AN INPUT-OUTPUT APPROACH TO THE MEASUREMENT OF PRODUCTIVITY DIFFERENCES*

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WP-AD 2001-32

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Editor: Instituto Valenciano de Investigaciones Económicas, S.A.
First Edition December 2001.
Depósito Legal: V-5309-2001

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* Thanks are due to Takao Fujimoto for helpful comments. Financial support from the Spanish *Ministerio de Ciencia y Tecnología*, under project BEC2001-0535 is gratefully acknowledged.

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ABSTRACT

This paper deals with the measurement of efficiency differences among countries within a federation. The key tool is the use of a suitable interpretation of the non-substitution theorem in order to get a natural technological standard. Two alternative indices are proposed. One provides a structural productivity measure, independent of consumption and prices. The other consists of a price index associated with the equilibrium aggregate demand.

KEYWORDS: Input-Output Analysis; Nonsubstitution Theorem; Measurement of Productivity.

1 INTRODUCTION

This paper addresses the problem of comparing the technological performance of a group of economies in terms of a common standard. By an "economy" it is meant here the economic system of a region or a country. By a "group" we refer to a wider system to which these economies belong, such as a country in the case of regions or a set of states that are members of some common organization (e.g. the countries within the European Union). To fix ideas let us identify the economies with "countries" and assume that they are part of an international economic system, called here "the Federation".

As it is common in the standard literature on productivity comparisons [e.g. Färe *et al.* (1994)], we first construct an overall efficient technology that defines our technological standard and then measure the efficiency of a country as some "distance" between the actual technology and our standard. The technological standard, that here corresponds to the Federation's efficient technology, is obtained from the information provided by the technologies used by different countries.

The production possibilities of each individual country will be modelled as a linear input-output system [see for instance Arrow & Hahn (1971, ch. 2) or Cornwall (1984, ch.2)]. Within this setting, we can naturally obtain our technological standard from a suitable version of the non-substitution theorem [Samuelson (1951), Georgescu-Roegen (1951)]. This theorem elicits an efficient technique among all possible combinations of the processes that are actually used in the different economies. It is a natural standard of comparison because all efficient production plans of the countries in the Federation can be realized by using this single technique. Moreover, associated with this technique there is a unique vector of efficiency prices. We shall develop this idea allowing for the presence of joint production [e.g. Bidard (1991, part II), Peris & Villar (1993), Villar (2000, ch. 11)], and making use of an appropriate version of the non-substitution theorem for joint-production models [Herrero & Villar (1988)].

Once the efficient technology has been identified and the efficiency prices calculated, there is a number of sensible ways of performing productivity comparisons. We suggest here two different indices:

(i) One is based on the comparison of the structural productivity of individual economies, that can be measured in terms of the maximum growth rate achievable, ϑ^i. This number is related to the eigen-vector associated with the Frobenius root of the ith country's input-output matrix. The way of constructing the input output coefficients proposed here implies that the differences in labour requirements are already taken into account. In this context we can measure the relative productivity of a country as a ratio

ϑ^i/ϑ^*, where ϑ^* is the maximum growth rate associated with the Federation's efficient technology. This ratio tells us the percentage of the maximum growth that this economy can achieve with the technique actually in use.

(ii) The other productivity index makes use of efficiency prices to quantify the extra cost that the economy incurs to realize its net production. Interestingly enough this approach starts from a sectorial measure of unitary productivity loss. We aggregate these sectorial measures into an overall productivity loss by taking the actual production as a reference. We shall see that this measure, which is nothing else than a Laspeyres *price* index, coincides with a suitable version of Debreu's (1951) coefficient of resource utilization (a Laspeyres *quantity* index)..

2 THE REFERENCE MODEL

We consider a *Federation* made of k different *countries*. There are exactly the same ℓ commodities in each country, consisting of n producible commodities and a single primary factor, that we call labour.[1] Hence, $\ell = n + 1$. The set of efficient production plans that are feasible for each country is described by a linear input output technology (constant returns to scale) with the same number of sectors as producible commodities. Joint production is the rule.

2.1 Individual economies

Consider, for the time being, the case of a single country. That will save us some notation at this early stage. A production plan that is feasible for the jth sector of this country is a vector $\mathbf{y}_j \in \mathbb{R}^\ell$ of the form:

$$\mathbf{y}_j = [(\mathbf{b}_j - \mathbf{a}_j),\ L_j]$$

where: $\mathbf{b}_j \in \mathbb{R}^n_+$ is a gross output vector, $\mathbf{a}_j \in \mathbb{R}^n_+$ is a gross input vector, and $L_j \leq 0$ is the amount of labour required. We shall assume throughout that labour is a necessary input for production, that is, $L_j < 0$ whenever $\mathbf{y}_j$ has some positive entry.

