0)\} \times A_3} Q(x, A_{12} \times A_3; R, w) \hat{\nu}(dx) = \int_{\alpha \in A_3} P(\alpha, A_3) \hat{\nu}_\alpha(d\alpha) = \hat{\nu}_\alpha(A_3)$ since $\hat{\nu}_\alpha$ is an invariant measure of $P(\cdot, \cdot)$. A similar argument applies to sets such that $(0, 0) \notin A_{12}$. This shows that the invariance requirement does hold. Therefore, $\hat{\nu} \in \mathcal{N}(R, w)$ as required. Using A.2 (iii a), $\mathcal{L}(R, w, \hat{\nu}) = 0$. This proves the first part of the result.

For the second part, by Theorem 2 (a), if $w \cdot \overline{\psi}_2^0 < R$ then $\hat{\gamma}_3(e, b, \alpha, R, w) = 0$. Under A.2 (iii a), $\psi(\alpha, 0) = 0$ for all $\alpha \in \mathcal{A}$. It follows that $\mathcal{L}(R, w, \nu) = 0$ for all $\nu \in \mathcal{N}(R, w)$ as required.

(c) Let us first prove that, under the stated conditions, $\hat{\gamma}_3(e, b, \alpha, R, w) > 0$ if $w \cdot \psi(\alpha, e) + R \cdot b > 0$.

We have assumed that $\beta \cdot w \cdot \underline{\psi}_2^0 > 1$. It follows that

$$\overline{u}(0, R) < \beta \cdot w \cdot \overline{u}(0, R) \cdot \underline{\psi}_2^0.$$

Since u is concave, $\overline{u}(y, R) \leq \overline{u}(0, R)$ showing that

$$\bar{u}(y, R) \leq \bar{u}(0, R) < \beta \cdot w \cdot \bar{u}(0, R) \cdot \underline{\psi}_2^0.$$

But then, since we have also assumed that A.2 (iii a) holds, i.e., $\psi(\alpha, 0) = 0$ for all $\alpha \in \mathcal{A}$, under our assumptions

$$\bar{u}(y, R) < \beta \cdot w \cdot \int \bar{u}(w \cdot \psi(\alpha', 0), R) \cdot \psi_2(\alpha', 0) \cdot P(\alpha, d\alpha')$$

must hold. Furthermore, $\beta \cdot w \cdot \underline{\psi}_2^0 > 1$ and $R < 1/\beta$ imply that $w \cdot \underline{\psi}_2^0 > R$ for $\beta > 0$. It follows that, under the maintained hypotheses, the conditions in Theorem 2 (b) are satisfied and $\hat{\gamma}_3(e, b, \alpha, R, w) > 0$ if $w \cdot \psi(\alpha, e) + R \cdot b > 0$ as required.

Assume that $\mathcal{L}(R, w, \tilde{\nu}) = 0$ for $\tilde{\nu} \in \mathcal{N}(R, w)$. Since $\hat{\gamma}_3$ is always nonnegative, and strictly positive if $w \cdot \psi(\alpha, e) + R \cdot b > 0$, it follows that $\tilde{\nu}((0, \bar{e}] \times [0, \bar{b}] \times A_3) = 0$ for all $A_3 \in \mathcal{B}_{\mathcal{A}}$. $\tilde{\nu}$ being a probability measure, $\tilde{\nu}(\{0\} \times [0, \bar{b}] \times A_3) = 1$ necessarily for all $A_3 \in \mathcal{B}_{\mathcal{A}}$. In this case, the invariance requirement takes the form

$$\begin{aligned} \tilde{\nu}((0, \bar{e}] \times [0, \bar{b}] \times A_3) &= \int_{x \in (0, \bar{e}] \times [0, \bar{b}] \times A_3} Q(x, (0, \bar{e}] \times [0, \bar{b}] \times A_3; R, w) \tilde{\nu}(dx) \\ &\quad + \int_{x \in \{0\} \times [0, \bar{b}] \times A_3} Q(x, (0, \bar{e}] \times [0, \bar{b}] \times A_3; R, w) \tilde{\nu}(dx) \end{aligned}$$

