NETWORK FORMATION AND
ANTI-COORDINATION GAMES *

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ABSTRACT

We study a setting in which individual players choose their partners as well as a mode of behavior in 2 x 2 anti-coordination games -- games where a player's best response is to behave differently than the opponent. We characterize the nature of equilibrium networks as well as study the effects of network structure on individual behavior. Our analysis shows that both the network architecture and the induced behavior are crucially dependent on the value of the cost of forming links. In general, the equilibrium configurations are found to be neither unique nor efficient. This conclusion continues to hold if the population game is embedded in a standard evolutionary model of learning, since all equilibria turn out to be stochastically stable.

KEYWORDS: networks, links, anti-coordination games, efficiency.

1   Introduction

In the past few years, there has been an extensive literature on social networks which shows that the structure of interaction between individuals can be decisive in determining the nature of the outcomes and, in particular, the players' action choices in an underlying game. In much of this literature, the structure of interaction is exogenously specified and the nature of the outcome under different specifications is examined (see e.g., Anderlini and Ianni (1996), Ellison (1993), Goyal (1996), Morris (2000), Young (2002)).

Recently, interest has grown in understanding the process through which the interaction structure itself develops. The earlier part of this literature (e.g. Aumann and Myerson (1989), Bala and Goyal (2000), Dutta, van den Nouweland and Tijs (1995), Jackson and Wolinsky (1996), among others) has focused on contexts where players choose links with others and there is no additional strategic dimension (i.e. there is no explicit game being played among connected players). Later contributions, such as Goyal and Vega-Redondo (2000) and Jackson and Watts (2000) have studied settings in which each agent plays a game with each of her 'partners' and therefore (in addition to connecting decisions) has to choose a mode of behavior in the accompanying game. This research has focused on a class of games where individuals have an incentive to choose the same action as their partners; these games are referred to as coordination games.

In the present paper, we wish to consider the role of network formation in the polar case, where individuals prefer to choose an action unlike that chosen by their partners. We shall refer to these interactions as games of anti-coordination[1]. Many interesting situations can be suitably conceived in this fashion, e.g. when the successful completion of a task requires that the individuals involved adopt complementary actions (or skills), or when a meaningful interaction can only be conducted when the agents adopt different roles (say, buyers and sellers), or when in the contest for a certain resource, an optimal response is not to respond with the same behavior (aggressive or peaceful, as in the Hawk-Dove game) as one's opponent.

We consider a model where individuals can form pair-wise links on their own individual initiative, i.e. the link formation is one-sided. In addition to the links, each player also chooses which of two actions to use in the interaction with his partners (the same action in all of them). Each bilateral interaction provides some gross return to the players involved according to the actions chosen. On the other hand, links are costly, with the players initiating each link paying for it. Thus, overall, the total net payoffs earned by a player consist of the sum

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[1] Bramoullé (2001) analyzes anti-coordination games played on a *fixed* structure. He shows that the structure has a much stronger impact on the equilibria than in the case of coordination games.

of the gross returns obtained from each of the pairwise interactions minus the costs of the links she initiates. For simplicity, we make the assumption that the gross returns accruing from each link are non-negative, so that no link initiated by an agent will ever be refused by her partner.

In this setting, we first provide a characterization of the strict Nash equilibria of the static game (Propositions 1-3). We find that as the costs of link formation increase the equilibrium network becomes more sparse. For low costs it is complete, for moderate costs it is a bipartite graph, while for high costs it is empty. The costs of link formation also have a profound impact on the number of players who choose the two actions. In particular, for low costs of forming links, the numbers of players choosing the two actions corresponds roughly to the proportions that would arise in the mixed strategy equilibrium of the two-person anti-coordination game, while for moderate costs of forming links a wide range of proportions can be sustained in equilibrium. The intuition for this latter multiplicity is as follows: consider the class of games where a player wishes only to form a link with another player who is doing the other action. In this setting, a player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action, since in this way she plays the largest number of games. However, in our setting, a player has to balance these considerations with the fact that costly links have to be formed in order to play the game. This argument also illustrates that as the costs of forming links increase, the distribution of links can have a bigger influence on the incentives to switch actions. Thus for larger costs levels, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her at equilibrium.

We then study the efficiency of different network structures (Propositions 4 and 5). In general, the architecture of efficient networks becomes less dense as the costs of link formation increase. For low costs of forming links, typically, the efficient network is complete, while for moderate costs the efficient network is a bipartite network. The costs of forming links also have an impact on the proportions of players choosing different actions. In an interesting class of cases, where the links are only worthwhile between players choosing different actions, efficient profiles have roughly equal proportions of players choosing the two actions. A comparison of efficient and equilibrium networks thus suggests that equilibrium and efficient networks are very different typically.

This variety of equilibria and their inefficiency motivates an examination of the dynamic stability of different outcomes (Propositions 6 and 7). Our analysis of the dynamics shows that all (strict) Nash equilibria are stochastically stable, i.e., they are robust to small but persistent perturbations. We thus find that stochastic stability has almost no refinement power in this setting.

The above results are in sharp contrast to the findings on coordination games reported in Jackson and Watts (2000) and Goyal and Vega-Redondo (2000).

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Our model with one-sided link formation is similar to the latter paper, and we now discuss the differences in results in some detail. Goyal and Vega-Redondo find that for all relevant values of the cost there is a unique stochastically stable network architecture: the complete network. By contrast, we find that, in anti-coordination games, the selected network architectures are generally incomplete and their qualitative structure depends in interesting ways on the underlying payoffs and linking cost. Similarly, the relationship between efficiency and equilibrium (or dynamic stability) is also very different. They show that there is a certain threshold for the linking cost below which risk dominance is selected, while in the present paper the relationship is much richer and, in some cases exactly the reverse: efficient outcomes are guaranteed for low linking costs but are not attainable beyond a certain cost threshold.

The rest of the paper is organized as follows: In Section 2 we set up the model. In Section 3 we discuss the Nash equilibrium results. The welfare analysis is taken up in Section 4. In Section 5 we study the dynamic framework and characterize the stochastically stable states. Finally, Section 6 summarizes the results and concludes. The proofs that do not appear in the body of the paper are contemplated in the Appendix.

2 The Model

2.1 Link formation

Let $N = \{1, 2, \ldots, n\}$ be a set of players where, for simplicity, $n \geq 3$ is assumed even. We are interested in modeling a situation where each of these players can choose the subset of other players with whom to play a fixed bilateral game. Formally, let $g_i = (g_{i1}, \ldots g_{ii-1}, g_{ii+1}, \ldots g_{in})$ be the set of links formed by player i. We suppose that $g_{ij} \in \{1, 0\}$, and say that player i forms a link with player j if $g_{ij} = 1$. The set of link options is denoted by $\mathcal{G}_i$. Any player profile of link decisions $g = (g_1, g_2 \ldots g_n)$ defines a directed graph, called a *network*.

Specifically, the network g has the set of players N as its set of *vertices* and its set of arrows, $\Gamma \subset N \times N$, defined as follows, $\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$. Graphically, the link (i, j) may be represented as an edge between i and j, a filled circle lying on the edge near agent i indicating that this agent has formed (or supports) that link. Every link profile $g \in \mathcal{G}$ has a unique representation in this manner. Figure 1 below depicts an example. In it, player 1 has formed links with players 2 and 3, player 3 has formed a link with player 1, while player 2 has formed no links.[2]

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[2] Since agents choose strategies independently of each other, two agents may simultaneously initiate a link, as seen in Figure 1.
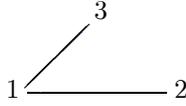
Figure 1

Given a network g, we say that a pair of players i and j are directly linked if at least one of them has established a link with the other one, i.e. if $\max\{g_{ij}, g_{ji}\} = 1$. To describe the pattern of players' links, it is useful to define a modified version of g, denoted by $\bar{g}$, that is defined as follows: $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$ for each i and j in N. Note that $\bar{g}_{ij} = \bar{g}_{ji}$ so that the index order is irrelevant.

Let $N^d(i; g) \equiv \{j \in N : g_{ij} = 1\}$ be the set of players in network g with whom player i has established links, while $\nu^d(i; g) \equiv |N^d(i; g)|$ is its cardinality. Similarly, let $N^d(i; \bar{g}) \equiv \{j \in N : \bar{g}_{i,j} = 1\}$ be the set of players in network g with whom player i is *directly linked*, while $\nu^d(i; \bar{g}) \equiv |N^d(i; \bar{g})|$ is the cardinality of this set.

A *bipartite graph* is a network in which there exists a partition of the players into two mutually exclusive and exhaustive sets, N_1 and N_2, such that $\overline{g}_{ij} = 1$ only if $i \in N_1$ and $j \in N_2$. A bipartite graph is *complete* if $\overline{g}_{ij} = 1$ for every pair of players $i \in N_1$ and $j \in N_2$ and *essential* if there are no redundant links, i.e. $g_{ij}g_{ji} = 0$ for every $i, j \in N$. A *semi-bipartite graph* is a network in which there exists a partition of the players into two mutually exclusive and exhaustive sets, N_1 and N_2 such that $\overline{g}_{ij} = 1$ only if $i \in N_1$ and $j \in N_2$ or if $i, j \in N_k$ where $k \in \{1, 2\}$. Analogously to the definitions mentioned above, a semi-bipartite graph will be complete and essential if all *possible* links are formed and they are not redundant.

2.2 Social Game

Individuals located in a social network play a 2×2 symmetric game in strategic form with a common action set. The set of partners of player i depends on her location in the network. We will assume that two individuals can play a game if and only if they have a direct link between them. Thus, player i will play a game with all other players in the set $N^d(i; \bar{g})$.

We now describe the two-person game that is played between any two partners. The set of actions is $A = \{\alpha, \beta\}$. For each pair of actions $a, a' \in A$, the payoff $\pi(a, a')$ earned by a player choosing a when the partner plays a' is given by the following table:

2 1	α	β
α	d	e
β	f	b

Table I

We shall assume that it is one of anti-coordination with two pure strategy equi-libria, (α, β) and (β, α). In other words we consider the following restrictions on the payoffs:

$$d < f \text{ and } b < e. \tag{1}$$

We shall assume that every player i is obliged to choose the same action in the (generally) several bilateral games that she is engaged in. This assumption is natural in the present context; if players were allowed to choose a different action for every two-person game this would make the behavior of players in any particular game insensitive to the network structure. The strategy space of a player can be identified with $S_i = \mathcal{G}_i \times A$, where $\mathcal{G}_i$ is the set of possible link decisions by i and A is the common action space of the underlying bilateral game.[3]

We start by deriving the payoffs of the social game. An important feature of our approach is that links are assumed *costly*. Specifically, every agent who establishes a link with some other player incurs a cost $c > 0$. Thus, we suppose that the cost of forming a link is independent of the number of links being established and is the same across all players.

Another important feature of our model is that links are *one-sided*. That is, players' decisions as to whether or not propose a link to some other player are adopted individually (i.e. independently by each of them). This aspect of the model allows us to use standard solution concepts from non-cooperative game theory in addressing the mechanism of link formation. It raises, however, the issue of whether a proposal to form a link might not be accepted by the player who receives it (even though she would bear no linking cost). In the present paper, we abstract from these complications by simply positing that the payoffs of the bilateral game are non-negative and, therefore, no player will ever have any incentives to refuse forming any proposed link.

This issue, however, might be addressed in more generality by an extension of the model where genuine objections could be raised in the face of any particular link proposal by other player. In turn, this generalization would allow not only for the possibility of negative game payoffs, but also of a less asymmetric cost

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[3] In our formulation of the static model, players choose links and actions in the anti-coordination game simultaneously.

sharing in the formation of links. By way of illustration, one such modelling extension could have the link formation mechanism consist of two stages. In a first stage, action choices and link proposals would be issued. Then, in a second stage, any agent who has received a unilateral link proposal could decide (already knowing the relevant action choices and, therefore, the entailed payoff consequences) whether to accept it or not. Clearly, such a model would be equivalent to the present formulation under the maintained assumption that the game payoffs are positive and the whole linking cost is incurred by the proposer. If these conditions where not met, however, it would certainly introduce novel considerations which, as indicated, we choose to ignore in the present paper.