Let $\mathbf{y}_j \neq \mathbf{0}$ be an efficient production plan. As we are assuming constant returns to scale, we can re-write production plans as follows:

$$\mathbf{y}_j = \begin{bmatrix} B_j - A_j \\ -1 \end{bmatrix} q_j$$

[1] We shall discuss on this assumption later.

where $B_j,\ A_j \in \mathbb{R}^n_+$ are column vectors, with $B_{hj} = \frac{b_{hj}}{q_j}$, $A_{hj} = \frac{a_{hj}}{q_j}$ for $h = 1, 2, ..., n$, and $q_j = -L_j > 0$. With this representation we can describe all production plans that are proportional to $\mathbf{y}_j$ as

$$\begin{bmatrix} B_j - A_j \\ -1 \end{bmatrix} q_j$$

for some $q_j \geq 0$. These are the jth sector's technical coefficients of production in the country. They express the production and input requirements *per unit of labour*. The number q_j can be identified with the jth sector activity level, measured in terms of labour demanded.

Now take an efficient production plan $\mathbf{y}_j$ for each sector, with $j = 1, 2, \ldots, n$. Making use of the above representation, the corresponding aggregate production plan can be expressed as follows:

$$\mathbf{y} = \sum_{j=1}^{n} \mathbf{y}_j = \begin{pmatrix} \mathbf{B} - \mathbf{A} \\ -\mathbf{1} \end{pmatrix} \mathbf{q}$$

in which:

(i) $\mathbf{q} \in \mathbb{R}^n_+$ is the vector of activity levels (measured in units of labour).

(i) $\mathbf{B}$ is an $n \times n$ non-negative matrix of gross output per unit of labour.

(ii) $\mathbf{A}$ is an $n \times n$ non-negative matrix whose entries describe the input coefficients per unit of labour.

(iv) $\mathbf{1} \in \mathbb{R}^n_+$ is the vector all whose components are equal to 1. Therefore, $\mathbf{1q}$ is the total amount of labour involved in the aggregate production $\mathbf{y}$.

Note that in the usual input-output model the construction of production coefficients is given by: $B_{jj} = 1$, $B_{jt} = 0$, for all $t \neq j$, and $A_{hj} = \frac{a_{hj}}{b_{jj}}$. In this case, the vector of activity levels coincides with the gross production vector. In our case, however, vector $\mathbf{q}$ describes the level at which each sector is operated as measured by the labour required. Therefore, the differences in labour requirements among sectors are incorporated into the technical coefficients of production. This is an important modelling choice in order to obtain sensible comparisons between countries with different technologies.

Consider now the following assumptions:

Axiom 1 *There exists* $\mathbf{q}^o \in \mathbb{R}^n_+$, *such that* $(\mathbf{B} - \mathbf{A})\mathbf{q}^o >> \mathbf{0}$.

Axiom 2 $\mathbf{B}^{-1}$ *exists and* $\mathbf{AB}^{-1} \geq \mathbf{0}$.

Axiom 3 $[(\mathbf{B} - \mathbf{A})\mathbf{z} \geq \mathbf{0}\ \&\ \mathbf{Bz} \geq \mathbf{0}] \Longrightarrow \mathbf{z} \geq \mathbf{0}$.

Axiom 1 is a *productivity* condition. It says that the economy is able to produce positive net amounts of *all* producible commodities.

There are two different requirements in axiom 2. The first says that matrix $\mathbf{B}$ is of full rank. This is partly a technical requirement that induces a one to one correspondence between activity levels and gross output vectors. The second, $\mathbf{AB}^{-1} \geq \mathbf{0}$ is a *monotonicity* requirement that amounts to saying that in order to increase the gross output vector, more inputs should be used up. To see this simply note that when $\mathbf{B}^{-1}$ exists, we can write $(\mathbf{B} - \mathbf{A})\mathbf{q} = \mathbf{q} - \mathbf{AB}^{-1}\mathbf{q}$.

