BANKRUPTCY RULES AND PROGRESSIVE TAXATION*

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WP-AD 2002-15

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Editor: Instituto Valenciano de Investigaciones Económicas, S.A.
First Edition August 2002
Depósito Legal: V-3093-2002

IVIE working papers offer in advance the results of economic research under way in order to encourage a discussion process before sending them to scientific journals for their final publication.

* Part of this research was conducted while the first author was visiting the Institute of Economics (Copenhagen), thanks to a Marie Curie Fellowship. We thank Jens L. Hougaard for suggesting the interest of analyzing some of these concepts. Financial support from the Ministerio de Ciencia y Tecnología, under project BEC2001-0535, and from the Generalitat Valenciana under project GV01-371, is gratefully acknowledged.

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A B S T R A C T

This paper explores the relative progressivity of the main bankruptcy rules in taxation problems. A rule *F* is more progressive than a rule *G* when the after-tax income vector generated by *F* Lorenz dominates that generated by *G.* We focus our analysis on four classical rules (proportional, equal-awards, equal-losses and Talmud) and on the TAL-family, introduced in Moreno-Ternero & Villar (2002). This family depends on a parameter $\theta \in [0,1]$ and encompasses the constrained equal awards rule, the constrained equal losses rules and the Talmud rule.

KEYWORDS: Bankruptcy Rules; Taxation Problems; Progressivity; TAL-Family.

1 INTRODUCTION

A bankruptcy problem is one in which one has to allocate a given amount of a divisible good when there is not enough to satisfy the demands of all the incumbent agents. It can be regarded as a particularly simple case of rationing situations in which the only information available refers to the number of agents, their corresponding claims, and the amount of the good to be distributed. To solve this type of problem one uses certain procedures or *rules* that implement some ethical and operational criteria. Different rules result from alternative combinations of these criteria. Therefore, a good deal of the literature on bankruptcy refers to the analysis of the properties that different rules satisfy and their characterizations in terms of well defined, intuitive and sensible principles. The reader is referred to Thomson (1995) and Moulin (2001) for a review of this literature.

In spite of its formal simplicity the bankruptcy problem is a model capable of accommodating a number of quite different situations. The best known examples are the bankruptcy of a firm, the execution of a will with insufficient assets, the allocation of equities in privatized firms, the distribution of commodities in a fixed-price setting, the allocation of food supplies in a refugees camp, sharing the cost of an indivisible public facility, etc. The ethical properties of different rules may help choosing the most suitable distribution method applicable to each different rationing situation. These ethical properties usually refer to the equitable treatment of agents, the behaviour of the rule with respect to very large or very small claims, the securement of minimal amounts, etc.

The bankruptcy problem can also be interpreted as a particular *taxation problem* [e.g. Young (1988)], which is the one we shall endorse here. It consists of collecting a given amount $X > 0$ of taxes out of a population N whose gross income vector is y. Therefore, a taxation problem can be identified with a triple (N, X, y), with $Y \equiv \sum_{i \in N} y_i > X$. A *tax rule* is a mapping F that applies the space of taxation problems into the space of allocations. The solution proposed by the rule F is an allocation $F(N, X, y)$ which specifies the amounts of taxes paid by the agents, with $0 \leq F(N, X, y) \leq y$ and $\sum_{i \in N} F_i(N, X, y) = X$. The after-tax income vector is given by $y - F(N, X, y)$.

We are interested in assessing the distributive impact of different tax rules, focussing on the notion of progressivity. A tax rule is called progressive when agents with larger incomes should contribute relatively more. Most of the actual tax schedules exhibit this property, which can be rationalized in a number of ways. The most direct justification refers to the fact that progressive taxation produces a reduction in the after-tax income inequality, something considered desirable when we interpret this feature as a sort of compensation for differences in agents' opportunities [e.g. Roemer (1998)]. A second line of defence comes from the application of John Stuart Mill's "equal sacrifice" criterion. An equitable tax rule is one that distributes equally the welfare loss induced by the tax burden. Therefore, if utilities are concave, the equal sacrifice principle implies that taxes are increasing with income, which in most cases implies progressive taxation (but not always). A step beyond in this direction came from Edgeworth, who suggested that taxes should be distributed so as to minimize the aggregate sacrifice. This principle yields a rather extreme progressive tax rule (the leveling tax). The reader is referred to Young (1994, ch. 6) and Thomson (1995) for a discussion.

Besides determining whether a tax rule is progressive or not, one can also be interested in comparing the relative progressivity of different rules in terms of their outcomes. A tax rule F will be declared "more progressive" than another tax rule G if, for any given taxation problem (N, X, y), F yields an after-tax income distribution that Lorenz dominates that generated by G. The assessment of the distributive properties of bankruptcy rules by means of the classical Lorenz dominance criterion can be found in Moulin (1988, ch. 6) and, more recently, in Hougaard & Thorlund-Petersen (2001).

There are in the literature four classical solutions to the bankruptcy problem, that are applicable to the taxation problem: the proportional rule, the constrained equal awards rule (that here corresponds to the *head tax*), the constrained equal losses rule (here the *leveling tax*), and the Talmud rule [see Herrero & Villar (2001) for a comparative analysis]. The first three rules implement the idea of equal division, with different reference variables (ratios, contributions, and after-tax incomes, respectively). The last three rules can be regarded as part of a parametric family of rules introduced in Moreno-Ternero & Villar (2002) under the name of the TAL-family. For each $\theta \in [0, 1]$ the rule F^θ in this family implements the following protective criterion: no agent will pay more (resp. less) than a fraction θ of her income if the tax burden X is below (resp. above) θ times the aggregate income $\sum_{i \in N} y_i$. The rule associated with $\theta = \frac{1}{2}$ is the Talmud rule whereas the extreme values $\theta = 1$ and $\theta = 0$ correspond to the head tax and the leveling tax, respectively. The proportional solution is not part of this family.