In the setup being considered, the payoff to a player i from playing some strategy $s_i = (g_i, a_i)$ when the strategies of other players are $s_{-i} = (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_n)$ can be written as follows:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N^d(i; \bar{g})} \pi(a_i, a_j) - \nu^d(i; g) \cdot c \tag{2}$$

We note that the individual payoffs are aggregated across the games played by him. In our framework, the number of games an individual plays is endogenous, and we want to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate payoff formulation.

The above payoff expression allows us to particularize the standard notion of Nash equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a *Nash equilibrium* for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{3}$$

The set of Nash equilibria will be denoted by S^*. A Nash equilibrium is said to be strict if every player gets a strictly higher payoff with his current strategy than she would with any other strategy. The set of strict Nash equilibria will be denoted by S^{**}.

2.3 Dynamics

Time is modeled as being discrete, $t = 1, 2, 3, \ldots$. At each t, the state of the system is given by the strategy profile $s(t) \equiv [(g_i(t), a_i(t))]_{i=1}^n$ specifying the action played, and links established, by each player $i \in N$. Let us suppose that, at every period t, with an independent probability p, a player revises over a particular component of her strategy, i.e. with probability p she revises a particular link g_{ij} or her action a_i. For simplicity this probability is independent across components and across individuals. Thus, for example, there is probability p^n that a player may revise her complete strategy (all her links and her action). In other words this dynamics includes the possibility of revising together links and actions, but also admits doing it separately. The intuition is that sometimes it is not feasible for an agent to change her whole strategy but only part of it. This

could also be understood as an additional expression of bounded rationality. An agent, once she gets a revision opportunity just considers part of her strategy. This approach resembles the model studied by Jackson and Watts (2000) but with a major difference: our dynamic allows for a revision of the complete strategy. On the other hand, we are in a non-cooperative one sided link model where decisions are taken unilaterally, i.e. revision opportunities over a particular link are independent for the two individuals forming the link.

Hence, with probability $p^k(1-p)^{n-k}$ a player i gets the chance to revise over k components of her strategy which, using standard notation, we write as $s_i = (s_{i_k}, s_{i_{-k}})$ to distinguish the components which can be revised from those that cannot. In that event, she is assumed to choose a myopic best response:

$$s_{i_k}(t) \in \arg \max_{s_{i_k} \in S_{i_k}} \Pi_i(s_{i_k}, s_{i_{-k}}(t-1), s_{-i}(t-1)). \tag{4}$$

That is, she selects a best response to what other players chose in the preceding period and what she chose in the $n-k$ components that are not open for revision. If there are several strategies that fulfill (4), then any of them is taken to be selected with, say, equal probability. This strategy revision process defines a Markov chain on $S \equiv S_1 \times ... \times S_n$.

In our setting, which will be seen to display multiple strict equilibria, there are several absorbing states of the Markov chain.[4] This motivates the examination of the relative robustness of each of them. To do so, we rely on the approach proposed by Kandori, Mailath and Rob (1993), and Young (1993). We suppose that, occasionally, players make mistakes, experiments, or simply disregard payoff considerations in choosing their strategies. Specifically, we assume that, conditional on receiving a revision opportunity, a player chooses her strategy at random with some small "mutation" probability $\epsilon > 0$. For any $\epsilon > 0$, the process defines a Markov chain that is aperiodic and irreducible and, therefore, has a unique invariant probability distribution. Let us denote this distribution by μ_ϵ. We analyze the form of μ_ϵ as the probability of mistakes becomes very small, i.e. formally, as ϵ converges to zero. Define $\lim_{\epsilon \to 0} \mu_\epsilon = \hat{\mu}$. When a state $s = (s_1, s_2, \ldots, s_n)$ has $\hat{\mu}(s) > 0$, i.e. it is in the support of $\hat{\mu}$, we say that it is *stochastically stable*. Intuitively, this reflects the idea that, even for infinitesimal mutation probability (and independently of initial conditions), this state materializes a *significant* fraction of time in the long run.

3 Nash equilibria

We now characterize the set of strict Nash equilibria of the social game. First, we describe the types of Nash networks and how they depend on the anti-coordination game and on the cost of link formation. Second, we characterize

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[4] Notice that the set of absorbing states of the Markov chain coincides with the set of strict Nash equilibria of the population game with payoffs given by (2).

the range of possible values for the number of agents playing each action (α or β) in equilibrium. Third, we ask how the costs of link formation are divided among agents at equilibrium.

Anti-coordination games have different possible payoffs configurations and we will see that they also lead to different types of Nash networks. By definition, we have $d < f$ and $e > b$. Without loss of generality, assume that $f > e$. In equilibrium, β-players (i.e. players who choose action β in the anti-coordination game) earn a higher payoff than α-players (i.e. players who choose action α in the anti-coordination game). If all the parameters are distinct (i.e. the non degenerate case), there are three possible payoffs ordering.

$$
\begin{array}{rcl}
\text{Case 1} & : & b < e < d < f \\
\text{Case 2} & : & b < d < e < f \\
\text{Case 3} & : & d < b < e < f
\end{array}
$$

Each ordering corresponds to a different type of anti-coordination game. In Case 1, the payoff of coordinating on α is higher than the payoff of an α-player in equilibrium. Therefore, Case 1 represents exploitation games akin to the Hawk-Dove game. In Case 2 and 3, equilibrium payoffs are higher than any other payoffs. Cases 2 and 3 represent situations of complementary, in which both players earn higher payoffs at equilibrium than out of it. In Case 2 the payoff of coordinating on α is higher than the payoff of coordinating on β, while the situation is reversed in Case 3.

Since link formation is one-sided, the cost of any link, at equilibrium, is supported by one and only one agent. Hence, Nash networks are essential, i.e., $g_{ij} = 1 \Rightarrow g_{ji} = 0$. Nash networks, on the other hand, depend on how the cost of link formation compares with the parameters of the game. For example, when $c > b$ (i.e. the cost of link formation is higher than the payoff obtained when both agents plays β), the β-players do not have an incentive to form links with other β-players. Therefore, in equilibrium there is no link among β-players, i.e., the network of links among β-players is empty. Instead, when $c < b$, the β-players are willing to form links and to support the cost of link formation with any other agent playing β. In equilibrium, therefore, all the β-players are directly linked with all the other β-players and the network of links among β-players is essential and complete.[5] The reasoning is similar for the other parameters. For example, if $c > e$, there is no link from α-players to β-players, whereas if $c < e$, all the α-players have links with all the β-players in a complete and essential way.

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[5] It is our assumption that payoffs depend linearly on the number of social neighbors playing a strategy that causes this 'all or nothing' result.

Figure 2: payoffs configurations

Case 1

b _____ e d _____ f

Case 2

d _____ f

b _____ e

Case 3

d _____ f

b _____ e

The following shorthand notation will allow us to refer to all the possible types of Nash networks. (Here, "to be linked to" is taken to mean that the links go in only one direction, whereas "to be linked with" signifies that the links may go in both directions.)

$\beta \; \emptyset \; \alpha :$ the empty graph

$\beta \rightarrow \alpha :$ all β-players are linked to all α-players,
but no α-player is linked to a β-player

$\beta \rightleftharpoons \alpha :$ all β-players are linked with all α-players

$\beta \rightarrow \vec{\alpha} :$ all β-players are linked to all α-players,
and all α-players are linked with all α-players

$\beta \rightleftharpoons \vec{\alpha} :$ all α-players are linked with all α-players and with all β-players

$\vec{\beta} \rightleftharpoons \vec{\alpha} :$ the complete graph

Hence, $\beta \rightarrow \alpha$ and $\beta \rightleftharpoons \alpha$ represent complete and essential bipartite graphs, while $\beta \rightarrow \vec{\alpha}$ and $\beta \rightleftharpoons \vec{\alpha}$ are semi-bipartite graphs. Using the above notation, the following result describes how the cost of link formation determines the type of Nash networks.

Proposition 1 *Suppose (1) holds. Then strict Nash equilibria exhibit the following pattern of link formation:*

Case 1		Case 2		Case 3	
$0 < c < b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$	$0 < c < b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$	$0 < c < d$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$b < c < e$	$\beta \rightleftharpoons \vec{\alpha}$	$b < c < d$	$\beta \rightleftharpoons \vec{\alpha}$	$d < c < b$	$\vec{\beta} \rightleftharpoons \alpha$
$e < c < d$	$\beta \rightarrow \vec{\alpha}$	$d < c < e$	$\beta \rightleftharpoons \alpha$	$b < c < e$	$\beta \rightleftharpoons \alpha$
$d < c < f$	$\beta \rightarrow \alpha$	$e < c < f$	$\beta \rightarrow \alpha$	$e < c < f$	$\beta \rightarrow \alpha$
$f < c$	$\beta \; \emptyset \; \alpha$	$f < c$	$\beta \; \emptyset \; \alpha$	$f < c$	$\beta \; \emptyset \; \alpha$

Several interesting points follow from the above result. First, it shows that (except for very low costs), the nature of links is quite complicated and the link initiation and hence the network architecture depends very much on the game that is being played. On the one hand, if the game is one of exploitation (Case 1), it supports $\beta \rightarrow \vec{\alpha}$ as Nash networks. In this case, α-players are willing to support the costs of link formation with themselves but not with β-players, while β-players are willing to support the costs of link formation with α-players but not with themselves. Thus, in an equilibrium, players choosing α link with each other and free-ride on the links that players choosing β form with them. On the other hand, if the game is one of strict complementarity (as in Cases 2 and 3), it supports bipartite graphs $\beta \rightleftharpoons \alpha$ as Nash networks. In this case, both α and β-players have an interest to be linked to players choosing the other strategy, while they do not wish to be linked with players choosing the same action.

A second point worth noting concerns the effects of increasing linking costs. In each of the three cases, the effect of higher costs is broadly similar. The payoffs of the anti-coordination game define cut-off values such that, as the costs of link

