THE TAL-FAMILY OF RULES FOR BANKRUPTCY PROBLEMS*

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ABSTRACT

This paper analyzes a family of solutions to bankruptcy problems that generalizes the Talmud rule (T) and encompasses both the constrained equal-awards rule (A) and the constrained equal-losses rule (L). The family is defined by means of a parameter $\theta \in [0,1]$ that can be interpreted as a measure of the distributive power of the rule. We study the structural properties of this family of rules and provide a characterization result.

KEYWORDS: Bankruptcy Problems; TAL-Family; Characterization Result.

1 Introduction

A bankruptcy problem describes a situation in which an arbitrator has to allocate a given amount $E > 0$ of a perfectly divisible commodity, referred to as the *estate,* among a group N of agents, when the available amount is not enough to satisfy all their claims $(c_i)_{i \in N}$. That is, $E \leq \sum_{i \in N} c_i$. A solution to a bankruptcy problem is a procedure or "rule" that exhibits some desirable properties and determines an allocation, for each specific problem, satisfying two elementary restrictions: (i) No agent gets more than she claims nor less than zero; and (ii) The entire estate is distributed. Note that most rationing problems can be given this form. Since this is a well known problem and examples of these situations abound, we shall not dwell on its relevance. The reader is referred to the works of Young (1994, ch. 4), Thomson (1995) and Moulin (2001) for a review of this literature.

There are four classical solutions to the bankruptcy problem: the proportional solution, usually associated with Aristotle, the constrained equal-awards rule and the constrained equal-losses rule, which can be traced back to Maimonides, and the Talmud rule, which extends the ancient "contested garment principle" [see Herrero & Villar (2001) for a comparative analysis of these solutions]. The first three of these rules implement the idea of equal division, with different reference variables (ratios, awards and losses, respectively). The Talmud rule is an alternative procedure that combines the principles that identify the former three rules. It can be interpreted as implementing a protective criterion that ensures that all agents suffer a rationing that is "of the same sort" as that experienced by the whole society. The distribution procedure depends on whether the estate E is higher or lower than half of the aggregate claim $\sum_{i \in N} c_i$. It can be justified on the psychological principle of "more than half is like the whole, whereas less than a half is like nothing". Thus, this rule considers the size of the awards when they are below half of the aggregate claim ($E \leq \frac{1}{2} \sum_{i \in N} c_i$) and the size of the losses above that amount ($E \geq \frac{1}{2} \sum_{i \in N} c_i$).

The *TAL-family* generalizes this idea by applying exactly the same principle to all possible shares of the estate in the aggregate claim.[1] That is, for any given value of the parameter $\theta \in [0, 1]$, a rule F^θ in this family considers whether E is higher or lower than $\theta \sum_{i \in N} c_i$, and distributes the estate accordingly. The rule associated with $\theta = \frac{1}{2}$ is precisely the Talmud rule, as expected, and the extreme values $\theta = 1$ and $\theta = 0$ correspond to the constrained equal awards rule and the constrained equal losses rule, respectively. The proportional solution, however, is not part of this family. The

[1] See Hokari & Thomson (2000) for a different extension of the Talmud rule.

parameter θ can be interpreted as an index of the distributive power of the rule. Higher values of θ imply that F^{θ} gives more satisfaction to those agents with lower claims, whereas lower values of θ imply that the rule favours those agents with larger claims. In particular, as θ increases the share of the estate proposed by the rule F^{θ} for the smallest claimant increases and the share corresponding to the largest claimant decreases.

The paper is organized as follows. Section 2 contains the model and the preliminary definitions. The structural properties of the TAL-family are discussed in Section 3. First we show that these rules exhibit a precise duality relationship: the dual of the rule associated with the parameter θ is that rule associated with the parameter $(1-\theta)$. Then we analyze how the TAL-family fares with respect to some standard properties that are satisfied by the T, A and L rules. We shall see that there are properties that are satisfied by all members of the family whereas some others are only satisfied by part of them. The characterization of the TAL-family is taken up in Section 4. We show that the TAL-family is made exactly of those rules which are consistent and satisfy equal treatment of equals and weaker versions of four known properties: independence of claims truncation, composition from minimal rights, exclusion and exemption. We also show that the characterization result is tight. A final comment in Section 5 concludes. Most of the proofs have been relegated to an Appendix.

2 The model

Let $\mathbb{N}$ represent the set of all potential agents (a set with an infinite number of members) and let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents $N = \{1, 2, ..., n\}$, where we take $|N| = n$. A **bankruptcy problem** [O'Neill (1982)] is a triple (N, E, c), where N is the set of agents, $E \in \mathbb{R}_+$ represents the **estate** (the amount to be divided), and $c \in \mathbb{R}^n_+$ is a **vector of claims** whose ith component is c_i. The very notion of bankruptcy problem requires $\sum_{i \in N} c_i \geq E > 0$. The family of all those bankruptcy problems is $\mathbb{B}$. To simplify notation we write, for any given problem $(N, E, c) \in \mathbb{B}$, $C = \sum_{i \in N} c_i$. We assume, without loss of generality, that agents are labelled so that $c_1 \leq c_2 \leq ... \leq c_n$.

Definition 1 *A* ***rule*** *is a mapping F that associates with every $(N, E, c) \in \mathbb{B}$ a unique point $F(N, E, c) \in \mathbb{R}^n$ such that:*
(i) $0 \leq F(N, E, c) \leq c$.
(ii) $\sum_{i \in N} F_i(N, E, c) = E$.

The point $F(N, E, c)$ represents a desirable way of dividing E among the agents in N. Requirement (i) is that each agent receives an award that is non-negative and bounded above by her claim. Requirement (ii) is that the entire estate be allocated. These two requirements imply that $F(N, E, c) = c$ whenever $E = \sum_{i \in N} c_i$.

We now consider three different rules. The constrained equal-awards rule distributes the estate equally among the agents, provided no agent ends up with more than she claims. The constrained equal-losses rule selects the point in the budget set that is closest (according to the Euclidean distance) to the claims vector. The outcome imposes equal losses for all the agents with one proviso: no one obtains a negative amount. Finally, the Talmud rule behaves like the constrained equal-awards or the constrained equal losses rule, depending on whether the estate exceeds or falls short of half of the aggregate claims.

Formally:

Definition 2 *The **constrained equal-awards*** $(\mathbf{A})$ *is the rule that, for all* $(N, E, c) \in \mathbb{B}$, *and all* $i \in N$, *yields:*

$$A_i(N, E, c) = \min\{c_i, \lambda\}$$

where $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Definition 3 *The **constrained equal-losses*** $(\mathbf{L})$ *is the rule that, for all* $(N, E, c) \in \mathbb{B}$, *and all* $i \in N$, *yields:*

$$L_i(N, E, c) = \max\{0, c_i - \lambda\}$$

where $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

Definition 4 *The **Talmud*** $(\mathbf{T})$ *is the rule that, for all* $(N, E, c) \in \mathbb{B}$, *and all* $i \in N$, *yields:*

$$T_i(N, E, c) = \begin{cases} \min\{\frac{1}{2}c_i, \lambda\} & \text{if } E \leq \frac{1}{2}C \\ \max\{\frac{1}{2}c_i, c_i - \mu\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ *and* μ *are chosen so that* $\sum_{i \in N} T_i(N, E, c) = E$.

Let us introduce a family of rules that generalizes the Talmud rule (T) and encompasses the constrained equal awards rule (A) and the constrained equal losses rule (L). The analysis of this family provides further insights into the relationship between these distributive criteria.

The Talmud rule is an allocation method that considers the size of the estate with respect to the aggregate claim. Nobody gets more than half of her claim if the estate is less than half of the aggregate claim and nobody gets less than half of her claim if the amount to be distributed exceeds half of the total demand. The TAL-family generalizes this idea by applying the same principle to all the rules that solve the bankruptcy problem depending on the relation between E and θC, for all values θ in the interval $[0,1]$. Formally:

Definition 5 *The* ***TAL-family*** *consists of all rules with the following form: For some $\theta \in [0,1]$, for all $(N,E,c) \in \mathbb{B}$, and all $i \in N$,*

$$F_i^{\theta}(N,E,c) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \leq \theta C \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \geq \theta C \end{cases}$$

where $C = \sum_{i \in N} c_i$ and λ, μ are chosen so that $\sum_{i \in N} F_i^{\theta}(N,E,c) = E$.

A rule F^{θ} in the TAL-family resolves bankruptcy problems according to the following principle: Nobody gets more than a fraction θ of her claim if the estate is less than θ times the aggregate claim and nobody gets less than a fraction θ of her claim if the amount to be distributed exceeds θ times the aggregate claim. Note that the constrained equal-losses rule corresponds to the case $\theta = 0$ ($F^0 = L$), whereas the constrained equal-awards rule corresponds to the other extreme value, $\theta = 1$ ($F^1 = A$). Obviously the Talmud rule is obtained for $\theta = \frac{1}{2}$ ($F^{1/2} = T$). Also observe that, for a given bankruptcy problem $(N,E,c) \in \mathbb{B}$, $\theta = \frac{E}{C}$ yields a solution $F^{E/C}(N,E,c)$ that coincides with the allocation provided by the proportional rule to this bankruptcy problem. Yet, there is no θ for which F^{θ} is the proportional rule (i.e. the proportional rule is *not* a member of the TAL-family).

The value of the parameter θ can be interpreted as a measure of the *distributive power* of the rule, in the following sense. Higher values of θ imply higher protection for those agents with lower claims (more redistribution), whereas lower values of θ entail higher shares for those with larger claims (less redistribution). Therefore, choosing θ means given a degree of priority in the distribution to those agents with lower claims and a degree of priority $(1-\theta)$ to those with higher demands. From this perspective, the Talmud rule is a balanced compromise among the different claimants.

Remark 1 *It can be shown that, fo any given a bankruptcy problem, F^{θ} proposes a distribution that gives the smallest claimant a share of the estate that increases with θ, and gives the largest claimant a share that decreases with θ [see Moreno-Ternero (2001) for further details].*

3 Structural properties of the TAL-family

This section provides a detailed analysis of the properties that are satisfied by the TAL-family, among those which are common in the literature. Since most of these properties are well known, we shall restrict their motivation and interpretation to the minimum.

3.1 Duality

Following Aumann & Maschler (1985) we can define the **dual rule** of F, denoted F^*, as follows: For all $(N,E,c) \in \mathbb{B}$, $F^*(N,E,c) = c - F(N, C-E, c)$. Note that F^* is also a rule defined on $\mathbb{B}$, which satisfies $(F^*)^* = F$. When a rule and its dual produce the same outcomes is called **self-dual**. That is, a rule F is self-dual if, for all $(N,E,c) \in \mathbb{B}$, $F(N,E,c) = F^*(N,E,c)$.

