POTENTIAL, VALUE AND PROBABILITY*

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ABSTRACT

This paper focuses on the probabilistic point of view and proposes a extremely simple probabilistic model that provides a single and simple story to account for several extensions of the Shapley value, as weighted Shapley values, semivalues, and weak (weighted or not) semivalues, and the Shapley value itself. Moreover, some of the most interesting conditions or notions that have been introduced in the search of alternatives to Shapley's seminal characterization, as 'balanced contributions' and the 'potential', are reinterpreted from this same point of view. In this new light these notions and some results lose their 'mystery' and acquire a clear and simple meaning. These illuminating reinterpretations strongly vindicate the complementariness of the probabilistic and the axiomatic approaches, and shed serious doubts about the achievements of the axiomatic approach since Nash's and Shapley's seminal papers in connection with the genuine notion of value.

Keywords: Coalitional games, value, potential, probabilistic models.

JEL classification number: C71, D84

1 Introduction

From Shapley's (1953a) seminal paper a copious family of 'solutions' (of what?) for transferable utility (TU) games has grown in different directions with many ramifications. Among others, weighted Shapley values (Shapley (1953b), Kalai and Samet (1987, 1988), Weber (1988)), probabilistic values (Weber, 1979, 1988), semivalues (Weber (1979, 1988), Dubey, Neyman and Weber (1981), Einy (1987)), weak semivalues and weighted weak semivalues (Calvo and Santos, 2000), as well as nontransferable utility (NTU) and non atomic extensions of some of these notions, or their restriction to special subdomains as, e.g., simple games. In fact, a lot of energy has been and is still devoted in cooperative game theory to the study of these ramifications, and to the search of new axiomatic characterizations of these objects, be it the very seminal concept (e.g., Dubey (1975), Myerson (1980), Young (1985), Hart and Mas-Colell (1989), Feltkamp (1995)) or any of its extensions, in different domains. Most of these extensions have been born out of axiomatic explorations: dropping axioms and seeing what happens, finding weaker or more appealing ones, recombining already existing ones, etc. Less attention has been paid to the meaning of the resulting families. As a result, the 'stories' behind some of these mathematical constructions do not seem very much compelling or even transparent.

As is well-known, the Shapley value and some of these extensions admit a more or less clear interpretation in probabilistic terms. From this point of view, Weber's (1979, 1988) concept of probabilistic value, or, better, of group value, is the concept that covers the notions formerly alluded with which we deal in this paper. A group value is a vector of probabilistic values, that is, a vector of expected marginal contributions to the worth of a coalition based on n different points of view. In other words, each player has his/her own assessment of the probability of joining different coalitions. Nevertheless the possibility of accommodating different extensions into this very general notion is achieved at the cost of a quite different probabilistic story for every case: the formation of the grand coalition in a certain order, all orders being equally probable (the Shapley value) or not (weighted Shapley values); or the probabilistic assessments of the different players of the coalition they will join satisfying different 'consistency' requirements (semivalues, weak semivalues and weighted weak semivalues). The same can be said about different notions, as for instance the potential (Hart and Mas-Colell, 1989), for which it is possible to give an *ad hoc* probabilistic interpretation. Perhaps the lack of a tightly fitting and really unifying probabilistic model has contributed to overlooking the possibilities of the probabilistic approach, or maybe it has been the other way round, the lack of attention to this approach explains the lack of a clear unifying model so far.

Be it as it may, in this paper we focus on the probabilistic point of view and propose a surprisingly simple probabilistic model that provides a single and simple story to account for these 'solutions'. Extremely simple variations (particular cases, in fact) of this basic story account for the above mentioned families supported in the literature by different axiomatic combinations. Moreover, some of the most interesting conditions and notions that have been introduced in the search of alternatives to Shapley's seminal characterization, as 'balanced contributions' (Myerson, 1980) and the 'potential' (Hart and Mas-Colell, 1989), are reinterpreted from this same point of view. In this new light these notions lose their 'mystery' and acquire a clear and simple meaning. One of the most remarkable results of this reinterpretation, apart from its simplicity, is the fact that in this new light the 'weights' present in *weighted* Shapley values, *weighted* weak semivalues, *weighted* balanced contributions or *weighted* potential, evaporate as an 'optical effect'. So, weighted weak semivalues appear in a precise sense as the Shapley value's 'natural family', while the other are narrower circles around the seminal concept that correspond to particular cases of the general model. Similarly, against the common appreciation, in this model it turns out that *weighted* potential and *weighted* balanced contributions are natural and general conditions with a clear meaning, while the corresponding non weighted notions appear as *ad hoc* variations of these notions devoid in general of a clear meaning, whose main virtue seems to be the fact that, along with efficiency, allow to single out the Shapley value.

On the other hand, the extremely simple probabilistic story that accounts for all these notions seems to have very little to do with Shapley's original interpretation of his 'value' for TU games, in the sense of von Neumann and Morgenstern's (1944) notion of value of a two-person zero sum game, and Nash's (1950) solution to the bargaining problem. Thus, the lack of a clear matching of the different combinations of axioms with the genuine notion of value, along with their perfect matching with the extreme simple model sheds serious doubts about the achievements of the axiomatic approach in connection with the notion of value as is discussed in the concluding remarks.

The paper is organized as follows. In section 2 the seminal concept of the Shapley value is briefly reviewed. In section 3 the unifying probabilistic model is formulated. In section 4 it is shown how the Shapley value and some of its extensions, as (weighted or not) weak semivalues, semivalues, and weighted Shapley values fit into the proposed probabilistic model. In section 5 the main properties that have been introduced in the literature in search of alternatives to linearity in Shapley's original characterization, as 'transfer' (Dubey, 1975), 'balanced contributions' (Myerson, 1980), 'strong monotonicity' (Young, 1985), and the 'potential' (Hart and Mas-Colell, 1989) are reinterpreted and discussed in terms of the probabilistic model introduced in section 3. Finally, section 6 summarizes the main conclusions along with some methodological remarks, and some lines of further research.

2 The Shapley value

An N-*coalitional game* is a pair (N, v), where $N = \{1, ..., n\}$ is a set of *players* and v is a map that assigns to each subset or *coalition* $S \subseteq N$ a real number or *worth* $v(S)$, such that $v(\emptyset) = 0$. The model was introduced by von Neumann and Morgenstern (1944). In the usual interpretation the worth of a coalition represents what this coalition can guarantee for itself if it forms. This 'worth' can be the amount of something objectively measurable as money, or some other physical entity (water, oil, land, mineral, etc.). A second possibility is interpreting it as 'utility', assuming that there exists some infinitely divisible good in which the players' utility is linear. In the first interpretation nothing about the players or their preferences is put into the model beyond the implicit assumption that for all of them 'the more the better'. In the second, these preferences are included at the cost of a very simplifying assumption according to which all players are virtually identical. Because of this second interpretation coalitional games are often called transferable utility (TU) games. G_N will denote the set of all coalitional games with N as the set of players. When this set is clear from the context we will refer by v to game (N, v). We will drop i's brackets in $S \backslash \{i\}$ and $S \cup \{i\}$.