Finally, axiom 3 prevents the possibility of increasing the gross and net output vectors by increasing the use of labour in some sector and reducing it in some other (if one lets $\mathbf{z} = \mathbf{q} - \mathbf{q}'$, then this axiom says that $(\mathbf{B} - \mathbf{A})\mathbf{q} \geq (\mathbf{B} - \mathbf{A})\mathbf{q}'$ and $\mathbf{Bq} \geq \mathbf{Bq}'$ together imply that $\mathbf{q} \geq \mathbf{q}'$). This condition is closely related to the existence of a dominant diagonal in the input-output matrix $(\mathbf{B} - \mathbf{A})$ and suggests that, in spite of joint production, each productive sector can still be identified as "the main producer" of a specific commodity.[2]

Remark 1 *Note that when* $\mathbf{B} = \mathbf{I}$ *(that is, single production prevails), axiom 1 implies axioms 2 and 3.*

Let $\mathbf{d} \in \mathbb{R}^n_+$ be a given final demand vector. To find an equilibrium in the quantity side of this economy consists of finding a solution to the following linear system:

$$(\mathbf{B} - \mathbf{A})\mathbf{q} = \mathbf{d}$$

When we are able to solve this system for all possible final demand vectors, we can calculate the changes in the corresponding activity levels and input requirements associated with a change in the final demand.

Let us consider now the price system. We know that, under constant returns to scale, the only price vectors that are compatible with profit maximization are those that entail zero profits. As all firms exhibit constant returns to scale, we can express the equilibrium price system of this economy as a solution to the following linear equation system:

$$\widetilde{\mathbf{p}} \begin{bmatrix} \mathbf{B} - \mathbf{A} \\ -\mathbf{1} \end{bmatrix} = \mathbf{0}$$

where $\widetilde{\mathbf{p}} \in \mathbb{R}^\ell_+$. We normalize prices by making unity the wage rate, that is, we take $p_\ell = 1$. Then, letting $\widetilde{\mathbf{p}} = (\mathbf{p}, 1)$, with $\mathbf{p} \in \mathbb{R}^n_+$, this equation system

[2] See Peris & Villar (1993) for an extension of this idea to the case of non-square joint production models.

can alternatively be written as:

$$\mathbf{pB} = \mathbf{pA} + \mathbf{1}$$

an expression that says that average revenues equal average costs in all sectors.

When we normalize prices by letting $p_\ell = 1$, the equilibrium price vector of producible commodities corresponds to the vector of labour values, that is, p_t tells us the amount of labour incorporated in one unit of commodity t obtained as a net output.

The following preliminary result is well known [see for instance Debreu & Herstein (1953), Berman & Plemmons (1979, ch. 9)]:

Lemma 1 *Let* $\mathbf{M} \geq \mathbf{0}$ *be a square n-matrix. The following statements are equivalent:*

(a) There exists a point $\mathbf{z} \in \mathbb{R}^n_+$ *for which* $(\mathbf{I} - \mathbf{M})\mathbf{z} >> \mathbf{0}$.

(b) $0 < \lambda(\mathbf{M}) < 1$ *(where* $\lambda(\mathbf{M})$ *is the Frobenius root of matrix* $\mathbf{M}$*).*[3]

(c) The inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$ *exists and it is non-negative.*

We shall use this Lemma to prove the following:[4]

Theorem 1 *Suppose that axioms 1, 2 and 3 hold. Then: (i) The linear production system* $(\mathbf{B} - \mathbf{A})\mathbf{q} = \mathbf{d}$ *has a unique non-negative solution for all parameter vectors* $\mathbf{d} \in \mathbb{R}^n_+$*. (ii) There is a unique (normalized) price vector* $\widetilde{\mathbf{p}} = (\mathbf{p}, 1)$ *that solves the system* $\mathbf{pB} = \mathbf{pA} + \mathbf{1}$, *with* $\widetilde{\mathbf{p}} \in \mathbb{R}^\ell_{++}$.

Proof. (i) For a given $\mathbf{d} \in \mathbb{R}^n_+$, consider the system $(\mathbf{I} - \mathbf{M})\mathbf{t} = \mathbf{d}$, where $\mathbf{M} = \mathbf{AB}^{-1} \geq \mathbf{0}$, by axiom 2. Call $\mathbf{t}^o = \mathbf{Bq}^o$, where $\mathbf{q}^o$ satisfies axiom 1. Clearly $\mathbf{t}^o \in \mathbb{R}^n_+$ with $(\mathbf{I} - \mathbf{M})\mathbf{t}^o >> \mathbf{0}$. Hence, (c) of Lemma 1 implies that $(\mathbf{I} - \mathbf{M})^{-1} \geq \mathbf{0}$. Therefore, there exists $\mathbf{t}^* \in \mathbb{R}^n_+$ such that $\mathbf{t}^* = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{d}$, with $\mathbf{t}^* = \mathbf{Bq}^*$. Axiom 3 trivially implies that $\mathbf{q}^* \geq \mathbf{0}$.