Figure 3: Nash networks
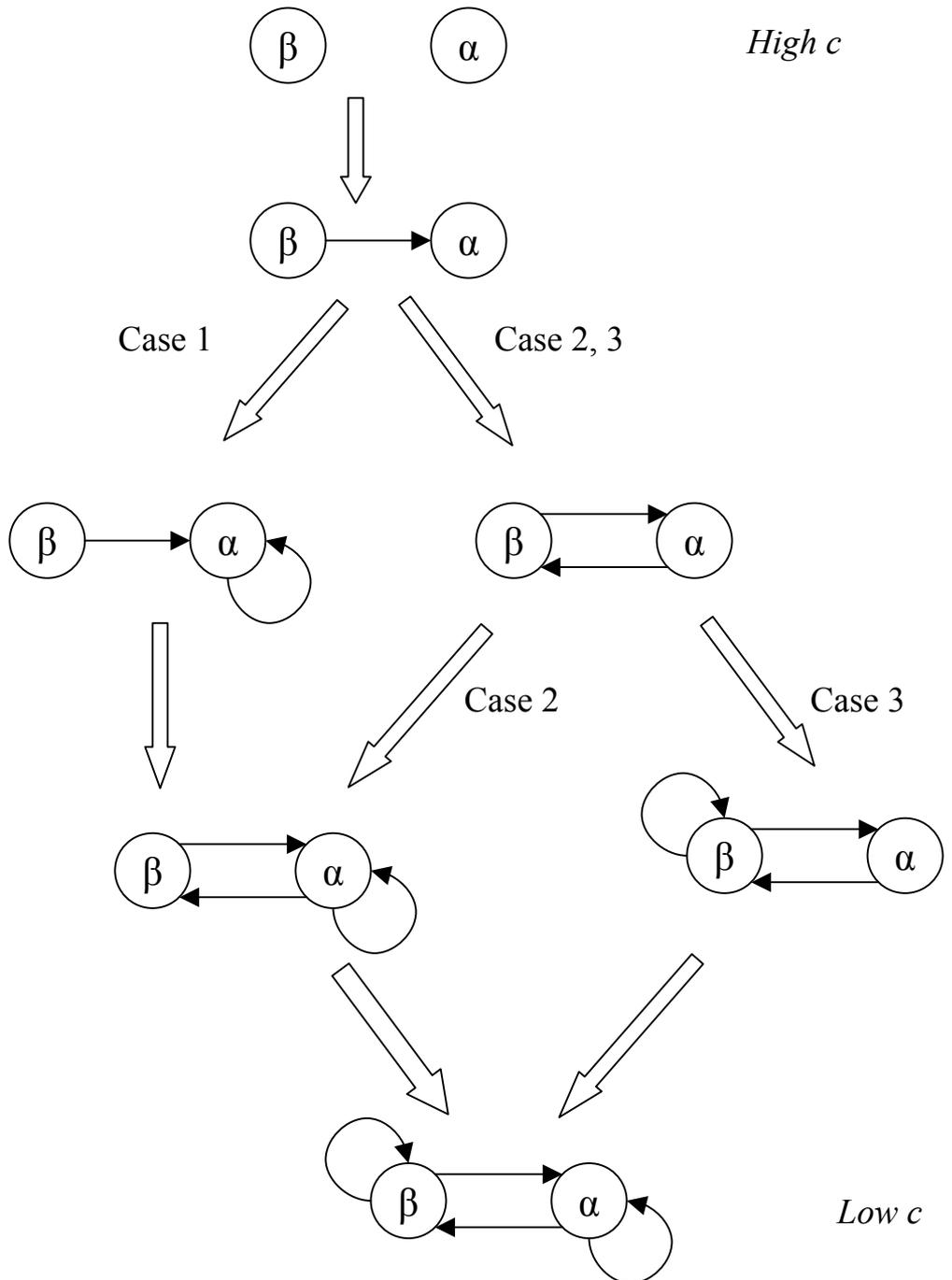
formation surpasses them, an economic opportunity disappears along with its corresponding type of link. In general, we find that, as the cost of link formation rises, the possible types of Nash networks become more sparse, going from the complete network to the empty network through three intermediary cases.

We now analyze how the number of players choosing each of the different actions in equilibrium depends on the linking cost c. Let s be any given strategy profile, and denote by $N_\alpha^s = \{i \in N | a_i = \alpha\}$ the set of α-players in it with $n_\alpha^s = card(N_\alpha^s)$ being its cardinality, while $N_\beta^s = \{i \in N | a_i = \beta\}$ is the set of β-players and $n_\beta^s = card(N_\beta^s)$ its cardinality. Moreover, let $q_i^{s,k}$ be the number of active links that a player i has with individuals choosing action k and $r_i^{s,k}$ be the number of passive links that a player i has with individuals choosing action k, where $k = \alpha, \beta$. We will avoid superscript s when there is no possible confusion. Our next result derives the lower and upper bounds for n_α^s and n_β^s in equilibrium. We derive this result by examining the best-responses for every possible case.[6] Denote $p_\beta = \frac{f-d}{f-d+e-b}$. Notice that p_β is the probability of playing β in the mixed strategy equilibrium of the anti-coordination game. It is useful to introduce two auxiliary functions φ and ψ as follows:

$$\varphi(c) = \begin{cases} p_\beta \text{ if } c < b \\ \frac{f-d}{f-d+e-c} \text{ if } b < c < e \\ 1 \text{ if } e < c \end{cases}$$

and

$$\psi(c) = \begin{cases} p_\beta \text{ if } c < d \\ \frac{f-c}{f-c+e-b} \text{ if } d < c < f \\ 0 \text{ if } f < c \end{cases}$$

Note that φ and ψ are continuous, φ is increasing and convex, and ψ is decreasing and concave. These functions bound the relative sizes of the different α- and β-parts of the network, as established by the following result.

Proposition 2 *Suppose (1) holds. Then there exists a strict Nash equilibrium with n_β players choosing β iff $(n-1)\psi(c) < n_\beta < (n-1)\varphi(c) + 1$.*

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[6] The best-response equations do not depend on the particular payoffs and cost configuration, but only on the type of Nash network to which this configuration leads. For example, situations where the payoffs correspond to Case 1 and $b < c < e$, and where the payoffs correspond to Case 2 and $b < c < d$ both support $\beta \rightleftharpoons \vec{\alpha}$ as Nash networks. Hence, both cases can be analyzed as one. This reduces the number of domains to analyze from 16 to 6.

Figure 4: group sizes in equilibrium

Case 1

Case 2

Case 3
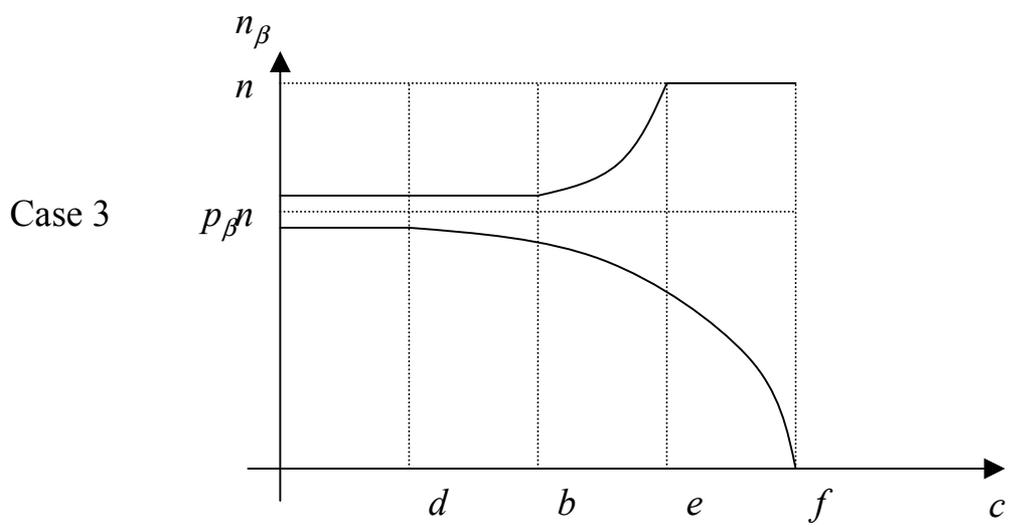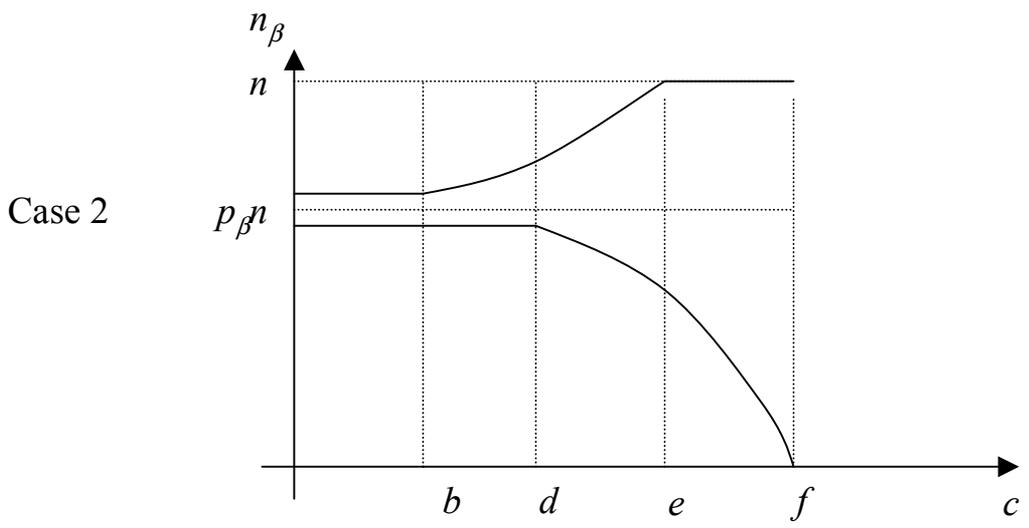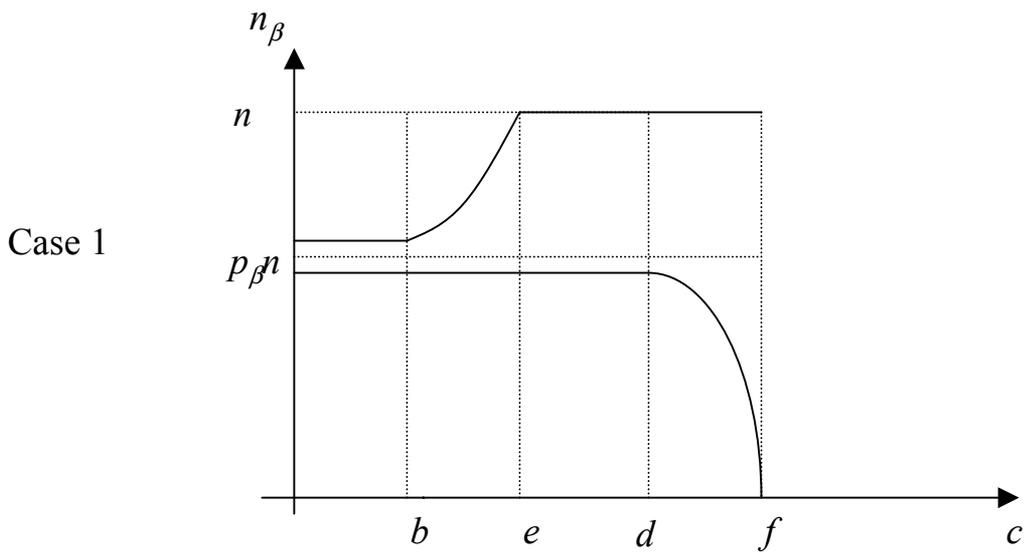
This result and Figure 2 illustrate the precise relationship between the linking cost and the range of equilibrium behavior in the respective game. In particular, it states that for a low cost of forming links, the proportion of players choosing actions α and β corresponds (roughly) to the mixed-strategy of the two-person anti-coordination game. This simply follows from the fact that, for low linking costs, players have incentives to form the complete network and hence the link formation mechanism has no particular influence on individual behavior. However, beyond this low range, the cost of link formation has a profound impact on individual choice of actions. In particular, a broader range of proportions of players choosing actions α and β becomes possible.

The intuition behind the latter conclusion is best explained in the context of strict complementarity where a player wishes to form a link only with a partner choosing a different action. In this setting, if both actions are symmetric, the player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action. In adjusting her behavior, however, she has to take into account that the creation of any new link on her part involves a cost. This implies that, for a fixed configuration of actions, the incentives for any given player to keep doing what she currently does are maximized when she is the "passive recipient" of all links to the players that are choosing the other action. This argument allows us to compute the bounds on the maximum and minimum number of players who can be playing each action at equilibrium. It also suggests that, as the costs of forming links increase, their distribution can have a bigger influence on the incentives to switching actions. In particular, for large costs levels, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her.

Propositions 1 and 2 characterize the Nash equilibria of the social game. However, they do not always provide information on how the cost of links is divided among players. For example when the type of Nash networks is $\beta \to \vec{\alpha}$, we know that β-players incur the costs of links with α-players but the cost distribution among α-players is not specified. Similarly, for the case $\beta \rightleftharpoons \alpha$, the distribution of costs of links formed between α and β-players is not specified. In general, when the number of β-players is high, one expects that more α-players will support the costs.

We now investigate a specific aspect of the issue of costs distribution. Consider a configuration of the parameters of the game. Given the type of Nash network and the range of possible values for n_β consistent with this configuration, we say that n_β is **distribution insensitive** if all the *possible* divisions of costs are sustainable in equilibrium. Distribution insensitiveness is a strong notion which captures cases where the allocation of costs of link formation does not affect equilibria. In general, the existence of distribution insensitive values is not guaranteed. We then ask: under what conditions the equilibrium values for n_β are distribution insensitive? This question is solved in the following Proposition.

