ON THE ECONOMETRIC ESTIMATION OF A VARIABLE RATE OF DEPRECIATION*

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ABSTRACT

Measuring the capital stock is crucial in some fields of economic research. Capital stock is not observable, and its estimation requires the knowledge of its rate of depreciation. In most of the cases, econometric is not used for this task. However this methodology is adecuate to generate measures consistent with its productivity and the technology of the economy. If we assume that the depreciation rate is not constant, its estimation poses some technical difficulties. In principle, it is not possible to use standard econometric packages. In this note we suggest two estimation methods of a variable rate of depreciation which are easily implementable in standard packages by means of NLS or ML. The formalization of these methods and empirical evidence on its implementation is shown.

KEYWORDS: Capital stock; Variable Depreciation Rate, Production Function Estimation.

1 INTRODUCTION

The stock of physical capital of an economy - a region, an industry or a whole country - is one of the basic economic aggregates. Among other uses it lies at the heart of potential GDP and total employment calculations, it is required to break down total output among contributing factors, it is necessary to provide a measure of the evolution of the productivity of the capital and it enters also as an argument in the labor productivity function. Yet, this variable is hard to measure directly, part of the problem deriving from the difficulty in assessing capital consumption[1]. The depreciation of physical capital is also important to set differences between net and gross measures of a macroeconomic variable - see, for instance Mauleón and Sardá (2000) for an application to the case of a welfare measure based on net, rather than gross income. In Nationals Accounts, measurement of capital consumption is usually solved by some *ad-hoc* method based on accounting regulations at the firm level. This is a crude measure, not linked with the technology of the economy neither with the productivity associated to the capital. It would be interesting, then, to devise alternative methods. Some of the alternative ways are based in the measurement of the weighted coefficients of present and past investment values for the stock of physical capital - see Hulten (1991) and Jorgenson (1991). This methodology does not produce satisfactory results when econometric is in use to identify a depreciation pattern – see Hulten and Wykoff (1981) – and poses serious difficulties. Some of these are solved by *ad-hoc* methods to calculate the mentioned weighted coefficients of the past investment based on age-efficiency and age-price functions. In this paper it is suggested an alternative method based on the econometric estimation of the stock of capital by mean of the estimation of the depreciation rate and the initial capital stock. In the end, the capital stock is computed as the weighted sum of current and lagged investment values, but the weights are derived in a more systematic way than previously (see also Dadkiah et. al. (1990), Nadiri et. al. (1993), Prucha (1995) and Prucha et. al. (1996) for related work along the same lines).

The way the econometric analysis is introduced in the problem of measuring the capital stock is based on the fact that the capital stock is one of the basic arguments of the production function. Thus, the capital stock can be estimated indirectly by mean of the estimation of the rate of depreciation -unknown parameter - jointly with the rest of parameters of the production function itself. It could be argued that, since the capital stock enters other economic relations (for example, demand for labour and capital stock), they could also be used to estimate the depreciation rate. However, these relations, being more behavioral in nature than the production function itself, are more subject to specification errors of several kinds, and therefore, would perhaps lead to less robust results. The idea of the estimation of the capital stock based in the estimation of the production function is the following: since the depreciation rate is unknown, the capital stock is not known either. Then, in econometric terms the problem can

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[1] This is one of the four main problems pointed out by the OECD in the elaboration of the National Accounts - see OECD (1992).

be solved as one of estimation with unobservable variables. Put it simply, this would amount to trying several values for the depreciation rate, calculating the corresponding capital stock variables, and finally, fitting the production function under every capital stock so derived: the depreciation rate and the capital stock, would then be selected on the basis of the best econometric fit.

As some authors have pointed out - see Prucha (1995), the econometric approach imposes some methodological restrictions and it is not immediate to apply the standard estimation methods, part of the problem deriving from the fact that the expression of the capital stock, in terms of the current and lagged investment values, have a time dependent number of arguments, what make unfeasible the use of standard packages like TSP. If additional assumptions are made related to the rate of depreciation parameter, as assuming variability in it, then the weights of lagged investment have more complex specifications and the use of standard packages becomes even more complicated. Yet, as theory suggest, it seems natural to consider that the rate of depreciation varies with technical shocks, with the induced economic depreciation associated to technical progress, with changes in the relative price of inputs, in the level of output - see Prucha and Nadiri, (1996), or in the interest rates or with the maintenance cost of capital stock or the capital utilization level - see Burnside and Eichenbaum, (1994). Dynamic factor demand models provide a context in which the methods could be applied, since the optimal demand of capital explicitly depend on the interest rate and on relative prices of inputs. The depreciation rate then has the role of driving the economic depreciation required to adjust the actual capital stock to its optimal level.