We analyze in this paper the progressivity of all these rules, both in absolute and relative terms. The domain of problems on which these rules are progressive is considered first. Clearly the proportional solution is not progressive (even though it satisfies a weak form of this property). We show that a rule F^θ in the TAL-family exhibits progressivity on the domain of tax problems (N, X, y) in which $\theta < \frac{X}{Y}$, and only on this domain. Indeed, the only rule that satisfies progressivity on an unrestricted domain is the leveling tax.

The analysis of the relative progressivity of these rules comes next. The proportional rule and the Talmud rule are more progressive than the head tax and less progressive than the leveling tax, as expected. More interestingly, we show that all the members of the TAL-family can be ordered in terms of relative progressivity, according to the value of the parameter θ. In particular, if θ_1, $\theta_2 \in [0, 1]$ are such that $\theta_1 \leq \theta_2$, then F^{θ_1} is more progressive than F^{θ_2}. Therefore, the parameter θ that generates the rules in the TAL-family can be given a very precise interpretation as an index of progressivity.

The rest of the paper is organized as follows. Section 2 contains the model. The behaviour of the rules with respect to the progressivity criteria is discussed in Section 3. The proof of the main theorem is relegated to an Appendix.

2 THE MODEL

Let $\mathbb{N}$ represent the set of all potential agents (a set with an infinite number of members) and let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents $N = \{1, 2, ..., n\}$, where we take $|N| = n$. A **taxation problem** is a triple (N, X, y), where N is the set of agents, $X \in \mathbb{R}_+$ represents the **tax burden** (the amount of taxes to be collected), and $y \in \mathbb{R}^n_+$ is the agents' gross income vector whose ith component is y_i. It will be assumed throughout the paper that $\sum_{i \in N} y_i > X > 0$. The family of all those taxation problems is $\mathbb{X}$. To simplify notation we write, for any given problem $(N, X, y) \in \mathbb{X}$, $Y = \sum_{i \in N} y_i$. We assume, without loss of generality, that agents are labelled so that $y_1 \leq y_2 \leq ... \leq y_n$.

Definition 1 *A **rule** is a mapping F that associates with every $(N, X, y) \in \mathbb{X}$ a unique point $F(N, X, y) \in \mathbb{R}^n$ such that:*
(i) $0 \leq F(N, X, y) \leq y$.
(ii) $\sum_{i \in N} F_i(N, X, y) = X$.

The point $F(N, X, y)$ represents a fair way of allocating the tax burden X among the agents in N. Requirement (i) is that each agent pays an amount that is non-negative and bounded above by her income. Requirement (ii) is that the total tax burden is to be covered. The set $\Omega(N, X, y) = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = X$, with $x_i \leq y_i$, for all $i = 1, ..., n\}$ is the set of **feasible allocations**. The **after-tax income** vector is given by $y - F(N, X, y) \in \mathbb{R}^n_+$.

Given a rule F we define its **dual** F^* as follows [Cf. Aumann & Maschler (1985)]: For all $(N, X, y) \in \mathbb{X}$, $F^*(N, X, y) = y - F(N, Y - X, y)$. When a rule and its dual produce the same outcomes is called self-dual. The notion of duality is extended to the properties a rule satisfies in an obvious way (namely, $\mathcal{P}^*$ is the dual property of $\mathcal{P}$ if for every rule F it is true that F satisfies $\mathcal{P}$ if and only if its dual rule F^* satisfies $\mathcal{P}^*$).

We now consider four classical tax rules: the proportional rule, the head tax, the leveling tax, and the Talmud rule. The first three rules apply an egalitarian criterion and differ in the variables they equalize. The proportional rule solves the problem dividing the tax burden so that each agent pays an equal share of her income. The head tax is basically a flat tax that distributes the tax burden uniformly, provided no agent ends up paying above her income. The leveling tax rule aims at equalizing after-tax income across the agents, with one proviso: all contributions are non-negative. Finally, the Talmud rule behaves like the head tax or the leveling tax, depending on whether the tax burden exceeds or falls short of half the total income. In particular, nobody pays more than half of her income if the tax burden is less than half of the aggregate income and nobody pays less than half of her income if the tax burden exceeds half of the total income.

Formally:

Definition 2 *The* ***proportional tax*** (P) *is the rule that, for all* $(N, X, y) \in \mathbb{X}$, *and all* $i \in N$, *yields:*

$$P_i(N, X, y) = \lambda \cdot y_i$$

where $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \lambda \cdot y_i = X$.

Definition 3 *The* ***head tax*** (A) *is the rule that, for all* $(N, X, y) \in \mathbb{X}$, *and all* $i \in N$, *yields:*

$$A_i(N, X, y) = \min\{y_i, \lambda\}$$

where $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \min\{y_i, \lambda\} = X$.

Definition 4 *The* ***leveling tax*** (L) *is the rule that, for all* $(N, X, y) \in \mathbb{X}$, *and all* $i \in N$, *yields:*

$$L_i(N, X, y) = \max\{0, y_i - \lambda\}$$

where $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \max\{0, y_i - \lambda\} = X$.

Definition 5 *The* ***Talmud*** (T) *is the rule that, for all* $(N, X, y) \in \mathbb{X}$, *and all* $i \in N$, *yields:*

$$T_i(N, X, y) = \begin{cases} \min\{\frac{1}{2}y_i, \lambda\} & \text{if } X \leq \frac{1}{2}Y \\ \max\{\frac{1}{2}y_i, y_i - \mu\} & \text{if } X \geq \frac{1}{2}Y \end{cases}$$

where λ *and* μ *are chosen so that* $\sum_{i \in N} T_i(N, X, y) = X$.