The notion of duality can also be applied to the properties a solution satisfies. That is, $\mathcal{P}^*$ is the **dual property of** $\mathcal{P}$ if for every rule F it is true that F satisfies $\mathcal{P}$ if and only if its dual rule F^* satisfies $\mathcal{P}^*$. It is easy to verify that if a rule F is characterized by a set of properties then the dual rule F^* is characterized by the corresponding set of dual properties.

Our first result shows that there exists a precise duality relationship between the members of the TAL-family. Namely, $(F^\theta)^* = F^{1-\theta}$.

Proposition 1 *Let F^θ be a rule in the TAL-family $\left\{F^\theta\right\}_{\theta\in[0,1]}$. Then, the dual rule of F^θ is $F^{1-\theta}$.*

Proof.

Let $\theta \in [0,1]$ be given and let (N,E,c) be a bankruptcy problem. First, suppose that $E \leq (1-\theta)C$, or equivalently $C - E \geq \theta C$. Let $i \in N$, be a particular claimant. Then, $\left(F_i^\theta\right)^*(N,E,c) = c_i - F_i^\theta(N, C-E, c) = c_i - \max\{\theta c_i, c_i - \mu\} = \min\{(1-\theta)c_i, \mu\} = F_i^{1-\theta}(N,E,c)$.[2]

Now, suppose that $E \geq (1-\theta)C$, or equivalently $C - E \leq \theta C$. In this case, for each $i \in N$, we have $\left(F_i^\theta\right)^*(N,E,c) = c_i - F_i^\theta(N, C-E, c) = c_i - \min\{\theta c_i, \lambda\} = \max\{(1-\theta)c_i, c_i - \lambda\} = F_i^{1-\theta}(N,E,c)$.[3] ■

The following results are immediate consequences:

[2]Note that μ is such that $\sum_{i\in N}\max\{\theta c_i, c_i - \mu\} = C - E$. Thus, $\sum_{i\in N}\min\{(1-\theta)c_i, \mu\} = \sum_{i\in N}(c_i - \max\{\theta c_i, c_i - \mu\}) = E$, and therefore, $\left(F_i^\theta\right)^*(N,E,c) = F_i^{1-\theta}(N,E,c)$.

[3]Note that λ is such that $\sum_{i\in N}\min\{\theta c_i, \lambda\} = C - E$. Thus, $\sum_{i\in N}\max\{(1-\theta)c_i, c_i - \lambda\} = \sum_{i\in N}(c_i - \min\{\theta c_i, \lambda\}) = E$, and therefore, $\left(F_i^\theta\right)^*(N,E,c) = F_i^{1-\theta}(N,E,c)$.

Corollary 1 *L and A are dual rules.*

Corollary 2 *T is a self-dual rule. In fact, there is no other self-dual rule in the TAL-family.*

3.2 Basic properties

Let us consider five basic properties that are satisfied by most bankruptcy rules. "Equal treatment of equals" refers to the impartiality of the rule with regard to those agents with the same characteristics (claims in our model).[4] A stronger requirement is that of "reasonableness" [Herrero (2001)], which states that agents with higher claims receive higher awards and face higher losses. "Scale invariance", implies that the units in which the estate and the claims are measured have no influence on the outcome. "Consistency", requires that if we apply a rule F to a given problem (N, E, c) or do so to any of the associated reduced problems, all incumbent agents get the same outcome. Finally, "continuity" establishes that small changes in the parameters of the problem induce small changes in the corresponding solution function.

Formally:

Definition 6 *A rule satisfies* ***equal treatment of equals*** *if, for all $(N, E, c) \in \mathbb{B}$, $c_i = c_j$ implies $F_i(N, E, c) = F_j(N, E, c)$.*

Definition 7 *A rule is* ***reasonable****, if for all $(N, E, c) \in \mathbb{B}$, all $i, j \in N$, $c_i \geq c_j$ implies that $F_i(N, E, c) \geq F_j(N, E, c)$ and $c_i - c_j \geq F_i(N, E, c) - F_j(N, E, c)$.*

Definition 8 *A rule F satisfies* ***scale invariance*** *if, for all $(N, E, c) \in \mathbb{B}$ and all $\alpha > 0$, we have $F(N, \alpha E, \alpha c) = \alpha F(N, E, c)$.*

Definition 9 *A rule F is* ***consistent*** *if, for all $(N, E, c) \in \mathbb{B}$, all $Q \subset N$, and all $i \in Q$, we have $F_i(N, E, c) = F_i(Q, E_Q, c_Q)$, where $E_Q = \sum_{i \in Q} F_i(N, E, c)$ and $c_Q = (c_i)_{i \in Q}$.*

Definition 10 *A rule is* ***continuous****, if for all $(N, E, c) \in \mathbb{B}$, all $(N, E_q, c_q) \in \mathbb{B}$ such that $\lim_{q \to \infty} E_q = E$, and $\lim_{q \to \infty} c_q = c$, we have $\lim_{q \to \infty} F(N, E_q, c_q) = F(N, E, c)$.*

All rules in the TAL-family satisfy these five properties. Formally:

[4] See Moulin (2000) for the analysis of asymmetric rules.

Proposition 2 *Let F^θ be a rule in the TAL-family $\{F^\theta\}_{\theta\in[0,1]}$. Then F^θ satisfies equal treatment of equals and scale invariance, and is a reasonable, consistent and continuous rule.*

(The proof is given in the Appendix)

Remark 2 *>From these results it follows that the TAL-family is a subset of the "parametric rules", identified by Young (1987) as those rules which satisfy equal treatment of equals, continuity and consistency [see Moreno-Ternero (2001) for a parametric representation of the TAL-family].*

3.3 Independence of claims truncation and composition from minimal rights

Dagan (1996) introduces the property of "independence of claims truncation" in order to characterize the constrained equal awards rule. This property establishes that one cannot claim more than there is. Hence, all claims that exceed the estate are truncated. Formally:

Definition 11 *A rule F satisfies* ***independence of claims truncation*** *if, for all $(N, E, c) \in \mathbb{B}$, $F(N, E, c) = F(N, E, c^T)$, where $c_i^T = \min\{E, c_i\}$ for all $i \in N$.*

Aumann & Maschler (1985) propose the notion of "composition from minimal rights" to deal with the characterization of the Talmud rule. This property ensures each agent a minimal amount $m_i(N, E, c) = \max\{0,\ E - \sum_{j\in N-\{i\}} c_j\}$, which is the portion of the estate that is left to the ith agent when the claims of all other agents are fully honored, provided this amount is nonnegative.

Definition 12 *A rule F satisfies* ***composition from minimal rights*** *if, for all $(N, E, c) \in \mathbb{B}$,*

$$F(N, E, c) = m(N, E, c) + F\left[N,\ E - \sum_{i\in N} m_i(N, E, c),\ c - m(N, E, c)\right]$$

where $m(N, E, c) = [m_i(N, E, c)]_{i\in N}$.

Independence of claims truncation and composition from minimal rights are dual properties [Herrero & Villar (2001, Claim 5)]. As the following proposition shows, $\theta = \frac{1}{2}$ is the precise value of the parameter that separates those rules in the family that satisfy independence of claims truncation from those that satisfy composition from minimal rights. A direct consequence of this result is that there is only one rule in the family that satisfies these two properties simultaneously: the Talmud rule, $T = F^{1/2}$. Formally:

Proposition 3 *Let F^{θ} be a rule in the TAL-family $\left\{F^{\theta}\right\}_{\theta\in[0,1]}$. Then:*
(i) F^{θ} satisfies independence of claims truncation if and only if $\theta \in \left[\frac{1}{2}, 1\right]$.
(ii) F^{θ} satisfies composition from minimal rights if and only if $\theta \in \left[0, \frac{1}{2}\right]$.

(The proof is given in the Appendix)

Corollary 3 *There is one and only one rule in the TAL-family that satisfies simultaneously independence of claims truncation and composition from minimal rights. It is the Talmud rule, $T = F^{1/2}$.*

This proposition shows that none of these two properties can be satisfied by all members of the family. One can informally say that independence of claims truncation "cuts too much" the rights of the agents with large claims whereas composition from minimal rights induces "excessive concessions" in favour of these agents. Therefore, to extend those principles in order to encompass all members of the TAL-family we have to relax the associated restrictions and so avoid both excessive and insufficient redistribution. That can be done by linking the extent of "cuts and concessions" to the smallest claim rather than to the agents' actual demands.

The notions of $\theta-$independence and $\theta-$composition formalize this approach and relate the strength of the principles of independence and composition to the distributive power of the rule (the parameter θ) and the size of the smallest claim (recall that $\theta \in [0, 1]$, and $c_1 \leq c_i$ for all $i \in N$, by assumption).

Axiom 1 ($\theta-$independence) *For all $(N, E, c) \in \mathbb{B}$ such that $\frac{E}{n} < \theta c_1$, $F(N, E, c) = F(N, E, c^{\theta})$, where $c_i^{\theta} = \theta c_1$ for all $i \in N$.*

This condition shares the spirit of independence from claims truncation. It establishes that when the estate is so small that equal division does not even cover a fraction θ of the smallest claim, then the solution to the problem coincides with that in which actual claims are truncated by θc_1. Note that for low values of θ this property imposes no restriction. In particular, every rule satisfies $0-$independence vacuously.

Axiom 2 ($\theta-$composition) *For all* $(N, E, c) \in \mathbb{B}$ *such that* $\frac{C-E}{n} < (1-\theta)c_1$, $F(N, E, c) = m^{\theta}(c) + F(N, E - \sum_{i \in N} m_i^{\theta}(c), c - m^{\theta}(c))$, *where* $m_i^{\theta}(c) = c_i - (1-\theta)c_1$ *for all* $i \in N$.

The axiom of $\theta-$composition states that those problems in which the average loss is smaller than a fraction $(1-\theta)$ of the smallest claim, are to be solved according to the following two-step procedure. First, concede each agent an amount which corresponds to her original claim reduced by a fraction $(1-\theta)$ of the smallest claim. Then distribute the remainder estate according to the outstanding claims. Notice that every rule satisfies $1-$composition, vacuously.[5]

The next result establishes a precise duality relationship between both concepts and shows the fulfillment of these properties within the TAL-family.

Proposition 4 *The following two statements hold:*

(i) For all $\theta \in [0, 1)$, $\theta-$*composition is the dual rule of* $(1-\theta)-$*independence.*

(ii) Let F^{θ} *be a rule in the TAL-family* $\left\{F^{\theta}\right\}_{\theta \in [0,1]}$. *Then* F^{θ} *satisfies* $\theta-$*independence and* $\theta-$*composition.*

(The proof is given in the Appendix)

3.4 Exemption and exclusion

The principles of "exemption" and "exclusion" are introduced in Herrero & Villar (2001) in order to characterize the constrained equal awards rule and the constrained equal losses rule, respectively. They are dual properties which impose restrictions on the behavior of a rule when claims are very unequal.