The demand for more sophisticated methods to measure the capital consumption, in order to estimate the capital stock, in the case that a variable rate of depreciation is considered, is the key argument that motivates the goals of this paper: first we describe two econometric-based estimation methods of a variable rate of depreciation that are easily implementable in standard econometric packages. The variability of the rate of depreciation rate is modelled as a linear dependence with the explanatory variable. In second place, an additional exercise is done by using prior information on the rate of depreciation, modelled as a stochastic restriction. This information is included in the estimation of the complete model following Theil and Goldberger (1961) procedure adapted to maximum likelihood (ML) estimation, and brings more acceptable results in the estimates and in the performance of the convergence when ML estimation is in use.

The effectiveness of the proposed methods is given by simulated and real data estimation results. It is applied to the Spanish economy for the period 1970 to 1997, yielding results in line with other international research, and rather different for those currently accepted. The validation of the proposed methods is also tested in a simulation exercise in which estimation results are statistically close to real values of the parameters.

The plan of this paper is as follows: In Section 2 we describe the baseline method of estimation of a constant rate of depreciation δ. In Section 3 we describe the methods proposed to estimate a variable rate of depreciation. In Section 4 estimation methods are extended to the general case in which the rate

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of depreciation depends on m variables. Empirical and simulation results are presented in Section 5 and 6. Finally, in the Appendix 1, some technical results are shown.

2 ESTIMATION OF A CONSTANT RATE OF DEPRECIATION

In this section we present the original method suggested by Prucha (1995) to estimate a constant rate of depreciation. It is the benchmark of the methods we later introduce in the following sections to estimate a variable rate of depreciation.

Consider a standard production function given by

$$Y_t = F(L_t, K_t, \theta) \tag{1}$$

where Y_t, L_t and K_t denote respectively output, labor and capital stock at the end of period t, and θ represents a vector of unknown parameters of the technology. The capital stock accumulates according to the perpetual inventory method equation:

$$K_t = I_t + \phi K_{t-1} \tag{2}$$

where I_t denotes gross investment at period t, $\phi = 1 - \delta$ and δ is the unknown rate of depreciation. Repeated substitution of the lagged capital stock in the original equation of K_t yields

$$K_t = \sum_{i=0}^{t-1} \phi^i I_{t-i} + \phi^t K_0 = G_t(I_1, ..., I_t, K_0, \delta) \tag{3}$$

which is a function of t, since the number of arguments of this function depends on the period index. Now we can substitute (3) into (1) to obtain:

$$Y_t = H_t(L_t, I_1, ..., I_t, K_0, \delta, \theta) \tag{4}$$

which is also a function of t, because of K. Note that $L_1, ..., L_T$ and $I_1, ..., I_T$ are observable variables, and, in principle, K_0, δ and θ can be estimated by nonlinear least squeares (NLS) or maximum likelihood (ML), for instance. However, it is not possible to use directly standard econometric packages to estimate (4) because of the fact that the number of arguments on each period change with the period. This difficulty can be easily overcome if we rewrite K_t with the use of dummy variables. First, taking $j = t - i$,

$$K_t = \sum_{j=1}^{t} \phi^{t-j} I_j + \phi^t K_0 \tag{5}$$

Now, for a given t, we define T new variables I_t^j, $j = 1, ..., T$ as follows

$$
\begin{aligned}
I_t^j &= I_j D_t^j, \\
D_t^j &= \begin{cases} 1 & j \leq t \\ 0 & j > t \end{cases}
\end{aligned}
\tag{6}
$$

This allows us to rewrite (5) as

$$K_t = \sum_{i=1}^{T} \phi^{t-i} I_t^i + \phi^t K_0 = G(I_t^1, ..., I_t^T, K_0, \delta) \tag{7}$$