Moreno-Ternero & Villar (2002) introduce a family of rules, called the TAL-family, that generalizes the Talmud rule and encompasses both the head tax and the leveling tax rules. This family is generated by applying the principle underlying the Talmud rule to all the rules that solve the taxation problem depending on the relation between X and θY, for $\theta \in [0, 1]$. Formally:

Definition 6 *The* ***TAL-family*** *consists of all rules with the following form: For some* $\theta \in [0, 1]$, *for all* $(N, X, y) \in \mathbb{X}$, *and all* $i \in N$,

$$F_i^{\theta}(N, X, y) = \begin{cases} \min\{\theta y_i, \lambda\} & \text{if } X \leq \theta Y \\ \max\{\theta y_i, y_i - \mu\} & \text{if } X \geq \theta Y \end{cases}$$

where λ *and* μ *are chosen so that* $\sum_{i \in N} F_i^{\theta}(N, X, y) = X$.

A rule F^{θ} in the TAL-family resolves taxation problems according to the following principle: Nobody pays more than a fraction θ of her income if the tax burden is less than θ times the gross national income and nobody pays less than a fraction θ of her income if the amount of taxes to be collected exceeds θ times the aggregate income. Note that the leveling tax rule corresponds to the case $\theta = 0$ ($F^0 = L$), whereas the head tax corresponds to the other extreme value, $\theta = 1$ ($F^1 = A$). Obviously the Talmud rule is obtained for $\theta = \frac{1}{2}$ ($F^{1/2} = T$). The parameter θ can be interpreted as a measure of the *distributive power* of the rule. The next section conveys a precise meaning to this interpretation of θ.

Note that the proportional tax schedule is not a member of the TAL-family. In other words, there is no θ for which F^{θ} is the proportional rule. Yet, for any given taxation problem $(N, X, y) \in \mathbb{X}$, the value $\theta = \frac{X}{Y}$ yields a solution $F^{\frac{X}{Y}}(N, X, y)$ that coincides with the allocation provided by the proportional tax schedule to this taxation problem. This fact will be important in the ensuing discussion.

Remark 1 *A rule in the TAL-family can also be expressed as a function of the head tax and the leveling tax rules, as follows:* $F^{\theta}(N, X, y) = A(N, X, \theta y)$ *if* $\theta Y \geq X$ *and* $F^{\theta}(N, X, y) = \theta y + L(N, X - \theta Y, (1 - \theta) y)$ *otherwise.*

3 PROGRESSIVITY ANALYSIS

The notions of progressivity and regressivity refer to the behaviour of the tax shares $F_i(.)/y_i$ with respect to the income level y_i. Progressivity (resp. regressivity) requires larger incomes to contribute proportionally more (resp. less) to the collection of an amount X of taxes.

The formal definition of these properties is as follows:

Definition 7 (Progressivity / Regressivity) *A rule F is* ***progressive*** *(resp.* ***regressive****) on $\mathbb{D} \subset \mathbb{X}$ if, for all $(N, X, y) \in \mathbb{D}$ and for all i, $j \in N$, $y_i > y_j$ implies $\frac{F_i(N,X,y)}{y_i} \geq \frac{F_j(N,X,y)}{y_j}$ (resp. $\frac{F_i(N,X,y)}{y_i} \leq \frac{F_j(N,X,y)}{y_j}$), with at least one strict inequality.*

It is easy to see that the head tax satisfies regressivity on $\mathbb{X}$ whereas the leveling tax satisfies progressivity on $\mathbb{X}$. The Talmud rule satisfies neither progressivity nor regressivity on this unrestricted domain. The proportional rule cannot satisfy any of these properties (even though it is *the* rule that satisfies simultaneously a weaker version of both).[1]

More generally, it is worth noting that progressivity and regressivity are dual properties (that is, a rule F exhibits regressivity if and only if its dual rule F^* exhibits progressivity). Moreno-Ternero & Villar (2002, Prop. 1) show that $F^{1-\theta}$ is the dual rule of F^{θ}, for all $\theta \in [0, 1]$, so that when applying these properties to the TAL-family we already know that F^{θ} exhibits progressivity if and only if $F^{1-\theta}$ exhibits regressivity. Moreover, as progressivity and regressivity are mutually exclusive concepts, there is no self-dual rule satisfying any of these properties on the unrestricted domain $\mathbb{X}$ of taxation problems.

Let $\tau(N, X, y) = \frac{X}{Y}$ stand for the share of the tax burden in the aggregate income, and define

$$\mathbb{D}^{\delta} = \{(N, X, y) \in \mathbb{X} : \tau(N, X, y) = \delta\}$$

for each $\delta \in (0, 1)$. In other words, $\mathbb{D}^{\delta}$ is the set of taxation problems whose tax share is δ. The following result shows a rule in the TAL-family is progressive on $\mathbb{D}^{\delta}$ if and only if $\theta < \delta$. Formally:

Theorem 1 *Let $\{F^{\theta}\}_{\theta \in [0,1]}$ denote the TAL-family, and let $\delta \in (0, 1)$ be given. Then:*

(i) If $\theta < \delta$ then F^{θ} exhibits progressivity on $\mathbb{D}^{\delta}$.

(ii) If $\theta > \delta$ then F^{θ} exhibits regressivity on $\mathbb{D}^{\delta}$.

(iii) If $\theta = \delta$ then F^{θ} exhibits neither progressivity nor regressivity on $\mathbb{D}^{\delta}$.

[1] By a weaker version we mean that $y_i > y_j$ implies $F_i(.)/y_i \geq F_j(.)/y_j$ (resp. $\leq$), without requiring any strict inequality.

Proof.

Let $\delta \in (0,1)$ be given and consider its corresponding domain of taxation problems $\mathbb{D}^\delta$. Let $(N,X,y) \in \mathbb{D}^\delta$ and consider i, $j \in N$ such that $y_i > y_j$. Let also $\theta \in [0,1]$.