(ii) The result in (i) amounts to saying that $(\mathbf{B} - \mathbf{A})^{-1}$ exists and is non-negative. Therefore, we can let $\mathbf{p} = \mathbf{1}(\mathbf{B} - \mathbf{A})^{-1}$ which is the unique solution to the price equilibrium equation system. Clearly $\mathbf{p} >> \mathbf{0}$. $\boxed{\textbf{Q.e.d.}}$

This result ensures that joint production models that satisfy axioms 1 to 3 behave the same way as the standard single-production ones, since $(\mathbf{B} - \mathbf{A})^{-1}$

[3] The Frobenius root is the greatest number λ for which the following relation holds: $\lambda\mathbf{x} = \mathbf{Mx}$, for $\mathbf{x} \in \mathbb{R}^n$. The numbers that solve this system are called eigen-values, and the associated vectors are called eigen-vectors. When $\mathbf{M} \geq \mathbf{0}$ and $\lambda(\mathbf{M}) \in (0, 1)$, $\mathbf{x}$ is non-negative.

[4] This is a slight simplification of the result in Peris & Villar (1993), that we present here for the sake of compleness in the exposition.

exists and is non-negative. Observe that this is ensured assuming neither $(\mathbf{B}-\mathbf{A})$ to be a Z-matrix nor $\mathbf{B}^{-1} \geq \mathbf{0}$.

Now let us consider the competitive equilibrium of this type of economies, when there are m consumers whose expenditure capacity is given by labour income, exclusively. A consumption plan for consumer c is a point

$$\mathbf{x}_c = (\mathbf{d}_c, \ -T_c)$$

where $\mathbf{d}_c \in \mathbb{R}^n_+$ is the demand vector of producible commodities, and T_c is the labour supply. A consumption plan is affordable for the consumer whenever $(\mathbf{p},1)(\mathbf{d}_c, -T_c) \leq 0$, that is, if $\mathbf{p}\mathbf{d}_c \leq T_c$. For a given price vector $(\mathbf{p},1) \in \mathbb{R}^\ell_+$, let $\xi_c(\mathbf{p})$ stand for consumer c's demand, and $\xi(\mathbf{p}) = \sum_{c=1}^m \xi_c(\mathbf{p})$ for the aggregate demand. A point in $\xi(\mathbf{p})$ will be denoted by $(\mathbf{d}, -T) = \sum_{c=1}^m (\mathbf{d}_c, -T_c)$.

An **equilibrium** is a price vector $\mathbf{p}$, a consumption allocation $(\mathbf{d}_c, -T_c)_{c=1}^m$, and a vector of activity levels $\mathbf{q} \in \mathbb{R}^n$ such that:

(i) $(\mathbf{d}_c, -T_c) \in \xi_c(\mathbf{p})$, for all c.

(ii) $\mathbf{p} = \mathbf{1}(\mathbf{B}-\mathbf{A})^{-1}$.

(iii) $\begin{pmatrix} \mathbf{B}-\mathbf{A} \\ -\mathbf{1} \end{pmatrix} \mathbf{q} = \begin{pmatrix} \mathbf{d} \\ -T \end{pmatrix}$

Part (i) says that all consumers choose equilibrium consumption plans at these prices. Parts (ii) and (iii) describe the equilibrium of industries and market balance.

Now consider the following assumption:

Axiom 4 *For all $c = 1, 2, ..., m$: (i) Consumer c's expenditure capacity is given by her labour income. (ii) $\xi_c(\mathbf{p}) \neq \emptyset$ for all $\mathbf{p} >> \mathbf{0}$, with $\mathbf{p}\mathbf{d}_c = T_c$, for all $(\mathbf{d}_c, -T_c) \in \xi_c(\mathbf{p})$.*

Part (i) of this axiom establishes that the consumers' initial endowments consist of their labour capacity, exclusively. Part (ii) says that the ith consumer's demand is non-empty valued, for all positive price vectors, and that the ith consumer expends all her income.