To illustrate what this transformation imply, vector $K = (K_1, .., K_T)'$ can be written as

$$
\begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ . \\ . \\ . \\ K_T \end{bmatrix}
=
\begin{bmatrix}
I_1 & K_0 & 0 & . & . & . & 0 & 0 \\
I_2 & I_1 & K_0 & . & . & . & 0 & 0 \\
I_3 & I_2 & I_1 & & & & 0 & 0 \\
. & . & . & & & & . & . \\
. & . & . & & & & . & . \\
. & . & . & & & & . & . \\
I_T & I_{T-1} & I_{T-2} & . & . & . & I_1 & K_0
\end{bmatrix}
\begin{bmatrix} 1 \\ \phi \\ \phi^2 \\ . \\ . \\ \phi^{T-1} \\ \phi^T \end{bmatrix}
$$

Substitution of (7) into (1) yields

$$Y_t = H(L_t, I_t^1, ..., I_t^T, K_0, \delta, \theta)$$

Now the number of variables for each period is constant and the parameters can be estimated using standard econometric packages like TSP. Note that K_t depends on T variables, but this does not give rise to multicollinearity, since the number of parameters is fix and the model is therefore identified. In linear model, this point was stressed by Greene and Seaks (1991).

3 ESTIMATION OF A VARIABLE RATE OF DEPRECIATION

In this section we describe two new methods proposed to estimate the parameters of the rate of depreciation equation, and hence to estimate the capital stock so determined. As in the previous case, this estimation is carried out together with the estimation of a set of parameters θ of the production function. The rate of depreciation is assumed to depend linearly on a variable x. Initially the two methods are described for a simple case given by a rate of depreciation that has two components: a constant term and a variable term that depends on the explanatory variable x. This feature does not impose any restriction on the validation of the methods, which can be described for more complex specifications – see Section 4 the description of the multidimensional case –. Variable x is suggested by the economic theory, and some examples were given in Section 1. Again, to solve the problem of the estimation of the capital stock we consider a production function

$$Y_t = \tilde{F}(L_t, K_t, \theta)$$

where K_t now depends on a variable rate of depreciation. By assuming a linear equation for δ we have $\delta_t = d_1 + d_2 x_t$. Note that since the goal of the method described below is the econometric estimation of d_1, d_2 and θ, one of

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the advantages of it is the possibility of testing the null hypothesis $H_0 : d_2 = 0$ to conclude whether δ is constant or not. Calling $f_1 = 1 - d_1$, and $f_2 = -d_2$, we have $\phi_t = 1 - \delta_t = f_1 + f_2 x_t$. If we substitute recursively K_{t-s} for all $s = 1, ..., t-1$, into (2), then, since ϕ_t is not constant, we have

$$K_t = I_t + \phi_t I_{t-1} + ... + \phi_t.\phi_{t-1}...\phi_2\phi_1 K_0 \tag{8}$$

Let $C_{t,s}$ be the coefficient of the gross investment at period $t - s$ for all $0 \leq s \leq t - 1$ and $C_{t,t}$ the coefficient of K_0. Then,

$$C_{t,s} = \begin{cases} \prod_{j=0}^{s-1} \phi_{t-j} & 1 \leq s \leq t \\ 1 & s = 0 \end{cases} \tag{9}$$

Now (8) can be written as

$$\begin{aligned} K_t &= \sum_{s=0}^{t-1} C_{t,s} I_{t-s} + C_{t,t} K_0 \\ &= \tilde{G}_t(I_1, ..., I_t, d_0, d_1, K_0) \end{aligned} \tag{10}$$

Note that when $\delta_t = \delta$ for all t, then $C_{t,s} = \phi^s$ and $\tilde{G}_t(.) = G_t(.)$ which is the case described in Section 2. Again, as shown in the δ constant case, equation (10) has a period-dependent number of arguments. We next describe two methods to solve this problem.

3.1 Method 1

This method is the complete extension to the variable case of the original method presented in Section 2. First taking $j = t - s$,

$$K_t = \sum_{i=1}^{t} C_{t,t-j} I_j + C_{t,t} K_0$$

and now using I_t^j as stated in (6),

$$K_t = \sum_{i=1}^{T} C_{t,t-i} I_t^i + C_{t,t} K_0 \tag{11}$$

Since now we have the same number of arguments for each period in the capital stock equation, also has the production function. Hence, parameters θ, d_0 and d_1 can be estimated by ML or NLS using standard packages.