for all $A_3 \in \mathcal{B}_{\mathcal{A}}$. So invariance and the hypothesised form of the measure lead to

$$\int_{x \in \{0\} \times [0, \bar{b}] \times A_3} Q(x, (0, \bar{e}] \times [0, \bar{b}] \times A_3; R, w) \tilde{\nu}(dx) = 0$$

for all $A_3 \in \mathcal{B}_{\mathcal{A}}$. It follows that, necessarily, $Q((0, b, \alpha), (0, \bar{e}] \times [0, \bar{b}] \times A_3; R, w) = 0$ for all $A_3 \in \mathcal{B}_{\mathcal{A}}$, for all $\alpha \in \mathcal{A}$ and for all b such that $(0, b, \alpha)$ is in the support of $\tilde{\nu}$. However, we have shown that $\hat{\gamma}_3(e, b, \alpha, R, w) > 0$ if $w \cdot \psi(\alpha, e) + R \cdot b > 0$. Hence, $\mathcal{L}(R, w, \tilde{\nu}) = 0$ implies that, necessarily, $\tilde{\nu}(A_{12} \times \mathcal{A}) = 0$ if $(0, 0) \notin A_{12}$. This completes the proof.

(d) Trivially, under A.2 (iii b), the quantity of efficiency units of labour is bounded below by $\min_{\alpha \in \mathcal{A}} \psi(\alpha, 0) > 0$. Since $\mathcal{L}(R, w, \nu)$ is obtained by integration, the property is preserved.

If, in addition, $w \cdot \overline{\psi}_2^0 < R$ then, by Theorem 2 (a), $\hat{\gamma}_3(e, b, \alpha, R, w) = 0$. Under A.2 (iii b), $\max_{\alpha \in \mathcal{A}} \psi(\alpha, 0) > 0$. The result follows.

(e) We prove the continuity properties.

Given a sequence of invariant measures, denote by $\nu_{\alpha, i}$ the marginal measure on $(\mathcal{A}, \mathcal{B}_{\mathcal{A}})$ induced by the invariant measure ν_i . If $\nu_i \xrightarrow{w} \bar{\nu}$ we have (Hildenbrand Chapter 1 D27) $\nu_{\alpha, i} \times \nu_i \xrightarrow{w} \bar{\nu}_{\alpha} \times \bar{\nu}$ so that the product measure also converges weakly.

For the first part, recall that

$$\mathcal{L}(R, w, \nu) := \int \int \psi(\alpha', \hat{\gamma}_3(x, R, w)) \nu_{\alpha}(d\alpha') \nu(dx).$$

Since $\hat{\gamma}_3$ is continuous by Theorem 2 (a), and $\psi(\alpha, \cdot)$ is continuous by A.2 (i), and the product measure also converges weakly, $\mathcal{L}(R, w, \nu_i) \rightarrow \mathcal{L}(R, w, \bar{\nu})$. For \mathcal{K} and ξ the result follows directly from the continuity of $\hat{\gamma}$ and the fact that $\nu_i \xrightarrow{w} \bar{\nu}$.

We prove the second part of the continuity property. Under the stated conditions Theorem 3 (b) can be invoked to show that if $(R_n, w_n) \rightarrow (R_0, w_0)$ and $\nu_n \in \mathcal{N}(R_n, w_n)$, then there exists $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ and a subsequence $\nu_{n(j)}$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$. Now consider a continuous function $f : \bar{X} \times [1 - \delta, \bar{R}] \times [w_{\beta}, \bar{W}]$. Notice that