Proposition 3 *Suppose (1) holds. If the type of the network is $\beta \rightarrow \alpha$, $\beta \rightarrow \vec{\alpha}$, or $\vec{\beta} \rightleftharpoons \vec{\alpha}$, all the possible equilibrium n_β are distribution insensitive.*[7]

If the type of the network is $\beta \rightleftharpoons \alpha$, n_β is distribution insensitive iff

$$(n-1)\frac{f-c}{f+e-2c} < n_\beta < (n-1)\frac{f-c}{f+e-2c} + 1$$

If the type of the network is $\beta \rightleftharpoons \vec{\alpha}$, n_β is distribution insensitive iff

$$(n-1)\frac{f-d}{f-d+e-c} < n_\beta < (n-1)\frac{f-d}{f-d+e-c} + 1$$

If the type of the network is $\overrightarrow{\beta} \rightleftharpoons \alpha$, n_β is distribution insensitive iff

$$(n-1)\frac{f-c}{f-c+e-b} < n_\beta < (n-1)\frac{f-c}{f-c+e-b} + 1$$

This result says that, in general, distribution insensitive values always exist. In addition, either all the possible equilibrium values are distribution insensitive, or a unique equilibrium value is distribution insensitive. In the first case, when the Nash network is of type $\beta \rightarrow \alpha$, $\beta \rightarrow \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \vec{\alpha}$, the best responses equations turn out to be independent on distribution considerations, which explains the result.

We briefly discuss the arguments behind the second case, focusing for concreteness on the case with $b, d < c < e, f$, where every equilibrium network is of type $\beta \rightleftharpoons \alpha$. Consider any distribution insensitive n_β and let $i \in N$ be an agent who chooses α in the underlying state and supports q_i^β links to β-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that

$$\eta_\beta > (n-1)\frac{f-c}{f-c+e-b} + q_i^\beta \frac{c-b}{f-c+e-b}. \tag{5}$$

Note that the right hand side of the above expression is increasing in q_i^β and therefore reaches a maximum at $q_i^\beta = \eta_\beta$. Therefore, substituting η_β for q_i^β in (5), we obtain the following condition:

$$\eta_\beta > (n-1)\frac{f-c}{f-c+e-c}, \tag{6}$$

which is necessary and sufficient for distribution insensitivity to apply to α-players. Turning now the attention to the counterpart condition for any agent j choosing β, note that, in order for this player to be playing a best response, a necessary and sufficient condition is:

$$\eta_\beta < (n-1)\frac{f-d}{f-d+e-c} + 1 - q_j^\alpha \frac{c-d}{f-d+e-c},$$

---

[7]In the network $\beta \rightarrow \alpha$ the direction of all the links is already determined and therefore distribution insensitivity is not an issue.

where q_j^α denotes the number of links to α-players supported by j. The right hand side of the above condition is decreasing in q_j^α and therefore it attains its minimum value at $q_j^\alpha = n_\alpha$. Thus substituting n_α for q_j^α we obtain:

$$n_\beta < (n-1)\frac{f-c}{f-c+e-c} + 1, \tag{7}$$

which is again a necessary and sufficient condition for distribution insensitivity concerning any player choosing β. Combining (6) and (7), the desired conclusion follows.

When α-players and β-players both want to link with the other type and the cost of link formation is not too low, distribution insensitiveness selects a unique equilibrium value. When the number of agents playing β is equal to this value, a strategy profile is an equilibrium no matter how the costs of links formation are allocated among agents (conditional on the fact that the network is Nash). In contrast, when the size of the population of β-players is not distribution insensitive, certain allocations of costs will not be sustained in equilibrium. The existence of distribution insensitive values will play an important role for the analysis of the dynamics of the game, see section 5.

4   Welfare analysis

There are many ways to measure the social welfare of a network structure. Here, we identify welfare with the sum of individuals' payoffs. More precisely, the welfare of a strategy profile $s = (s_1, ..., s_n)$, denoted as $W(s)$, is made equal to the sum of the individuals' payoffs,

$$W(s) = \sum_{i=1}^{n} \Pi_i(s).$$

Furthermore, we say that a state s is efficient if and only if $W(s) \geqslant W(s')$, for all $s' \in S$.

First of all, notice that the welfare contribution of a link is $2b - c$ in the case of two β-players, $2d - c$ in the case of two α-players, and $e + f - c$ in case of an α-player linked to a β-player.[8] This implies that the appropriate classification of payoffs configuration for the welfare analysis is different from the classification used for the equilibrium analysis. This is worth keeping in mind in the following results on welfare. It is also worth noting that, given any particular pattern of connections, the division of passive and active links does not affect its welfare. Therefore, in order to characterize an efficient profile it is enough to find the undirected structure of its network and the number of players choosing each action. Given a particular state s, denote $n_{\alpha\alpha}$ to be the number of links in s

---

[8] Since the cost of a link is incurred only by one of the agents forming the link, the formation of it can be optimal in terms of welfare, yet not feasible at equilibrium. This occurs, for example, between two β players if $b < c < 2b$.

between two α-players, $n_{\beta\beta}$ to be the number of links in s between two β-players, and $n_{\alpha\beta}$ to be the number of links in s between players choosing different actions (α, β).

If $2b < e + f < 2d$ then a link involving two players choosing α provides the highest payoff. The following result then follows:

Proposition 4 *Suppose $2b < e + f < 2d$. Then if $c < 2d$ a profile is efficient iff its network is complete and essential and all players choose α. If $c > 2d$ then an efficient profile has an empty network.*

The above class of games exhibit a severe form of "exploitation" in which the welfare of the anti-coordination game is highest off-equilibrium. The other two parameter configurations, $2b < 2d < e + f$ and $2d < 2b < e + f$ are more complicated and we now take them up. Since they are symmetric across actions, we focus here on the first case. To state the result, it is useful to introduce an auxiliary function g as follows:

$$
g(c) = \left\{
\begin{array}{l}
\lceil \frac{e+f-2d+\frac{d-b}{n}}{e+f-(d+b)} \rfloor \text{ if } c < 2b \\
\lceil \frac{e+f-2d+\frac{d-c/2}{n}}{e+f-d-c/2} \rfloor \text{ if } 2b < c < 2d \\
1 \text{ if } 2d < c,
\end{array}
\right.
$$

where $\lceil a \rfloor$ refers to the integer closest to a. It is straightforward to see that $g(c)$ is piece-wise linear and increasing.

Proposition 5 *Suppose (1) holds and in addition $2b < 2d < e + f$. Then the following statements hold: (i) If $c < 2b$ then a profile is efficient iff it is of type $\vec{\beta} \rightleftharpoons \vec{\alpha}$, and $n_{\beta}^* = g(c)\frac{n}{2}$. (ii) If $2b < c < 2d$ a profile is efficient iff it is of type $\beta \rightleftharpoons \vec{\alpha}$, and $n_{\beta}^* = g(c)\frac{n}{2}$. (iii) If $2d < c < e + f$ a profile is efficient iff it is of type $\beta \rightleftharpoons \alpha$, and $n_{\beta}^* = g(c)\frac{n}{2}$. (iv) If $e + f < c$ a profile is efficient iff its network is empty.*

This Proposition tells us that, as the linking cost increases, efficient networks become less connected going from the complete network to the empty network through two intermediary cases. Moreover, efficiency generally selects a unique relative size of the two parts. Especially, when the efficient network is bipartite, the efficient profile is perfectly balanced for all values of the parameters (i.e., $n_{\beta} = n_{\alpha} = \frac{n}{2}$). The reason is that, when the efficient network is bipartite, each link provides the same welfare contribution $e + f - c$. Therefore, in order to maximize welfare the number of links must be maximized, which is obtained when the two groups of players have the same size.

Figure 5: group sizes for an efficient profile.
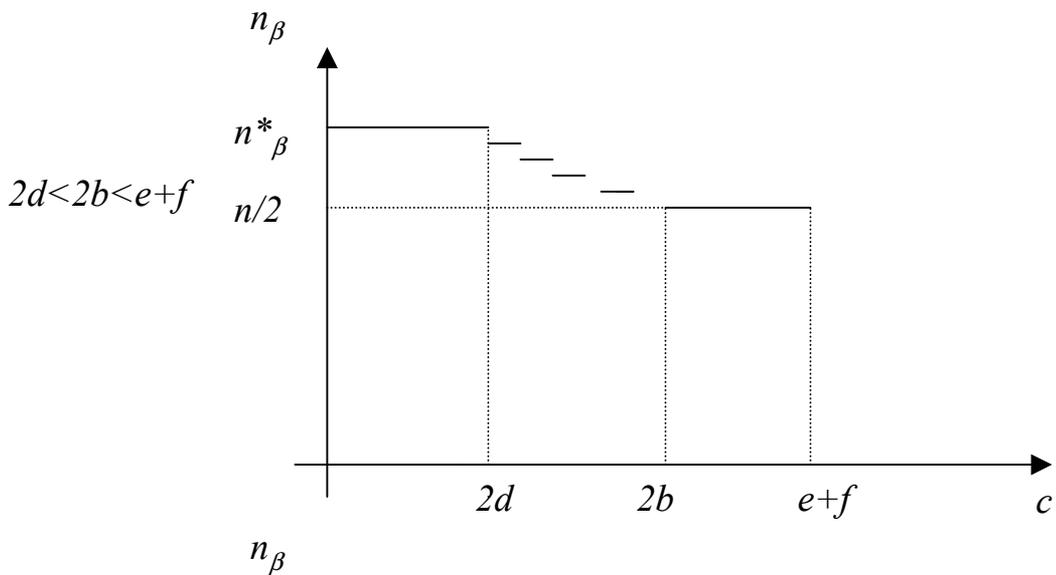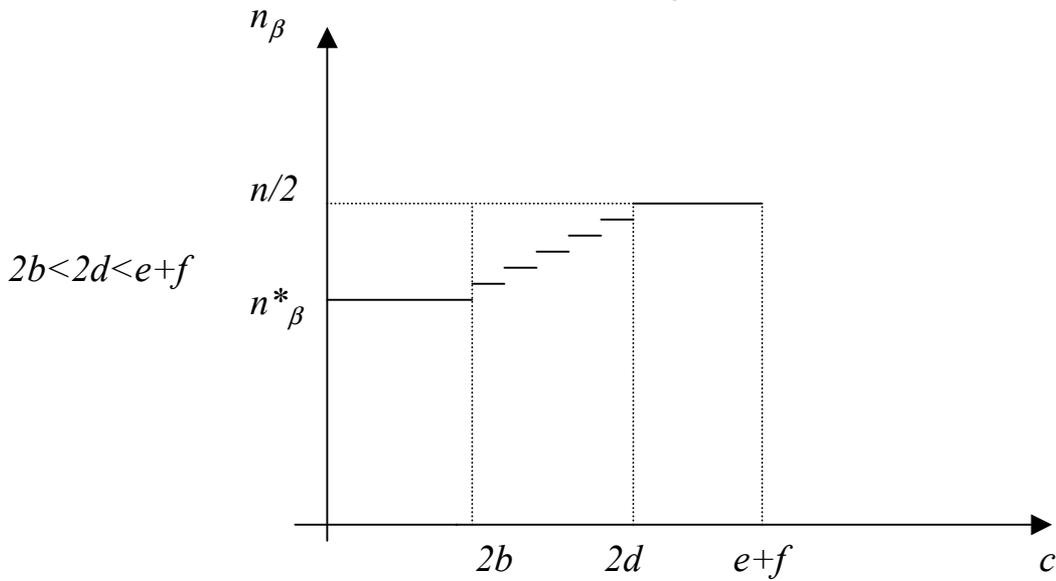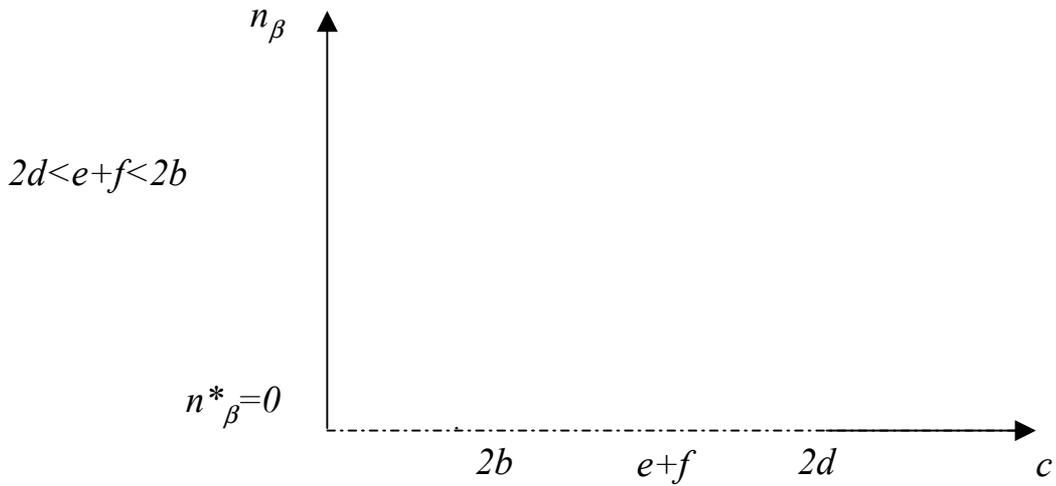
If we compare Propositions 4 and 5 with Propositions 1 and 2 we conclude that, in general, Nash profiles are not efficient and vice versa. The essential basis for this negative conclusion is twofold.