The following result is obtained:

Theorem 2 *An equilibrium exists under axioms 1 to 4.*

Proof. Let $(\mathbf{p}, 1) >> \mathbf{0}$ be the unique equilibrium price vector for this economy [(ii) of Theorem 1]. Part (ii) of axiom 4 ensures that we can find some point $(\mathbf{d}, -T)$ in $\xi(\mathbf{p})$. The vector of equilibrium activity levels is uniquely obtained as $\mathbf{q} = (\mathbf{B}-\mathbf{A})^{-1}\mathbf{d}$. Therefore, it only remains to check that $\sum_{j=1}^{n} q_j = T$. But this is immediate since budget balance implies that $\mathbf{pd}_c = T_c$, for all c, so that $\mathbf{pd} = T$. Now substituting $\mathbf{p}$ by its value gives us $\mathbf{1}(\mathbf{B} - \mathbf{A})^{-1}\mathbf{d} = T$, that is, $\mathbf{1q} = T$. $\boxed{\textbf{Q.e.d.}}$

This general equilibrium model provides us with a useful representation of the economy of a country, in which economic sectors take the role of firms. The strength of axioms 1 to 4 permits one to prove the existence of competitive equilibrium, assuming neither the continuity nor the convexity of preferences. It also ensures the uniqueness of the equilibrium prices (a very appealing property of equilibrium outcomes, hard to get on the basis of the fundamentals of the economy).

2.2 The technological standard

Once we have modelled the economies of individual countries let us consider the measurement of their relative efficiency. For that we start by identifying the *Federation's efficient technology* (FETCH, for short) by means of a suitable interpretation of the non-substitution theorem. This result also gives us a well defined vector of efficiency prices.

Consider a Federation made of k different countries, each of which is characterized by the technology $(\mathbf{B}^i, \mathbf{A}^i)$ actually used, for $i = 1, 2, ..., k$. To define a technological standard in order to perform efficiency comparisons we have to identify "the best" technology. This can be naturally done in this framework by means of the non-substitution theorem. This theorem says that, given a collection of alternative techniques $\left[(\mathbf{B}^i, \mathbf{A}^i)\right]_{i=1}^{k}$, there is a single one $(\mathbf{B}^*, \mathbf{A}^*)$ that can be taken as a sufficient representation of the *Federation's efficient technology* (FETCH), in spite of the having many alternative production configurations. This follows from two complementary outcomes. The first one says that all efficient production plans can be obtained by means of a single technique, that is to say, by means of a single pair of matrices $(\mathbf{B}^*, \mathbf{A}^*)$. The second one shows that, even if there are several techniques that can be used to generate all these efficient production plans, all of them are equivalent from an economic viewpoint because equilibrium prices are uniquely determined.

Here comes the theorem:[5]

[5] A detailed proof can be seen in Herrero & Villar (1988) or Villar (2000, ch. 11). Note that the requirement of axioms 1 to 3 for all individual techniques is not required for this theorem to hold.

Theorem 3 (The Non-Substitution Theorem) *Suppose that we have k alternative techniques $[(\mathbf{B}^i, \mathbf{A}^i)]_{i=1}^k$, all of which satisfy axioms 1, 2 and 3. Then:*

(i) There is a technique $(\mathbf{B}^*, \mathbf{A}^*)$ that permits to obtain all possible efficient production plans.

(ii) Let $\mathbf{p}^* \in \mathbb{R}_+^n$ be the unique (normalized) equilibrium price vector associated with $(\mathbf{B}^*, \mathbf{A}^*)$, and let $(\mathbf{B}^i, \mathbf{A}^i)$ be the technique used by the ith country, $i = 1, 2, ..., k$. Then $\mathbf{p}^*(\mathbf{B}^i - \mathbf{A}^i) \leq \mathbf{p}^*(\mathbf{B}^* - \mathbf{A}^*) = \mathbf{1}$.

This theorem establishes that there is a technique that permits to obtain all efficient production plans (changes in the level or composition of the final demand will affect the activity levels at which the technique $(\mathbf{B}^*, \mathbf{A}^*)$ operates, but not the technology in use). Moreover, it determines uniquely the equilibrium price vector for this efficient technology. The average cost of sector j in country i, evaluated at prices $\mathbf{p}^*$, is always larger than or equal to that associated with the efficient technique, for all $j = 1, 2, ..., n$, all $i = 1, 2, ..., k$. Therefore, we shall refer to $\mathbf{p}^*$ in the sequel as the vector of *efficiency prices.*

3 EFFICIENCY COMPARISONS

Time is ripe to discuss about the measurement of relative efficiency within the countries of the Federation. We propose here two alternative efficiency measures. The first one compares the maximum growth rate of a country with respect to that of the FETCH. The second is an estimate of the cost reduction that would result from the use of the efficiency prices.