(i) If $\theta < \delta$, it follows from the definition, that $F_i^\theta(N,X,y) = \max\{\theta y_i, y_i - \lambda\}$, for all $i \in N$. Suppose first that $\lambda \leq (1-\theta)y_j$. In such a case $F_i^\theta(N,X,y) = y_i - \lambda$, and $F_j^\theta(N,X,y) = y_j - \lambda$, which implies $\frac{F_i(N,X,y)}{y_i} > \frac{F_j(N,X,y)}{y_j}$. On the other hand, if $(1-\theta)y_j < \lambda < (1-\theta)y_i$, then $\frac{F_i(N,X,y)}{y_i} = 1 - \frac{\lambda}{y_i} > \theta = \frac{F_j(N,X,y)}{y_j}$. Finally, suppose that $\lambda \geq (1-\theta)y_i$. Then, $\frac{F_i(N,X,y)}{y_i} = \theta = \frac{F_j(N,X,y)}{y_j}$.

To conclude this case, let us see that there exists at least a pair of agents i, $j \in N$ such that $y_i > y_j$ and $\frac{F_i(N,X,y)}{y_i} > \frac{F_j(N,X,y)}{y_j}$. Otherwise, the above reasoning would imply $\lambda \geq (1-\theta)y_i$ for all $i \in N$. This ensures that $F_i^\theta(N,X,y) = \theta y_i$ for all $i \in N$, and therefore $X = \theta Y$, which is a contradiction, since $\theta < \delta$. The proof of case (i) is in this way completed. Moreover, it follows that $F^0 = L$ exhibits progressivity on $\cup_{\delta \in (0,1)} \mathbb{D}^\delta = \mathbb{X}$.

(ii) If $\theta > \delta$ (i.e. $X < \theta Y$) then, $F_i^\theta(N,X,y) = \min\{\theta y_i, \lambda\}$, for all $i \in N$. Suppose first that $\lambda \leq \theta y_j$. In such a case $F_i^\theta(N,X,y) = \lambda = F_j^\theta(N,X,y)$, which implies $\frac{F_i(N,X,y)}{y_i} < \frac{F_j(N,X,y)}{y_j}$. Secondly, if $\theta y_j < \lambda < \theta y_i$, then $\frac{F_i(N,X,y)}{y_i} = \frac{\lambda}{y_i} < \theta = \frac{F_j(N,X,y)}{y_j}$. Finally, suppose that $\lambda \geq \theta y_i$. Then, $\frac{F_i(N,X,y)}{y_i} = \theta = \frac{F_j(N,X,y)}{y_j}$.

To conclude this case, let us see that there exists at least a pair of agents i, $j \in N$ such that $y_i > y_j$ and $\frac{F_i(N,X,y)}{y_i} < \frac{F_j(N,X,y)}{y_j}$. Otherwise, the above reasoning would imply $\lambda \geq \theta y_i$ for all $i \in N$. This ensures that $F_i^\theta(N,X,y) = \theta y_i$ for all $i \in N$, and therefore $X = \theta Y$, which is a contradiction, since $\theta > \delta$. The proof of case (ii) is in this way completed. Moreover, it follows that $F^1 = A$ exhibits regressivity on $\cup_{\delta \in (0,1)} \mathbb{D}^\delta = \mathbb{X}$.

(iii) Finally, it is straightforward to see that if $\theta = \delta$, then $F_i^\theta(N,X,y) = \theta y_i$ for all $i \in N$, which implies $\frac{F_i(N,X,y)}{y_i} = \theta$ for all $i \in N$. ■

Remark 2 *The income tax burden in real economies is usually well below 20 % of the national income. Therefore, the Talmud rule is typically a regressive tax schedule.*

The following is an immediate consequence of Theorem 1:

Corollary 1 *Let $\{F^\theta\}_{\theta \in [0,1]}$ denote the TAL-family. Then:*

(i) F^θ exhibits progressivity on $\mathbb{X}$ if and only if $\theta = 0$.

(ii) F^θ exhibits regressivity on $\mathbb{X}$ if and only if $\theta = 1$.

Redressing inequality is one of the basic goals of progressive taxation. A progressive tax schedule yields an after-tax income which is more egalitarian than the original income distribution. It is then natural to consider that a tax schedule F is more progressive than a tax schedule G if it generates a more egalitarian after-tax income distribution. Fortunately, it is well established in the literature on economic inequality that one can speak safely of income distribution x being "more egalitarian" than income distribution x', when the Lorenz curve associated with x lies everywhere above that associated with x'. The Lorenz dominance criterion can be regarded as the most fundamental principle for the evaluation of income inequality, even though it is only a partial ordering. It is therefore sensible to apply this principle to the evaluation of the relative progressivity of tax schedules.[2]

Consider now the following definition:

Definition 8 *Let $x, z \in \mathbb{R}^n$ be two given vectors whose components are increasingly ordered, i.e. $x_1 \leq x_2 \leq ... \leq x_n$ and $z_1 \leq z_2 \leq ... \leq z_n$, and such that $\sum_{i=1}^n x_i = \sum_{i=1}^n z_i$. We say that vector x **Lorenz dominates** vector z, which is written as $x \succsim_L z$, if and only if $\sum_{i=1}^k x_i \geq \sum_{i=1}^k z_i$, for all $k = 1, ..., n-1$.*

The Lorenz dominance criterion induces a partial ordering on the allocations which reflects their relative spread. The expression $x \succsim_L z$ means that the distribution x is unambiguously "more egalitarian" than the distribution z. It is well known that this property is equivalent to saying that z can be obtained from x by means of a finite collection of transfers "from the richer to the poorer", and that $I(z) > I(x)$ for any sensible inequality index $I(\cdot)$ [see Atkinson (1970), Dasgupta, Sen & Starret (1973), and Rostchild & Stiglitz (1973)].