1. First, let us consider the effect of the cost of forming links in the tension between efficiency and equilibrium. Given that the cost is incurred only by the proposer, the formation of a link can be welfare improving, even if no agent wants to form it. For example, when $b < c < 2b$, β-players do not have an incentive to form links with themselves, even if this would increase the collective welfare. This problem could be somewhat alleviated under alternative assumptions on link formation. For instance, it could be supposed that link formation is two-sided and both players pay the same cost $c/2$ to establish a link. In this case, when $b < c < 2b$, two β-players would have an incentive to form a link. However, such a two-sided link formation also has its drawbacks. For example, under equal cost sharing, when $2e < c < f$, α-players do not want to form links with β-players, even though β-players would be willing to pay the entire cost of link formation under a one-sided rule. Therefore, in general, there are cases where a link would be formed under one-sided link formation but not under two-sided link formation, and vice versa. Apart from these considerations, the cost of link formation also has implications over the distribution of passive and active links sustained in equilibrium. As the cost increases the range of possible sizes in equilibrium extends. This is because when costs are high the positive externalities induced by passive links are higher. Nevertheless, passive and active links have no role in the welfare analysis. This is why there is just a single relative size of the two parts in efficient profiles.

2. A more primitive reason for the discrepancy between efficiency and equilibrium is the fact that actions in the anti-coordination game are typically asymmetric. To distinguish this most clearly from previous considerations, it is useful to consider a situation where the cost of link formation is low. Hence, assume that c is close to 0. Both equilibrium and efficient networks are complete. Equilibrium requires that $\frac{n_\beta}{n_\alpha} \approx \frac{f-d}{e-b}$ in every case. In contrast, efficiency requires that $\frac{n_\beta}{n_\alpha} = 0$ when $2b < e + f < 2d$ and that $\frac{n_\beta}{n_\alpha} \approx \frac{e+f-2d}{e+f-2b}$ otherwise. In the first case, efficiency and equilibrium requirements can never be reconciled, while in the other cases, efficiency and equilibrium are compatible only when $f - e$ is close to 0. Hence, when costs are low, equivalence between equilibrium and efficient profiles is only obtained for degenerated cases of anti-coordination games, in which actions are symmetric in equilibrium (i.e. $f - e \approx 0$). In general, therefore, collective welfare and individual incentives are not aligned.

5 Dynamics

The analysis of the static model shows that there is a wide range of outcomes that can arise in equilibrium. It is worth noting that these equilibria display very different numbers of players choosing the two different actions and therefore also have very different welfare properties – thus such a diversity is substantive. This leads us to examine the dynamic stability of different outcomes. In this section, we shall show that all (strict) equilibria of the static model are stochastically stable. In this sense, therefore, we may conclude that the whole range of equilibria identified in the static analysis of the model are, all of them, equally robust configurations.

We start with a preliminary result which shows that the best-response dynamic converges to one of the Nash equilibria whose main features are specified in Propositions 1 and 2.

Proposition 6 *Suppose (1) holds. The unperturbed dynamic process converges to a (strict) Nash equilibrium, with probability one.*

The proof of this result is given in the Appendix. We now proceed to the analysis of the perturbed dynamic. The above result allows us to restrict our attention to the set of strict Nash equilibria of the static game. Our analysis of stochastic stability is summarized in the following result.

Proposition 7 *Suppose (1) holds. Then all strict Nash equilibria are stochastically stable.*

Proof: From the previous proposition we know that the unperturbed dynamic converges to a strict Nash equilibrium. We will now show that all strict Nash equilibria are in the same recurrent set, i.e. for all $s, s' \in S^{**}$ there exists a path of one step mutations that leads from s to s' and vice versa. This will lead to the conclusion that all strict Nash equilibria are stochastically stable (see Samuelson, 1994).

To this effect, it is useful to define an equivalence relation $\sim$ in the following way: $s \sim s'$ if and only if one of them is obtained from the other just by a permutation of the indices of the nodes. It is easy to show that $\sim$ satisfies all the properties required for it to be an equivalence relation. This induces a partition of S^{**} in equivalence classes that we will denote by Ω. First, let us show that, for our purpose, it will be enough to prove the following two statements.

a) There exists an equivalence class $c^* \in \Omega$ satisfying that, any two of its states are connected by a path of one step mutations, i.e. $\forall s, s' \in c^*$ there exists a one-step mutation path going from s to s' and vice versa.

b) There exists a one-step mutation path connecting any two equivalence classes, i.e., $\forall c, c' \in \Omega$ and $\forall s \in c$ there exists a state $s' \in c'$ such that we can reach s' from s by a path of one step mutations and vice versa.

Assuming that a) and b) are true, we can now argue that there exists a path of one step mutations connecting any two strict Nash equilibria. The key property used in order to prove this is that, a composition of one-step mutation paths generates a one-step mutation path. Making use of this fact, we observe that property a) is satisfied by any other equivalence class $c' \in \Omega$. This is because, if we take $s, s' \in c'$ then, using b), there exist two states $\overline{s}, \overline{s}' \in c^*$ connected with s and s' respectively by their corresponding bidirectional one-step mutation paths. Moreover, using a) the states $\overline{s}$ and $\overline{s}'$ are also connected in both directions by a path of one step mutations and therefore, by composition of these paths, we are able to connect s and s'. In order to complete the proof, we have to show that there exists a path between any two states belonging to different equivalence classes. As before, this path can also be constructed by composition of two paths, one path connecting the two equivalence classes that exists due to b) and, the other one, connecting two states inside the corresponding equivalence class that exists due to the preceding argument.

First, to establish a), we shall rely on the following Lemma.

Lemma 1 *Given an equivalence class $c^* \in \Omega$ formed by distributive insensitive states then, for any two states belonging to c^*, we can reach one from the other by a one-step mutation path.*

The proof of Lemma 1 will be presented in the Appendix. Interestingly, notice that, the concept of distributive insensitiveness plays a crucial role for the proof of a).

To illustrate the arguments involved in the proof of b), we present in the text the proof for the first two parts of the classification on cases provided in Section 3 and provide the remaining four parts in the Appendix.

1: $c < b, d, e, f$. We have to show that, $\forall c, c' \in \Omega$ and $\forall s \in c$ there exists a state $s' \in c'$ such that we can reach s' from s by a path of one step mutations and vice versa. All strict Nash equilibria are complete and essential networks with a number of β-players satisfying the following: $(n-1)p_\beta < n_\beta < (n-1)p_\beta + 1$. Given $s \in c$, there exists a state $s' \in c'$ where the only difference between them is the distribution of passive and active links. Thus, there must exist two players $i, j \in N$ such that $g_{ij} = 1$ in s, whereas $g_{ij} = 0$ in s'. We know that s' is a complete graph, therefore $g_{ji} = 1$ in s'. Assume player i mutates and deletes her link with j. Then, with a positive probability, player j gets a revision opportunity and initiates the link with player i. This is due to the fact that, j's best response, with respect to the action (α or β), only depends on the number of players doing each action but not on her distribution of active and passive links. Then, we would reach another Nash equilibrium "closer" to s'. If we do the same for all the links that differ between s and s' we would reach s'. Therefore, by a sequence of one step mutations, we can go from s to s'. Analogously, we can reach s from s' with a path of one step mutations.

2: $b < c < d, e, f$. Consider $c, c' \in \Omega$ and $s \in c$. We will first make the assumption that the proportion of players doing each action in c and c' coincides, that is, $n_\beta(c) = n_\beta(c') = \overline{n}_\beta$. We want to show that we can go from s to a state $s' \in c'$. Recall, from Proposition 2, that β-players' best response, with respect to the action, is independent of the distribution of active and passive links. On the other hand, α-players' best response depends on the number of active links they have with β-players but not on the number of active links they have with other α-players. More precisely, given a number n_β of β-players n_β, there is a maximum number of active links that an α-player can sustain with β-players in equilibrium. Let this number be denoted by $q_{n_\beta}^{\max}$.

Consider $s' \in c'$ such that it differs with s only in the distribution of active and passive links.[9] There exist two players $i, j \in N$ such that $g_{ij} = 1$ in s, but $g_{ij} = 0$ in s' (indeed it has to be the case that $g_{ji} = 1$ in s'). If i and j are both α-players the process is straightforward. If we want to go from s to s' we do the following. With positive probability i mutates and deletes her link with player j. By best response, player j forms the link with i. A similar argument can be made for the reverse transition, from s' to s. Specifically, with positive probability j deletes her link with i and then by best response i forms the link back with j. This is a Nash equilibrium because any possible division of passive and active links between α-players is sustained in equilibrium.

What would happen if $i \in N_\alpha$ and $j \in N_\beta$? If we want to construct a path from s to s', we do the following. By mutation, player i deletes her link with j. Then, if j gets an opportunity of revising her strategy, she will form the link with i. This is due to the fact that, in equilibrium, a β-player can be sustaining all the links with α-players.

Assume now that, $i \in N_\beta$ and $j \in N_\alpha$, then the argument is slightly more complicated. If the number of active links of player j is less than $q_{\overline{n}_\beta}^{\max}$, i.e., the maximum number of active links with β-players allowed in order for an α-player to be doing a best response, then we can reason as before. That is, player i deletes her link with j by mutation and, by best response, j forms the link back with i. If, on the other hand, $q_j^\beta = q_{\overline{n}_\beta}^{\max}$ we cannot use the same sequence of mutations and best responses as before because, if by mutation, player i deletes her link with j, forming the link back with i is no longer player j's best response. By assumption, we know that s' is also a Nash equilibrium and therefore $q_j^{s',\beta} \leq q_{\overline{n}_\beta}^{\max}$. This tells us that there must exist $l \in N_\beta$ such that $g_{jl} = 1$ in s and $g_{jl} = 0$ in s'. If this were not the case, then $q_j^{s',\beta} > q_{\overline{n}_\beta}^{\max}$ because j would have in s' all the active links that she has in s plus the one with i. This would contradict the assumption that s' is an equilibrium. Now, let us describe a path of positive probability that leads from s to s'. First, j deletes her link with l. By best response, l would form the link with j. This

_____

[9] In fact, what we are saying is that the indices of the players doing each of the actions coincide for s and s'.

would leads us to a state that we will denote by $\widetilde{s}$, which is also a strict Nash equilibrium and such that $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$. We can now conclude the argument. By mutation, player i deletes her link with j. Then, j's best response is to form the link back with i because $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$.

We can do this with all the links that differ (with respect to the direction) between states s and s'. Therefore, with a process based on one step mutations, we can go from s to s'. The reverse path can be calculated in an analogous way.