On this basis Shapley (1953a) proposed the seminal concept that, as commented in the introduction, is the starting point for a number of extensions. For a game (N, v) the *Shapley value* of player i is given by

$$Sh_i(v) = \sum_{S:S \subseteq N} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \backslash i)). \tag{1}$$

There are several interpretations of the map $Sh_i : G_N \rightarrow R^N$, characterized by the well-known conditions of efficiency, anonymity, null player and additivity[1]. Shapley's original interpretation is that of a 'value' for TU games in the sense of von Neumann and Morgenstern's (1944) notion of value of a two-person zero sum games, and that of Nash's (1950) solution to the bargaining problem. That is to say, the value that would mean for a rational player the prospect of engaging into any situation of the class considered with other rational players, TU games in this case. An alternative interpretation is as an allocation rule, or a way of distributing the worth of the grand coalition among the n players in any situation describable as a coalitional game. It is the only allocating rule

[1] In fact Shapley's original characterization is slightly different as the original framework involves a 'universe' of players. In the literature there are a variety of alternative characterizations to some of which we will refer later.

that satisfies the above mentioned conditions, that can be interpreted (at least three of them) in normative terms. A third interpretation is as an expected payoff of each player for a specific random procedure to form the grand coalition sequentially and assigning to each player his/her marginal contribution to the worth of the coalition formed when incorporating[2]. A forth interpretation is as a utility function $\varphi(i, v) = Sh_i(v)$ representing von Neumann-Morgenstern preferences over 'roles' or positions (i, v) in coalitional games and lotteries over them[3].

The second interpretation is the one that seems to prevail nowadays and the one that has attracted more attention. And it is the axiomatic approach that has absorbed most of the attention in search of alternative characterizations of the Shapley value and its extension in different directions and domains. In this paper we concentrate in the domain of n-person coalitional games and in the probabilistic interpretation of some of the extensions of the seminal concept in this domain, as well as on the interpretation of some of the conditions which have been introduced in the literature as alternatives to additivity with characterizing purposes.

3 A unifying probabilistic model

In the standard interpretation of a coalitional game (N, v), this pair is all what is put into the model. Here, consistently with the interpretation of a coalitional game according to which no information about the players is included, we see a coalitional game as only one of the ingredients in a more general model. Here $v(S)$ is seen as what coalition S would get *if* it forms. So far so good, this in accordance with the usual orthodox story. The difference is this: *we assume that only a coalition will form.* This still seems in accordance with orthodoxy, in which (at least for the Shapley value and some of its 'efficient' variations) the grand coalition will finally form, though players *do not* receive their marginal contribution to its worth. But here we include a second independent ingredient: a probability distribution over all coalitions. In other terms, the coalition that will form is a *random coalition.* This in fact amounts to admit the insufficiency of the information contained in the sole coalitional game. In a first step it is as if the probability distribution that specifies the random coalition were *exogenous.* At first sight this may give rise to a certain reluctance, as usually closed models in which everything is endogenous are preferred however

[2]This is also in Shapley (1953), where is presented as a bargaining model in which a single random move settles the issue.

[3]This interpretation is the motivating idea in Shapley (1953), although his treatment does not involve preferences explicitly. For a more explicit study from this point of view see Roth (1977a), and also Roth (1977b) and Laruelle and Valenciano (2000) in the context of simple games.

unconvincing stories might support them. A second motive of uneasiness may arise from the assumption that only one coalition will form: why if S forms, the rest of the players in $N \backslash S$ cannot form other coalitions in their turn? We will come back to these points.

Thus, a second input enters the model: a distribution of probability p that associates with each coalition S its probability of forming $p(S)$. In other words and to avoid misunderstandings, the elementary events are the coalitions in 2^N. As the number of coalitions is finite, any such a probability distribution can be represented by a map $p : 2^N \to R$, where $0 \leq p(S) \leq 1$ for any $S \subseteq N$, and $\sum_{S \subseteq N} p(S) = 1$. We assume that the probability of any player belonging to the resulting coalition is strictly positive, that is, for all $i \in N$

$$\textit{Prob}\,\{\text{the coalition that forms contains } i\} = \textit{Prob}\,\{S \ni i\} = \sum_{T : i \in T} p(T) > 0.$$

This rules out the case of players who will never belong to the resulting coalition. Such cases can be dealt with by eliminating this player and reformulating appropriately the coalitional game and the probability distribution for the remaining $n-1$ players. Let $\mathfrak{P}_N$ denote the set of all such probability distributions.

A few remarks are worth here. First note that these probability distributions include the possibility of no coalition forming if $p(\emptyset) > 0$, but the trivial case $p(\emptyset) = 1$ is excluded[4]. Second, the events 'i belongs to the resulting coalition' and 'j belongs to the resulting coalition' are *not* necessarily independent in this model, in which the independence is only a particular case.

It is important to stress also that this distribution of probability is *not* meant to be interpreted as a subjective probability of any particular player, as it is the case in the 'probabilistic values' (Weber, 1979) and usual interpretations of other cooperative game theoretic concepts that will be reviewed later. These probabilities are to be interpreted either as an objective probability or as some external probabilistic assessment, possibly based on the frequencies of coalitions in previous cases, or whatever available data. This leaves open the door both to descriptive applications and, in the theoretical level, to the enrichment of the model involving the players' preferences, or their relative proximity or affinities dependent on additional information, which may influence the probabilities of different coalitions.

Thus we have two inputs: a coalitional game $(N, v) \in G_N$ and a probability distribution over coalitions $p \in \mathfrak{P}_N$. Given these two data, *the expected marginal contribution of any*

[4] Mind here '$\emptyset$' does not denote the empty event, but the event of no coalition forming, or better the empty coalition being the one formed. In the example of voting later commented this corresponds to the case in which everybody votes 'no'.

player i to the coalition that will form is given by

$$E_p[v(S) - v(S\backslash i)] = \sum_{S:i\in S} p(S)(v(S) - v(S\backslash i)). \tag{2}$$

One can also calculate the expected marginal contribution of any player i to the coalition that will form, *conditional to such player belonging to it,* that will be denoted by $\Phi_i(v,p)$ and is given for player i by

$$\Phi_i(v,p) := E_p[v(S) - v(S\backslash i) \mid S \ni i] = \frac{E_p[v(S) - v(S\backslash i)]}{Prob\,\{S \ni i\}}$$

$$= \frac{1}{\sum\limits_{T:i\in T} p(T)} \sum_{S:i\in S} p(S)(v(S) - v(S\backslash i)) = \sum_{S:i\in S} \frac{p(S)}{\sum\limits_{T:i\in T} p(T)}(v(S) - v(S\backslash i)). \tag{3}$$

Or, denoting p^i the probability distribution over coalitions resulting from p conditional to '$i \in S$' (i.e., the coalition that will form contains i), that is,

$$p^i(S) := Prob\,\{S \mid S \ni i\} = \begin{cases} \frac{p(S)}{\sum\limits_{T:i\in T} p(T)} & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases}$$

(3) can be rewritten as

$$\Phi_i(v,p) = E_{p^i}[v(S) - v(S\backslash i)] = \sum_{S:i\in S} p^i(S)(v(S) - v(S\backslash i)). \tag{4}$$

Which is the interest of the data provided by formulae (2) and (3) or (4)? If a coalition is forming it is ingenuous for players to expect receiving their marginal contribution. If players are not stupid they should know this will probably be too much or too little to ask for. Nevertheless, the marginal contribution of each player to the coalition which forms seems a reasonable *assessment* of the importance or relevance of this player, and a possible basis for claims, relative to other members of such coalition. This is exactly what these formulae give (conditionally or unconditionally) for a random coalition based on the two inputs.