Concerning taxation note that, for any two rules F, G and a problem $(N, X, y) \in \mathbb{X}$, $F(N, X, y) \succsim_L G(N, X, y)$ if and only if $y - G(N, X, y) \succsim_L y - F(N, X, y)$. That is to say, if the vector of contributions with a tax rule F Lorenz dominates that with a tax rule G, then the after-tax income vector generated by G Lorenz dominates that generated by F. This motivates the following:

Definition 9 (Relative progressivity) *A rule F is said **more progressive** than a rule G **on a domain** $\mathbb{D} \subset \mathbb{X}$, which we write $F \succsim_{\mathbb{D}}^* G$, when for*

[2]Some of the best known contributions to the theory of economic inequality are Lorenz (1905), Dalton (1920), Sen (1973), and Atkinson (1975).

all taxation problems $(N, X, y) \in \mathbb{D}$, $y - F(N, X, y) \succsim_L y - G(N, X, y)$. If F is more progressive than G on the whole domain $\mathbb{X}$ of taxation problems, we simply say that F is ***more progressive*** *than G, and write $F \succsim^* G$.*

The content of this notion is clear. A tax rule F is more progressive than a tax rule G on a domain $\mathbb{D} \subset \mathbb{X}$ when F yields an after-tax income vector that is more egalitarian than that generated by G, for each admissible problem. Alternatively, F is more progressive than G if, for any given problem $(N, X, y) \in \mathbb{D} \subset \mathbb{X}$, the after-tax income distribution $y - F(N, X, y)$ can be obtained from the after-tax income distribution $y - G(N, X, y)$, by a sequence of income transfers from the richer to the poorer.

Let $(N, X, y) \in \mathbb{X}$ be a taxation problem and let z be an arbitrary point in the feasible set $\Omega(N, X, y)$. It is easy to see that $A(N, X, y) \succsim_L z$ and $y - L(N, X, y) \succsim_L y - z$. This simply follows from the fact that $L(N, X, y)$ is the closest point to y in the feasible set $\Omega(N, X, y)$ whereas $A(N, X, y)$ is the closest point to equal division in that set [see for instance Moulin (1988, ch. 6)]. As a consequence, we immediately deduce that $L \succsim^* P \succsim^* A$, and also that $L \succsim^* T \succsim^* A$. That is, the leveling tax is more progressive than either the proportional or the Talmud rule, which in turn are more progressive than the head tax.

The main result of the paper, which is presented next, extends these relations. In particular, it shows that all the rules in the TAL-family can be ranked according to the relative progressivity principle on the unrestricted domain $\mathbb{X}$, so that $\theta_1 \leq \theta_2$ implies that F^{θ_1} is more progressive than F^{θ_2}. Formally:

Theorem 2 *Let F^{θ_1}, F^{θ_2} be two rules in the TAL-family, with $\theta_1, \theta_2 \in [0, 1]$. Then, $F^{\theta_1} \succsim^* F^{\theta_2}$ when $\theta_1 \leq \theta_2$.*

(The proof is given in the Appendix)

Theorem 2 conveys a very precise content to the interpretation of the parameter θ as an index of the distributive power of the rule F^{θ}. For any given taxation problem $(N, X, y) \in \mathbb{X}$, the allocation proposed by the F^{θ} yields an after-tax income vector that becomes more egalitarian as θ decreases. Therefore, the rules in the TAL-Family are fully ranked according to the progressivity principle, depending monotonically on the parameter θ.

>From this result we can also provide a clear assessment on the relative progressivity of the proportional rule with respect to other rules within the TAL-family, recurring once more to the $\mathbb{D}^{\delta}$ sets.[3]

[3] This corollary generalizes Theorem 1 (b) in Hougaard & Thorlund-Petersen (2001).

Corollary 2 *Let* $\{F^\theta\}_{\theta\in[0,1]}$ *denote the TAL-family, and let* $\delta \in (0,1)$ *be given. Then:*

(i) If $\theta < \delta$ *then* $F^\theta \succsim^*_{\mathbb{D}^\delta} P$.

(ii) If $\theta > \delta$ *then* $P \succsim^*_{\mathbb{D}^\delta} F^\theta$.

(iii) If $\theta = \delta$ *then* F^θ *coincides with* P *on* $\mathbb{D}^\delta$.

In other words, Corollary 2 says the following. Given a taxation problem $(N,X,y) \in \mathbb{X}$, $y - P(N,X,y) \succsim_L y - F^\theta(N,X,y)$ for all $\theta \in (\tau(N,X,y),1]$ and $y - F^\theta(N,X,y) \succsim_L y - P(N,X,y)$ for all $\theta \in [0,\tau(N,X,y))$. Finally, if $\theta = \tau(N,X,y)$ then $P(N,X,y) = F^\theta(N,X,y)$. In particular P is more (resp. less) progressive than the Talmud rule $T = F^{1/2}$ when $\tau(N,X,y) < \frac{1}{2}$ (resp. $> \frac{1}{2}$). According to the remark above, this implies that the Proportional rule is typically more progressive than the Talmud rule.

The following corollary provides some additional information on the distributive consequences of these rules. In particular, it identifies the preferred rules in the TAL-family for the poorest and the richest agent in N. Formally:

Corollary 3 *Let* $(N,X,y) \in \mathbb{X}$*, and* $\theta_1,\ \theta_2 \in [0,1]$*, where* $\theta_1 \leq \theta_2$*. Then* $F_1^{\theta_1}(N,X,y) \leq F_1^{\theta_2}(N,X,y)$ *and* $F_n^{\theta_1}(N,X,y) \geq F_n^{\theta_2}(N,X,y)$.

Proof.

Let $(N,X,y) \in \mathbb{X}$ and $\theta_1,\ \theta_2 \in [0,1]$, where $\theta_1 \leq \theta_2$. Theorem 1 says $F^{\theta_2}(N,X,y) \succsim_L F^{\theta_1}(N,X,y)$. In particular, $F_1^{\theta_1}(N,X,y) \leq F_1^{\theta_2}(N,X,y)$ and $\sum_{i=1}^{n-1} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{n-1} F_i^{\theta_2}(N,X,y)$.