- (1) $\int f(x, R_0, w_0) \nu_{n(j)}(dx) \rightarrow \int f(x, R_0, w_0) \bar{\nu}(dx)$ since $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$,
- (2) $\int f(x, R_{n(j)}, w_{n(j)}) \nu_{n(j)}(dx) \rightarrow \int f(x, R_0, w_0) \nu_{n(j)}(dx)$

by the Dominated Convergence Theorem (SL Theorem 7.10) since f is continuous on a compact set and we can take $\|f\|$, the norm of f in the sup norm, as the dominating function. The triangle inequality and (1) and (2) imply that

$$\int f(x, R_{n(j)}, w_{n(j)}) \nu_{n(j)}(dx) \rightarrow \int f(x, R_0, w_0) \bar{\nu}(dx).$$

We have show that, under the stated conditions, for every continuous function $f : \bar{X} \times [1 - \delta, \bar{R}] \times [w_\beta, \bar{W}]$, $\int f(x, R_{n(j)}, w_{n(j)}) \nu_{n(j)}(dx) \rightarrow \int f(x, R_0, w_0) \bar{\nu}(dx)$.

Since $\hat{\gamma}$ is continuous, the result we just proved lets us conclude that $\mathcal{K}(R_{n(j)}, w_{n(j)}, \nu_{n(j)}) \rightarrow \mathcal{K}(R_0, w_0, \bar{\nu})$, and $\xi(R_{n(j)}, w_{n(j)}, \nu_{n(j)}) \rightarrow \xi(R_0, w_0, \bar{\nu})$ since we have shown that there exists $\bar{\nu} \in \mathcal{N}(R_0, w_0)$ and a subsequence $\nu_{n(j)}$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$. But we have also shown that $\nu_{\alpha, n(j)} \times \nu_{n(j)} \xrightarrow{w} \bar{\nu}_\alpha \times \bar{\nu}$. The same argument given for the function $\mathcal{K}(\cdot)$ now shows that $\mathcal{L}(R_{n(j)}, w_{n(j)}, \nu_{n(j)}) \rightarrow \mathcal{L}(R_0, w_0, \bar{\nu})$. \blacksquare

In what follows, we study the existence problem in parametric form where the parameter is the capital-labour ratio. To simplify the notation, we use $\mathcal{N}(k)$ instead of $\mathcal{N}(R(k), w(k))$. Similarly, $\mathcal{L}(k, \nu)$ is used instead of $\mathcal{L}(R(k), w(k), \nu)$ with $\nu \in \mathcal{N}(k)$, $\mathcal{K}(k, \nu)$ is used for mean capital demand, and $\xi(k, \nu)$ for demand for the produced good.

Lemma 11: *Assume A.1 (i)-(iii), A.2 (i), (ii), (iii b), (iv), A.3 (i)-(iv), A.4, and A.5. Consider $\underline{k} > k_\beta$ and \bar{k} such that $w(\bar{k}) \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e}) < 1/\beta$ for some $\tilde{e} \in R_{++}$. Assume further that for some $\bar{y} \in R_+$ and for all $(\alpha, y) \in \mathcal{A} \times [\bar{y}, \infty)$, and for all $(R, w) \in [1 - \delta, R(\underline{k})] \times [w_\beta, w(\bar{k})]$, V satisfies*

$$V^-(y, \alpha, R, w) > \beta \cdot \max\{R(\underline{k}), w(\bar{k}) \cdot \sup_{\alpha \in \mathcal{A}} \psi_2(\alpha, \tilde{e})\} \int V^-(y, \alpha', R, w) P(\alpha, d\alpha').$$

Define a correspondence $\Phi : [\underline{k}, \bar{k}] \rightarrow R_+$ as follows:

$$\Phi(k) := \{\kappa \in R_+ : \text{there exists } \nu \in \mathcal{N}(k) \text{ such that } \kappa = \mathcal{K}(k, \nu) / \mathcal{L}(k, \nu)\}.$$

$\Phi(k)$ is nonempty, compact and convex valued. If $k_n \rightarrow k_0$ and $\kappa_n \in \Phi(k_n)$ for all $n = 1, 2, \dots$, then there exists a subsequence and a $\kappa_0 \in \Phi(k_0)$, such that $\kappa_{n(j)} \rightarrow \kappa_0$.