3.1 A structural productivity measure

For each country $i = 1, 2, ..., k$, let $\mathbf{t}^i \geq \mathbf{0}$ denote the right eigen-vector of matrix $\mathbf{A}^i(\mathbf{B}^i)^{-1}$ associated with its Frobenius root. That is, $\mathbf{t}^i$ solves the following equation:

$$\lambda^i \mathbf{t}^i = \mathbf{A}^i(\mathbf{B}^i)^{-1}\mathbf{t}^i$$

Note that under axioms 1 to 3 we can ensure that $0 < \lambda^i < 1$, as $\mathbf{A}^i(\mathbf{B}^i)^{-1}$ turns out to be a productive non-negative matrix (Lemma 1). Define now

$$\vartheta^i := \frac{1}{\lambda^i} - 1$$

which is a positive scalar. This number provides a structural *productivity measure* for the ith country, because it corresponds to the maximum growth rate that is attainable with this technology. To see this note that we can rewrite the first equation as:

$$\mathbf{t}^i = (1+\vartheta^i)\mathbf{A}^i(\mathbf{B}^i)^{-1}\mathbf{t}^i$$

or, equivalently,

$$[\mathbf{I} - \mathbf{A}^i(\mathbf{B}^i)^{-1}]\mathbf{t}^i = \vartheta^i\mathbf{A}^i(\mathbf{B}^i)^{-1}\mathbf{t}^i$$

Let $\mathbf{s}^i = \mathbf{t}^i - \mathbf{A}^i(\mathbf{B}^i)^{-1}\mathbf{t}^i$. Theorem 1 ensures the existence of a unique vector $\mathbf{g}^i \in \mathbb{R}^n_+$ such that $(\mathbf{B}^i - \mathbf{A}^i)\mathbf{g}^i = \mathbf{s}^i$, with $\mathbf{g}^i = (\mathbf{B}^i)^{-1}\mathbf{t}^i$. Then, we can rewrite the last equation as follows:

$$\mathbf{B}^i\mathbf{g}^i = \mathbf{A}^i\mathbf{g}^i(1+\vartheta^i) \qquad [1]$$

This shows that the parameter ϑ^i is precisely the common proportion between the net output vector associated with the eigenvector $\mathbf{t}^i$ and the vector of input requirements associated to the realization of this outcome.

A simple interpretation of this measure can be obtained by multiplying both sides of equation [1] by the equilibrium price vector $\mathbf{p}^i = \mathbf{1}[\mathbf{B}^i - \mathbf{A}^i]^{-1}$ and normalizing the eigen-vector $\mathbf{g}^i$ by letting $\mathbf{p}^i\mathbf{A}^i\mathbf{g}^i = 1$, for all i. That gives us:

$$\vartheta^i = \frac{\mathbf{p}^i(\mathbf{B}^i - \mathbf{A}^i)\mathbf{g}^i}{\mathbf{p}^i\mathbf{A}\mathbf{g}^i} = \sum_{j=1}^{n} g^i_j \qquad [2]$$

As $\mathbf{A}^i\mathbf{g}^i$ is an input vector and $\mathbf{p}^i\mathbf{A}^i\mathbf{g}^i$ represents the labour incorporated in this vector, this normalization of $\mathbf{g}^i$ amounts to saying that we devote one unit of labour to produce intermediate inputs. Therefore, equation [2] tells us that the productivity measure ϑ^i is given by *the employment level that is generated by the production capacity derived from the application of one unit of labour to the production of intermediate inputs.*

Let us call ϑ^* the productivity measure associated with the FETCH. Clearly, $\vartheta^* \geq \vartheta^i$ for all $i = 1, 2, ..., k$. Also note that even though there might be more than one FETCH, the scalar ϑ^* is uniquely determined. Then we can define, for all $i = 1, 2, ..., k$ the ratio:

$$\phi_i = \frac{\vartheta^i}{\vartheta^*} \qquad [3]$$

that gives the ith country's productivity loss as the percentage of "full efficiency". It is as if we were measuring country i's internal productivity in terms of the "efficiency units" defined by the FETCH. When we compare the *difference* between two countries, we get the difference between their corresponding internal productivity coefficients in therms of these "efficiency units". That is,

$$\phi_i - \phi_t = \frac{\vartheta^i - \vartheta^t}{\vartheta^*}$$

Clearly, the productivity *ratio* between any two countries becomes independent on the efficiency units.