We want to see that this is also true for classes whose states differ in the number of players doing each action, that is, $n_\beta(c) \neq n_\beta(c')$. First, we consider $c, c' \in \Omega$ such that $n_\beta(c) < n_\beta(c')$ and $s \in c$. We want to show that, there exists a state $s' \in c'$ such that, by a sequence of one step mutations, we can go from s to s'. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Let us assume that s' has the same indices of players choosing each action than s, except for a given player i who is doing α in s and β in s'.

By mutation, i in state s switches to β, deletes her links with β-players and maintains all her links with the α-players. Then, by best response, the remaining β-players delete the possible links they had with player i. They will not switch to α because their best response does not depend on the distribution of active and passive links, it simply depends on the proportion of players doing each action and our assumption indicates that $n_\beta^s + 1$ is sustainable as a Nash equilibrium. Note that, α-players are also doing a best response because there is now one more player choosing β, therefore their incentives to switch actions has diminished. This state will be denoted by $\widetilde{s}$ and has the property that the players doing each of the actions coincide with the ones in s'. Now, using what we have already proved, by a process of one step mutations we can go from $\widetilde{s}$ to s'.

To conclude, we consider that the opposite holds, i.e., $n_\beta(c) > n_\beta(c')$. More precisely, we assume $n_\beta(c) = n_\beta(c') + 1$. Take $s' \in c'$ such that, the indices of players choosing each action coincides with s except for a given player i who is doing β in s and α in s'. Given state s, by a sequence of one step mutations, we can reach a strict Nash equilibrium $\widehat{s}$ in which all β-players are actively linked with all α-players. This is due to the fact that, in equilibrium, β players are choosing a best response independently of the distribution of active and passive links. Now, we will show that we can reach s'. Player i, by mutation, chooses α and deletes all her links. Then, by best response, all β-players get a revision opportunity and form links with i. This new state has one more player choosing α and all links between α and β- players are formed actively by the β-players. Hence, α-players are in the most favorable situation with respect to the direction of the links. Moreover, given that n_β^s is sustainable in equilibrium, this new state is an equilibrium we will denote it by $\overline{s}$. Notice that, the set of players choosing each action in coincides for $\overline{s}$ and s'. Finally, using the preceding argument, via a process of one step mutations, we can appropriately change the direction of some links and reach s'. Analogously, we can show that there exists a path going in the opposite direction, i.e., leading from s' to s. $\qquad\square$

6 Summary

In this paper, we study a model of social interaction between partner choice and individual behavior in anti-coordination games i.e., games where choosing dissimilar actions is individually optimal. In our context, two players interact only if at least one of them has invested in establishing a costly pair-wise link, i.e. links are one-sided. As the linking cost varies, we find that there is a wide range of possible (strict) Nash networks structures: complete graphs, semi-bipartite graphs, bipartite graphs and empty graphs. The relative numbers of individuals doing each action depends crucially on the cost of forming links. More specifically we observe that, for low costs, the only stable network is complete, with the proportion of individuals doing each action coinciding with the mixed strategy equilibrium proportions of the anti-coordination game. As the cost of link formation increases, a wider set of relative proportions become sustainable in equilibrium, some of them representing very asymmetric bipartite graphs. This effect arises due to the trade-off faced by any player between the advantages of cheap passive links and the gains from being on the shorter side of the population. In addition, the payoffs in an anti-coordination game are such that players have an incentive to be on the short side of the 'market' even if aggregate welfare is enhanced when all players choose the same action. This strategic conflict is a second source of inefficiency. These two considerations imply that efficiency and equilibrium requirements typically conflict in anti-coordination games. Finally, we show that these features of equilibrium outcomes are robust with respect to the dynamics: all strict Nash equilibria of the static game are stochastically stable.

The importance of the cost of forming links in this analysis suggests the interest of exploring alternative formulations. We could consider the case where a link is formed only if both players make an investment in the link. A preliminary analysis of this model suggests that a much larger class of network structures can arise in equilibrium than the ones derived with one-sided link formation.

It is also left for further research to analyze the concept of *distribution insensitive* configurations in more general settings. This notion reflects a stronger criterion of stability that provides a selection with respect to the standard analysis. We feel that *distribution insensitiveness* could be seen as one way of introducing 'two-sided considerations' in a one-sided link formation model.

7 Appendix

Proof of Proposition 2

We proceed by successive examination of all the possible domains. For each domain, the first step is to derive the two strict best-response equations, one for the α-players, denoted by BRα, and one for the β-players, denoted by BRβ. The second step is to understand how the best response equations allow one to compute the lower and upper bounds on n_β. In general, BRβ leads to the upper bound, whereas BRα leads to the lower bound. The reason is intuitive:

for anti-coordination games, the higher the number of people playing β, the lower the utility of playing β compared to the utility of playing α. Hence, when β players are too numerous, BRβ will not hold.

1: $c < b, d, e, f$. Nash networks are complete.

$$BR\alpha \Leftrightarrow (n_\alpha - 1)d + n_\beta e - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + n_\beta b - c(q_i^\alpha + q_i^\beta)$$

The left term of the inequality is the utility obtained by an agent playing α. The right term is the utility that an agent playing α would obtain if he changed to β. Through elementary algebraic manipulations, we obtain

$$
\begin{aligned}
BR\alpha &\Leftrightarrow n_\beta(e - b) > (n_\alpha - 1)(f - d) \\
BR\alpha &\Leftrightarrow n_\beta(e - b + f - d) > (n - 1)(f - d)
\end{aligned}
$$

Similarly, we show that BRβ is given as follows:

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e - b + f - d) < (n - 1)(f - d) + (e - b + f - d).$$

2: $b < c < d, e, f$. Equivalently, Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$. The α-players are linked (actively or passively) with every other agent. Thus, they obtain e with β players and d with all α players, while they have to pay for all the links they support. Hence, the utility of an α player is $n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta)$. If she changed to β, she would sever her active links with β players, but keep her active links with α players. She would still be linked (actively or passively) with all the α players, but would now only be passively linked with β players. The number of passive links she has with β players is equal to n_β minus the number of active links she has with them. Therefore,

$$BR\alpha \Leftrightarrow n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + (n_\beta - q_i^\beta)b - cq_i^\alpha$$

which yields us:

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(e - b + f - d) > (n - 1)(f - d) + q_i^\beta(c - b)$$

Similarly, we can show that

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e - c + f - d) < (n - 1)(f - d) + (e - c + f - d)$$

Hence, BRβ directly gives the upper bound for n_β. To find the lower bound for n_β, first note that the lowest possible value of n_β is equal to $(n - 1)p_\beta$ and that it is attained for a state such that $\forall i \in N_\alpha, q_i^\beta = 0$. Second, let us show that this state indeed leads to the lower bound. By definition, this state satisfies BRα. This state satisfies BR β iff

$$(n - 1)\frac{f - d}{f - d + e - b} < (n - 1)\frac{f - d}{f - d + e - c} + 1$$

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Since $b < c$, we have $e - b > e - c$ and $\frac{f-d}{f-d+e-b} < \frac{f-d}{f-d+e-c}$. Thus, the state leading to the lowest possible lower bound is a strict Nash equilibrium.

3: $d < c < b, e, f$ and Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$.
By symmetry, we can apply the previous result to n_α by exchanging f with e and d with b. This leads to

$$(n-1)p_\alpha < n_\alpha < (n-1)\frac{e-b}{e-b+f-c} + 1$$

Since $n_\beta = n - n_\alpha$, we obtain that, in this case, there exists a strict Nash equilibrium iff

$$(n-1)\frac{f-c}{f-c+e-b} < n_\beta < (n-1)p_\beta + 1$$

4: $b, d < c < e, f$. In this case, Nash networks are of the type $\beta \rightleftharpoons \alpha$.
As in part 2, the proof for the upper bound unfolds in three steps. First, as usual, we derive the best-response equations for α and β. After simplification, both equations depend on the number of active links of the agent. Second, we use BRβ to show that the highest possible upper bound for n_β is obtained for a state where no β has an active link (all the active links are supported by α). Third, we show that this state satisfies BRα, hence is a valid Nash equilibrium, and thus leads to the actual upper bound for n_β. The lower bound of n_β can be computed in a similar fashion.

Finally, it remains to check that all the values between these two bounds can be sustained as a Nash equilibrium. To show this, we prove that the ranges of values of n_β that sustain the two most asymmetric states overlap. This means that any n_β between the two bounds can sustain one of these two states, which completes the proof.

First, let us derive the best-response equations. The α-players are linked (actively or passively) with all the β-players. Hence, they earn e times the number of β-players, while they have to pay for the active links they support. If they changed to β, they would sever their active links with β players and form active links with all the α-players. They would still be passively linked with some β-players. These considerations yield:

$$BR\alpha \Leftrightarrow n_\beta(e-b+f-c) > (n-1)(f-c) + q_i^\beta(c-b)$$

Similarly, it can shown that,

$$BR\beta \Leftrightarrow n_\beta(f-d+e-c) < (n-1)(f-d) + f - d + e - c - q_i^\alpha(c-d)$$

Hence, BRβ shows that the highest possible upper bound for n_β is equal to $\bar{n}_\beta = (n-1)\frac{(f-d)}{(f-d+e-c)} + 1$ and it is obtained for the state such that $\forall i \in N_\beta, q_i^\alpha = 0$.

In this state, the agents playing α support all the links, hence $\forall i \in N_\alpha, q_i^\beta = n_\beta$. Therefore, this state satisfies BRα iff

$$(n-1)\frac{(f-d)}{(f-d+e-c)} + 1 > (n-1)\frac{(f-c)}{f-c+e-c}$$

which is satisfied.

Similarly, the lowest possible lower bound is equal to $\check{n}_\beta = (n-1)\frac{(f-c)}{(f-c+e-b)}$ and it is obtained for the state such that the agents playing β support all the links, hence $\forall i \in N_\beta, q_i^\alpha = n_\alpha$. This state satisfies BRα by construction, and satisfies BRβ iff

$$(n-1)\frac{(f-c)}{(f-c+e-b)} < (n-1)\frac{(f-c)}{f-c+e-c} + 1$$

which is satisfied.

5: $b, e < c < f, d$ and the network is of type $\beta \to \vec{\alpha}$. Standard considerations suggest that

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(e - b + f - d) > (n-1)(f-d)$$

and

$$BR\beta \Leftrightarrow n_\alpha(f-c) > n_\alpha(d-c)$$

hence $BR\beta$ is always satisfied if $n_\alpha \neq 0$.

6: $b, d, e < c < f$. In this case, Nash networks are of type $\beta \to \alpha$. It follows that

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(f - c + e - b) > (n-1)(f-c)$$

and

$$BR\beta \quad \Leftrightarrow \quad n_\alpha(f-c) > 0$$
$$BR\beta \quad \Leftrightarrow \quad n > n_\beta$$

$\square$

Proof of Proposition 3:

1: $c < b, d, e, f$. Directly from Proposition 2's proof we observe that a player's best response does not depend on her distribution of active and passive links. This indicates that all Nash equilibria are structure insensitive.

2: $b < c < d, e, f$. Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$. As shown in Proposition 2's proof an α- player with q_i^β active links with β-players and q_i^α active links with α-players is doing a best response if and only if,

$$n_\beta > (n-1)\frac{(f-d)}{(e-b+f-d)} + q_i^\beta \frac{(c-b)}{(e-b+f-d)}$$

Note that the RHS of the above expression does not depend on q_i^α and it is increasing in q_i^β Therefore it reaches a maximum at $q_i^\beta = n_\beta$. Substituting n_β for q_i^β we obtain the following condition:

$$n_\beta > (n-1)\frac{(f-d)}{(f-d+e-c)}$$

which is necessary and sufficient for distribution insensitive to apply to the α-players. Considering now the counterpart condition for a β-player, note that, in order for this player to be playing a best response, a necessary and sufficient condition is:

$$n_\beta < (n-1)\frac{(f-d)}{(f-d+e-c)} + 1$$

BRβ does not depend on the number of active links with the α-players. Hence, this condition is again a necessary and sufficient condition for distribution insensitivity, but now it concerns any β-player. Combining these two conditions we obtain the desired result.