Definition 13 *A rule* F *satisfies* ***exemption*** *if, for all* $(N, E, c) \in \mathbb{B}$, $c_i \leq \frac{E}{n} \Longrightarrow F_i(N, E, c) = c_i$.

This property can be interpreted as an extreme form of protection of those agents with very small claims: Those claims that are below equal division of the estate should be fully honored.

Definition 14 *A rule* F *satisfies* ***exclusion*** *if, for all* $(N, E, c) \in \mathbb{B}$, *if* $c_i \leq \frac{C-E}{n} \Longrightarrow F_i(N, E, c) = 0$.

[5] When $\theta = \frac{1}{2}$, we shall talk about mid-independence and mid-composition.

The principle of exclusion, on the contrary, aims at protecting those agents with very large deficits: Those agents whose claims are smaller than the average loss are to be disregarded. The constrained equal-awards rule and the constrained equal-losses rule represent well these two principles.

The Talmud rule provides a protective criterion of a different nature: all agents suffer a rationing that is "of the same sort" of that experienced by the whole society. Therefore, the rights of those agents with small claims and the rights of those agents with large deficits are only partially protected by T.

The following result is straightforward:

Proposition 5 *The only rule in the TAL-family that satisfies exclusion is* $F^0 = L$. *The only rule in the TAL-family that satisfies exemption is* $F^1 = A$.

This result suggests the idea of introducing the concepts of $\theta-$exemption and $\theta-$exclusion as extensions of the original notions that can encompass other rules, including the Talmud. Let us present these notions and see how these new concepts apply to the analysis of the TAL-family.

"$\theta-$exemption" shares with the original notion of exemption the idea that when the resources to be divided are large enough, relative to the agents' claims, those individuals with larger claims are to be rationed more intensively. More precisely, $\theta-$exemption establishes that when a fraction θ of an agent's claim is smaller than equal division of the estate, the rule should grant her at least a share θ of her claim.

Formally:

Axiom 3 ($\theta-$exemption) *For all* $(N, E, c) \in \mathbb{B}$ *such that* $\theta c_i \leq \frac{E}{n}$ *we have* $F_i(N, E, c) \geq \theta c_i$.

The property of exemption corresponds to the extreme case $\theta = 1$. Note that $\theta = 0$ imposes no restriction on a rule, as it simply says that awards cannot be negative.

The following property, $\theta-$exclusion, conveys the opposite message: If a fraction $(1 - \theta)$ of an agent's claim is below the average loss, this agent will get at most a share θ of her claim. Formally:

Axiom 4 ($\theta-$exclusion) *For all* $(N, E, c) \in \mathbb{B}$ *such that* $(1 - \theta)c_i \leq \frac{C-E}{n}$ *we have* $F_i(N, E, c) \leq \theta c_i$.

The property of exclusion corresponds to the extreme case $\theta = 0$. In this case the value $\theta = 1$ imposes no restriction on a rule, as it amounts to saying that no agent can receive more than she claims. When $\theta = \frac{1}{2}$, we shall talk about **mid-exemption** and **mid-exclusion,** respectively.

The next result tells us about the bite of these properties.

Proposition 6 *The following two statements hold:*

(i) For all $\theta \in (0,1]$, $\theta-$exemption is the dual property of $(1-\theta)-$exclusion.

(ii) Let F^{θ} be a rule in the TAL-family $\left\{F^{\theta}\right\}_{\theta\in[0,1]}$. Then F^{θ} satisfies $\theta-$exemption and $\theta-$exclusion.

(The proof is given in the Appendix)

4 Characterization results

We provide in this section a number of characterization results based on the four $\theta-$axioms presented above. We show first that the only rules that satisfy $\theta-$independence, $\theta-$composition, $\theta-$exemption, $\theta-$exclusion and consistency, for $\theta \in (0,1)$, are precisely those rules in the TAL-family with parameter θ. Since we have already demonstrated that all the rules in the TAL-family satisfy those five properties, the characterization follows.

Interestingly enough we do not have to assume equal treatment of equals to characterize the TAL-family. For two-person bankruptcy problems, however, this property can be deduced from $\theta-$independence, $\theta-$composition and $\theta-$exemption, provided $\theta \in (0,1)$ (resp. $\theta-$independence, $\theta-$composition and $\theta-$exclusion). This is the content of Lemma 1 below.

Next we address the characterization of the L rule and the A rule, that correspond to the extreme values $\theta = 0$ and $\theta = 1$, respectively. The characterization of these two rules must be taken up separately because, as already noticed, the $\theta-$axioms loose their bite for the extreme values $\theta = 0$ and $\theta = 1$. As a consequence, equal treatment of equals cannot be inferred from these properties. In other words, Lemma 1 does not hold for $\theta = 0$ and $\theta = 1$.

Finally, we provide a specific characterization of the Talmud rule making use of self-duality.

It is worth mentioning that all characterization results are tight. That is, if we drop one of the axioms then they are no longer true.

The following preliminary result plays a relevant role in the proof of our first characterization theorem.

Lemma 1 *For two-agent bankruptcy problems, $\theta-$independence, $\theta-$composition and $\theta-$exemption, imply equal treatment of equals, for each $\theta \in (0,1)$ (resp. $\theta-$independence, $\theta-$composition and $\theta-$exclusion, imply equal treatment of equals, for each $\theta \in (0,1)$).*

(The proof is given in the Appendix)

Lemma 1 fails for the extreme values $\theta = 0$ and $\theta = 1$ as the following example shows.

Example 1 *Let* $\overline{B}$ *denote the subset of two-agent bankruptcy problems* $\overline{B} = \{(\{1,2\},\ E,\ (1,E)) : E \geq 1\}$ *and consider the following rule:*

$$f(N,E,c) = \begin{cases} L(N,E,c) & \text{if } (N,E,c) \notin \overline{B} \\ \left(\frac{1}{3}, E-\frac{1}{3}\right) & \text{if } (N,E,c) \in \overline{B} \end{cases}.$$

The rule f *is well defined and satisfies consistency, exclusion and* $0-$*composition (see the Appendix for the details). To see that this rule does not satisfy equal treatment of equals it suffices to take the problem* $(N,E,c) = (\{1,2\},1,(1,1))$, *whose solution according to the rule* f *is given by:* $f(N,E,c) = (\frac{1}{3},\frac{2}{3})$. *This shows that equal treatment of equals fails for* $\theta = 0$ *under the assumptions of the Lemma. To see that the result does not hold for* $\theta = 1$ *it is enough to consider the rule* f^* *(the dual rule of* f*).*

We now present our main result in this section:

Theorem 1 *A rule satisfies consistency,* $\theta-$*independence,* $\theta-$*composition,* $\theta-$*exemption and* $\theta-$*exclusion, for* $\theta \in (0,1)$, *if and only if it is the member of the TAL-family with parameter* θ.

Proof.

Propositions 4 and 6 show that, for all $\theta \in (0,1)$, F^θ satisfies $\theta-$independence, $\theta-$composition, $\theta-$exclusion and $\theta-$exemption. Furthermore, they also satisfy consistency (Proposition 2).

Now let us see the converse.

Let $\theta \in (0,1)$ be fixed. Let f be a rule that satisfies consistency, $\theta-$independence, $\theta-$composition, $\theta-$exclusion and $\theta-$exemption. By consistency it is enough to prove the result for the two-agent case. In this case Lemma 1 ensures that f also satisfies equal treatment of equals. Without loss of generality let (N,E,c) be a bankruptcy problem with $N = \{1,2\}$, and $c_1 \leq c_2$. In these circumstances, it is straightforward to show that F^θ can be expressed as:

$$F^\theta(N,E,c) = \begin{cases} \left(\frac{E}{2},\frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E-\theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 + (2\theta-1)\,c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 + (2\theta-1)\,c_1 \leq E \end{cases}.$$

There are several cases to be discussed.

Case 1.- $2\theta c_1 \leq E \leq c_2 + (2\theta-1)\,c_1$.

It is straightforward to see that in this case, $\theta c_1 \leq \frac{E}{2}$, and $(1-\theta)c_1 \leq \frac{C-E}{2}$. The first upper bound and $\theta-$exemption imply that $f_1(N,E,c) \geq \theta c_1$. The second upper bound and $\theta-$exclusion imply that $f_1(N,E,c) \leq \theta c_1$. As a result, $f(N,E,c) = (\theta c_1, E-\theta c_1) = F^\theta(N,E,c)$.

Case 2.- $E < 2\theta c_1$.

It follows from $\theta-$independence that $f(N,E,c) = f(N,E,(\theta c_1,\theta c_1))$. Then, equal treatment of equals implies $f(N,E,(\theta c_1,\theta c_1)) = \left(\frac{E}{2},\frac{E}{2}\right) = F^{\theta}(N,E,c)$.

Case 3.- $c_2 + (2\theta - 1)\,c_1 < E$.

First note that the restriction that defines this case can be rewritten as: $\frac{C-E}{2} < (1-\theta)c_1$. Then, $\theta-$composition says that $f(N,E,c) = m^{\theta}(c) + f(N, 2(1-\theta)c_1 - (C-E), c - m^{\theta}(c))$, where $m^{\theta}(c) = (\theta c_1, c_2 - (1-\theta)c_1)$. By equal treatment of equals we know that $f(N, 2(1-\theta)c_1 - (C-E), c - m^{\theta}(c)) = ((1-\theta)c_1 - \frac{C-E}{2}, (1-\theta)c_1 - \frac{C-E}{2})$. Therefore, $f_i(N,E,c) = c_i - \frac{C-E}{2} = F_i^{\theta}(N,E,c)$, for $i = 1,2$. ■

Let us show that all the properties in Theorem 1 are independent, with some examples. In each case, we mention the property that is not satisfied. The reader may consult the Appendix to verify that in every case the proposed rule satisfies all the properties in the theorem except the one that is explicitly mentioned.

- $\theta-$*exemption*: Suppose $|N| = 2$. Let $g^{\theta}(N,E,c) = (\frac{\theta}{4},\frac{7\theta}{4})$ if $(N,E,c) = (\{1,2\}, 2\theta, (\frac{1}{2},\frac{5}{2}))$, and $F^{\theta}(N,E,c)$ otherwise.
- $\theta-$*exclusion*: Suppose $|N| = 2$. Let $g^{\theta}(N,E,c) = (\frac{1}{2},\frac{1}{2}+2\theta)$ if $(N,E,c) = (\{1,2\}, 2\theta+1, (\frac{1}{2},\frac{5}{2}))$, and $F^{\theta}(N,E,c)$ otherwise.
- $\theta-$*independence*: Suppose $|N| = 2$. Let $g^{\theta}(N,E,c) = (0,1)$ if $(N,E,c) = (\{1,2\}, 1, (\frac{1}{\theta},\frac{2}{\theta}))$, and $F^{\theta}(N,E,c)$ otherwise.
- $\theta-$*composition*: Suppose $|N| = 2$. Let $g^{\theta}(N,E,c) = (\theta, 2)$ if $(N,E,c) = (\{1,2\}, 2+\theta, (1,2))$, and $F^{\theta}(N,E,c)$ otherwise.
- *Consistency*: Let $g^{\theta}(N,E,c) = (\theta,\theta,2\theta)$ if $(N,E,c) = (\{1,2,3\}, 4\theta, (1,2,3))$, and $F^{\theta}(N,E,c)$ otherwise.