3.2 An economic efficiency measure

Let $\mathbf{p}^*$ denote the equilibrium price vector for the Federation's efficient technology $(\mathbf{B}^*, \mathbf{A}^*)$. We know (Theorem 3) that this is vector is uniquely determined by the family of technologies $[(\mathbf{B}^i, \mathbf{A}^i)]_{i=1}^k$. It follows from (ii) of Theorem 1 and (ii) of Theorem 3 that:

$$\begin{aligned} \mathbf{1} - \mathbf{p}^*[\mathbf{B}^* - \mathbf{A}^*] &= \mathbf{0} \\ \mathbf{1} - \mathbf{p}^*[\mathbf{B}^i - \mathbf{A}^i] &\geq \mathbf{0}, \qquad \forall \;\; i = 1, 2, ..., k \end{aligned}$$

This suggests that the efficiency prices $\mathbf{p}^*$ provide us with a suitable system of shadow prices that permits one to evaluate inefficiency costs. More precisely, for each sector $j = 1, 2, ..., n$, the number

$$1 - \sum_{t=1}^{n} p_t^* \left(B_{jt}^i - A_{jt}^i \right)$$

tells us the (unitary) loss incurred by the jth sector for *not* using the FETCH. This loss is measured by the (average) profits that would result from the application of the efficiency prices $\mathbf{p}^*$. Therefore, we can take the expression

$$\mathbf{1} - \mathbf{p}^*[\mathbf{B}^i - \mathbf{A}^i]$$

as a disaggregate measure of the efficiency loss per unit of output.

In order to get an overall measure of the inefficiency cost in the economy we have to multiply the vector of individual efficiency losses by that of production levels. Measuring the total cost with respect to the actual activity levels gives us:

$$\begin{aligned} C^i(\mathbf{p}^*, \mathbf{q}^i) &= \mathbf{1}\mathbf{q}^i - \mathbf{p}^*[\mathbf{B}^i - \mathbf{A}^i]\mathbf{q}^i \\ &= (\mathbf{p}^i - \mathbf{p}^*)\mathbf{d}^i \end{aligned} \qquad [4]$$

$C^i(\mathbf{p}^*, \mathbf{q}^i)$ tells us how much we would save by using the efficiency prices to pay for the cost of the actual demand, instead of using current market prices. Since prices correspond to labour values, this expression also gives us the extra labour we are using in order to satisfy the actual demand. This is easily seen if we rewrite the last equation as:

$$C^i(\mathbf{p}^*, \mathbf{q}^i) = \sum_{j=1}^{n} q_j^i - \mathbf{p}^*\mathbf{d}^i \qquad [4']$$

Associated with this index one can also define a relative productivity measure, that turns out to be independent on the production levels, as follows:

$$k^i(\mathbf{p}^*, \mathbf{q}^i) = \frac{\mathbf{p}^*\mathbf{d}^i}{\mathbf{p}^i\mathbf{d}^i} \qquad [5]$$

which is nothing else than the Laspeyres price index associated with the actual net production bundle $\mathbf{d}^i$. This coefficient provides a measure of relative productivity that is given by *the percentage of labour that would suffice to achieve the actual production pattern* (or the percentage of the aggregate expenditure required when using the efficiency prices instead of the current prices).

In a classical contribution Debreu (1951) proposes a measure of relative efficiency that compares the cost of the actual production with the closest efficient production, with a common vector of efficiency prices. Under the conditions of our model the efficiency price vector is uniquely determined and does not change with the consumption schedule. The only source of inefficiency that appears derives from not using the best available technology in the Federation. Therefore, in order to calculate Debreu's *coefficient of resource utilization*, we only need to define an efficient production plan which results from using the Federation's efficient technology at the actual production levels. So, let $\mathbf{d}^{*i} = [\mathbf{B}^* - \mathbf{A}^*]\mathbf{q}^i$ denote the net production that would result from using the FETCH at the current production levels for country $i = 1, 2, ..., k$. The coefficient of resource utilization for the ith country, ρ^i, is given by:

$$\rho^i(\mathbf{p}^*, \mathbf{q}^i) = \frac{\mathbf{p}^*\mathbf{d}^i}{\mathbf{p}^*\mathbf{d}^{*i}} \qquad [6]$$

That is, ρ^i is the Laspeyres quantity index in which we compare the worth of current net production and that corresponding to the use of the FETCH, evaluated at the efficiency prices.

The next result tells us that our economic efficiency measure coincides with Debreu's coefficient of resource utilization:

Theorem 4 *Under axioms 1 to 3,* $k^i(\mathbf{p}^*, \mathbf{q}^i) = \rho^i(\mathbf{p}^*, \mathbf{q}^i)$.

Proof.