3: $d < c < b, e, f$. Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$. By symmetry, we can apply the previous result to n_α by exchanging f with e and d with b. This leads to the following condition:

$$(n-1)\frac{(e-b)}{(e-b+f-c)} < n_\alpha < (n-1)\frac{(e-b)}{(e-b+f-c)} + 1$$

Since $n_\alpha = n - n_\beta$, we can substitute in the above expression and calculate the necessary and sufficient conditions for distributive insensitivity in terms of n_β, which is the following:

$$(n-1)\frac{(f-c)}{(f-c+e-b)} < n_\beta < (n-1)\frac{(f-c)}{(f-c+e-b)} + 1$$

4: $b, d < c < e, f$. This case has been described precisely in the paper.

5: $b, e < c < f, d$ and the network is of type $\beta \rightarrow \vec{\alpha}$. In this part, all n_β sustained as a Nash equilibrium will also be structure insensitive. Notice that, the best response of a player choosing α does not depend on the distribution of active and passive with other α-players.

6: $b, d, e < c < f$. In this part, Nash networks are of type $\beta \to \alpha$ therefore the direction of all the links is already determined and therefore distribution insensitivity is not an issue. $\square$

Proof of Proposition 4:

The welfare of a complete and essential graph with every agent choosing α is $\binom{n}{2}(2d - c)$. Any other possible profile would provide a lower welfare because $\binom{n}{2}(2d-c) \geq n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2d-c) + n_{\alpha\beta}(2d-c) \geq n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2b-c) + n_{\alpha\beta}(f + e - c)$ given that $n_{\alpha\alpha} + n_{\beta\beta} + n_{\alpha\beta} \leq \binom{n}{2}$. Thus, if $c < 2d$ the efficient profile is a complete and essential graph of agents choosing α. $\square$

Proof of Proposition 5:

(i) If $c < 2b$ then all links are profitable and therefore the efficient network must be complete. In order to obtain the size of n_β that maximizes the welfare we must work out the following maximization problem:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d - c) + n_{\beta\beta}(2b - c) + n_{\alpha\beta}(f + e - c).$$

Taking into account that in a complete and essential network

$$n_{\alpha\alpha} = \binom{n - n_\beta}{2} = \frac{(n - n_\beta)(n - n_\beta - 1)}{2},$$

$$n_{\beta\beta} = \binom{n_\beta}{2} = \frac{(n_\beta)(n_\beta - 1)}{2},$$

$$n_{\alpha\beta} = (n_\beta)(n - n_\beta).$$

the above expression reaches the maximum in,

$$n_\beta^* = \lceil \frac{e + f - 2d + \frac{d-b}{n}}{e + f - (d + b)} \rceil \frac{n}{2} = g(c)\frac{n}{2}.$$

(ii) If $2b < c < 2d$ then the links between two players choosing β are not profitable, which implies that $n_{\beta\beta} = 0$. Apart from these links all other links will be formed. The maximization problem we have to solve is the following:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d - c) + n_{\alpha\beta}(f + e - c),$$

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It is easily seen that:

$$n_{\alpha\alpha} = \binom{n - n_\beta}{2} = \frac{(n - n_\beta)(n - n_\beta - 1)}{2},$$

and

$$n_{\alpha\beta} = (n_\beta)(n - n_\beta).$$

the solution of this maximization problem is:

$$n_\beta^* = \lceil \frac{e + f - 2d + \frac{d - c/2}{n}}{e + f - d - c/2} \rceil \frac{n}{2} = g(c)\frac{n}{2},$$

where $g(c)$ is piece-wise linear and increasing.

(iii) If $2d < c < e + f$ then the only links that will be profitable are the ones between players choosing different actions. Thus $n_{\alpha\alpha} = n_{\beta\beta} = 0$ and .the maximization problem we have to solve is the following:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\beta}(f + e - c),$$

It is easily seen that

$$n_{\alpha\beta} = (n_\beta)(n - n_\beta).$$

The solution of this problem is simply

$$n_\beta^* = n/2.$$

$\square$

Proof of Proposition 6: It is sufficient to show that from any initial state s_0, there is a positive probability of reaching a strict Nash equilibrium. We have to study 6 different cases depending on the relation between the cost c and the parameters from the payoff table of the anti-coordination game (that is, depending on the type of network that arises in equilibrium).

1: $c < f, d, e, b$. Given a state s_0 we have to show that with a positive probability we can reach a strict Nash equilibrium. Consider the following process. One after the other, individuals have the opportunity of revising their links (that is, one individual per period.) All links will be formed because the linking cost is lower than all possible payoffs of the anti-coordination game. This will leads us to a complete and essential network $(\vec{\beta} \rightleftharpoons \vec{\alpha})$ that we denote by s_1. If the proportion of players doing each action coincides with the one required in

equilibrium, i.e., $(n-1)p_\beta < n_\beta^{s_1} < (n-1)p_\beta+1$, then s_1 would be a strict Nash equilibrium and therefore the proof would be completed. Assume that this is not the case, that is $n_\beta^{s_1} < (n-1)p_\beta$.[10] In this network there are more α- players than what the equilibrium prescribes. Thus, we deduce that α-players are not choosing a best response. With a positive probability a player choosing α gets a revision opportunity. She would maintain her links and switch to action β. This would leads us to a complete and essential state s_2 with one more player choosing α . If we still have $n_\beta^{s_2} < (n-1)p_\beta$, then players choosing α are still not doing a best response. As described before, there is a positive probability that one of them switches to action β. After a finite number of periods we would reach a state s_k satisfying, $(n-1)p_\beta < n_\beta^{s_k} < (n-1)p_\beta+1$ and therefore we would reach a Nash equilibrium.

2: $b < c < f,d,e$. Consider an initial state s_0. One after the other, individuals have the opportunity of revising their links. All links will be formed except those between two β-players. This would leads us to a state that we will denote by s_1 in which the network is of the type $\beta \rightleftharpoons \vec{\alpha}$. If s_1 is an equilibrium this would complete the proof. If this is not the part, then there is at least one individual who is not doing a best response. Recall from Proposition 3 that there exists a number of β-players, n_β^*, which is distribution insensitive. We are assuming that s_1 is not an equilibrium, therefore $n_\beta^* \neq n_\beta^{s_1}$. Assume that $n_\beta^{s_1} < n_\beta^*$.[11] There are less players choosing β than what distribution insensitive prescribes, thus, all β-players would be choosing a best response. Since s_1 is not an equilibrium, at least one α-player is not choosing a best response. Consider a player i choosing α who is not doing a best response. If i gets a revision opportunity, she would switch to action β and delete the possible links she might have with β-players. Then, all players in N_β would get a revision opportunity and they would delete their links with i. This leads us to a network with the structure $\beta \rightleftharpoons \vec{\alpha}$ that we will denote as s_2. Notice that s_2 has one more β-player than s_1, i.e., $n_\beta^{s_2} = n_\beta^{s_1} + 1$. If s_2 is an equilibrium, then the proof would be completed. If it is not an equilibrium, then $n_\beta^{s_2} < n_\beta^*$. Using the same argument described above, we can construct a positive probability path that leads us to a Nash equilibrium. This is due to the fact that, after a finite number of steps, we would reach a state s_k such that $n_\beta^{s_k} = n_\beta^*$ and we know that this state is an equilibrium, no matter how passive and active links are distributed among players. □

3: $d < c < f,b,e$. It is analogous to the proof of part 2. We simply have to exchange the roles of n_β, d and f by n_α, b and e, respectively.

4: $d,b < c < e,f$. Consider an initial state s_0. All players get a revision opportunity, one at a time, just over their links. This will leads us to a state s_1 which is a complete and essential bipartite graph ($\beta \rightleftharpoons \alpha$). Recall, from

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[10]If, on the contrary, $n_\beta^{s_1} > (n-1)p_\beta+1$ the proof would be analogous.

[11]If $n_\beta^{s_1} > n_\beta^*$ the proof goes along the same lines.

Proposition 3, that there is a particular value n_β^* that is distribution insensitive. In other words, any state with that particular proportion of players doing each action, satisfies that all possible distributions of active and passive links between α and β-players determine an equilibrium network. This indicates that, if s_1 is such that $n_\beta^{s_1} < n_\beta^*$ the incentives for a β-player to switch to action α diminishes Therefore, if s_1 is not an equilibrium, then a player choosing α is not doing a best response. This is because the β-players are doing a best response independently of the distribution of active and passive links. If, by contrary, $n_\beta^{s_1} > n_\beta^*$ and s_1 is not an equilibrium, then a player choosing β is not doing a best response. Let us suppose that we are in a state s which satisfies the inequality $n_\beta^{s_1} < n_\beta^*$ (if the reversed inequality holds the proof would be analogous). We want to prove that with positive probability (using the unperturbed best response dynamics) we can reach a strict Nash equilibrium. If s_1 is an equilibrium, the proof would be completed. If it is not, at least one player choosing α, say i, is not doing a best response. With positive probability, i gets an opportunity of revising her strategy. If this is the part, she would switch to action β, delete all her links with the β-players and form links with the α-player. If then, all β-players get a revision opportunity, they would delete the possible links they might have with i. This new state, that we will denote as s_2 has one more β-player i.e. $n_\beta^{s_2} = n_\beta^{s_1} + 1$. If s_2 is an equilibrium the proof would be completed. If it is not an equilibrium then $n_\beta^{s_2} < n_\beta^*$. Using the same process described previously, we can construct a positive probability path that leads us to a Nash equilibrium. This is due to the fact that after a finite number of steps we would reach a state s_k satisfying $n_\beta^{s_k} = n_\beta^*$ and we know that this will be an equilibrium, no matter how passive and active links are distributed among players.

5: $b, e < c < d, f$. Consider an initial state s_0. All players, one at each time, get a revision opportunity over their links. This would leads us to a semi-bipartite graph $(\beta \to \vec{\alpha})$. Denote this state by s_1. Recall from Propositions 2 and 3, that s_1 will be a strict Nash equilibrium if and only if $(n-1)p_\beta < n_\beta^{s_1} < n$. If s_1 satisfies this inequality, then s_1 would be a Nash equilibrium and therefore the proof would be completed. If, on the contrary, s_1 is not an equilibrium then $n_\beta^{s_1} < (n-1)p_\beta$. In this part, α-player are not choosing a best response. With a positive probability, one of them, say i, gets a revision opportunity and therefore switches to action β. Then, if all α-player get a revision opportunity, just over their links, they would delete the possible links they had with i. Also, β-players would delete their links with i. Then, if i gets a new revision opportunity, she will form the links with the α-player. This process leads us to a directed semi-bipartite graph that we will denote by s_2. This state has one more β-player i.e. $n_\beta^{s_2} = n_\beta^{s_1} + 1$. If $n_\beta^{s_2} < (n-1)p_\beta$ then we would repeat the process. After a finite number of steps, we would reach a state s_k such that the inequality is reversed, i.e. $n_\beta^{s_k} > (n-1)p_\beta$. Therefore, we would have reached a Nash equilibrium.

6: $b, d, e < c < f$. Consider an initial state s_0. All players get a revision opportunity, one at a time, just over links. This would leads us to a state s_1 which is a complete and essential bipartite graph $(\beta \to \alpha)$. Notice that, from

Proposition 2, s_1 would be an equilibrium if and only if $\frac{(n-1)(f-c)}{(f-c)+(e-b)} < n_\beta^{s_1} < n$. If $n_\beta^{s_1}$ satisfies the inequality, then s_1 would be an equilibrium, and therefore we would have finished with the proof. If not, i.e., $n_\beta^{s_1} < \frac{(n-1)(f-c)}{(f-c)+(e-b)}$ all α-player are not choosing a best response. Thus, with positive probability one of them, say i, gets the opportunity of revising. In such a case, she would switch to action β and would form all the links with α-player. If then, all β-players get a revision opportunity, they would delete their links with player i. This would leads us to a state s_2 that has one more β-player. If s_2 is not an equilibrium, then α-player are still not doing a best response. We then repeat the process as described above. We know that, after a finite number of steps, we would reach a state s_k such that $\frac{(n-1)(f-c)}{(f-c)+(e-b)} < n_\beta^{s_k} < n$, and therefore we would reach a Nash equilibrium.$\square$

Proof of Lemma 1:
Consider $c^* \in \Omega$ an equivalence class composed by distribution insensitive states. That is, the number of β-players coincides with one of the distribution insensitive number of β-players i.e. n_β^*.