Let us now consider the rules $L = F^0$ and $A = F^1$. To characterize these rules one has to assume that equal treatment of equals holds (see Example 1 above). The following results are obtained:

Theorem 2 *The constrained equal losses rule (F^0) is the only rule satisfying consistency, equal treatment of equals,* $0-$*composition and exclusion.*

Theorem 3 *The constrained equal awards rule (F^1) is the only rule satisfying consistency, equal treatment of equals,* $1-$*independence and exemption.*

(Proofs are omitted since they are easily adapted from that in Theorem 1).

As a direct consequence of Theorems 1, 2 and 3, and Lemma 1, we can provide a characterization result for the entire family, as follows:

Corollary 4 *A rule f satisfies consistency, equal treatment of equals, $\theta-$composition, $\theta-$independence, $\theta-$exemption and $\theta-$exclusion, if and only if it is the member of the TAL-family F^{θ}.*

We conclude with some considerations about the Talmud rule, one of the most special members of the TAL-family. Since $F^{1/2} = T$ is the unique self-dual rule within the TAL-family [c.f. Corollary 2], we can state some similar characterization results to Theorem 1, for this rule, upon replacing some of the partial conditions by self-duality.

Theorem 4 *A rule satisfies consistency, self-duality, mid-exclusion (or mid-exemption) and mid-composition (or mid-independence), if and only if it is the Talmud rule.*

Note that self-duality is an essential property in Theorem 4, and also that the axioms stated in this Theorem are independent from those stated in Theorem 1, as shown in the Appendix.

5 Final Remarks

We have presented in this paper a family of bankruptcy rules, the TAL-family, that generalizes the Talmud rule and encompasses the constrained equal awards rule and the constrained equal losses rule. This family depends on a parameter $\theta \in [0, 1]$ that refers to the relative magnitude of the estate with respect to the aggregate claim and can be interpreted in terms of an index of the distributive power of the rule. When the ratio between both magnitudes is below the parameter θ, nobody gets more than a share θ of her claim. When that ratio exceeds the value of the parameter θ nobody gets less than a share θ of her claim. The rule associated with the parameter $\theta = \frac{1}{2}$ is precisely the Talmud rule, as expected, and the extreme values $\theta = 1$ and $\theta = 0$ correspond to the constrained equal awards rule and the constrained equal losses rule, respectively.

We have analyzed the behavior of the rules in the TAL-family with respect to some standard properties and have also provided some characterization results. In particular, we have shown that the TAL-family is made exactly of those rules which are consistent and satisfy equal treatment of equals, $\theta-$exemption, $\theta-$exclusion, $\theta-$independence and $\theta-$composition. The last four properties are weaker versions of four known principles which turn out to be too sharp to cover all rules within the TAL-family.

Let us conclude by summarizing our main findings in the following table, which shows the properties satisfied by the members of the TAL family. An asterisk indicates that those properties characterize the entire family.

Properties	**Rules that satisfy the properties**
(∗) *Consistency*	F^θ for all $\theta \in [0,1]$
(∗) *Equal treatment of equals*	F^θ for all $\theta \in [0,1]$
Reasonableness	F^θ for all $\theta \in [0,1]$
Continuity	F^θ for all $\theta \in [0,1]$
Scale Invariance	F^θ for all $\theta \in [0,1]$
Self-duality	$F^{1/2} = T$
Independence of claims truncation	F^θ for all $\theta \in [\frac{1}{2},1]$
Composition from minimal rights	F^θ for all $\theta \in [0,\frac{1}{2}]$
Exemption	$F^1 = A$
Exclusion	$F^0 = L$
(∗) *$\theta-$independence*	F^θ for all $\theta \in [0,1]$
(∗) *$\theta-$composition*	F^θ for all $\theta \in [0,1]$
(∗) *$\theta-$exemption*	F^θ for all $\theta \in [0,1]$
(∗) *$\theta-$exclusion*	F^θ for all $\theta \in [0,1]$

6 Appendix 1: Proofs

Proposition 2 *Let F^θ be a rule in the TAL-family $\left\{F^\theta\right\}_{\theta\in[0,1]}$. Then F^θ satisfies equal treatment of equals and scale invariance, and is a reasonable, consistent and continuous rule.*

Proof.

Since equal treatment of equals and continuity are trivial, we only discuss explicitly reasonableness, scale invariance and consistency.

(i) *F^θ satisfies scale invariance.*

Case 1.- Let $\theta \in [0,1]$ and $(N,E,c) \in \mathbb{B}$ with $E < \theta C$. Since $\alpha E < \alpha\theta C$ for all $\alpha > 0$, it follows that $F_i^\theta(N,E,c) = \min\{\theta c_i, \lambda\}$, and $F_i^\theta(N,\alpha E,\alpha c) = \min\{\theta\alpha c_i, \lambda'\}$, for all $i \in N$, where λ and λ' are such that $\sum_{i\in N} F_i^\theta(N,E,c) = E$ and $\sum_{i\in N} F_i^\theta(N,\alpha E,\alpha c) = \alpha E$, respectively. Now, there exists $\overline{\lambda} > 0$, such that $\lambda' = \alpha\overline{\lambda}$. Thus, $\min\{\theta\alpha c_i, \lambda'\} = \alpha\min\left\{\theta c_i, \overline{\lambda}\right\}$. Assume, without loss of generality, that $c_n = \max_{i\in N}\{c_i\}$. Then it is straightforward to see that the function $H^\theta : [0,\theta c_n] \to \mathbb{R}_+$ such that $H^\theta(\lambda) = \sum_{i\in N}\min\{\theta c_i,\lambda\}$ is a piecewise linear and strictly increasing function. Therefore, for all $E \in [0,\theta C)$ there exists a unique λ_0 such that $H^\theta(\lambda_0) = E$. This implies $\lambda = \overline{\lambda} = \lambda_0$, and therefore, $F_i^\theta(N,\alpha E,\alpha c) = \alpha F_i^\theta(N,E,c)$, for all $i \in N$.

Case 2.- Let $\theta \in [0,1]$ and $(N,E,c) \in \mathbb{B}$ with $E > \theta C$. As $\alpha E > \alpha\theta C$ for all $\alpha > 0$, we have $F_i^\theta(N,E,c) = \max\{\theta c_i, c_i - \mu\}$, and $F_i^\theta(N,\alpha E,\alpha c) = \max\{\theta\alpha c_i, \alpha c_i - \mu'\}$, for all $i \in N$, where μ and μ' are such that $\sum_{i\in N} F_i^\theta(N,E,c) = E$ and $\sum_{i\in N} F_i^\theta(N,\alpha E,\alpha c) = \alpha E$, respectively. Now, there exists $\overline{\mu} > 0$, such that $\mu' = \alpha\overline{\mu}$. Thus, $\max\{\theta\alpha c_i, \alpha c_i - \mu'\} = \alpha\max\{\theta c_i, c_i - \overline{\mu}\}$. Take again $c_n = \max_{i\in N}\{c_i\}$. It is immediate to check that the function $G^\theta : [0,(1-\theta)c_n] \to \mathbb{R}_+$ such that $G^\theta(\mu) = \sum_{i\in N}\max\{\theta c_i, c_i - \mu\}$ is a piecewise linear and strictly increasing function. Therefore, for all $E \in (\theta C, C]$ there exists a unique μ_0 such that $G^\theta(\mu_0) = E$. This implies $\mu = \overline{\mu} = \mu_0$, and therefore, $F_i^\theta(N,\alpha E,\alpha c) = \alpha F_i^\theta(N,E,c)$, for all $i \in N$.

Case 3.- Let $\theta \in [0,1]$ and $(N,E,c) \in \mathbb{B}$ with $E = \theta C$. Then $F_i^\theta(N,E,c) = \theta c_i$, for each $i \in N$, and therefore, scale invariance is trivially satisfied.

(ii) *F^θ is a reasonable rule.*

Case 1.- Let $\theta \in [0,1]$ and $(N,E,c) \in \mathbb{B}$ with $E \leq \theta C$. Thus, $F_i^\theta(N,E,c) = \min\{\theta c_i, \lambda\}$, for all $i \in N$. Let $i,j \in N$, such that $c_i \geq c_j$. It is straightforward that $F_i^\theta(N,E,c) \geq F_j^\theta(N,E,c)$. Now, suppose that $\lambda < \theta c_j \leq \theta c_i$. In this case, $c_i - F_i^\theta(N,E,c) = c_i - \lambda \geq c_j - \lambda = c_j - F_j^\theta(N,E,c)$. On the other hand, if $\theta c_i \geq \lambda \geq \theta c_j$ then $c_i - F_i^\theta(N,E,c) = c_i - \lambda \geq (1-\theta)c_i \geq (1-\theta)c_j = c_j - F_j^\theta(N,E,c)$. Finally, if $\lambda > \theta c_i$ then $c_i - F_i^\theta(N,E,c) = c_i - \theta c_i \geq c_j - \theta c_j = c_j - F_j^\theta(N,E,c)$.

Case 2.- Let $\theta \in [0,1]$ and $(N,E,c) \in \mathbb{B}$ with $E \geq \theta C$. Thus, $F_i^\theta(N,E,c) = \max\{\theta c_i, c_i - \mu\}$, for all $i \in N$. Let $i,j \in N$, such that $c_i \geq c_j$. It is straightforward that $F_i^\theta(N,E,c) \geq F_j^\theta(N,E,c)$. Now, suppose that $\mu < (1-\theta)c_j$. In this case, $c_i - F_i^\theta(N,E,c) = \mu = c_j - F_j^\theta(N,E,c)$. On the other hand, if $(1-\theta)c_i \geq \mu \geq (1-\theta)c_j$ then $c_i - F_i^\theta(N,E,c) = \mu \geq c_j - \theta c_j = c_j - F_j^\theta(N,E,c)$. Finally, if $\mu > (1-\theta)c_i$ then $c_i - F_i^\theta(N,E,c) = c_i - \theta c_i \geq c_j - \theta c_j = c_j - F_j^\theta(N,E,c)$.