Now, since $\sum_{i=1}^{n} F_i^{\theta_1}(N,X,y) = X = \sum_{i=1}^{n} F_i^{\theta_2}(N,X,y)$, it follows that $X - F_n^{\theta_1}(N,X,y) \leq X - F_n^{\theta_2}(N,X,y)$, or what is equivalent, $F_n^{\theta_1}(N,X,y) \geq F_n^{\theta_2}(N,X,y)$. ■

Corollary 3 says the following. Let $(N,X,y) \in \mathbb{X}$ be given and consider the rules in the TAL-family F^θ, for θ varying in $[0,1]$. Higher values of θ yield an allocation with larger (smaller) shares of the tax burden for the poorest (resp. richest) agent. Therefore, the most preferred rules by the poorest and the richest agents are the leveling tax and the head tax, respectively. And vice versa. These outcomes are of some import for the non-cooperative game theoretic approach to these type of problems [e.g. Chun (1989), Dagan, Serrano & Volij (1997), Herrero (2001), Herrero, Moreno-Ternero & Ponti (2001)].

4 Appendix: Proof of Theorem 2

To prove this result we have to introduce some additional notation and invoke the results by Hougaard & Thorlund-Petersen (2001).

For each taxation problem $(N, X, y) \in \mathbb{X}$ we define the following sets:

- $\Omega^0(N, X, y) = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = X, \text{ and } x_i \leq y_i, \text{ for all } i = 1, ..., n\}$.
- $\Omega_0(N, X, y) = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = X, \text{ and } x_i \geq 0, \text{ for all } i = 1, ..., n\}$.

$\Omega^0(N, X, y)$ is the portion of the hyperplane $\mathbf{x1} = X$ which is bounded above by the gross income vector y (where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{1} = (1, 1, ..., 1)$, and $\mathbf{x1}$ is the inner product in $\mathbb{R}^n$). Similarly, $\Omega_0(N, X, y)$ corresponds to the portion of the same hyperplane which intersects the non-negative orthant. Note that $\Omega^0(N, X, y) \cap \Omega_0(N, X, y) = \Omega(N, X, y)$ (the set of feasible allocations).

The following lemma summarizes some of the findings in Hougaard & Thorlund-Petersen (2001).

Lemma 1 *Let $(N, X, y) \in \mathbb{X}$ a taxation problem. The following statements hold:*
(a) $A(N, X, y) \succsim_L \mathbf{x}$ for all $\mathbf{x} \in \Omega^0(N, X, y)$.
(b) $y - L(N, X, y) \succsim_L y - \mathbf{x}$ for all $\mathbf{x} \in \Omega_0(N, X, y)$.

We now can prove our theorem:

Theorem 2 *Let F^{θ_1}, F^{θ_2} be two rules in the TAL-family, with $\theta_1, \theta_2 \in [0, 1]$. Then, $F^{\theta_1} \succsim^* F^{\theta_2}$ when $\theta_1 \leq \theta_2$.*

Proof.

Let θ_1, $\theta_2 \in [0, 1]$, where $\theta_1 \leq \theta_2$, and fix a particular taxation problem $(N, X, y) \in \mathbb{X}$. We need to show that $y - F^{\theta_1}(N, X, y) \succsim_L y - F^{\theta_2}(N, X, y)$, or equivalently, $F^{\theta_2}(N, X, y) \succsim_L F^{\theta_1}(N, X, y)$. Several cases are to be considered.

Case 1.- $X \leq \theta_1 Y$. In this case, by the definition of the TAL-family, $F_i^{\theta_1}(N, X, y) \leq \theta_1 y_i \leq \theta_2 y_i$, for all $i \in N$. Therefore, $F^{\theta_1}(N, X, y) \in \Omega^0(N, X, \theta_2 y)$. Now, Lemma 1 (a) implies $A(N, X, \theta_2 y) \succsim_L F^{\theta_1}(N, X, y)$. Furthermore, since $X \leq \theta_2 Y$, we have $A(N, X, \theta_2 y) = F^{\theta_2}(N, X, y)$. All together says that $F^{\theta_2}(N, X, y) \succsim_L F^{\theta_1}(N, X, y)$, as desired.

Case 2.- $X \geq \theta_2 Y$. Now, by the definition of the TAL-family, $F_i^{\theta_2}(N, X, y) \geq \theta_2 y_i \geq \theta_1 y_i$, for all $i \in N$. Equivalently, $y_i - F_i^{\theta_2}(N, X, y) \leq (1 - \theta_2) y_i \leq (1 - \theta_1) y_i$, and therefore, $F^{\theta_2}(N, X, y) - \theta_1 y \in \Omega_0(N, X - \theta_1 Y, (1 - \theta_1) y)$. As a consequence, Lemma 1 (b) implies $(1 - \theta_1) y - L(N, X - \theta_1 Y, (1 - \theta_1) y) \succsim_L$

$(1-\theta_1)y - (F^{\theta_2}(N,X,y) - \theta_1 y) = y - F^{\theta_2}(N,X,y)$. Now, since $X \geq \theta_1 Y$, we know that $\theta_1 y_1 + L(N, X-\theta_1 Y, (1-\theta_1)y) = F^{\theta_1}(N,X,y)$, which shows that $y - F^{\theta_1}(N,X,y) \succsim_L y - F^{\theta_2}(N,X,y)$, as desired.

Case 3.- $\theta_1 Y < X < \theta_2 Y$. In this case the above arguments are no longer valid.[4] Let r_1 be the minimum non-negative integer in $\{0,...,n-1\}$ such that $X \geq \theta_1 Y + (1-\theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})$.[5] Furthermore, let r_2 be the minimum non-negative integer in $\{0,...,n-1\}$ such that $X \leq \theta_2((\sum_{i=1}^{r_2} y_i) + (n-r_2)y_{r_2+1})$. As a consequence,

$$F^{\theta_1}(N,X,y) = (\theta_1 y_1, ..., \theta_1 y_{r_1}, y_{r_1+1} - \mu, ..., y_n - \mu),$$

and

$$F^{\theta_2}(N,X,y) = (\theta_2 y_1, ..., \theta_2 y_{r_2}, \lambda, ..., \lambda),$$

where λ and μ are determined to achieve feasibility.[6] Several cases need to be discussed.