3.2 Method 2

Note that the number of terms in $C_{t,s}$ is s and that for each t such elements are different. This fact together with the highly nonlinearities in the original parameters could bring some difficulties in the convergence of the estimation procedure, possibly due to the intractability of the criteria used to optimize –

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NLS or ML, for instance. In that case it would be interesting to investigate an expression for K_t where the weights of the current and lagged investment values has a simpler form and where the cost in terms of efficiency of the estimates were low. In this context another method is proposed, with the mentioned properties, as we see in the Monte Carlo experiment - see Section 6. This method is based on the substitution of the set of coefficients given by (9) by their linear approximation around $x = 0$ in the capital stock equation. The idea is to simplify the nonlinearities associated to the cross products of the coefficients f_1 and f_2. It is shown in the Appendix 1 that the linear approximation yields

$$\tilde{C}_{t,s} = f_1^s + f_1^{s-1} f_2 \sum_{j=0}^{s-1} x_{t-j}$$

and back to (11) we have

$$
\begin{aligned}
K_t \quad = \quad & I_t + (f_1 + f_2 x_t) I_{t-1} + f_1 (f_1 + f_2(x_t + x_{t-1})) I_{t-2} + ... \\
& ... + f_1^{t-2}(f_1 + f_2 \sum_{j=0}^{t-2} x_{t-j}) I_1 + f_1^{t-1}(f_1 + f_2 \sum_{j=0}^{t-1} x_{t-j}) K_0
\end{aligned}
$$

Now, taking $j = t - s$, we can rewrite the above equation as

$$K_t = \sum_{j=1}^{t} \tilde{C}_{t,t-j} I_j + \tilde{C}_{t,t} K_0$$

Again, using I_t^j as defined in (6),

$$K_t = \sum_{i=1}^{T} \tilde{C}_{t,t-i} I_t^i + \tilde{C}_{t,t} K_0 = \tilde{G}(\tilde{C}_{t,t}, ...\tilde{C}_{t,t-T}, I_t^1, ..., I_t^T, K_0)$$

where $\tilde{C}_{t,t-i}$ and x_{t-i} are arbitrary numbers if $i > t$ since for this case we have I_t^i equals zero. Finally, the above equation is plugged into the production function to estimate the parameters of the model.

4   MULTIDIMENSIONAL CASE

If variable x has dimension $m > 2$, the suggested Methods 1 and 2 can also be extended to estimate the parameters involved in the production function together with those behind the rate of depreciation. Method 1 extension is immediate, since the modification to take into account is simply to write down the full coefficients in terms of the lagged investment and then apply the dummy variable approach to fix the number of arguments in the equation of the capital stock. Method 2 is also easy to extend, simply by computing the linear approximation of $C_{t,s}$ when δ has dimension m.
When $x \in \mathbb{R}^m$ the rate of depreciation is given by

$$\delta_t = \delta_1 + \delta_2 x_{2t} + ... + \delta_m x_{mt}$$

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and the term $\phi_t = 1 - \delta_t = f_1' + f_2 x_{2t} + ... + f_m x_{mt}$, where $f_1 = 1 - \delta_1$ and $f_j = -\delta_j$, $j = 2, ..., m$. The original coefficients of lagged investment is given by

$$C_{t,s} = \prod_{j=0}^{s-1} \phi_{t-j} = \prod_{j=0}^{s-1}(f_1 + f_2 x_{2,t-j} + ... + f_m x_{m,t-j})$$

for all $s = 1, ..., t$ and $C_{t,0} = 1$. The first order linear approximation of $C_{t,s}$ around $x = 0$ yields -see Appendix 1 -

$$\tilde{C}_{t,s} = f_1^{s-1}(f_1 + f_2 \sum_{j=0}^{s-1} x_{2,t-j} + ... + f_m \sum_{j=0}^{s-1} x_{m,t-j})$$

Replacing $C_{t,s}$ by $\tilde{C}_{t,s}$ in the equation of K_t and at the same time calling $j = t - s$, we have,

$$K_t = \sum_{i=j}^{t} \tilde{C}_{t,t-j} I_j + \tilde{C}_{t,t} K_0$$

At this point, again, it is applied the dummy variable transformation described in Section 2 to have the same number of argument for K_t for all t. Then it is included in the production function to finally estimate the parameters under a standard estimation method.

5   EMPIRICAL RESULTS

In this section we present the empirical results obtained fitting a Cobb-Douglas production function to the Spanish economy data. Results are given in Table 1.