(iii) F^θ *is a consistent rule.*

Consider a problem (N,E,c) and let $F^\theta(N,E,c)$ stand for the solution given by rule F^θ. Let $Q \subset N$ and define a new problem (Q,E_Q,c_Q), where $E_Q = \sum_{i\in Q} F_i^\theta(N,E,c)$, $c_Q = (c_i)_{i\in Q}$, and has a solution $F^\theta(Q,E_Q,c_Q)$. First observe that if $E \leq \theta C$, then $E_Q \leq \theta C_Q = \theta \sum_{i\in Q} c_i$. Therefore, in this case, for all $i \in Q$, we have $F_i^\theta(N,E,c) = \min\{\theta c_i, \lambda\}$, with $\sum_{i\in N} \min\{\theta c_i, \lambda\} = E$, and $F_i^\theta(Q,E_Q,c_Q) = \min\{\theta c_i, \lambda_Q\}$, with $\sum_{i\in Q} \min\{\theta c_i, \lambda_Q\} = E_Q$. Several cases are to be considered:

Case 1.- $\lambda = \lambda_Q$. In this case, $F_i^\theta(N,E,c) = F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$, and consistency follows.

Case 2.- When $\lambda < \lambda_Q$, $F_i^\theta(N,E,c) \leq F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$. Now, suppose that there exists some $i_0 \in Q$ such that $F_{i_0}^\theta(N,E,c) < F_{i_0}^\theta(Q,E_Q,c_Q)$. Thus, we would have $E_Q = \sum_{i\in Q} F_i^\theta(N,E,c) < \sum_{i\in Q} F_i^\theta(Q,E_Q,c_Q) = E_Q$, which is a contradiction. Therefore, $F_i^\theta(N,E,c) = F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$.

Case 3.- Let now $\lambda > \lambda_Q$. Then, $F_i^\theta(N,E,c) \geq F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$. Suppose, for the sake of contradiction, that there exists $i_0 \in Q$ such that $F_{i_0}^\theta(N,E,c) > F_{i_0}^\theta(Q,E_Q,c_Q)$. As in the previous case, we would have $E_Q = \sum_{i\in Q} F_i^\theta(N,E,c) > \sum_{i\in Q} F_i^\theta(Q,E_Q,c_Q) = E_Q$, which is a contradiction. Therefore, $F_i^\theta(N,E,c) = F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$.

A similar argument can be applied to the case in which $E \geq \theta C$. Under this new framework, $E_Q \geq \theta C_Q$. Thus, for all $i \in Q$, we have $F_i^\theta(N,E,c) = \max\{\theta c_i, c_i - \mu\}$, with $\sum_{i\in N} \max\{\theta c_i, c_i - \mu\} = E$, and $F_i^\theta(Q,E_Q,c_Q) = \max\left\{\theta c_i, c_i - \mu_Q\right\}$, with $\sum_{i\in Q} \max\left\{\theta c_i, c_i - \mu_Q\right\} = E_Q$. As before, we have several cases to be considered:

Case 1.- $\mu = \mu_Q$. In this case, $F_i^\theta(N,E,c) = F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$.

Case 2.- $\mu > \mu_Q$. Now, $F_i^\theta(N,E,c) \leq F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$. Now, suppose that there exists some $i_0 \in Q$ such that $F_{i_0}^\theta(N,E,c) < F_{i_0}^\theta(Q,E_Q,c_Q)$. Thus, we would have $E_Q = \sum_{i\in Q} F_i^\theta(N,E,c) < \sum_{i\in Q} F_i^\theta(Q,E_Q,c_Q) = E_Q$, which is a contradiction. Therefore, $F_i^\theta(N,E,c) = F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$.

Case 3.- $\mu < \mu_Q$. Thus, $F_i^\theta(N,E,c) \geq F_i^\theta(Q,E_Q,c_Q)$ for all $i \in Q$. Now, for the sake of contradiction, suppose that there exists some

$i_0 \in Q$ such that $F^{\theta}_{i_0}(N, E, c) > F^{\theta}_{i_0}(Q, E_Q, c_Q)$. Then, we would have $E_Q = \sum_{i \in Q} F^{\theta}_i(N, E, c) > \sum_{i \in Q} F^{\theta}_i(Q, E_Q, c_Q) = E_Q$, which is a contradiction. Therefore, $F^{\theta}_i(N, E, c) = F^{\theta}_i(Q, E_Q, c_Q)$ for all $i \in Q$.

The proof is in this way complete. ■

Proposition 3 *Let F^{θ} be a rule in the TAL-family $\{F^{\theta}\}_{\theta \in [0,1]}$. Then:*

(i) F^{θ} satisfies independence of claims truncation if and only if $\theta \in \left[\frac{1}{2}, 1\right]$.

(ii) F^{θ} satisfies composition from minimal rights if and only if $\theta \in \left[0, \frac{1}{2}\right]$.

Proof.

(ia) *Independence of claims truncation implies $\theta \in [\frac{1}{2}, 1]$.*

For $\theta = 0$ we have $F^0 = L$ which does not satisfy this property [e.g. Dagan (1996, Prop. 1)]. Take now θ in the open interval $\left(0, \frac{1}{2}\right)$ and consider the following two-person bankruptcy problem: $\left[\{1,2\}, E, \left(E, \frac{E}{\theta}\right)\right]$. It is straightforward to check that $F^{\theta}(\{1,2\}, E, c) = (\theta E, (1-\theta) E)$. The associated bankruptcy problem in which claims are truncated by the estate is $[\{1,2\}, E, (E, E)]$, whose solution is $F^{\theta}_i\left(\{1,2\}, E, c^T\right) = \frac{E}{2}$, for $i = 1, 2$. Therefore, $F^{\theta}(\{1,2\}, E, c) \neq F^{\theta}\left(\{1,2\}, E, c^T\right)$. As a consequence, F^{θ} does not satisfy independence of claims truncation for $\theta \in [0, \frac{1}{2})$.

(ib) $\theta \in [\frac{1}{2}, 1]$ *implies independence of claims truncation.*

Let now $\theta \in \left[\frac{1}{2}, 1\right]$, and let $(N, E, c) \in \mathbb{B}$ be given. We will prove this part by induction in the cardinality of N.

Suppose first $|N| = 2$, i.e. the two-claimant case. Without loss of generality let us suppose that $N = \{1, 2\}$, and $c_1 \leq c_2$. As we mentioned above, in these circumstances the rule F^{θ} can be expressed as:

$$F^{\theta}(N, E, c) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E - \theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 + (2\theta - 1) c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 + (2\theta - 1) c_1 \leq E \end{cases}.$$

There are several cases to be discussed.

Case 1.- $E \leq c_1$. Since $\theta \geq \frac{1}{2}$, it follows that $E \leq 2\theta c_1$. Thus, $F^{\theta}(N, E, c) = \left(\frac{E}{2}, \frac{E}{2}\right)$. Moreover, in this case, $c^T = (E, E)$. Now, every rule in the TAL-family satisfies equal treatment of equals, which implies $F^{\theta}\left(N, E, c^T\right) = \left(\frac{E}{2}, \frac{E}{2}\right)$.

Case 2.- $c_1 < E < c_2$. In this case, $c^T = (c_1, E)$. Now, if $E \leq 2\theta c_1$, then $F^{\theta}(N, E, c) = (\frac{E}{2}, \frac{E}{2}) = F^{\theta}\left(N, E, c^T\right)$. If, on the other hand, $E > 2\theta c_1$, and since $\theta \geq \frac{1}{2}$, we would have $E \leq c_2 + (2\theta - 1) c_1$, which implies $F^{\theta}(N, E, c) = (\theta c_1, E - \theta c_1)$. Similarly, since $\theta \geq \frac{1}{2}$, it follows that $E \leq E + (2\theta - 1) c_1 = c^T_2 + (2\theta - 1) c_1$, and therefore $F^{\theta}\left(N, E, c^T\right) = (\theta c_1, E - \theta c_1)$.

Case 3.- $E \geq c_2$. This would imply $c^T = c$, and therefore independence of claims truncation would hold trivially.

As a consequence, for every two-claimant bankruptcy problem, F^θ satisfies independence of claims truncation, when $\theta \in \left[\frac{1}{2}, 1\right]$. Let us now assume that it is also true when $|N| = k > 2$, and we will prove it for the case $|N| = k+1$. Without loss of generality, assume that $N = \{1, 2, ..., k+1\}$ and $c_1 \leq c_2 \leq ... \leq c_{k+1}$.

Let us show first that $F_1^\theta(N, E, c) = F_1^\theta\left(N, E, c^T\right)$. It is straightforward to see that

$$F_1^\theta(N, E, c) = \begin{cases} \frac{E}{k+1} & \text{if } E \leq (k+1)\theta c_1 \\ \theta c_1 & \text{if } (k+1)\theta c_1 \leq E \leq C - (k+1)(1-\theta)c_1 \\ c_1 - \frac{C-E}{k+1} & \text{if } C - (k+1)(1-\theta)c_1 \leq E \end{cases}.$$

Suppose that $c_1 < E$, i.e. $c_1 = c_1^T$.[6] As a result, $c_1 \leq c_j^T$, for all $j = 2, ..., k+1$. We can also assume that $c_{k+1} \geq E$, i.e. $c_{k+1}^T = E$.[7]

If $E \leq (k+1)\theta c_1$, then $F_1^\theta(N, E, c) = \frac{E}{k+1} = F_1^\theta\left(N, E, c^T\right)$. Now, assume that $E > (k+1)\theta c_1$. Let us show that $C^T - (k+1)(1-\theta)c_1 > E$, where $C^T = \sum_{i \in N} c_i^T$. This would imply $F_1^\theta\left(N, E, c^T\right) = \theta c_1$. To see this, note the following chain of inequalities, where it has been used that $\theta \geq \frac{1}{2}$, $c_1 = c_1^T$, $c_{k+1}^T = E$ and $c_1 \leq c_j^T$, for all $j = 2, ..., k+1$.

$$\begin{aligned} C^T - (k+1)(1-\theta)c_1 &= E + \sum_{i=2}^{k} c_i^T + [1 - (k+1)(1-\theta)]c_1 \\ &\geq E + \sum_{i=2}^{k} c_i^T - (k-1)\frac{c_1}{2} \\ &\geq E + \sum_{i=2}^{k} c_i^T - \frac{1}{2}\sum_{i=2}^{k} c_i^T \\ &= E + \frac{1}{2}\sum_{i=2}^{k} c_i^T > E. \end{aligned}$$

Consequently, it is also true that $C - (k+1)(1-\theta)c_1 > E$, and we have $F_1^\theta(N, E, c) = \theta c_1 = F_1^\theta\left(N, E, c^T\right)$.

Once it is shown that $F_1^\theta(N, E, c) = F_1^\theta\left(N, E, c^T\right)$, we conclude the proof, making use of consistency. Among the set of claimants N, consider the following subset $S = \{2, ..., k+1\}$, which implies $|S| = k$. Denote $E_S =$

[6] Otherwise, $c^T = (E, E, ..., E)$. In such a case, equal treatment of equals would imply $F^\theta\left(N, E, c^T\right) = \left(\frac{E}{k+1}, \frac{E}{k+1}, ..., \frac{E}{k+1}\right)$. Moreover, if $\theta \geq \frac{1}{2}$, then $c_1 \leq (k+1)\theta c_1$, and therefore $F^\theta(N, E, c) = \left(\frac{E}{k+1}, \frac{E}{k+1}, ..., \frac{E}{k+1}\right)$.