First, we will show how to go from one state to another of the equivalence class c^* when the difference between states just relies on permutation of indices between players choosing the same action. For instance, consider two states $s_1, s_2 \in c^*$ with the only difference between them being that strategies for player 1 and 2 are permuted but both players are choosing action α. Consider state s_1, player 1, by mutation, imitates precisely the strategy done by player 2.[12] Then, by best response, and because we are considering a distribution insensitive state, all other players will form or delete their links ,depending on the case, with player 1. After this, we would reach a strict Nash equilibrium where player 1 will have the exact same strategy as player 2 had previously. Even though the structure of the network might have changed, given that we are in a distribution insensitive state, we still have a Nash equilibrium. Analogously, player 2 can imitate the original strategy of player 1 and end up having the exact same strategy as player 1 had in state s_1. This will leads us to a state in the same equivalence set but in which there has been a permutation between the strategies of players 1 and 2. This is precisely the state we have denoted by s_2.

To finish, we would like to consider a case in which the players for which we want to permute strategies are doing different actions. Say, for instance, that player 1 is doing α and player 2 is doing β. Consider two states $s_1, s_2 \in c^*$ with the only difference between them being that the strategies for players 1 and 2 are permuted. In contrast with the previous part, players 1 and 2 are choosing different actions. First of all, we have to distinguish between the following cases:

    i) n_β^* is in the upper bound obtained for the strict Nash equilibrium,
    ii) n_β^* is in the lower bound obtained for the strict Nash equilibrium, and
    iii) n_β^* is in the interior.

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[12] The imitation is only relevant with respect to the link formation because the action by assumption is already the same for both players.

Let us construct the proof just for case i).[13] Notice that, for this case, Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$.

Given state s_1, by a path of one step mutations, we can reach a state in which all β-players are forming the links. This will be a strict Nash equilibrium because we are in a distribution insensitive state. Then, by mutation, player 2 switches to action α. Then, by best response, given that now there is one more α-player, all β-players form the link back with player 2. Note that α-players will be doing a best response because all the links are passive links for them and given that n_β^* is in the upper bound obtained for the strict Nash equilibrium, we will still be in the range in which the number of β-players can be sustained in equilibrium. Therefore, we would have reached another strict Nash equilibrium. Then, player 1, by mutation, switches to action β and forms actively all the links with the α-players. By best response, the α-players will delete the possible active links they have with player 1. This will also be a distribution insensitive state that will be denoted by $\widetilde{s}$. Although $\widetilde{s}$ might not be in c^* it is easy to show that, by a path of one step mutation on the direction of the links, we can reach a state $\widehat{s} \in c^*$. Notice that if $\widehat{s} \neq s_2$ it must be because of permutations in the indices of nodes that are choosing the same action. To conclude, using what we showed at the beginning of the proof, we can con reach s_2 from $\widehat{s}$ by a path of one step mutations. $\qquad\square$

Proof of Proposition 7:
Proof of part b)
The first two parts have been proved in the paper. Here we present the proofs of the remaining parts.

3: $d < c < b, e, f$. The proof is analogous to part 2. We simply have to exchange the roles of n_β, d and f by n_α, b and e, respectively.

4: $d, b < c < e, f$. Recall from Proposition 3 that there is a particular value n_β^* that is distribution insensitive. This indicates that, if we consider a state s such that $n_\beta^s < n_\beta^*$, then β-players can be incurring in the cost of all their links, whereas α-player have a maximum number of active links they can support in equilibrium. On the other hand, if we consider a state s such that $n_\beta^s > n_\beta^*$, then α-player are the ones that can be incurring in the cost of all their links in equilibrium, whereas β-players have a maximum number of active links they can support in equilibrium.

Taking this into account, let us consider two equivalence classes $c, c' \in \Omega$ and $s \in c$, we want to show that, there exists a state $s' \in c'$ such that there exist a one-step mutation path connecting s with s'. Let us assume that $n_\beta(c) = n_\beta(c') = \overline{n}_\beta < n_\beta^*$. [14]

Consider $s' \in c'$ such that the indices of nodes choosing each action coincides with s. Hence, the differences between s and s' must be in the distribution of

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[13] The remaining cases can be proved in an analogous way.
[14] The proof when $\overline{n}_\beta \geq n_\beta^*$ goes along the same lines.

active and passive links. Then, there exist players $i, j \in N$ such that $g_{ij} = 1$ in s, but $g_{ij} = 0$ in s' (indeed it has to be the case that $g_{ji} = 1$ in s'). Suppose that $i \in N_\alpha$ and $j \in N_\beta$. If we want to go from s to s', we do the following. By mutation, player i deletes her link with j. Then if j gets an opportunity of revising her strategy she will form the link with i. This is due to the fact that, in equilibrium, a β-player can be sustaining all the links with α-players.

Assume, by contrary that, $i \in N_\beta$ and $j \in N_\alpha$, then the argument is more subtle. If the number of active links of player j is less than $q_{\overline{n}_\beta}^{\max}$, i.e., the maximum number of active links allowed in order for an α-player to be doing a best response, then we can reason as before. That is, player i deletes her link with j by mutation, and by best response, j forms the link back with i. If, on the other hand, $q_j^\beta = q_{\overline{n}_\beta}^{\max}$ we cannot use the same sequence of mutations and best responses as before, because if, by mutation, player i deletes her link with j, forming the link back with i is no longer player j's best response. By assumption, we know that s' is also a Nash equilibrium and therefore $q_j^{s',\beta} \leq q_{\overline{n}_\beta}^{\max}$. This tells us that there must exist $l \in N_\beta$ such that $g_{jl} = 1$ in s and $g_{jl} = 0$ in s'. If this were not the case, then $q_j^{s',\beta} > q_{\overline{n}_\beta}^{\max}$ because j would have in s' all the active links that she has in s plus the one with i. This would contradict the assumption that s' is an equilibrium. Now, let us describe a path of positive probability that leads from s to s'. First, j deletes her link with l. By best response, l would form the link with j. This would leads us to a state that we will denote by $\widetilde{s}$, which is also a strict Nash equilibrium and such that $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$. We can now conclude the argument. By mutation, player i deletes her link with j. Then, j's best response is to form the link back with i because $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$.

We can do this with all the links that differ (in the sense of the direction of the link) between states s and s'. Therefore, with a process based on one step mutations, we can go from s to s'.

Now, we assume that the number of players choosing each action in c and c' does not coincide. For instance, assume that $n_\beta(c) < n_\beta(c')$. We want to show that we can still find a path of one step mutations going from s to a state $s' \in c'$. Also suppose that, $n_\beta(c') < n_\beta^*$.[15] Without loss of generality, we can assume that $n_\beta(c') = n_\beta(c) + 1$. Consider s' such that the indices of nodes choosing different actions coincides with s.except for a player i that is doing β in s' and α in s. Given s, by mutation, player i switches to action β, deletes all her links with β-players and forms links with all α-players. Then, by best response, all β-players delete links with i. This leads us to a Nash equilibrium, that we will denote by $\overline{s}$, satisfying that $n_\beta^{\overline{s}} = n_\beta(c')$. It is a Nash equilibrium because, now the number of β-players is larger and therefore, α-players have less incentives to mutate than before. Also, β-players are doing a best response because $n_\beta^{\overline{s}} \leq n_\beta^*$. As we have previously proved, we can go from $\overline{s}$ to $s'$by a process of one step mutations.

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[15] If $n_\beta^* < n_\beta(c')$ the proof would be analogous.

Let us assume by contrary that $n_\beta(c') < n_\beta(c)$. Also suppose that $n_\beta(c) < n_\beta^*.$[16] First, by a process of one step mutations, we can go from s to a state, denoted by $\tilde{s}$, that has the property that all β-players are incurring in the cost of the links with α-players. That is, $q_j^{\tilde{s},\alpha} = n_\alpha$ for all $j \in N_\beta$. Now, by mutation, one player j doing β deletes her links and switches to action α. By best response, all β-players form links with j. Now, the α-players are choosing a best response because they are in the most favorable situation with respect to links and the number of β-players coincides with the one in c' which, by hypothesis, is sustainable as a Nash equilibrium. Thus, this path leads us to a Nash equilibrium denoted by $\hat{s}$ satisfying $n_\beta^{\hat{s}} = n_\beta(c')$. To finish, using what we have already shown, by a process of one step mutations, we can reach state s' from $\hat{s}$.

5: $b, e < c < d, f$. Recall from Proposition 3 that, in this part, all possible n_β in equilibrium are distribution insensitive. Similarly to what we have already done in previous proofs, let us start by considering two equivalence classes $c, c' \in \Omega$ such that $n_\beta(c) = n_\beta(c')$ and $s \in c$. We will show that, by a path of one step mutations, we can go from s to a state $s' \in c'$. Take s' such that the indices of the players choosing each action are the same than in s, but, the distribution of active and passive links between α-players differs (this must be the case because $c \neq c'$). Hence, there exist $i, j \in N_\alpha$ such that $g_{ij} = 1$ in s but $g_{ij} = 0$ in s'. If player i, in state s, mutates and deletes her link with j. Then, by best response, player j forms the link again with i because all possible distribution of active and passive links between α-players are sustained in equilibrium. If this is done for all the links in which s and s' differ, this would leads us to s'.

Next, let us consider c and c' such that $n_\beta(c) < n_\beta(c')$. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Take $s' \in c'$ such that the indices of nodes doing each action is the same than in s except for a given player i that is doing β in s' and α in s. We want to describe a process of one step mutations that leads us from s to s'. By mutation, player i doing α mutates, switches to action β and forms links with all α-player. Then, by best response, α-player delete all the possible links they had with i. Also, by best response, all β-players delete their links with i. This would leads us to a Nash equilibrium denoted by $\tilde{s}$, such that $n_\beta^{\tilde{s}} = n_\beta^{s'}$. Using what we have proved in the previously, we can construct a one-step mutation path from $\tilde{s}$ to s'.

To finish, consider that the inverse inequality holds, that is, $n_\beta(c) > n_\beta(c')$. Now, $s' \in c'$ is such that the indices of nodes doing each action is the same than in s except for a given player i that is doing α in s' and β in s. We also have to show that we can go from s to s' by a process of one step mutations. Player i doing β in s, mutates and switches to action α. Then, by best response, all other β-players form links with i. This would leads us to a Nash equilibrium denoted by $\hat{s}$, satisfying that $n_\beta^{\hat{s}} = n_\beta^{s'}$. We know, by what we have already proved, that we can go from $\hat{s}$ to s' by a process of one step mutations.

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[16] If $n_\beta^* < n_\beta(c)$ the proof would be analogous.

6: $b, d, e < c < f$. In this particular part, there is no flexibility in the distribution between active and passive links because the cost of the links has to be incurred by β-players. If two equivalence classes c and c' are different, it must be because, the number of β-players in each class is different, that is $n_\beta(c) \neq n_\beta(c')$. First, let us assume that $n_\beta(c) < n_\beta(c')$. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Consider $s \in c$ and $s' \in c'$ such that the indices of nodes doing each action coincides except for a given player i that is doing α in s and β in s'. By mutation, player i in s switches to action β and forms links with all α-players. Then, all the other β-players, by best response, delete the links they have with i. This leads us precisely to state s'. To finish, assume that the inverse inequality holds, that is, $n_\beta(c) > n_\beta(c')$. As before, we can suppose that $n_\beta(c) = n_\beta(c') + 1$. Now, $s' \in c'$ differs from s in that there is a player i doing β in s that is doing α in s'. By mutation, player i doing β in s deletes all her links with the α-players and switches to action α. By best response, all β-players form links with i. This leads us precisely to state s'.

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