Returns to scale are assumed constant on theoretical grounds, and because they yield the most coherent empirical fit. Taking logs the function becomes,

$$y_t = c + dD_{1t} + \alpha l_t + (1 - \alpha)k_t + u_t \tag{12}$$

where y_t is GDP, c, the constant term, l_t, employment, k_t, the capital stock, and u_t, a random shock. A dummy variable $D_{1t} = 0, t \leq 1983$, and $D_{1t} = 1, t > 1983$, is detected to be significant. The breaking point is picked selecting the best fit among all possible dates, but it seems reasonable to argue that since technological progress is not included in the model and returns are assumed to be constant, the structural change in the constant term captures the modernization process initialized in the Spanish economy started in the early eighties. As a key element of this process, could be named the strong industrial restructuring initialized by the government at 1982.

The agricultural sector is excluded, since it only accounts for a small fraction of GDP (less than 5%), and investment is negligible (according to input-output tables). We only consider non residential investment, and data are measured at 1986 prices (where relevant). The sample data spans from 1970 to 1997, and is taken from the Spanish National Statistical Institute (the Tempus data

base, available in the internet). The data require to allow the random error to follow an AR(1) in some specifications, which could be understood as a result of productivity shocks. The notation is as follows,

$$
\begin{aligned}
u_t &= \rho u_{t-1} + \varepsilon_t \\
\varepsilon_t &\sim N(0, \sigma_\varepsilon)
\end{aligned} \tag{13}
$$

A variable depreciation rate has been considered in order to test the validity of the method proposed. Since convergence were achieved in the estimation and results were satisfactory only method 1 was used. Two possible explanatory variables have been tested, a dummy, and the GDP growth rate. The notation follows:

$$
\begin{aligned}
K_t &= I_t + (1 - \delta_t)K_t \\
\delta_t &= d_0 + d_1 z_t
\end{aligned} \tag{14}
$$

and two cases were considered: firs taking $z_t = \Delta y_t$, and second, $z_t = D_{2t}$, ($D_{2t} = 0, t \leq 1989, D_{2t} = 1, t > 1989$). Again, the breaking point is picked selecting the best fit among all possible dates and its economic interpretation is given next. The main empirical results are presented in Table 1. The estimation method used in all models is full maximum likelihood, assuming a Gaussian distribution [2]. The initial capital stock could be understood as a further parameter (see the previous section), so that beyond entering the model in a non linear way, it does not pose any special technical problem. Nevertheless, it is not identified (see Appendix 2), so that it has been judged a more sensible solution to estimate it from alternative accounting sources. Accordingly, it has been set equal to $1.3 x 10^7$ millions of ptas. at 1986 prices.

We turn now to the discussion of the results themselves. The average depreciation rate is calculated and given in the last line of Table 1 for all of the cases. Column I provides the first results, with an estimated value for the constant depreciation rate of 3.74%. Since this value must be greater than zero, a one sided significance test is in order, and it turns out to be significant at the 95% level (this also applies to the remaining results in columns II to III). This value is rather low when compared to international estimates, and to conventional values (see Corrales et.al.(1989)). Therefore, more complex specifications seem in order. The remaining results pursue this point, making the depreciation rate dependent on some variables. Column II shows the results obtained allowing for a dummy variable, D_{2t} (see above). The dummy is statistically significant, and it points to an increased depreciation rate in the second sub sample, yielding an average value for the whole sample of 4.61%. This is more admissible on a priori grounds, as commented above. An specially significant economic argument to explain the selected breaking year of D_2 is the fact that the interest rate changes its growing trend in 1990, year in which reached the 14.7 per cent. From that year on, the interest rate decreases to 10 per cent in 1994. This changing trend explains the change in the user cost of capital, that start to decrease in the early

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[2] In particular, in models involving an AR(1), the first observation is never dropped (see, for example, Hamilton (1994)).

nineties and hence in the intensification of the amortization process, due to the obsolescence of the vintage stock of capital. As it is shown in Table 1, the sign of d_1 verifies this conjecture to explain the structural change in the value of the estimates of the depreciation rate. It is also worth noticing, that the AR(1) becomes non significant, and that the fit improves (σ decreases). This may suggest that the AR(1) in column I captures a structural break, rather than a genuine productivity shock. However, it must be admitted that a dummy, although required by the data, precludes a full economic explanation. That is why it was considered appropriate to replace it by some variable, notably the GDP growth rate. Column III gives the results of this procedure. The growth rate is significant, yielding an average value for the depreciation rate of 5.48%, higher than previously. The AR(1) is now significant, although the value of the coefficient is smaller than in column I. It is interesting to note that the growth rate coefficient is negative, meaning that a decrease in the GDP growth rate increases the depreciation rate (and conversely), or in other words, that old capital is scrapped when production and demand decrease: investment has, thus, a cost saving feature, rather than being focused only on capacity increases. Since the GDP growth rate decreased in the second subsample, implying an increased depreciation rate, it should be pointed out that this is coherent with the previous result of column II (the dummy points to an increased depreciation rate in the second subsample, as well). This is a perturbing result and, unfortunately, one that fully matches other evidence related to the almost nil investment on R&D conducted by Spanish private firms.