[7] Otherwise we would have $c = c^T$, and therefore independence of claims truncation would hold trivially.

$\sum_{i\in S} F_i^\theta(N,E,c)$, $c_s=(c_i)_{i\in S}$, $E_S^T=\sum_{i\in S} F_i^\theta\left(N,E,c^T\right)$, and $c_s^T=(c_i^T)_{i\in S}$. Now, since all rules within the TAL-family are consistent [Proposition 2], $F_i^\theta(N,E,c)=F_i^\theta(S,E_S,c_s)$, for all $i\in S$. For the sake of induction hypothesis, $F_i^\theta(S,E_S,c_s)=F_i^\theta\left(S,E_S,c_s^T\right)$. Notice that $E_S=E-F_1^\theta(N,E,c)=E-F_1^\theta\left(N,E,c^T\right)=E_S^T$. Thus, $F_i^\theta\left(S,E_S,c_s^T\right)=F_i^\theta\left(S,E_S^T,c_s^T\right)=F_i^\theta\left(N,E,c^T\right)$, where the last equality holds, again, thanks to consistency. In other words, for all $i\in S$, or what is equivalent, for all $i=2,...,k+1$, $F_i^\theta(N,E,c)=F_i^\theta\left(N,E,c^T\right)$, which concludes the proof.

(ii) Independence of claims truncation and composition from minimal rights are dual properties [Herrero & Villar (2001, Claim 5)]. Thus, as F^θ satisfies independence of claims truncation if and only if $\theta\in\left[\frac{1}{2},1\right]$, and $F^{1-\theta}$ is the dual rule of F^θ (Proposition 1), F^θ must satisfy composition from minimal rights if and only if $\theta\in\left[0,\frac{1}{2}\right]$. ■

Proposition 4 *The following two statements hold:*
(i) For all $\theta\in[0,1)$, $\theta-$composition is the dual rule of $(1-\theta)-$independence.
(ii) Let F^θ be a rule in the TAL-family $\left\{F^\theta\right\}_{\theta\in[0,1]}$. Then F^θ satisfies $\theta-$independence and $\theta-$composition.

Proof.
(i) Fix some $\theta\in[0,1)$, and let F be a rule satisfying the axiom $\theta-$composition, and let $(N,E,c)\in\mathbb{B}$ such that $\frac{E}{n}<(1-\theta)c_1$. By definition, $F^*(N,E,c)=c-F(N,C-E,c)$. Now, since F satisfies $\theta-$composition, and $\frac{C-(C-E)}{n}<(1-\theta)c_1$, we have $F(N,C-E,c)=m^\theta(c)+F(N,C-E-\sum_{i\in N} m_i^\theta(c),c-m^\theta(c))$. As a result, and taking into account that $c-m^\theta(c)=c^{1-\theta}$, $F^*(N,E,c)=c^{1-\theta}-F(N,n(1-\theta)c_1-E,c^{1-\theta})=F^*(N,E,c^{1-\theta})$, which means that F^* satisfies $(1-\theta)-$independence.

Conversely, suppose that F^* satisfies $(1-\theta)-$independence, and let $(N,E,c)\in\mathbb{B}$ such that $\frac{C-E}{n}<(1-\theta)c_1$. Then, $F(N,E,c)=c-F^*(N,C-E,c)=c-F^*\left(N,C-E,c^{1-\theta}\right)=c-(c^{1-\theta}-F(N,n(1-\theta)c_1-C+E,c^{1-\theta}))$. As a result, and taking into account that $c-m^\theta(c)=c^{1-\theta}$, we have $F(N,E,c)=m^\theta(c)+F(N,E-\sum_{i\in N} m_i^\theta(c),c-m^\theta(c))$, which means that F satisfies $\theta-$composition.

(ii) Let $\theta\in[0,1]$ and $(N,E,c)\in\mathbb{B}$.[8] Let us see first that F^θ satisfies $\theta-$independence. If $\theta=0$, then it is vacuously satisfied. Assume that $\theta\in(0,1]$. In order to prove this result, suppose that $E<n\theta c_1$. In such a case, it is straightforward to show that $F_1^\theta(N,E,c)=\frac{E}{n}$. Now, since

[8] Recall that, without loss of generality, we suppose $c_1\leq c_2\leq...\leq c_n$, where n is the cardinality of N.

$E < n\theta c_1$, we have $E < \theta C$. This would imply, $F_i^{\theta}(N, E, c) = \min\{\theta c_i, \lambda\}$, where λ is chosen so that $\sum_{i \in N} F_i^{\theta}(N, E, c) = E$. Therefore, $F_1^{\theta}(N, E, c) = \lambda = \frac{E}{n} < \theta c_1 \leq \theta c_i$ for all $i \in N$. As a consequence, $F_i^{\theta}(N, E, c) = \lambda = \frac{E}{n}$, for all $i \in N$. Finally, since F^{θ} satisfies equal treatment of equals, it is also true that $F_i^{\theta}(N, E, c^{\theta}) = \frac{E}{n}$, for all $i \in N$, which shows that F^{θ} satisfies $\theta-$independence.

According to (i) above, $\theta-$composition and $(1-\theta)-$independence are dual properties. Since $F^{1-\theta}$ satisfies $(1-\theta)-$independence and the dual rule of F^{θ} is $F^{1-\theta}$, according to Proposition 1, F^{θ} also satisfies $\theta-$composition. ■

Proposition 6 *The following two statements hold:*

(i) For all $\theta \in (0, 1]$, $\theta-$exemption is the dual property of $(1-\theta)-$exclusion.

(ii) Let F^{θ} be a rule in the TAL-family $\{F^{\theta}\}_{\theta \in [0,1]}$. Then F^{θ} satisfies $\theta-$exemption and $\theta-$exclusion.

Proof.

(i) Let $\theta \in (0, 1)$ be given. Let us suppose that a rule F satisfies $\theta-$exemption. Now, consider some bankruptcy problem $(N, E, c) \in \mathbb{B}$, and suppose that $\theta c_i \leq \frac{C-E}{n}$, for some $i \in N$. By hypothesis, F satisfies $\theta-$exemption and, because $(N, C-E, c) \in \mathbb{B}$, we have $F_i(N, C-E, c) \geq \theta c_i$. Now, this is equivalent to $F_i^*(N, E, c) \leq (1-\theta)c_i$, which shows that F^*, the dual rule of F, satisfies $(1-\theta)-$exclusion.

Conversely, suppose that F^* satisfies $(1-\theta)-$exclusion. Let $(N, E, c) \in \mathbb{B}$, and suppose that $\theta c_i \leq \frac{E}{n}$, for some $i \in N$. By hypothesis, F^* satisfies $(1-\theta)-$exclusion and, since $(N, C-E, c) \in \mathbb{B}$, we have $F_i^*(N, C-E, c) \leq (1-\theta)c_i$. Now, this is equivalent to $F_i(N, E, c) \geq \theta c_i$, which shows that F satisfies $\theta-$exemption.

Herrero & Villar (2001) show that exemption and exclusion are dual properties [Claim 3]. This proves the proposition in the case in which $\theta = 1$.

(ii) $F^1 = A$ (resp. $F^0 = L$) satisfies exemption (exclusion) [Herrero & Villar (2001)], which proves the result in the extreme cases. Now, let $\theta \in (0, 1)$ and $(N, E, c) \in \mathbb{B}$. Let us see first that F^{θ} satisfies $\theta-$exemption. To do this, let $i \in N$ be such that $\theta c_i \leq \frac{E}{n}$. Two cases are to be considered:

Case 1.- $E \geq \theta C$. In this case, $F_i^{\theta}(N, E, c) = \max\{\theta c_i, c_i - \mu\} \geq \theta c_i$, which shows that F^{θ} satisfies $\theta-$exemption.

Case 2.- $E \leq \theta C$. In this case, $F_i^{\theta}(N, E, c) = \min\{\theta c_i, \lambda\}$. For the sake of contradiction, let us suppose that $F_i^{\theta}(N, E, c) = \lambda < \theta c_i$. Thus, $\lambda < \frac{E}{n}$. Now, $c_k \geq c_i$ for all $k \geq i$, which implies that $\theta c_k > \lambda$, and therefore $F_k^{\theta}(N, E, c) = \lambda < \frac{E}{n}$.

As a consequence, $E = \sum_{k\in N} F_k^\theta(N,E,c) = \sum_{k=1}^{i-1} F_k^\theta(N,E,c) + \sum_{k=i}^{n} \lambda < \sum_{k=1}^{i-1} F_k^\theta(N,E,c) + (n-i+1)\frac{E}{n}$. Thus, $\sum_{k=1}^{i-1} F_k^\theta(N,E,c) > \frac{i-1}{n}E$. Since $F_k^\theta(N,E,c) \geq 0$, for all k, then there exists some $k_0 \leq i-1$ such that $F_{k_0}^\theta(N,E,c) > \frac{E}{n} > \lambda = F_n^\theta(N,E,c)$, which is a contradiction, since F^θ is a reasonable rule [Proposition 2]. Therefore, F^θ satisfies $\theta-$exemption.

Part (i) above shows that $\theta-$exemption and $(1-\theta)-$exclusion are dual properties. Furthermore, the dual rule of F^θ is $F^{1-\theta}$ [Proposition 1]. As a result, F^θ also satisfies $\theta-$exclusion. ■

Lemma 1 *For two-agent bankruptcy problems, $\theta-$independence, $\theta-$composition and $\theta-$exemption, imply equal treatment of equals, for each $\theta \in (0,1)$ (resp. $\theta-$independence, $\theta-$composition and $\theta-$exclusion, imply equal treatment of equals, for each $\theta \in (0,1)$).*

Proof.

Consider $(N,E,(c_1,c_2))$ a two-claimant bankruptcy problem, whose claims are equal, i.e. $c_1 = c_2 = c$. Let $\theta \in (0,1)$ be fixed and let f be a rule satisfying $\theta-$independence, $\theta-$composition and $\theta-$exemption. Let us see that $f_i(N,E,(c_1,c_2)) = \frac{E}{2}$, for $i = 1,2$. Several cases are to be considered.