First note that the equilibrium conditions in Theorems 1 and 3 imply:

$$\mathbf{1} - \mathbf{p}^*[\mathbf{B}^* - \mathbf{A}^*] = \mathbf{0} = \mathbf{1} - \mathbf{p}^i[\mathbf{B}^i - \mathbf{A}^i]$$

so that $\mathbf{p}^*[\mathbf{B}^* - \mathbf{A}^*] = \mathbf{p}^i[\mathbf{B}^i - \mathbf{A}^i,]$ for all $i = 1, 2, ..., k$. That allows us to write:

$$\mathbf{p}^*[\mathbf{B}^* - \mathbf{A}^*] - \mathbf{p}^*[\mathbf{B}^i - \mathbf{A}^i] \geq \mathbf{0}$$

Multiplying both terms by $\mathbf{q}^i$ we get:

$$\mathbf{p}^*(\mathbf{d}^{*i} - \mathbf{d}^i) \geq \mathbf{0}$$

This together with [4] implies $\mathbf{p}^*(\mathbf{d}^{*i} - \mathbf{d}^i) = (\mathbf{p}^i - \mathbf{p}^*)\mathbf{d}^i$. Therefore, $\mathbf{p}^*\mathbf{d}^{*i} = \mathbf{p}^i\mathbf{d}^i$ and, consequently,

$$k^i(\mathbf{p}^*, \mathbf{q}^i) = \frac{\mathbf{p}^*\mathbf{d}^i}{\mathbf{p}^i\mathbf{d}^i} = \frac{\mathbf{p}^*\mathbf{d}^i}{\mathbf{p}^*\mathbf{d}^{*i}} = \rho^i(\mathbf{p}^*, \mathbf{q}^i)$$

$\boxed{\textbf{Q.e.d.}}$

4 COMMENTS AND REMARKS

We have presented two alternative measures that allow us to perform efficiency comparisons in a system of economies whose production possibilities are described by input-output matrices. The first index, ϕ_i, measures efficiency as the percentage of the maximum growth attainable. The second index, k^i, is a price index that compares the worth of aggregate demand at efficiency prices and current market prices, respectively. Interestingly enough, the structure of the model implies that this price index coincides with the coefficient of resource utilization (a quantity index).

These two indices have some common features worth stressing. First, they are obtained out of the same original data and the same theoretical model. Second, they produce values in the interval $[0, 1]$, where 1 is obtained if and only if full efficiency is achieved. And third, both measures exhibit good invariance properties. In particular, they are invariant with respect to the choice of units in which commodities are measured and with respect to the aggregation level.

They also exhibit some relevant differences. ϕ_i is an index that only depends on the structural properties of the input-output matrices and not on the equilibrium allocation or the equilibrium prices. k^i, on the contrary, is a measure that reflects the equilibrium conditions.

We can relate these two measures by computing the index $C^i(\mathbf{p}^*, \mathbf{g}^i)$, where $\mathbf{g}^i$ is the vector defined through equation [1] and normalized so that $\mathbf{p}^i\mathbf{A}^i\mathbf{g}^i = 1$. Using equations [1], [3] and [4′] we can write:

$$C^i(\mathbf{p}^*, \mathbf{g}^i) = \vartheta^i \left[1 - \mathbf{p}^*\mathbf{A}^i\mathbf{g}^i\right] \qquad [7]$$

Alternatively, by letting $\widehat{\mathbf{d}}^i = [\mathbf{B}^i - \mathbf{A}^i]\mathbf{g}^i$, we can easily deduce:

$$k^i(\mathbf{p}^*, \mathbf{g}^i) = \frac{\mathbf{p}^*\widehat{\mathbf{d}}^i}{\vartheta^i} \qquad [8]$$

Equation [7] says that the ith country's aggregate efficiency loss, when measured with the standard commodity $\mathbf{g}^i$, is equal to the maximum growth multiplied by a factor that tells us the distance between the actual aggregate cost and that associated with the efficiency prices. Equation [8] expresses this relationship in relative terms. It says that the relative efficiency loss can be measured as the ratio between the labour value of the net output vector associated with the standard commodity, evaluated at the efficiency prices, and the maximum growth rate.

Finally, let us remark that the simplicity of the model partly depends on the preliminary assumption of "the same ℓ commodities in each country". This assumption is not admissible when the economic sectors are highly disaggregated and the countries under study very different. Yet, in many empirical applications the economies are modelled in terms of a reduced number of sectors (10 to 15, say). In this context this assumption is more likely to hold. Moreover, if countries are very different they will hardly be part of a Federation and/or fine comparative evaluations of efficiency might be of little interest.

Be as it may, we should point out that there are natural extensions of the model that can cover economies with different goods and/or more than one primary input [e.g. Johansen (1972), Manning (1981), Peris & Villar (1993)], at the cost of loosing the uniqueness of the technological standard and the associated efficiency prices.

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