The final and complementary aspect of the results concerns the statistical checking of the estimated equations. Several tests have been conducted to that end: a) functional form, b) dynamic specification, c) cointegration, d) stability, e) heteroscedasticity, and, f) omitted variables. They are discussed next in more detail.

The Cobb-Douglas specification has been tested against a more general CES production function, given by

$$Y = A \left[\gamma K^{-\rho} + (1 - \gamma) L^{-\rho} \right]^{-\nu/\rho}$$

Since $\rho = 0$ yields a Cobb-Douglas function, a maximum likelihood ratio test is easily conducted estimating both forms of the function. Under the null of $\rho = 0$ the test follows a $\chi^2(1)$. The estimate $\hat{\rho} = 0.0025$ and $P(\hat{\rho} \leq 0.0025) = 0.004$. Then H_0 is not rejected at the confidence level 0.95 since the critical value $\chi^2_\alpha(1)$ is 3.84. Then, the Cobb-Douglas specification is accepted.

Several tests have been conducted to check the integration order of all variables. Overall, and after accounting for possible breaks, and stochastic and deterministic trends, all variables involved seem to be I(1) (see Perron, 1989). The errors of the estimated models do not have a unit root, so that the estimated equations are cointegrated, as required (in all cases). Finally, and as for the number of cointegrated vectors, the usual analysis yields only one such a vector (besides, it would not make any economic sense if there were two, since there is no economic link, behavioral, technical, or otherwise, among the variables considered).

The estimated equations are stable, once the constant dummy is added. This has been tested by means of two type of tests: a) estimating the model after splitting the sample in two equal subsamples, and, b) testing the forecasting ability of the equations for the last two observations. With the a) type test, the null of constancy is accepted, although the parameters apparently change somewhat. This can be due to a lack of power, because of the small size of both sub samples, and as a result of a changing depreciation rate (the parameters are more stable, once this feature is incorporated into the model; see columns II and III in Table 1).

Heteroscedasticity has been tested and rejected in several ways (ARCH tests, and making the variance depend on a set of assorted explanatory variables). As for omitted variables, there are more complex theoretical specifications that would grant their introduction (for example, demand shocks). However, they are of second order explanatory power, and could make the estimation results less robust.

The main empirical results given in this section can be summarized as follows: 1) the method 1 proposed and implemented in this paper work well, and yield sensible values for the depreciation rate - around 5.5% -, more in line with international results, than are accepted by current practice; 2) the depreciation rate seems to depend on the GDP growth rate inversely: i.e., it increases with decreased GDP growth rates, implying that old capital is scrapped when production and demand decrease; investment has, thus, a cost saving feature, rather than being focused only on capacity increases.

6 MONTE CARLO EXERCISE

In this section we describe and implement a simulation exercise designed to validate the effectiveness of the methods proposed. The fitted model is the one that generates the relation between labor, investment and total output. The goal of this exercise is to evaluate the effectiveness of using the tools described above in the task of estimating such model. In order to do it, a simulated exercise is designed in such a way that the real simulated capital stock series is generated by a variable rate of depreciation, but is not observable and then not used directly in the estimation process. In fact, this is the restriction that the methods proposed are designed to overcome. Once a simulated path is generated, only observable series – output, labor, investment and x – are taking into account in the estimation. This is carried out by both methods described in Section 3.