Before proceeding with the reexamination of some cooperative game theoretic notions from the point of view provided by the present model, an example will be of some help. A context in which this basic model makes a clear sense is provided by simple games when they are interpreted as models of voting procedures[5]. A voting procedure for n voters may be represented by a simple game, a particular class of coalitional games, by assigning worth 1 to coalitions that can pass a decision and worth 0 to those that cannot. In strict terms this is only a way of representing the voting rule itself. In other words,

[5]In fact it was revising the foundations of the theory of voting power as the basic idea of the model presented in this paper emerged.

it includes no information about the 'players' or voters who might use the rule to make decisions. Different voters with different preferences and behavior may use the same voting rule. In Laruelle and Valenciano (2002) a revision of the foundations of the theory of voting power is undertaken basing it on two separate inputs: the voting rule, that can be specified by a simple game (N, v), and a 'voting behavior', described by a probability distribution p over the coalitions, or what seems a more appropriate term in that context, the 'voting configurations' or possible results of a vote. That is, $p(S)$ represents in this case the probability of the configuration of votes in which voters in S vote 'yes' and voters in $N \backslash S$ vote 'no'. The necessary separation between game-procedure on one hand and players-voters on the other seems obvious in this case, and formulae (2) and (4) represent, respectively, the probability of voter i being decisive in passing a decision, and the conditional probability of voter i being decisive in passing a decision, given that i supports it.

Back to the general case, note that different distributions of probability can lead to the same conditional expected marginal contribution. The reason is that whatever the probability of the empty coalition, it does not affect the expected marginal contribution of player i *conditional to the coalition that forms contains* i, as far as the probability of the nonempty coalitions remains proportional. Therefore one can modify the probability of the empty coalition and re-scale proportionally the probability of the others without modifying the expected conditional marginal contributions. But this is the only degree of freedom.

Proposition 1 *Let* p,$p' \in \mathfrak{P}_N$, *then* $\Phi_i(v, p) = \Phi_i(v, p')$ *for any player* i *and any coalitional game* (N, v), *if and only if*

$$\frac{p(S)}{1 - p(\emptyset)} = \frac{p'(S)}{1 - p'(\emptyset)} \quad \textit{for all } S \neq \emptyset.$$

Observe that the trivial case $p(\emptyset) = 1$ is excluded in the model. On the other hand, the degree of indeterminacy for probability distributions that yield a same assessment would disappear if it is required $p(\emptyset) = 0$. But this would rule out, for instance, the case in which every player independently joins the coalition that will form with a certain probability. Thus we will not exclude the case $p(\emptyset) > 0$.

4 Some cooperative 'solutions' revisited

In this section we study the relationships between the probabilistic model presented in the previous section and some game theoretic concepts to be found in the literature on coalitional games, as weighted weak semivalues, weak semivalues, semivalues, and weighted

Shapley values, and the Shapley value itself. As will be shown, this simple probabilistic model gives a clear conceptual common basis to reinterpret coherently from a unified point of view these game theoretic concepts as particular cases of (3) or its equivalent (4), for particular probability distributions. We will proceed from the widest to the narrower classes of 'solutions', concluding with the Shapley value itself. But previously let us briefly compare our model with the notion of probabilistic value[6].

4.1 Probabilistic values

Probabilistic values were introduced by Weber (1979, 1988). For any $i \in N$, a *probabilistic value* is a map that associates with each game (N, v) a real number given by

$$\Psi_i(v) = \sum_{S:i\in S} p_S^i(v(S) - v(S\backslash i)), \tag{5}$$

for some p_S^i such that $0 \leq p_S^i \leq 1$ and $\sum\limits_{S:i\in S} p_S^i = 1$. Where the p_S^i are interpreted as player i's subjective distribution of probability over the coalitions containing this player, based on his/her own subjective assessment of the probability of *joining* different coalitions[7]. Thus a probabilistic value involves and concerns only one player. Weber characterized these *real valued* maps by the linearity, dummy and positivity axioms.

¿From formula (4) it is immediate to check that for any distribution of probability $p \in \mathfrak{P}_N$, and any player i, $\Phi_i(\text{-}, p)$ is, formally speaking, a 'probabilistic value'. Thus, formally speaking, for any distribution p, $\Phi(\text{-}, p)$, is a *vector* of probabilistic values. Nevertheless the interpretation underlying either concept is different. A probabilistic value, as first introduced by Weber, was interpreted as a *subjective* evaluation of a player's marginal contribution from the point of view *of that particular player.* The only vectors of probabilistic values, or 'group values' considered by Weber are the semivalues and the random-order values, whose relations with our probabilistic model will be also established in this section. But our vectors of probabilistic values are *not* to be interpreted as vectors of subjective evaluations based on n independent points of view. On the contrary, they

[6]We will not discuss here Straffin's (1977, 1982 and 1988) probabilistic model, extended by Dubey, Neyman and Weber (1981) to all semivalues. In Laruelle and Valenciano (2002) it is shown how this sophisticated model fits within the one considered here, but is not more general.

[7]In fact, Weber's notation was slightly different. He writes

$$\Psi_i(v) = \sum_{S:i\notin S} p_S^i(v(S \cup i) - v(S)),$$

with $0 \leq p_S^i \leq 1$ and $\sum\limits_{S:i\notin S} p_S^i = 1$. It is immediate to see that both formulations are equivalent: the coefficient p_S^i in this formula is just the $p_{S\cup i}^i$ in (5).

should be interpreted as *assessments of every player's marginal contribution based on a single (and same for all players) distribution of probability over coalitions.*

But of course not all vectors of probabilistic values can be interpreted as particularizations of (3)-(4). This is not surprising: one cannot expect consistency from n independent subjective points of view.

4.2 Weighted weak semivalues

In Calvo and Santos (2000) two classes of 'solutions' emerge axiomatically by adding to the properties that characterize probabilistic values (linearity, positivity and dummy axiom (Weber, 1979, 1988)), either 'balanced contributions' or the weaker 'w-balanced contributions' (both conditions are discussed in section 5). They call 'weighted weak semivalues' to the class of solutions that emerges by adding the second condition.

A *weighted weak semivalue* is a map $\Psi : G_N \to R^N$ which assigns to each game (N, v) a vector in R^N, given by

$$\Psi_i^{p^N,w^N}(v) = \sum_{S:i\in S} w_i \; p_S \; (v(S) - v(S\backslash i)), \tag{6}$$

where $w^N = (w_i)_{i\in N}$ is an N-*weight* vector s.t., $w_i > 0$, for all $i \in N$, and $p^N = (p_S)_{S\subseteq N,S\neq\emptyset}$ is a family of coefficients such that $p_S \geq 0$ for all $S \neq \emptyset$, and $\sum\limits_{S:i\in S} w_i p_S = 1$ for every $i \in N$. Calvo and Santos (2000) prove that map (6) is characterized by linearity, positivity, dummy axiom and w-balanced contributions, but no clear interpretation of this family, born axiomatically, nor of the coefficients p_S and the 'weights', is given. They interpret weighted weak semivalues just as a subclass of (vectors of) probabilistic values (formula (5)) in which the players' beliefs (in Weber's terms) satisfy a weak form of 'consistency'. Namely, for some system of weights w, they satisfy $\frac{p_S^i}{p_S^j} = \frac{w_i}{w_j}$, for all $S \subseteq N$ and all $i, j \in S$.