Proof: Since the assumptions of Theorem 3 (a) are verified, $\mathcal{N}(R(k), w(k)) \neq \emptyset$.

Under A.2 (iii b), by Lemma 10 (d), $\mathcal{L}(k, \nu) \geq \min_{\alpha \in \mathcal{A}} \psi(\alpha, 0)$ for all $\nu \in \mathcal{N}(k)$ and, by Lemma 10 (a), both $\mathcal{L}(k, \nu)$ and $\mathcal{K}(k, \nu)$ are uniformly bounded from above. Hence, Φ is nonempty, bounded from above by $\bar{y} / \min_{\alpha \in \mathcal{A}} \psi(\alpha, 0)$, and bounded from below by 0.

To prove convexity of $\Phi(k)$, consider $\kappa_i \in \Phi(k)$, $i = 1, 2$. Necessarily, there exist ν_i , $i = 1, 2$, such that $\nu_i \in \mathcal{N}(k)$ for $i = 1, 2$ and $\kappa_i = \mathcal{K}(k, \nu_i) / \mathcal{L}(k, \nu_i)$ where $\mathcal{K}(k, \nu_i) \geq 0$ for $i = 1, 2$, and $\mathcal{L}(k, \nu_i) > 0$ for $i = 1, 2$. Let $\theta \in (0, 1)$. We wish to show that for some $\nu \in \mathcal{N}(k)$, $\mathcal{L}(k, \nu) > 0$ and $\theta \cdot \kappa_1 + (1 - \theta) \cdot \kappa_2 = \mathcal{K}(k, \nu) / \mathcal{L}(k, \nu)$.

So set

$$\lambda(\theta) := \frac{\theta \cdot \mathcal{L}(k, \nu_2)}{\theta \cdot \mathcal{L}(k, \nu_2) + (1 - \theta) \cdot \mathcal{L}(k, \nu_1)}.$$

Clearly, $\lambda(\theta) \in (0, 1)$. Also

$$\theta = \frac{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1)}{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1) + (1 - \lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)}.$$

Consider $\nu^\theta := \lambda(\theta) \cdot \nu_1 + (1 - \lambda(\theta)) \cdot \nu_2$. Since $\mathcal{N}(k)$ is a convex set by Theorem 3 (a), $\nu^\theta \in \mathcal{N}(k)$. Now define $\kappa^\theta := \mathcal{K}(k, \nu^\theta) / \mathcal{L}(k, \nu^\theta)$. By using the linearity of the integral in the measure of integration we get $\mathcal{L}(k, \nu^\theta) > 0$ and

$$\kappa^\theta = \frac{\lambda(\theta) \cdot \mathcal{K}(k, \nu_1) + (1 - \lambda(\theta)) \cdot \mathcal{K}(k, \nu_2)}{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1) + (1 - \lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)}$$

$$\begin{aligned}
&= \frac{\lambda(\theta) \cdot \mathcal{K}(k, \nu_1)}{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1)} \cdot \frac{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1)}{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1) + (1-\lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)} + \frac{(1-\lambda(\theta)) \cdot \mathcal{K}(k, \nu_2)}{(1-\lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)} \cdot \frac{(1-\lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)}{\lambda(\theta) \cdot \mathcal{L}(k, \nu_1) + (1-\lambda(\theta)) \cdot \mathcal{L}(k, \nu_2)} \\
&= \kappa_1 \cdot \theta + \kappa_2 \cdot (1 - \theta)
\end{aligned}$$

as required.