We generate labor and gross investment vectors of size $T = 30$. Labor series follows a process given by a lagged component, a trend and a white noise error. Gross investment was generated with a trend and a lagged component. For these variables, values of the coefficients were taken to be close to values estimated in similar models and the variance of the white noise processes involved was to bring a R^2 close to 0.6, a reasonable value for empirical works. The variable rate of depreciation was generated by setting $d_0 = 0.05$ and $d_1 = 0.002$. The explanatory variable is $x = \log(t)$. Finally, the endogenous rate of depreciation

is given by $\delta_t = d_0 + d_1 x_t$. After that, from I_t and δ_t, the capital stock series is generated following the perpetual inventory method. Finally, taking $c = -10$ and $\alpha = 0.4$, we generate $S = 1000$ simulated y_t vectors from the technology specification. Each replicated output vector is given by a simulated disturbance u_t^s as shown in the following equation

$$y_t^s = c + \alpha l_t + (1 - \alpha)k_t + u_t^s$$

Here l_t and k_t are labor and capital stock in logarithms, u_t^s and y_t^s are respectively the simulated error and the corresponding output at the period t, for simulation s. Note that in each simulation path the disturbance is included only in the production function error term. It means that labor and capital stock remain constant across simulations and hence across each estimation of the model.

Next step, once simulated values have been generated is estimation of the model, whose equations are, simply, the production function and the capital stock equation. Parameters of the model are (c, α, d_0, d_1), and have been estimated by Methods 1 and 2. Results are shown in Table 2.

The estimation of the simulated data leads to estimates of the model close to the real value of the parameters. In average, Method 1 estimates has smaller bias and bigger standard deviation, than method 2 estimates and the quadratic error is smaller in Method 1, as expected since the method 1 estimation is based on the complete specification of the model. Note that the bias – taking the average of the estimates as the estimate – of d_0 by method 2 seems to capture the variability not detected by estimates of d_1, since the average of this parameter is smaller than real value 0.002. Finally, this results test the feasibility and effectiveness of the proposed methods, although a natural bias should be admitted when method 2 is used, due to the fact that such method is based on an approximation to the real weights of lagged investment. Nevertheless, this method is still useful where convergence is hard to achieve due to the high nonlinearities of the function to optimize in the parameters.

7 CONCLUSIONS

In this note we show two methods to estimate a variable rate of depreciation of the capital stock with standard econometric packages. Models in which the depreciation rate is an endogenous variable require estimation techniques for this parameter non feasible in principle in standard econometric packages. To illustrate the applicability of such methods simulated exercise and real data estimation are carried out and provide good results. In particular, it has been estimated the rate of depreciation of the capital stock of the Spanish economy and results leads to values of this parameter between 5 and 6 percent, values that are rather different from the traditional national account based measures. Also a simulated exercise is designed and implemented, and results lead to good results in terms of the effectiveness of the methods proposed.

APPENDIX 1: Linear approximation of $C_{t,s}$

The first order linear approximation of a function $C_{t,s} = c(x_t, ..x_{t-j+1})$ around $x = x^0$ yields

$$C_{t,s} \simeq c(x^0) + \left[c'(x)|_{x=x^0} \right]' (x - x^0)$$

When $\delta \in \mathbb{R}^2$, we want to approximate the coefficient of the lagged investment:

$$C_{t,s}(x_t, ..., x_{t-j+1}) = \prod_{j=0}^{s-1} (f_1 + f_2 x_{t-j})$$

First, valued at $x = 0$, we have,

$$C_{t,s}(0) = f_1^{s-1}$$

and for all $j = 0, ..., s-1$,

$$\frac{\partial C_{t,s}}{\partial x_{t-j}} \mid_{x=x^0} = f_2 \prod_{i=0}^{s-1} (f_1 + f_2 x_{t-i})|_{x=x^0} = f_2 f_1^{s-1}$$

Then,

$$\frac{\partial C_{t,s}}{\partial x} = (f_2 f_1^{s-1}, \overset{s)}{...}, f_2 f_1^{s-1})'$$

and,

$$\tilde{C}_{t,s} = f_1^{s-1}(1 + f_2 \sum_{j=0}^{s-1} x_{t-j})$$

The omitted part is $f_2^s \prod_{j=0}^{s-1} x_{t-j} + f_1 f_2^{s-1} (\sum_{i,j=0}^{s-1} x_{t-i} x_{t-j})$, negligible when correlation in x_t is low.