Case 3.1.- $r_1 > r_2$.

(i) If $k \in \{1,...,r_2\}$, then $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{k} y_i) \leq \theta_2(\sum_{i=1}^{k} y_i) = \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$.

(ii) Let $k \in \{r_2+1,...,r_1\}$. In this case,

$$\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{k} y_i),$$

and

$$\sum_{i=1}^{k} F_i^{\theta_2}(N,X,y) = \theta_2(\sum_{i=1}^{r_2} y_i) + (k-r_2)\cdot(\frac{X - \theta_2(\sum_{i=1}^{r_2} y_i)}{n-r_2}).$$

As a result, $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ if and only if

$$(k-r_2)X \geq (n-r_2)\theta_1(\sum_{i=1}^{k} y_i) - (n-k)\theta_2(\sum_{i=1}^{r_2} y_i)$$

[4] Consider the taxation problem $(N,X,y) = (\{1,2\}, 11, (10,15))$. It is straightforward to see that $F^{\frac{1}{2}}(N,X,y) = (5,6)$ and $F^{\frac{1}{4}}(N,X,y) = (3,8)$. Then, $\theta_2 = \frac{1}{2} < \frac{1}{4} = \theta_1$, but $F_2^{\frac{1}{4}}(N,X,y) > \frac{1}{2}\cdot 15$, which shows that $F^{\theta_1}(N,X,y) \notin \Omega^0(N,X,\theta_2 y)$. Similarly, consider the taxation problem $(N,X,y) = (\{1,2\}, 2.2, (3,5))$. It is straightforward to see that $F^{\frac{1}{2}}(N,X,y) = (1.1, 1.1)$. Then, $\theta_2 = \frac{1}{2} < \frac{1}{4} = \theta_1$, but $F_2^{\frac{1}{2}}(N,X,y) < \frac{1}{4}\cdot 5$, which shows that $F^{\theta_2}(N,X,y) - \theta_1 y \notin \Omega_0(N, X-\theta_1 y, (1-\theta_1)y)$

[5] For the sake of completeness, assume $y_0 = 0$.

[6] More precisely, $\mu = \frac{\theta_1(\sum_{i=1}^{r1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - X}{n-r}$, and $\lambda = \frac{X - \theta_2(\sum_{i=1}^{r2} y_i)}{n-r_2}$.

Now, since $X \geq \theta_1 Y + (1-\theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})$, it suffices to show that $(k-r_2)\cdot[\theta_1 Y + (1-\theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})] \geq (n-r_2)\theta_1(\sum_{i=1}^{k} y_i) - (n-k)\theta_2(\sum_{i=1}^{r_2} y_i)$, or equivalently,

$$\begin{aligned}
& (n-k)\,(\theta_2-\theta_1)\,(\sum_{i=1}^{r_2} y_i) + (k-r_2)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1}) \\
\geq\ & (n-k)(\sum_{i=r_2+1}^{k} y_i) - (k-r_2)[(\sum_{i=k+1}^{r_1} y_i) + (n-r_1)y_{r_1+1}].
\end{aligned}$$

Now, the right hand side of the last inequality is bounded above by $(n-k)((\sum_{i=r_2+1}^{k} y_i) - (k-r_2)y_{k+1}) \leq 0$. Since $\theta_2 \geq \theta_1$, and $k-r_2$, the desired inequality holds.

(iii) Let $k \in \{r_1+1, ..., n-1\}$. Under such a case,

$$\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i) - (k-r)(\frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - X}{n-r}),$$

and

$$\sum_{i=1}^{k} F_i^{\theta_2}(N,X,y) = \theta_2(\sum_{i=1}^{r_2} y_i) + (k-r_2)\cdot(\frac{X-\theta_2(\sum_{i=1}^{r_2} y_i)}{n-r_2}).$$

As a result, $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ if and only if

$$\begin{aligned}
(n-k)(r_1-r_2)X \ \geq\ & (n-k)(n-r_2)[\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i)] \\
& -(n-r_2)(k-r_1)(\sum_{i=k+1}^{n} y_i) - (n-r_1)(n-k)\theta_2(\sum_{i=1}^{r_2} y_i)
\end{aligned}$$

As mentioned above, $X \geq \theta_1 Y + (1-\theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})$. Then, it suffices to show that $(n-k)(r_1-r_2)[\theta_1 Y + (1-\theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})]$ is an upper bound for the right term in the above inequality. Or, equivalently, $(n-r_1)(n-k)\theta_2(\sum_{i=1}^{r_2} y_i) \geq (n-r_1)\theta_1((\sum_{i=1}^{r_1} y_i) - (r_1-r_2)y_{r_1+1}) + f(n,k,r_1,r_2,y)$, where

$$\begin{aligned}
f(n,k,r_1,r_2,y) \ =\ & (r_1-r_2)(n-k)[(n-r_1)y_{r_1+1} - (\sum_{i=r_1+1}^{n} y_i)] \\
& +(n-r_2)(n-k)(\sum_{i=r_1+1}^{k} y_i) - (n-r_2)(k-r_1)(\sum_{i=k+1}^{n} y_i).
\end{aligned}$$

Since $\sum_{i=r_1+1}^{k} y_i \leq (k-r_1)y_k$, we know that $f(n,k,r_1,r_2,y) \leq 0$. Finally, $(\sum_{i=1}^{r_1} y_i) - (r_1 - r_2)y_{r_1+1} \leq \sum_{i=1}^{r_2} y_i$ and $\theta_2 \geq \theta_1$, implies the desired inequality.

As a result, we have shown that $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ for all $k \in \{1,...,n-1\}$, which concludes the proof of Case 3.1.

Case 3.2.- $r_1 < r_2$.

(i) If $k \in \{1,...,r_1\}$, then $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{k} y_i) \leq \theta_2(\sum_{i=1}^{k} y_i) = \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$.