Case 1.- $E = 2\theta c$. In this case, $c_1 = c_2 = \frac{E}{2\theta}$. Since f satisfies $\theta-$exemption, $f_i(N,E,(c_1,c_2)) \geq \theta c_i = \frac{E}{2}$, for $i = 1,2$. Now, $f_1(N,E,(c_1,c_2)) + f_2(N,E,(c_1,c_2)) = E$, implies that $f_i(N,E,(c_1,c_2)) = \frac{E}{2}$, for $i = 1,2$.[9]

Case 2.- $E < 2\theta c$. Let us denote by n_1 the minimum positive integer for which $2\theta^{n_1+1}c < E$.[10] Now, let us denote by m_1 the minimum positive integer for which $E < 2\theta^{n_1+1}c \cdot \sum_{j=0}^{m_1+1}(1-\theta)^j$. Since f satisfies $\theta-$independence and $\theta-$composition, $f(N,E,(c,c)) = (a_1,a_1) + f(N,E_1,(d_1,d_1))$, where $a_1 = \theta^{n_1+1} \cdot c \cdot \sum_{j=0}^{m_1}(1-\theta)^j$, $E_1 = E - 2\theta^{n_1+1}c \cdot \sum_{j=0}^{m_1}(1-\theta)^j$, and $d_1 = \theta^{n_1} \cdot (1-\theta)^{m_1+1} \cdot c$. Since $\theta \in (0,1)$, $1 \leq \min\{n_1, m_1+1\}$. Thus, $d_1 \leq \theta(1-\theta)c \leq \frac{1}{4}c$. Notice that now, $E_1 < 2\theta d_1$, and we can apply the same argument. After a finite number of iterations, say K, we would have $f(N,E,(c,c)) = (\sum_{k=1}^{K} a_k, \sum_{k=1}^{K} a_k) + f(N,E_K,(d_K,d_K))$, where $d_K \leq \left(\frac{1}{4}\right)^K \cdot c$. As a result, $f_i(N,E_K,(d_K,d_K)) \leq \left(\frac{1}{4}\right)^K \cdot c$, for $i = 1,2$. If we take limits, when $K \to \infty$, then $f(N,E,(c,c)) = (\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} a_k)$. Now, since $f_1(N,E,(c_1,c_2)) + f_2(N,E,(c_1,c_2)) = E$, we have $\sum_{k=1}^{\infty} a_k = f_i(N,E,(c,c)) = \frac{E}{2}$, for $i = 1,2$.

Case 3.- $E > 2\theta c$. Let us denote by r_1 the minimum positive integer for which $2\theta c \sum_{j=0}^{r_1+1}(1-\theta)^j > E$. Now, let us denote by s_1 the minimum positive integer for which $E > 2\theta^{s_1+1} \cdot (1-\theta)^{r_1+1} \cdot c + 2\theta c \sum_{j=0}^{r_1}(1-\theta)^j$. Since

[9] It is also true that $c_1 = c_2 = \frac{C-E}{2(1-\theta)}$. Thus, if f satisfies $\theta-$exclusion (instead of $\theta-$exemption), $f_i(N,E,(c_1,c_2)) \leq \theta c_i = \frac{E}{2}$, for $i = 1,2$. Now, $f_1(N,E,(c_1,c_2)) + f_2(N,E,(c_1,c_2)) = E$, would also imply that $f_i(N,E,(c_1,c_2)) = \frac{E}{2}$, for $i = 1,2$.

[10] As a result, $2\theta^{n_1+1}c < E \leq 2\theta^{n_1}c$.

f satisfies $\theta-$independence and $\theta-$composition, $f(N, E, (c, c)) = (b_1, b_1) + f(N, E_1, (e_1, e_1))$, where $b_1 = \theta c \cdot \sum_{j=0}^{r_1}(1-\theta)^j$, $E_1 = E - 2\theta c \cdot \sum_{j=0}^{r_1}(1-\theta)^j$, and $e_1 = \theta^{s_1} \cdot (1-\theta)^{r_1+1} \cdot c$. Since $\theta \in (0, 1)$, $1 \leq \min\{r_1 + 1, s_1\}$. Thus, $e_1 \leq \theta(1-\theta)c \leq \frac{1}{4}c$. Notice that now, $E_1 > 2\theta e_1$, and we can apply the same argument. After a finite number of iterations, say K, we would have $f(N, E, (c, c)) = (\sum_{k=1}^{K} b_k, \sum_{k=1}^{K} b_k) + f(N, E_K, (e_K, e_K))$, where $e_K \leq \left(\frac{1}{4}\right)^K \cdot c$. As a result, $f_i(N, E_K, (e_K, e_K)) \leq \left(\frac{1}{4}\right)^K \cdot c$, for $i = 1, 2$. If we take limits, when $K \to \infty$, then $f(N, E, (c, c)) = (\sum_{k=1}^{\infty} b_k, \sum_{k=1}^{\infty} b_k)$. Now, since $f_1(N, E, (c, c)) + f_2(N, E, (c, c)) = E$, we have $\sum_{k=1}^{\infty} b_k = f_i(N, E, (c, c)) = \frac{E}{2}$, for $i = 1, 2$, and the proof is in this way completed. ■

7 Appendix 2: On the tightness of the characterization results

7.1 Essential properties in Theorem 1

Let us fix some $\theta \in (0, 1)$. We give examples of rules outside the TAL-family satisfying all but one of the properties mentioned in Theorem 1. We mention in each case the property that is not fulfilled.

$\theta-$**exemption.** Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the particular bankruptcy problem $\overline{B} = (\{1, 2\}, 2\theta, (\frac{1}{2}, \frac{5}{2}))$. Now, take the following bankruptcy rule g^θ:

$$g^\theta(N, E, c) = \begin{cases} F^\theta(N, E, c) & \text{if } (N, E, c) \neq \overline{B} \\ (\frac{\theta}{4}, \frac{7\theta}{4}) & \text{if } (N, E, c) = \overline{B} \end{cases}.$$

It is straightforward to see that g^θ is well defined. Since g^θ is only defined for two-claimant bankruptcy problems, consistency is vacuously satisfied. Obviously, if $(N, E, c) \neq \overline{B}$, the remaining properties are also satisfied. Let us now turn to $\overline{B}$. Since $2\theta c_1 < E < c_2 + (2\theta - 1)c_1$, $\theta-$composition and $\theta-$independence, are trivially satisfied. Moreover, it is clear that $g_i^\theta(\overline{B}) < \theta c_i$, for $i = 1, 2$, which implies that $\theta-$exclusion is satisfied. However, since $\theta c_1 \leq \frac{E}{2}$, and $g_1^\theta(\overline{B}) < \theta c_1$, $\theta-$exemption is not satisfied.

$\theta-$**exclusion.** Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the particular bankruptcy problem $\overline{B} = (N, 2\theta + 1, (\frac{1}{2}, \frac{5}{2}))$. Now, take the following bankruptcy rule g^θ:

$$g^\theta(N, E, c) = \begin{cases} F^\theta(N, E, c) & \text{if } (N, E, c) \neq \overline{B} \\ (\frac{1}{2}, \frac{1}{2} + 2\theta) & \text{if } (N, E, c) = \overline{B} \end{cases}.$$

It is straightforward to see that g^θ is well defined. Since g^θ is only defined for two-claimant bankruptcy problems, consistency is vacuously satisfied. Obviously, if $(N,E,c) \neq \overline{B}$, the remaining properties are satisfied, due to the fact that g^θ coincides with the member of the TAL-family F^θ. Let us now turn to $\overline{B}$. Since $2\theta c_1 < E < c_2 + (2\theta - 1)c_1$, $\theta-$composition and $\theta-$independence, are trivially satisfied. Moreover, it is clear that $g_i^\theta(\overline{B}) > \theta c_i$, for $i = 1, 2$, which implies that $\theta-$exemption is satisfied. However, since $(1-\theta)c_1 \leq \frac{C-E}{2}$, and $g_1^\theta(\overline{B}) > \theta c_1$, $\theta-$exclusion is not satisfied.

$\theta-$independence. Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the particular bankruptcy problem $\overline{B} = (N, 1, (\frac{1}{\theta}, \frac{2}{\theta}))$. Now, take the following bankruptcy rule g^θ:

$$g^\theta(N,E,c) = \begin{cases} F^\theta(N,E,c) & \text{if } (N,E,c) \neq \overline{B} \\ (0,1) & \text{if } (N,E,c) = \overline{B} \end{cases}.$$

It is straightforward to see that g^θ is well defined. Since g^θ is only defined for two-claimant bankruptcy problems, consistency is vacuously satisfied. Obviously, if $(N,E,c) \neq \overline{B}$, the remaining properties are satisfied, due to the fact that g^θ coincides with the member of the TAL-family F^θ. Let us now turn to $\overline{B}$. Since $E < c_2 + (2\theta - 1)c_1$, $\theta-$composition is trivially satisfied. It is also clear that $g_i^\theta(\overline{B}) < \theta c_i$, for $i = 1, 2$, which implies that $\theta-$exclusion is satisfied. Moreover, $\theta c_i > \frac{E}{2}$, for $i = 1, 2$, and therefore $\theta-$exemption is vacuously satisfied. However, if g^θ would satisfy $\theta-$independence, then we would have $g^\theta(\overline{B}) = g^\theta(N, 1, (1,1)) = (\frac{1}{2}, \frac{1}{2}) \neq (0,1)$.

$\theta-$composition. Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the particular bankruptcy problem $\overline{B} = (N, 2+\theta, (1,2))$. Now, take the following bankruptcy rule g^θ:

$$g^\theta(N,E,c) = \begin{cases} F^\theta(N,E,c) & \text{if } (N,E,c) \neq \overline{B} \\ (\theta,2) & \text{if } (N,E,c) = \overline{B} \end{cases}.$$

It is straightforward to see that g^θ is well defined. Since g^θ is only defined for two-claimant bankruptcy problems, consistency is vacuously satisfied. Obviously, if $(N,E,c) \neq \overline{B}$, the remaining properties are satisfied, due to the fact that g^θ coincides with the member of the TAL-family F^θ. Let us now turn to $\overline{B}$. Since $E > 2\theta c_1$, $\theta-$independence is trivially satisfied. It is also clear that $g_i^\theta(\overline{B}) \geq \theta c_i$, for $i = 1, 2$, which implies that $\theta-$exemption is satisfied. Moreover, $(1-\theta)c_i > \frac{C-E}{2}$, for $i = 1, 2$, and therefore $\theta-$exclusion is vacuously satisfied. However, if g^θ would satisfy $\theta-$composition, then we would have $g^\theta(\overline{B}) = (\theta, 1+\theta) + (\frac{1-\theta}{2}, \frac{1-\theta}{2}) \neq (\theta, 2)$.