The following result provides a clear interpretation of this family, showing that the family of maps generated by (3)-(4) for different probability distributions *is* the family of weighted weak semivalues. It provides as well of a meaning for these coefficients and these weights.

Theorem 1 *The family of maps $\Phi(-, p) : G_N \to R^N$, generated by formula (4) for different probability distributions p in $\mathfrak{P}_N$, is the family of weighted weak semivalues.*

Proof. Confronting formula (6) with formulae (3)-(4), it is clear that for any probability distribution $p \in \mathfrak{P}_N$, defining $p^N = (p_S)_{S\subseteq N,S\neq\emptyset}$, and $w^N = (w_i)_{i\in N}$ by

$$p_S := p(S), \quad \text{and} \quad w_i := \frac{1}{Prob\,\{S \ni i\}} = \frac{1}{\sum\limits_{T:i\in T} p(T)}, \tag{7}$$

it holds

$$\Phi_i(v,p) = \Psi_i^{p^N,w^N}(v)$$

for any player i and any game (N,v).

Reciprocally, any weighted weak semivalue can be generated in this way. Let Ψ^{p^N,w^N} be a weighted weak semivalue given by (6), with associated coefficients p_S and weights w_i. Mind there is a degree of indeterminacy both in the weights and the coefficients: the weights can be all multiplied by any positive constant and the coefficients divided by this same constant so that the constraint $\sum_{S:i\in S} w_i p_S = 1$ is satisfied and the associated weighted weak semivalue remains the same. Thus, defining $p'^N = (p'_S)_{S\subseteq N, S\neq\emptyset}$, and $w'^N = (w'_i)_{i\in N}$, by

$$p'_S := \frac{p_S}{\sum_{T:T\subseteq N, T\neq\emptyset} p_T} \quad \text{and} \quad w'_i := w_i \sum_{T:T\subseteq N, T\neq\emptyset} p_T,$$

we have $\Psi^{p^N,w^N} = \Psi^{p'^N,w'^N}$. Moreover, as $\sum_{S:S\subseteq N, S\neq\emptyset} p'_S = 1$ and $\sum_{S:i\in S} w'_i p'_S = \sum_{S:i\in S} w_i p_S = 1$, taking $p \in \mathfrak{P}_N$, given by $p(\emptyset) := 0$, and $p(S) := p'_S$ for $S \neq \emptyset$, we obtain

$$\Phi(v,p) = \Psi^{p'^N,w'^N}(v) = \Psi^{p^N,w^N}(v),$$

for any game (N,v). ■

Thus, weighted weak semivalues *are* the vectors of players' expected marginal contributions to the coalition which will form for arbitrary probability distributions in $\mathfrak{P}_N$, conditional to the player belonging to it. What is specially remarkable in this equivalence is that in our model both the coefficients p_S and the 'weights' w_i acquire a precise meaning (7). Once eliminated the indeterminacy factor as in the second part of the proof, for each $S \neq \emptyset$, p_S is the probability of S being the coalition formed, while the 'weight' w_i of player i *is not a weight any more*, but just the inverse of the probability of such player entering the coalition that will form. Thus 'weights' disappear in this model as just an 'optical effect'. On the other hand, under the point of view provided by our general model the nature of the 'consistency' alluded by Calvo and Santos becomes clear: the players' assessments are based on a single and same probability distribution over coalitions. Thus from this point of view it turns out artificial and superfluous to speak any more of a vector of assessments resulting from n points of view.

4.3 Weak semivalues

In the same paper Calvo and Santos (2000) introduce a particular class of weighted weak semivalues that is a generalization of the concept of semivalue (see next subsection). A *weak semivalue* is a weighted weak semivalue in which all players have the same weight.

Thus a weak semivalue is a map Ψ which assigns to each game (N, v) a vector in R^n, given by

$$\Psi_i(v) = \sum_{S:i\in S} p_S(v(S) - v(S\backslash i)), \tag{8}$$

for some p_S such that $0 \le p_S \le 1$ and $\sum_{S:i\in S} p_S = 1$ for every $i \in N$. Calvo and Santos (2000) prove that this family is characterized by linearity, positivity, dummy axiom and balanced contributions. They form obviously a subfamily of the weighted weak semivalues, the one corresponding to those whose associated weights are the same for all players.

Confronting formulae (3) and (8), it is clear that when the probability of being a member of the coalition that will form is the same for all players, that is, when for any two players $i, j \in N$, it holds

$$\sum_{T:i\in T} p(T) = \sum_{T:j\in T} p(T),$$

what emerges from the general model presented in section 3 are weak semivalues. Moreover, in view of the correspondence established in Theorem 1, we have as a corollary the following equivalence

Proposition 2 *The family of maps $\Phi(-, p) : G_N \to R^N$, generated by formula (4) for different probability distributions p in $\mathfrak{P}_N$ such that for any two players the probability of entering the coalition which will form is the same, is the family of weak semivalues.*

Thus, weak semivalues are the vectors of players' conditional expected marginal contributions to the coalition which will form for probability distributions over coalitions for which the probability of any player entering the coalition which will form is the same.

4.4 Semivalues

Semivalues were introduced by Weber (1979) (see also Weber (1988), Dubey, Neyman and Weber (1981), and Einy (1987)). A *semivalue* is a vector of probabilistic values (formula (5)) such that for all $i, j \in N$ and all $S \subseteq N$ containing i and j it holds that $p^i_S = p^j_S$ if $s = t$. In other words, coefficients in (5) only depend on the size of S. Thus a semivalue is a map Ψ which assigns to each game (N, v) a vector in R^n, given by

$$\Psi_i(v) = \sum_{S:i\in S} p_s(v(S) - v(S\backslash i)) \tag{9}$$

for some p_s such that $0 \leq p_s \leq 1$ for $s = 1, ..., n$, and $\sum_{S:i \in S} p_s = 1$[8]. They are characterized by anonymity, null player, and linearity (Weber, 1979, 1988). Again confronting (3) and (9), and in view of the previous equivalences, we have the following result whose proof we omit.

Proposition 3 *The family of maps $\Phi(-,p) : G_N \rightarrow R^N$, generated by formula (4) for probability distributions p in $\mathfrak{P}_N$ for which the probability of a coalition to form depends only on its size, is the family of semivalues. Moreover, if Ψ is the semivalue given by formula (9), then*

$$\Psi(v) = \Phi(v, p)$$

for any $p \in \mathfrak{P}_N$ such that

$$p(S) = (1 - p(\emptyset)) \frac{p_s}{\sum_{t=1}^{n} \binom{n}{t} p_t}, \qquad \textit{for any } S \neq \emptyset. \tag{10}$$

Thus, semivalues are the vectors of players' conditional expected marginal contributions to the coalition which will form when the probability of a coalition forming only depends on its size[9].