To prove that Φ is compact valued it suffices to show that $\Phi(k)$ is a closed set since we have already shown that it is uniformly bounded from above and below. So consider $\kappa_i := \mathcal{K}(k, \nu_i) / \mathcal{L}(k, \nu_i)$ where $\nu_i \in \mathcal{N}(k)$ and $i = 0, 1, 2, \dots$, and suppose that $\kappa_n \rightarrow \kappa_0$. κ_0 is finite since Φ is uniformly bounded from above. As in the proof of Lemma 9 (b), there exists a subsequence and $\bar{\nu} \in \mathcal{M}(\bar{X})$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$. Since, by Theorem 3 (a), $\mathcal{N}(k)$ is closed, we have $\bar{\nu} \in \mathcal{N}(k)$. By Lemma 10 (e), $\mathcal{K}(k, \nu_{n(j)}) \rightarrow \mathcal{K}(k, \bar{\nu})$ and $\mathcal{L}(k, \nu_{n(j)}) \rightarrow \mathcal{L}(k, \bar{\nu})$. Since $\mathcal{L}(k, \bar{\nu}) > 0$, by the defining property of Φ , $\mathcal{K}(k, \bar{\nu}) / \mathcal{L}(k, \bar{\nu}) \in \Phi(k)$ and $\mathcal{K}(k, \nu_{n(j)}) / \mathcal{L}(k, \nu_{n(j)}) \rightarrow \mathcal{K}(k, \bar{\nu}) / \mathcal{L}(k, \bar{\nu})$. So $\kappa_{n(j)} \rightarrow \mathcal{K}(k, \bar{\nu}) / \mathcal{L}(k, \bar{\nu})$. Since $n(j)$ defines a subsequence, and $\kappa_n \rightarrow \kappa_0$, where κ_0 is finite, it follows that $\kappa_0 = \mathcal{K}(k, \bar{\nu}) / \mathcal{L}(k, \bar{\nu})$; consequently, $\kappa_0 \in \Phi(k)$ as required.

We prove that Φ has the appropriate kind of continuity property. Consider $k_n \rightarrow k_0$ and note first that since, under A.5, F is continuously differentiable, $R(k_n) \rightarrow R(k_0)$ and $w(k_n) \rightarrow w(k_0)$. Let $\kappa_n \in \Phi(k_n)$ for all $n = 1, 2, \dots$, and let $\nu_n \in \mathcal{N}(k_n)$ be the corresponding sequence of invariant measures. By Theorem 3 (b) there exists a subsequence and $\bar{\nu} \in \mathcal{N}(k_0)$ such that $\nu_{n(j)} \xrightarrow{w} \bar{\nu}$. Also, by Lemma 10 (e), $\mathcal{K}(k_{n(j)}, \nu_{n(j)}) \rightarrow \mathcal{K}(k_0, \bar{\nu})$ and $\mathcal{L}(k_{n(j)}, \nu_{n(j)}) \rightarrow \mathcal{L}(k_0, \bar{\nu})$. As in the earlier proof of closure, $\mathcal{L}(k_0, \bar{\nu}) > 0$ and, by the defining property of Φ , $\mathcal{K}(k_0, \bar{\nu}) / \mathcal{L}(k_0, \bar{\nu}) \in \Phi(k_0)$. Set $\kappa_0 := \mathcal{K}(k_0, \bar{\nu}) / \mathcal{L}(k_0, \bar{\nu})$ and proceed as in the earlier proof of closure of $\Phi(k)$ to show that there exists a subsequence such that $\kappa_{n(j)} \rightarrow \kappa_0$ where $\kappa_0 \in \Phi(k_0)$ as required. \blacksquare