Consistency. Let us consider the particular bankruptcy problem $\overline{B} = (\{1,2,3\}, 4\theta, (1,2,3))$. Now, we define the following bankruptcy rule g^θ:

$$g^\theta(N,E,c) = \begin{cases} F^\theta(N,E,c) & \text{if } (N,E,c) \neq \overline{B} \\ (\theta,\theta,2\theta) & \text{if } (N,E,c) = \overline{B} \end{cases},$$

where F^θ is the corresponding rule in the TAL-family. It is straightforward to see that g^θ is well defined. Obviously, if $(N,E,c) \neq \overline{B}$, the $\theta-$axioms are satisfied. Let us now turn to $\overline{B}$. Since $3\theta c_1 < E < C - 3(1-\theta)c_1$, $\theta-$independence and $\theta-$composition, are trivially satisfied. It is also clear that $g_i^\theta(\overline{B}) \leq \theta c_i$, for $i = 1,2,3$, which implies that $\theta-$exclusion is satisfied. Moreover, $g_1^\theta(\overline{B}) = \theta c_1$, and $\theta c_i > \frac{E}{3}$, for $i = 2,3$, which shows that $\theta-$exemption is satisfied. Finally, g^θ is not a consistent rule. Otherwise, we would have $g^\theta\left(\overline{B}\right) = F^\theta\left(\overline{B}\right) = (\theta, \frac{3\theta}{2}, \frac{3\theta}{2})$, which is not the case.

7.2 Essential properties in Theorems 2 and 3

Now we focus in the case $\theta = 0$. We give examples of rules, different from the L rule, satisfying all of the properties mentioned in Theorem 2, except one. We mention in each case the property that is not fulfilled.

Remark 3 *Concerning Theorem 3 it is enough to consider the dual rules of the ones mentioned here.*

Equal treatment of equals. Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the subset of such problems $\overline{B} = \{(E,(1,E)) : E \geq 1\}$. Now, consider the following bankruptcy rule f.

$$f(N,E,c) = \begin{cases} L(N,E,c) & \text{if } (N,E,c) \notin \overline{B} \\ \left(\frac{1}{3}, E - \frac{1}{3}\right) & \text{if } (N,E,c) \in \overline{B} \end{cases}.$$

Notice that f is well defined. Since f is only defined when $|N| = 2$, then obviously satisfies consistency. Moreover, since L satisfies exclusion and $0-$composition, then so does f, for all bankruptcy problems out of $\overline{B}$. Let us now consider some $(N,E,c) \in \overline{B}$. Observe that in this case $c_1 \geq \frac{C-E}{2}$, and therefore, f vacuously satisfies exclusion. About $0-$composition, $m^0(c) = (0, c_2 - c_1)$. Let $E' = E - m_1^0(c) - m_2^0(c) = 1$. Then, $f(N,E',c-m^0(c)) = f(N,1,(1,1)) = (\frac{1}{3},\frac{2}{3})$. Moreover, $f(N,E,c) = \left(\frac{1}{3}, E - \frac{1}{3}\right) = (0, E-1) + (\frac{1}{3},\frac{2}{3}) = m^0(c) + f(N,E',c-m^0(c))$, which concludes the proof. However, as we mentioned above, f fails to satisfy equal treatment of equals. Take the bankruptcy problem $(N,1,(1,1))$, and observe that $f(N,1,(1,1)) = (\frac{1}{3},\frac{2}{3})$.

Exclusion. Let us consider the set of two-claimant bankruptcy problems, i.e. $|N| = 2$, and denote by $\overline{B}$ the particular bankruptcy problem $\overline{B} = (N, 1, (\frac{1}{2}, \frac{5}{2}))$. Now, take the following bankruptcy rule g:

$$g\left(N,E,c\right)=\begin{cases} L\left(N,E,c\right) & \text{if } \left(N,E,c\right)\neq\overline{B} \\ \left(\frac{1}{2},\frac{1}{2}\right) & \text{if } \left(N,E,c\right)=\overline{B} \end{cases}.$$

It is straightforward to see that g is well defined. Since g is only defined for two-claimant bankruptcy problems, consistency is vacuously satisfied. Obviously, if $(N, E, c) \neq \overline{B}$, g coincides with L and the remaining properties are satisfied. Let us now turn to $\overline{B}$. Since $E < c_2 - c_1$, $0-$composition is vacuously satisfied. However, since $c_1 < \frac{C-E}{2}$, and $g_1(\overline{B}) = \frac{1}{2} > 0$, exclusion is not satisfied.

$0-$**Composition.** Let $N = \{1, 2\}$, be the set of claimants, and (N, E, c) a bankruptcy problem, where without loss of generality, assume that $c_1 \leq c_2$. Now, consider the following bankruptcy rule f^2.

$$f^2\left(N,E,c\right)=\begin{cases} \left(\frac{E}{2},\frac{E}{2}\right) & \text{if } c_1=c_2 \\ \left(E-M_2(N,E,c),M_2(N,E,c)\right) & \text{if } c_1\neq c_2 \end{cases},$$

where $M_2(N, E, c) = \max\{c_2, E\}$. Notice that $0 \leq E - M_2(N, E, c) \leq c_1$, which ensures that f^2 is well defined. Since f^2 is only defined when $|N| = 2$, then obviously satisfies consistency. By definition, it satisfies equal treatment of equals. Let us see that it also satisfies exclusion. Suppose that $c_1 \leq \frac{C-E}{2}$. Thus, $c_2 > c_1$. If $c_2 < E$, then $C = c_1 + c_2 < E + \frac{C-E}{2}$, which implies $C - E < \frac{C-E}{2}$, a contradiction. Hence, $c_2 \geq E$, and therefore $f^2 (N, E, c) = (0, E)$, which proves that f^2 satisfies exclusion. Finally, it does not satisfy partial $0-$composition. To see this, it suffices to take the bankruptcy problem $(N, E, c) = (\{1, 2\}, 2, (1, 2))$. Then, $m^0(c) = (0, 1)$. Let $E' = E - m^0_1(c) - m^0_2(c) = 1$. Thus, $m^0(c) + f^2 (N, E', c - m^0(c)) = (\frac{1}{2}, \frac{3}{2}) \neq (0, 2) = f^2 (N, E, c)$.

Consistency. Let us consider the particular bankruptcy problem $\overline{B} = (\{1, 2, 3\}, 4, (1, 3, 5))$. Now, we define the following bankruptcy rule g:

$$g\left(N,E,c\right)=\begin{cases} L\left(N,E,c\right) & \text{if } \left(N,E,c\right)\neq\overline{B} \\ (0,2,2) & \text{if } \left(N,E,c\right)=\overline{B} \end{cases}.$$

It is straightforward to see that g is well defined and satisfies equal treatment of equals. It also satisfies exclusion and $0-$composition. Obviously, if $(N, E, c) \neq \overline{B}$, g coincides with L and the remaining properties are satisfied.

Let us now turn to $\overline{B}$. Since $E < C - 3c_1$, $0-$composition is trivially satisfied. It is easily observed that $c_i > \frac{C-E}{3}$, for $i = 2, 3$, and $g_1(\overline{B}) = 0$, which shows that exemption is satisfied. Finally, g is not a consistent rule. Otherwise, we would have $g\left(\overline{B}\right) = L\left(\overline{B}\right) = (0, 1, 3)$, which is not the case.

7.3 Essential properties in Theorem 4

Since mid-exclusion and mid-exemption are dual properties, if a self-dual rule satisfy one of them, then it satisfies both properties. The converse is not true. To show this, it suffices to take, the rule which coincides with the Talmud rule in each bankruptcy problem, except on $(\{1, 2\}, 1, (2, 4))$, for which the proposed allocation is $(0, 1)$.

A similar argument can be used for the properties of mid-composition and mid-independence. Let us consider, for instance, the rule which coincides with the Talmud rule in each bankruptcy problem, except on $(\{1, 2\}, 1, (\frac{1}{2}, \frac{5}{2}))$, for which the proposed allocation is $(\frac{1}{8}, \frac{7}{8})$. That is a rule which satisfies mid-composition and mid-independence, but it is not a self-dual rule.

Let us see now that Theorem 4 is tight. To do so, we give examples of rules, different from the Talmud rule, satisfying all of the properties mentioned in Theorem 4, except one. We mention in each case the property that is not fulfilled.

Mid-exclusion (or mid-exemption). Consider the universe of two-creditor bankruptcy problems, and take, for instance, the rule which coincides with the Talmud rule in such problems, except on $(\{1, 2\}, 2, (2, 4))$, and $(\{1, 2\}, 4, (2, 4))$. In the first case, the proposed allocation is $(0, 2)$, and in the second case $(2, 2)$. Such a rule, is self-dual, satisfies mid-composition (and therefore, mid-independence), and (vacuously) consistency, but neither mid-exclusion nor mid-exemption.

Mid-composition (or mid-independence). Consider the universe of two-creditor bankruptcy problems, and take, for instance, the rule which coincides with the Talmud rule in such problems, except on $(\{1, 2\}, 1, (2, 4))$, and $(\{1, 2\}, 5, (2, 4))$. In the first case, the proposed allocation is $(0, 1)$, and in the second case $(2, 3)$. Such a rule, is a self-dual rule which satisfies mid-exclusion (and therefore, mid-exemption) and (vacuously) consistency, but neither mid-composition nor mid-independence.

Consistency. Let us consider two particular bankruptcy problems $\overline{B_1} = (\{1, 2, 3\}, 2, (1, 2, 3))$, and $\overline{B_2} = (\{1, 2, 3\}, 4, (1, 2, 3))$. Now, we define the following bankruptcy rule g:

$$g(N,E,c)=\begin{cases} T(N,E,c) & \text{if } (N,E,c)\neq \overline{B_1},\ \overline{B_2} \\ (\frac{1}{2},\frac{1}{2},1) & \text{if } (N,E,c)=\overline{B_1} \\ (\frac{1}{2},\frac{3}{2},2) & \text{if } (N,E,c)=\overline{B_2} \end{cases}.$$

It is straightforward to see that g is well defined. Obviously, if $(N,E,c)\neq \overline{B_1}$, $\overline{B_2}$, the remaining properties are also satisfied, as T satisfies them. Let us now turn to $\overline{B_j}$, for $j=1,2$. Since $g(\overline{B_1})+g(\overline{B_2})=(1,2,3)$, it is straightforward to see that g is a self dual rule. Moreover, since $\frac{c_1}{2}<\frac{E}{3}$, in both cases, mid-independence is trivially satisfied. It is also clear that $g_i(\overline{B_1})\leq\frac{c_i}{2}$, for $i=1,2,3$. Moreover, in the case of $\overline{B_2}$, $g_1(\overline{B_2})=\frac{c_1}{2}$, and $\frac{c_i}{2}>\frac{(C-E)}{3}$, for $i=2,3$. All together says that mid-exemption is satisfied. Finally, g is not a consistent rule. Otherwise we would have $g\left(\overline{B_1}\right)=T\left(\overline{B_1}\right)=(\frac{1}{2},\frac{3}{4},\frac{3}{4})$, which is not the case.[11]

Self-duality. Each of the examples mentioned at the beginning of the section are valid, if they are only defined for two-claimant bankruptcy problems, which would ensure that they vacuously satisfy consistency. Both of them are not self-dual rules but they satisfy either mid-exclusion or mid-exemption and either mid-composition or mid-independence.

[11] Since g is a self-dual rule, satisfying mid-independence and mid-exemption, then it also satisfies mid-composition and mid-exclusion.

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