(ii) Let $k \in \{r_1+1,...,r_2\}$. In this case,

$$\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i) - (k-r_1)(\frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - X}{n-r_1}),$$

and $\sum_{i=1}^{k} F_i^{\theta_2}(N,X,y) = \theta_2(\sum_{i=1}^{k} y_i)$. As a result, $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ if and only if

$$(k-r_1)X \leq (n-r_1)\theta_2(\sum_{i=1}^{k} y_i) - (n-k)[\theta_1(\sum_{i=1}^{r_1} y_i) + \sum_{i=r_1+1}^{k} y_i] + (k-r_1)(\sum_{i=k+1}^{n} y_i)$$

Now, $X \leq \theta_2((\sum_{i=1}^{r_2} y_i) - (n-r_2)y_{r_2+1})$. Thus, as before, it is enough to show that $(k-r_1) \cdot [\theta_2((\sum_{i=1}^{r_2} y_i) - (n-r_2)y_{r_2+1})] \leq (n-r_1)\theta_2(\sum_{i=1}^{k} y_i) - (n-k)[\theta_1(\sum_{i=1}^{r_1} y_i) + \sum_{i=r_1+1}^{k} y_i] + (k-r_1)(\sum_{i=k+1}^{n} y_i)$, or equivalently,

$$\begin{aligned} & (n-k)\,(\theta_2-\theta_1)\,(\sum_{i=1}^{r_1} y_i) \\ \geq\ & (n-k)\,(1-\theta_2)\,(\sum_{i=r_1+1}^{k} y_i) - (k-r_1)[(\sum_{i=k+1}^{n} y_i) - \theta_2((n-r_2)y_{r_2+1} + \sum_{i=k+1}^{r_2} y_i)] \end{aligned}$$

Now, the second term in the above inequality is bounded above by $(n-k)\,(1-\theta_2)\,(k-r_1)(y_k - y_{k+1})$, which is a negative amount. Since $\theta_2 \geq \theta_1$, the result follows.

(iii) Let $k \in \{r_2+1,...,n-1\}$. Under such a case,

$$\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i) - (k-r_1)(\frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - X}{n-r_1}),$$

and

$$\sum_{i=1}^{k} F_i^{\theta_2}(N,X,y) = \theta_2(\sum_{i=1}^{r_2} y_i) + (k-r_2) \cdot (\frac{X - \theta_2(\sum_{i=1}^{r_2} y_i)}{n-r_2}).$$

As a result, $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ if and only if

$$\begin{aligned}(n-k)(r_2-r_1)X &\leq (n-r_2)(k-r_1)(\sum_{i=k+1}^{n} y_i) + (n-r_1)(n-k)\theta_2(\sum_{i=1}^{r_2} y_i)\\ &\quad -(n-k)(n-r_2)[\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i)]\end{aligned}$$

As before, since $X \leq \theta_2((\sum_{i=1}^{r_2} y_i)-(n-r_2)y_{r_2+1})$, it suffices to show that $(n-k)(r_1-r_2)\,[\theta_2((\sum_{i=1}^{r_2} y_i)-(n-r_2)y_{r_2+1})]$ is a lower bound for the right term in the above inequality. Or, equivalently, $(n-r_2)(n-k)(\theta_2-\theta_1)\,(\sum_{i=1}^{r_2} y_i) \geq (n-r_2)\cdot g(n,k,r_1,r_2,y)$, where

$$\begin{aligned}g(n,k,r_1,r_2,y) &= (n-k)[(r_2-r_1)y_{r_2+1} + (1-\theta_2)(\sum_{i=k}^{r_1+1} y_i)]\\ &\quad -\theta_2(n-k)(\sum_{i=k+1}^{r_2} y_i) - (k-r_1)(\sum_{i=k+1}^{n} y_i).\end{aligned}$$

Since $\left[(r_2-r_1)y_{r_2+1} + (1-\theta_2)(\sum_{i=k}^{r_1+1} y_i)\right] \leq [(k-r_1)+\theta_2(r_2-k)]\,y_k$, it is straightforward to see that $g(n,k,r_1,r_2,y) \leq 0$, which concludes the proof.

As a result, we have shown that $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ for all $k \in \{1,...,n-1\}$, which concludes the proof of Case 3.2.

Case 3.3.- $r_1 = r_2$.

(i) If $k \in \{1,...,r_1\}$, then $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1\left(\sum_{i=1}^{k} y_i\right) \leq \theta_2\left(\sum_{i=1}^{k} y_i\right) = \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$.

(ii) Let $k \in \{r_1+1,...,n-1\}$. In this case,

$$\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) = \theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i) - (k-r_1)(\frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - X}{n-r_1}),$$

and

$$\sum_{i=1}^{k} F_i^{\theta_2}(N,X,y) = \theta_2(\sum_{i=1}^{r_2} y_i) + (k-r_2)\cdot(\frac{X-\theta_2(\sum_{i=1}^{r_2} y_i)}{n-r_2}).$$

As a result, and since $r_1 = r_2$, $\sum_{i=1}^{k} F_i^{\theta_1}(N,X,y) \leq \sum_{i=1}^{k} F_i^{\theta_2}(N,X,y)$ if and only if

$$0 \geq (n-k)[\theta_1(\sum_{i=1}^{r_1} y_i) + \sum_{i=r_1+1}^{k} y_i] - (k-r_1)(\sum_{i=k+1}^{n} y_i) - (n-k)\theta_2(\sum_{i=1}^{r_1} y_i).$$

Or equivalently, if and only if

$$(k-r_1)(\sum_{i=k+1}^{n} y_i) \geq (n-k)[(\theta_1-\theta_2)(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{k} y_i)].$$

Now, $\left(\sum_{i=k+1}^{n} y_i\right) \geq (n-k)y_{k+1} \geq (n-k)(\sum_{i=r_1+1}^{k} y_i)$, which gives the desired inequality.

The proof is in this way completed. ■

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