Let us briefly indicate what happens to the result in Lemma 11 if instead of assuming that A.2 (iii b) holds, we were to assume that A.2 (iii a) holds. We would modify the definition of Φ by including the requirement that the denominator be positive; hence, to guarantee that Φ is nonempty valued, we would need to assume that $\beta \cdot w \cdot \underline{\psi}_2^0 > 1$ and that for some $\tilde{\nu} \in \mathcal{N}(R, w)$, $\tilde{\nu}((0, 0) \times \mathcal{A}) < 1$ so that $\mathcal{L}(R, w, \nu) > 0$ (see Lemma 10 c). This allows us to show that Φ is well defined and convex valued. For the continuity properties we can modify the statement to allow for the fact that Φ need not be bounded above; this can be accomplished by restricting attention to bounded sequences in the graph. However, we run into a different sort of problem. It could be the case that we are looking at a sequence ν_i with the property that it converges to a measure with support concentrated on the set $\{(0, 0)\} \times \mathcal{A}$. So, even though all along the sequence κ_i behaves nicely, since the values of both \mathcal{K} and \mathcal{L} go to zero in a nice way, the limit point does not satisfy the requirement that the denominator be positive. Hence, both closure and continuity can fail. To get around the problem we would need to consider a closed and convex subset of $\mathcal{N}(R, w)$ which does not include any measure with support concentrated on the set $\{(0, 0)\} \times \mathcal{A}$. Proving the existence of such a set requires that we first construct a compact invariant set for the state variables which does not include the point $(0, 0)$. In the i.i.d. case we can follow the argument in Brock and Mirman (1972) to construct such a set. However, for factor prices induced by k near k_β ,

since $\beta \cdot R(k) < 1$, the prerequisite for such a construction is that $\gamma_3 > 0$ and that it be expanding close to zero, a requirement which translates into the property that the function $\beta \cdot w(k) \cdot \psi(\cdot, 0)$ have slope exceeding one. This is in conflict with being able to satisfy our boundary condition (b) in Theorem 4 which, we recall, might be easiest to meet when $\gamma_3 = 0$. In the Markov case which interests us more, the problem of constructing such an invariant set seems to be much more difficult. In either case, if one insists on assuming A.2 (iii a) then one has to ensure that, even so, the model behaves as it does under A.2 (iii b); consequently, we prefer to assume A.2 (iii b) instead.

Proof of Theorem 4: We continue our study of the existence problem in parametric form where the parameter is the capital-labour ratio with the notation introduced earlier: $\xi(k, \nu) := \xi(R(k), w(k), \nu)$ with $\nu \in \mathcal{N}(k) := \mathcal{N}(R(k), w(k))$, and $\mathcal{K}(k, \nu)$ and $\mathcal{L}(k, \nu)$ are used for mean capital demand and mean labour supply respectively.

Let us first verify that if, for some domain $[\underline{k}, \bar{k}]$, there exists k^* such that $k^* \in \Phi(k^*)$, then an equilibrium can be identified. To see this note that any such k^* has associated with it an invariant measure $\nu^* \in \mathcal{N}(k^*)$ which induces $\Phi(k^*)$. Clearly, factor markets will clear if we define factor usage by $L^* := \mathcal{L}(k^*, \nu^*)$ and $K^* = L^* \cdot k^*$ since, by the fixed point property, $k^* \in \Phi(k^*)$ so

$$k^* = \mathcal{K}(k^*, \nu^*) / \mathcal{L}(k^*, \nu^*).$$

We have an equilibrium if we can verify that the market for the produced good also clears. To do so, consider Walras Law which can be written in terms of k in the form

$$\xi(k, \nu) = w(k) \cdot \mathcal{L}(k, \nu) + (R(k) - 1) \cdot \mathcal{K}(k, \nu).$$

Also, under A.5,

$$F(K, L) = K \cdot [R(k) - 1] + K \cdot \delta + L \cdot w(k),$$

since $F(K, L) = K \cdot F_1(K, L) + L \cdot F_2(K, L)$. It follows that if K^* and L^* are such that $\mathcal{K}(k^*, \nu^*) = K^*$ and $\mathcal{L}(k^*, \nu^*) = L^*$, where $k^* = K^*/L^*$ and $\nu^* \in \mathcal{N}(k^*)$, so that factor markets clear, then

$$\xi(k^*, \nu^*) = F(K^*, L^*) - \delta \cdot K^*$$

so the market for the produced good also clears (see (i) in Definition 1) as required.