4.5 Weighted Shapley values

Weighted Shapley values result by eliminating symmetry in Shapley's (1953a) characterizing system. This extension was introduced by Shapley himself in (1953b) associating a positive weight w_i with each player and distributing payoffs in unanimity games proportionally to these weights. In this way, keeping the other conditions (linearity, efficiency and null-player), a unique payoff vector for every game is determined[10].

[8] In fact, consistently with the variant chosen by Weber to define the probabilistic values (footnote 7), he defines semivalues in an obviously equivalent way as follows:

$$\Psi_i(v) = \sum_{S:i \notin S} p_s(v(S \cup i) - v(S))$$

for some p_s such that $0 \leq p_s \leq 1$ for $s = 0, ..., n-1$ and $\sum_{S:i \notin S} p_s = 1$. Note the coefficient p_s in the second formula is just the p_{s+1} in the first one.

[9] In contrast with the usual interpretation, in Laruelle and Valenciano (2001) semivalues, in the context of simple games and within the axiomatic approach, are interpreted as assessments of the relative capacity to influence the outcome of a vote attached to different roles in different decision rules from *a single point of view.*

[10] Kalai and Samet (1987) extended the notion to 'weight systems' enabling a weight zero for some players, and characterized the resulting family of 'random order' values, which in Weber (1988) are characterized as the unique vectors of probabilistic values that satisfy efficiency. As we will see only the weighted Shapley values within this family fit in (3)-(4).

Hart and Mas-Colell (1987) characterized weighted Shapley values by means of efficiency and the existence of w-potential (see section 5). Then, in view of Calvo and Santos's (2000) characterization of weighted weak semivalues (linearity, positivity, dummy player and w-balanced contributions), and the equivalence of w-potential and w-balanced contributions (Calvo and Santos, 1997), it follows that weighted Shapley values are weighted weak semivalues, and consequently also fit in this probabilistic model. Moreover, weighted Shapley values are the *efficient* weighted weak semivalues. The point is: which probability distributions generate these values? The following theorem, similar to Theorem 11 in Weber (1988) and that we prove for completeness, answers the question.

Theorem 2 *A map $\Phi(-,p) : G_N \to R^N$, generated by formula (4) satisfies 'efficiency', (i.e., $\sum_{i\in N} \Phi_i(v,p) = v(N)$ for all v), if and only if the probability distribution $p \in \mathfrak{P}_N$, verifies the following conditions:*

$$\textit{(i)} \quad \sum_{i\in N} p^i(N) = 1 \quad \textit{and} \quad \textit{(ii)} \quad \sum_{i\in S} p^i(S) = \sum_{j\in N\setminus S} p^j(S\cup j) \quad (\forall S \subsetneq N).$$

Proof. ($\Leftarrow$) Assume $p \in \mathfrak{P}_N$ verifies conditions *(i)* and *(ii)*. Then for any game (N,v),

$$\sum_{i\in N} \Phi_i(v,p) = \sum_{i\in N} \sum_{S:i\in S} p^i(S)(v(S) - v(S\backslash i))$$

$$= \sum_{S\subseteq N} v(S) \left(\sum_{i\in S} p^i(S) - \sum_{j\in N\setminus S} p^j(S\cup j) \right)$$

$$= v(N) \sum_{i\in N} p^i(N) + \sum_{S\subsetneq N} v(S) \left(\sum_{i\in S} p^i(S) - \sum_{j\in N\setminus S} p^j(S\cup j) \right) = v(N)$$

Where the last equality follows form *(i)* and *(ii)*.

($\Rightarrow$) Now assume $\sum_{i\in N} \Phi_i(v,p) = v(N)$, for any game (N,v). Then let (N,u^S) denote for any $S \subseteq N$, the unanimity game such that $u^S(T) = 1$ if $T \supseteq S$ and $u^S(T) = 0$ otherwise. Then $\sum_{i\in N} \Phi_i(u^N,p) = \sum_{i\in N} p^i(N)$, and *(i)* follows. And for any $S \subsetneq N$, let $\hat{u}^S$ the game such that $\hat{u}^S(T) = 1$ if $T \supsetneq S$ and $\hat{u}^S(T) = 0$ otherwise. Then we have

$$\Phi_i(u^S,p) - \Phi_i(\hat{u}^S,p) = \begin{cases} p^i(S) & \text{if } i \in S, \\ -p^i(S\cup i) & \text{if } i \in N \setminus S. \end{cases}$$

Thus, from (4) and the efficiency of $\Phi(-,p)$,

$$\sum_{i\in N} \Phi_i(u^S,p) - \sum_{i\in N} \Phi_i(\hat{u}^S,p) = \sum_{i\in S} p^i(S)) - \sum_{j\in N\setminus S} p^j(S\cup j) = 0.$$

Hence *(ii)* follows too. ■

The following result, the proof of which we leave to the reader, shows the connection between the 'weights' and the associated probability distribution for weighted Shapley values.

Theorem 3 *A map $\Phi(-,p) : G_N \to R^N$, generated by formula (4) satisfies 'efficiency' and coincides with the w-weighted Shapley value for a vector of positive weights $w \in R^n_+$, if and only if the probability distribution $p \in \mathfrak{P}_N$, verifies the following conditions:*

$$\frac{\sum\limits_{S:T\subseteq S} p(S)}{\sum\limits_{R:i\in R} p(R)} = \frac{w_i}{\sum\limits_{j\in T} w_j} \quad (\forall T \subseteq N,\ \forall i \in T).$$

As one can take $w_i = \frac{1}{Prob\{S\ni i\}}$, it is possible to eliminate w_i in the equations of the above system. Thus, Theorem 3 yields an alternative characterization of weighted Shapley values.

Corollary 1 *A map $\Phi(-,p) : G_N \to R^N$, generated by formula (4) satisfies 'efficiency' if and only if the probability distribution $p \in \mathfrak{P}_N$, verifies the following conditions:*

$$\frac{1}{Prob\,\{S \supseteq T\}} = \sum_{i\in T} \frac{1}{Prob\,\{S \ni i\}} \quad (\forall T \subseteq N).$$

The system specified by conditions (i) and (ii) in Theorem 2, as the one connecting weights and probabilities in the Theorem 3, or the characterizing condition in the corollary, show how unnatural is requiring or expecting 'efficiency' in the setting provided by this probabilistic model. Only for very special probability distributions over coalitions the players' conditional expected marginal contributions to the coalition that will form add up to the worth of the grand coalition. In Weber (1988), referring to the random order values, it is stated that "a collection of individual probabilistic values is efficient for all games in its domain *precisely* when the players' probabilistic views of the world are consistent; that is, *only* when the various.. [individual probabilistic views] arise from a single distribution [over the n! possible orderings of the players]" (italics and brackets are ours). But it should be noted that a similar type of consistency is to be found as we have shown in *all* the previous notions *without efficiency*. Moreover, all of them arise from one and the same probabilistic model.

4.6 The Shapley value

As is well-known, the Shapley value is the only element of the intersection of all the precedent families. In view of formula (10), which gives the probability distributions over coalitions p that yield every semivalue as a particular case of $\Phi(-,p)$, we have the following result.