The multidimensional case is given by the assumption $\delta_t = d_1 + d_2 x_{2t} + ... + d_m x_{mt}$. Now $C_{t,s}$ is

$$C_{t,s} = \prod_{j=0}^{s-1} (f_1 + f_2 x_{2,t-j} + ... + f_m x_{m,t-j})$$

where notation follows that used in the simple case. Since $C_{t,s}(0) = f_1^{s-1}$ we only need to compute the derivative of $C_{t,s}$ with respect to each variable in x at period $t - j$, which is

$$\left| \frac{\partial C_{t,s}}{\partial x_{h,t-j}} \right|_{x=x^0} = f_1^{s-1}(f_h, ..., f_h)$$

and has dimension s, where $h = 2, ..., m$. It is easy to check that

$$\left| \frac{\partial C_{t,s}}{\partial x} \right|_{x=x^0} x = f_1^{s-1} \left(f_2 \sum_{j=0}^{s-1} x_{2,t-j} + ... + f_m \sum_{j=0}^{s-1} x_{m,t-j} \right)$$

and

$$\tilde{C}_{t,s} \simeq f_1^{s-1}(1 + f_2 \sum_{j=0}^{s-1} x_{2,t-j} + ... + f_m \sum_{j=0}^{s-1} x_{m,t-j})$$

as shown in Section 4.

APPENDIX 2: The lack of identification of the initial capital stock

Consider the simple model

$$y_t = a\phi^t + u_t$$

where a is an unknown parameter, ϕ is a given constant, and u_t is a white noise error. It is immediate that,

$$Var(\widehat{a}_{OLS}) = \sigma_u^2 \left[\phi^2(1 - \phi^{2T})/(1 - \phi^2)\right]^{-1}$$

If $|\phi| > 1$, then this variance tends to zero as the sample size increases. But if $|\phi| < 1$, it approaches a positive constant larger than zero. In both cases the estimator is unbiased, i.e., $E(\widehat{a}_{OLS}) = a$, although in the second case the parameter cannot be consistently estimated, and is not identified: in other words, as the sample size gets larger, the information about the parameter does not, just because the associated regressor, ϕ^t, goes to zero (see Schmidt (1976)). Consider, now, the slightly more complex model

$$y_t = a(\phi^t/t) + u_t$$

It is easily checked that,

$$Var(\widehat{a}_{OLS}) = \sigma_u^2 \cdot \left[\sum_{t=1}^{T}(\phi^t/t)^2\right]^{-1} \tag{A1}$$

so that the parameter a is less identifiable (loosely speaking), because the regressor goes to zero at a faster rate than in the previous case.

Finally, consider the simple production function,

$$\log(y_t) = A + \alpha \cdot \log(K_t) + u_t$$

where the stock of capital, K_t, is given by,

$$K_t = K_0\phi^t + \sum_{s=0}^{t} \phi^s I_{t-s}$$

and $\phi = (1 - \delta)$. From asymptotic theory, and after straightforward manipulations one obtains,

$$
\begin{aligned}
Var(\widehat{K_0}_{OLS}) &\approx \sigma_u^2 \left[\frac{\partial^2\left(\sum_{t=1}^{T} u_t^2\right)}{\partial^2(K_0)}\right]^{-1} \\
&= \sigma_u^2 \cdot \left[(2\alpha^2)\sum_{t=1}^{T}(\phi^t/K_t)^2\right]^{-1}
\end{aligned}
$$

Since K_t is a trending variable, and from the discussion following (A1), we conclude that the stock of initial capital, K_0, is not identified, in the sense that, although it can be estimated, its variance does not approach zero as the sample size increases. In practice, what this result means is that we can expect poor estimates for this value.

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TABLES

	I	II	III
c	5.6	5.56	5.33
	(20.7)	(20.3)	(18.4)
d	.03	.045	.032
	(2.56)	(4.42)	(3.26)
α	.71	.71	.68
	(19.8)	(19.2)	(17.4)
δ_0	.0374	.0384	.0597
	(1.64)	(1.52)	(2.66)
δ_1		.0077	-.178
	()	(3.95)	(1.96)
ρ	.51		.39
	(2.89)	(n.s)	(2.1)
σ	.0102	.00938	.00973
$\widehat{\bar{\delta}}$	3.74%	4.03%	5.48%

Table 1: Estimation results

	Method 1	Method 2
$\bar{c}$	-10.0342	-9.8991
std	(-0.0468)	(-0.0289)
qe	0.0033659	0.011007
$\bar{\alpha}$	0.4010	0.4238
std	(0.0090)	(0.0057)
qe	0.000083040	0.00060087
$\bar{\delta_0}$	0.0456	0.0743
std	(0.0074)	(0.0049)
qe	0.000074918	0.00061626
$\bar{\delta_1}$	0.0019	0.0016
std	(1.3547e-4)	(1.0608e-4)
qe	2.15359e-08	1.61180e-07

Table 2: Simulation results

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