We have reduced our problem to that of proving the existence of such a k^* .

Define the correspondence $\tilde{\Phi} : [0, \bar{k}] \rightarrow [0, \bar{k}]$ where

$$\begin{aligned} \tilde{\Phi}(k) &:= \mathcal{K}(\underline{k}, \underline{\nu}) / \mathcal{L}(\underline{k}, \underline{\nu}) \quad \text{for } k \in [0, \underline{k}], \\ \tilde{\Phi}(k) &:= \Phi(k) \cap [0, \bar{k}] \quad \text{if } \Phi(k) \cap [0, \bar{k}] \neq \emptyset \quad \text{for } k \in [\underline{k}, \bar{k}] \\ \tilde{\Phi}(k) &:= \bar{k} \quad \text{if } \Phi(k) \cap [0, \bar{k}] = \emptyset \quad \text{for } k \in [\underline{k}, \bar{k}]. \end{aligned}$$

By the properties specified in Lemma 11 and by construction, $\tilde{\Phi}$ is nonempty valued, convex valued, compact valued, and upper hemicontinuous, and maps a compact and convex set to itself. By Kakutani's Fixed Point there exists $\tilde{k} \in \tilde{\Phi}(\tilde{k})$ where $\tilde{k} \in [0, \bar{k}]$. It is easy to check that if

$$\Phi(\bar{k}) \cap [0, \bar{k}] \neq \emptyset \quad \text{and} \quad \Phi(\underline{k}) \cap [\underline{k}, \infty) \neq \emptyset,$$

so that two boundary properties are satisfied, then the fixed points of $\tilde{\Phi}$ coincide with those of Φ . Hence, the existence proof can be completed by verifying that the boundary properties hold.

We proceed to show that, under the conditions labeled (a) and (b) in the statement of Theorem 4, the boundary properties mentioned above do hold. Since, under A.2 (iii b), $\mathcal{L}(k, \nu) > 0$, Walras Law implies that

$$\frac{\mathcal{K}(k, \nu)}{\mathcal{L}(k, \nu)} = w(k) + R(k) \cdot \frac{\mathcal{K}(k, \nu) - \xi(k, \nu)}{\mathcal{L}(k, \nu)} \leq w(k) + R(k) \cdot \frac{\bar{y}}{\mathcal{L}(k, \nu)}$$

where we also use the fact that ξ is nonnegative and that, by Lemma 10 (a), \bar{y} is the upper bound on \mathcal{K} . Since we have assumed that

$$w(\bar{k}) + R(\bar{k}) \cdot \frac{\bar{y}}{\mathcal{L}(R(\bar{k}), w(\bar{k}), \bar{\nu})} \leq \bar{k} \quad \text{for some } \bar{\nu} \in \mathcal{N}(R(\bar{k}), w(\bar{k}))$$

we have

$$\frac{\mathcal{K}(\bar{k}, \bar{\nu})}{\mathcal{L}(\bar{k}, \bar{\nu})} \leq \bar{k} \quad \text{for some } \bar{\nu} \in \mathcal{N}(\bar{k})$$

verifying the first of the two boundary conditions. For the other boundary condition, recall that by assumption (b) in Theorem 4,

$$\xi(\underline{k}) \geq \mathcal{L}(\underline{k}, \underline{\nu}) \cdot [F(\underline{k}, 1) - \delta \cdot \underline{k}].$$

By substituting for ξ from Walras Law, and using the fact that $F(\underline{k}, 1) = \underline{k} \cdot [R(\underline{k}) - 1] + \underline{k} \cdot \delta + w(\underline{k})$, we get

$$\mathcal{K}(\underline{k}, \underline{\nu}) \geq \underline{k} \cdot \mathcal{L}(\underline{k}, \underline{\nu})$$

thus verifying the other boundary condition. ■

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