Proposition 4 $\Phi(v,p) = Sh(v)$ *for any coalitional game* (N,v) *if and only if* $p \in \mathfrak{P}_N$ *is such that*

$$p(S) = (1 - p(\emptyset)) \frac{\frac{1}{s}}{\sum_{t=1}^{n} \frac{1}{t}} \binom{n}{s}^{-1}, \qquad \textit{for any } S \neq \emptyset.$$

Thus, the Shapley value of player i gives the expected marginal contribution of this player to the coalition that will form conditional to this player belonging to it if the coalition is chosen as follows. With probability $p(\emptyset) < 1$ the empty coalition is chosen; with probability $1 - p(\emptyset)$ proceed as follows: a coalition's size (from 1 to n) is chosen with probability inversely proportional to the size, then one coalition of size s is chosen at random, all of them being equally probable. Note this probabilistic model is completely different from the usual one reviewed in section 2 in which players enter sequentially according to an order chosen at random up to the grand coalition is formed, but they *do not* receive their marginal contribution to its worth.

5 Some properties reexamined

Since Shapley's seminal paper some attention has been paid to the search of new 'axiomatic' characterizations of the Shapley value. We examine and reinterpret now some of the most popular conditions introduced by different authors to provide alternative characterizations, from the point of view provided by this probabilistic model. We review the properties introduced by Dubey (1975) (see also Feltkamp (1995)), Young (1985), Myerson (1980), and Hart and Mas-Colell (1987)) to replace additivity.

5.1 Transfer and strong monotonicity

Dubey (1975), in order to characterize the Shapley-Shubik index, that is to say, the Shapley value in the domain of simple games, replaced the linearity, that does not make sense in this domain, by what here, following Weber (1988), we will call the 'transfer' property. Later Feltkamp (1995) showed that this condition, weaker that linearity, can replace it, and characterized the Shapley value in G_N along with efficiency, anonymity and null player.

Transfer (T): For any $v, w \in D \subseteq G_N$, s.t. $v \wedge w$, $v \vee w \in D$:

$$\Psi(v) + \Psi(w) = \Psi(v \wedge w) + \Psi(v \vee w),$$

where $(v \wedge w)(S) := \min\{v(S), w(S)\}$ and $(v \vee w)(S) := \max\{v(S), w(S)\}$.

It is immediate to check that as is well-known all of the families of variants of $\Phi(-,p)$ considered in the previous section satisfy additivity and consequently also this weak form of additivity. Nevertheless our model does not provide any alternative interpretation to this condition.

Young (1985) characterized the value Shapley as the only value in the domain of monotonic games which verifies efficiency, anonymity and strong monotonicity.

Strong Monotonicity (SMon): A value $\Psi : D \subseteq G_N \to R^N$ satisfies *strong monotonicity* (SMon) if for any two games v and w in D, and any player $i \in N$,

$$MC_i(v) \leq MC_i(w) \quad \Rightarrow \quad \Psi_i(v) \leq \Psi_i(w).$$

Where $MC_i(v) := (v(S) - v(S \backslash i))_{S \ni i}$.

The meaning of this condition is clear and it is obviously satisfied by any evaluation $\Phi_i(-, p)$ according to formula (4), to which is in fact inherent, whatever the probability distribution $p \in \mathfrak{P}_N$.

In sum, both conditions, transfer and strong monotonicity, are rather general properties that are satisfied by all evaluations of the form $\Phi(-, p)$ for any probability distribution $p \in \mathfrak{P}_N$. As to their meaning in our model's terms, it is clear only that of strong monotonicity, which is inherent to this model.

5.2 Balanced contributions

In Myerson (1980) the Shapley value is characterized by means of efficiency and 'balanced contributions'. The formulation of this condition requires the restriction of a coalitional game to a subset of players. Let $(N, v) \in G_N$, for any $i \in N$, we will denote by $(N \backslash i, v^{N \backslash i})$ the game in $G_{N \backslash i}$, defined by $v^{N \backslash i}(S) := v \mid_{N \backslash i} (S) = v(S)$ for any $S \subseteq N \backslash i$. To simplify the notation we will write $v^{N \backslash i}$ instead of $(N \backslash i, v^{N \backslash i})$ if no confusion arises. For $N \backslash i = \emptyset$, $(\emptyset, v^{N \backslash i})$ is the trivial 0-game.

Balanced Contributions (BC): For any $(N, v) \in D \subseteq G_N$, and all $i, j \in N$,

$$\Psi_i(v) - \Psi_i(v^{N \backslash j}) = \Psi_j(v) - \Psi_j(v^{N \backslash i}). \tag{11}$$

The usual interpretation is that for any two players the benefit of each of them from the participation of the other is the same. Calvo and Santos (2000) show that the weak semivalues are the vectors of probabilistic values that satisfy balanced contributions. They also show that the *weighted* weak semivalues is the family of group values that satisfy the following weaker version of this property, with a similar meaning, but in which every player's benefits are weighted according to a 'weight system' $w = (w_i)_{i \in N}$.

w-Balanced Contributions (w-BC): For any game (N, v) and all $i, j \in N$,

$$\frac{1}{w_i}(\Psi_i(v) - \Psi_i(v^{N\backslash j})) = \frac{1}{w_j}(\Psi_j(v) - \Psi_j(v^{N\backslash i})). \tag{12}$$

It is not possible to discuss the meaning of any of these conditions in the framework of this probabilistic model unless we specify a way of restricting to $N\backslash i$ players the second ingredient in it, that is, the probability distribution over coalitions. A natural way of doing it, consistent with the meaning of this probability, is the following. If only the information concerning $N\backslash i$ matters, coalitions S and $S \cup i$ are indistinguishable form this (i.e., $N\backslash i$'s) point of view. Thus given $p \in \mathfrak{P}_N$, we define the restriction of p to $N\backslash i$ as the distribution $p^{N\backslash i} \in \mathfrak{P}_{N\backslash i}$ given by[11]

$$p^{N\backslash i}(S) := p(S) + p(S \cup i) \quad \text{for all } S \subseteq N\backslash i. \tag{13}$$

Remark 1: It is easy to check from (13) that for all $j \in N \setminus i$,

$$Prob_p \{S \ni j\} = \sum_{T: j \in T \subseteq N} p(T) = \sum_{T: j \in T \subseteq N\backslash i} p^{N\backslash i}(T) = Prob_{p^{N\backslash i}} \{S \ni j\}. \tag{14}$$

That is to say, for any player in $N \setminus i$ the probability of belonging to the coalition that will form in N and in $N \setminus i$ according to p and $p^{N\backslash i}$, respectively, is the same.

Adopting (13) as the way of restricting probability distributions to $N\backslash i$ players, we can reformulate the balanced contributions condition (11) in terms of our probabilistic model as follows. For all $i, j \in N$,

$$\Phi_i(v, p) - \Phi_i(v^{N\backslash j}, p^{N\backslash j}) = \Phi_j(v, p) - \Phi_j(v^{N\backslash i}, p^{N\backslash i}). \tag{15}$$

Now one can wonder if for any p the evaluation $\Phi_i(-, p)$ given by (4) satisfies this condition or not for any two players and any game. The following lema gives the answer and yields as corollaries all mentioned characterizing results, giving them a clear meaning within this probabilistic model.

Lemma 1 *For all $(N, v) \in G_N$, all $p \in \mathfrak{P}_N$, and all $i, j \in N$, it holds*

$$\begin{aligned} &E_p[v(S) - v(S\backslash i)] - E_{p^{N\backslash j}}[v^{N\backslash j}(S) - v^{N\backslash j}(S\backslash i)] \\ &= E_p[v(S) - v(S\backslash j)] - E_{p^{N\backslash i}}[v^{N\backslash i}(S) - v^{N\backslash i}(S\backslash j)]. \end{aligned} \tag{16}$$

[11] Observe this condition yields as a particular case the well-known condition of consistency for semivalues (Dubey, Neyman and Weber, 1981) $p_s^n = p_s^{n+1} + p_{s+1}^{n+1}$.

Proof. Taking into account (13), we have

$$\begin{aligned}
& E_p[v(S)-v(S\backslash i)] - E_{p^{N\backslash j}}[v^{N\backslash j}(S) - v^{N\backslash j}(S\backslash i)] \\
= & \sum_{S:i\in S\subseteq N} p(S)(v(S)-v(S\backslash i)) - \sum_{S:i\in S\subseteq N\backslash j} p^{N\backslash j}(S)(v^{N\backslash j}(S) - v^{N\backslash j}(S\backslash i)) \\
= & \sum_{S\subseteq N} p(S)(v(S)-v(S\backslash i)) - \sum_{S\subseteq N\backslash j} (p(S)+p(S\cup j))\ (v(S)-v(S\backslash i)) \\
= & \sum_{S\subseteq N} p(S)\ (v(S)-v(S\backslash i) - v(S\backslash j) + v(S\backslash \{i,j\}).
\end{aligned}$$

Then (16) follows from the symmetry of the last expression with respect to i and j. ■

Note expression (16) makes a clear sense: the mutual contributions of any two players to each other's expected marginal contribution are the same. As we will see it deserves the title of generalized balanced contributions. In view of Lemma 1, Remark 1 and (3) the equality which holds *for any* $p \in \mathfrak{P}_N$ is (16), which can be rewritten as

$$\begin{aligned}
& (\sum_{T:i\in T} p(T))\ (\Phi_i(v,p) - \Phi_i(v^{N\backslash j}, p^{N\backslash j})) \\
= & (\sum_{T:j\in T} p(T))\ (\Phi_j(v,p) - \Phi_j(v^{N\backslash i}, p^{N\backslash i})).
\end{aligned} \tag{17}$$

Thus in general (15) will *not* hold for all p's, only when $\sum_{T:i\in T} p(T) = \sum_{T:j\in T} p(T)$ for any two players. So it follows immediately the following

Corollary 2 *$\Phi(-,p): G_N \to R^N$ verifies 'probabilistic' balanced contributions, that is, (15), if and only for any two players the probability of belonging to the coalition that will form is the same.*

Which is consistent with the above alluded result of Calvo and Santos (2000) in which balanced contributions characterize (along with the conditions characterizing probabilistic values) the weak semivalues, that as we have seen in subsection 4.3 are precisely what the conditional marginal contributions vector becomes when any two players have the same the probability of belonging to the coalition that will form.

Note that calling again $w_i := \frac{1}{\sum_{T:i\in T} p(T)}$, as in (7) in subsection 4.2, (17) becomes the probabilistic counterpart in our framework of the w-balanced contributions condition (12), which as established in Calvo and Santos (2000) characterizes (along with the conditions characterizing probabilistic values) the weighted weak semivalues. Thus we have also the following

Corollary 3 *For any $p \in \mathfrak{P}_N$, the evaluation given by the map $\Phi(-,p): G_N \to R^N$ satisfies condition (17) or its equivalent (16).*

Thus, again, in exact correspondence with what happened with the 'weighted' weak semivalues, 'weights' disappear as an 'optical effect' or a misunderstanding. It can be then concluded that in our setting condition (16), the counterpart of 'weighted' balanced contributions in the usual setting , is the really general condition satisfied by *all* evaluations $\Phi(-,p)$, while the counterpart of balanced contributions holds only in especial cases.

5.3 The potential

In Hart and Mas-Colell (1989) the Shapley value is characterized as the unique efficient solution that admits a 'potential'. A solution $\Psi : G_N \to R^N$ admits a potential if there exists a map $P : G_N \to R$, called then the 'potential' of Ψ, such that

$$\Psi_i(v) = P(v) - P(v^{N\setminus i}) \quad \text{(for all } i \in N). \tag{18}$$

They show that the Shapley value's potential is given by[12]

$$P(v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}\, v(S) = E_p[\frac{|N|}{|S|} v(S)], \tag{19}$$

where p is a 'standard' though rather *ad hoc* probability distribution over coalitions.

Calvo and Santos (2000) prove that semivalues and weak semivalues also admit a potential. But this property needs to be relaxed to cover other solutions, as is the case of the weighted Shapley values, that for a vector of weights w admit a 'w-potential' (Hart and Mas-Colell, 1989). In fact Hart and Mas-Colell's w-potential is a combination of efficiency and something else. The amalgamation of these two conditions allows them to characterize the weighted Shapley values with it. Calvo and Santos (1997) disentangle these two conditions and formulate the condition in the following terms. A solution $\Psi : G_N \to R^N$ admits a w-potential if there exists a map $P : G_N \to R$, and a vector of positive weights w, such that

$$\Psi_i(v) = w_i \ (P(v) - P(v^{N\setminus i})) \quad \text{(for all } i \in N). \tag{20}$$

The same authors in Calvo and Santos (2000) prove that weighted weak semivalues are the only vectors of probabilistic values that admit a w-potential.

For a precise interpretation of the former notions and the mentioned results we will introduce a general concept of 'potential' with an obvious meaning in our model (so much that it will make superfluous and inadequate that term). For any $(N,v) \in G_N$ and any $p \in \mathfrak{P}_N$, let

$$\mathbf{P}(v,p) := E_p[v(S)], \tag{21}$$

[12]They require $P(\emptyset, v) = 0$.

that is to say, $\mathbf{P}(v,p)$ is *the expected worth of the coalition that will form, given the game* (N,v) *and the probability distribution over coalitions* p. Then, the consistency relation (13) to restrict $p \in \mathfrak{P}_N$ to $N\backslash i$, yields the following relation:

Lemma 2 *For any* $(N,v) \in G_N$*, and all* $p \in \mathfrak{P}_N$*,*

$$E_p[v(S) - v(S\backslash i)] = E_p[v(S)] - E_{p^{N\backslash i}}[v^{N\backslash i}(S)].$$

Proof. According to (13), from (3) we have

$$E_p[v(S)] - E_{p^{N\backslash i}}[v^{N\backslash i}(S)] = \sum_{S\subseteq N} p(S)\ v(S) - \sum_{S\subseteq N\backslash i} p^{N\backslash i}(S)\ v^{N\backslash i}(S)$$

$$= \sum_{S\subseteq N} p(S)\ v(S) - \sum_{S\subseteq N\backslash i} (p(S) + p(S\cup i))\ v(S)$$

$$= \sum_{S:i\in S\subseteq N} p(S)\ (v(S) - v(S\backslash i)) = E_p[v(S) - v(S\backslash i)]. \blacksquare$$

That is to say, the expected marginal contribution of every player to the worth of the coalition which will form, coincides with the marginal contribution of his/her participation to the expected worth. Then, combining (3) and the lemma, results

$$\Phi_i(v,p) = E_p[v(S) - v(S\backslash i) \mid i \in S] = \frac{E_p[v(S) - v(S\backslash i)]}{Prob\{S \ni i\}}$$

$$= \frac{E_p[v(S)] - E_{p^{N\backslash i}}[v^{N\backslash i}(S)]}{Prob\{S \ni i\}}. \tag{22}$$

Formula (22) includes as particular cases all results involving the different variations of the potential notion. First notice that for any $p \in \mathfrak{P}_N$ *such that for any player the probability of belonging to the coalition that will form is the same* (as for weak semivalues), the denominators in (22) for different i's become 'hidden' in the sense that they do not depend on i. Thus, calling

$$\alpha := Prob\{S \ni i\},$$

and denoting

$$P(v) := \frac{\mathbf{P}(v,p)}{\alpha} = \frac{E_p[v(S)]}{\alpha} = E_p[v(S) \mid i \in S],$$

a mysterious function (the 'potential' in fact) emerges, so that (22) can be rewritten *in this particular case* for any i as

$$\Phi_i(v,p) = \frac{E_p[v(S)]}{\alpha} - \frac{E_{p^{N\backslash i}}[v^{N\backslash i}(S)]}{\alpha} = P(v) - P(v^{N\backslash i}).$$

That is, yielding Hart and Mas-Colell's condition (18), which is guaranteed only for weak semivalues, as established by Calvo and Santos (2000). But mind that in general

$Prob\{S \ni i\}$ depends on i and consequently the above formula not even makes any sense. While (22) *makes sense and holds for any* p. And notice that again denoting by w_i the inverse of $Prob\{S \ni i\}$, that is, $w_i := \frac{1}{\sum\limits_{T:i\in T} p(T)}$, condition (22) becomes

$$\Phi_i(v,p) = w_i(E_p[v(S)] - E_{p^{N\setminus i}}[v(S)]) = w_i(\mathbf{P}(v,p) - \mathbf{P}(v^{N\setminus i}, p^{N\setminus i})). \tag{23}$$

In other words, the general condition satisfied by all evaluations of the form $\Phi(-,p)$ is (22), or its equivalent (23), whose counterpart in the usual setting is 'weighted' potential. And again, as before, the 'weighted' character of the notion evaporates[13]. Similarly, formula (21) seems that of a general, natural[14] and transparent notion of potential, even if that name is out of place in this context, given its obvious meaning: the expected worth of the coalition which will form.

6 Concluding remarks

We have provided an extremely simple model with great unifying power, in which a variety of notions acquire a transparent meaning. On the one hand, several families of 'solutions' born out of different departures from Shapley's seminal characterization appear as particular cases of a general and simple story: the evaluation of the expected marginal contribution of every player to the coalition that will form conditional to that player belonging to it. And the disgusting 'weights' associated with some of these notions evaporate as pure misunderstandings. On the other hand, some interesting notions from the theory of value, as the existence of potential and the balanced contributions condition, lose their 'mystery' and appear as obvious properties of some particular cases within a general model. While the 'weighted' variants of both notions turn out to be the true general and transparent properties with a clear meaning.

A 'disappointingly' simple story? The tight fitting of a variety of notions born out of axiomatic exploration into so simple a model cannot be dismissed without reflection. You may choose: the fascinating obscurities of the axiomatic approach to value or the

[13]This is not that surprising after having seen what happens with balanced contributions, given the equivalence of w-balanced contributions and the existence of a w-potential in the usual setting (Calvo and Santos, 1997)). But we have preferred discussing separately both notions and their meaning in our setting, for we are mainly concerned in this paper with the 'meaning' of notions and results.

[14]Compare the simplicity and transparence of (21) with (19), or that of the w-potential in Hart and Mas-Colell (1989):

$$P_w(N,v) = E[\int_0^1 \frac{1}{t} v(N(t))dt].$$

Where for every $t \in [0,1]$, $N(t) = \{i \in N \mid X^i \leq t\}$, X^i being, for each $i \in N$, a random variable (all them independent) with distribution function $\mathrm{Prob}(X^i \leq t) = t^{w_i}$ for all $t \in [0,1]$.

transparency of the probabilistic tale to account for them. To keep thinking in terms of 'solutions' axiomatically characterized and ignore that everything is *a if* it were the simple story we have presented in this paper, or to adopt this as the story we have been turning around but have failed so far grasping. Is it not simpler to think that the earth moves rather than only that everything is *as if* it moved?

On the other hand, the simple story of our probabilistic model seems to have very little to do with Shapley's original interpretation of his 'value' for TU games, as the value for a rational player of the prospect of engaging into any situation of this class with other rational players. But the lack of a clear matching of the different combinations of axioms with the genuine notion of value, along with the perfect matching with the extreme simple probabilistic model presented here, sheds serious doubts about the achievements of the axiomatic approach since Nash (1950) and Shapley (1953) in connection with the notion of value. Maybe it has ended by even shedding more shadows than light on the issue. If additivity has raised suspicion from the very beginning, its substitutes (balanced contributions, strong monotonicity, transfer, or the existence of potential) or their weighted variants, shared by *all* the axiomatic systems that generate the different families considered here, are in no more clear way connected with the original notion of value. Transfer is a bit weaker than additivity, but cannot be compellingly motivated. Balanced contributions can be justified on fairness grounds (which makes sense only for an allocation rule). As to its equivalent, the existence of potential, it may only raise some in our opinion not especially illuminating fascination. There is only left Young's strong monotonicity, with a transparent meaning, again compelling for an allocation rule, but completely dependent (as null player) on the notion of 'marginal contribution', so cherished by economists but with no clear translation into the more general NTU terms. In sum, a convincing axiomatization of the Shapley value in the genuine sense is still missing in our opinion in spite of many people thinking it is an exhausted issue. But this should be obvious: only a compelling characterization which made sense and worked on a domain wider than that of TU games could claim to be thoroughly acceptable, and this is still missing.

The results presented in this paper yield thus a critical conclusion about the excesses of the axiomatic approach and the unjustified disregard of the probabilistic interpretation. It seems that comparatively speaking the probabilistic approach is considered less worthy than the axiomatic one. As a result much less attention has been paid to this approach. Is this regard justified? The least this paper permits to conclude is the complementariness of the probabilistic and the axiomatic approaches. Even if one were exclusively interested in the axiomatic point of view, this paper shows how a probabilistic model contributes to a better understanding of the meaning of axioms.

Finally, however illuminating the model might be, it is too simple a model. A single coalition forms, why not more? Another line of further development may be a richer model in which the probability distribution were endogenously generated. Further developments from the starting point provided by the model seem promising. Maybe the transparent story presented in this paper can be a more inspiring starting point than a set of more or less beautiful and more or less obscure abstract axioms.

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