MULTIDIMENSIONAL INEQUALITY AND SOCIAL WELFARE*

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A B S T R A C T

This paper provides a cardinal welfare measure for the allocation of a bundle of goods among a group of people. Social welfare is measured as the sum of *n* partial indices, one for each good, each of which consists of a function of the amount of the good availabe and its dispersion, measured by Theil's first inequality index. Some applications of this welfare measure are also suggested.

KEYWORDS: Inequality and Welfare; Social Goods; Cardinal Welfare Measures.

JEL classification number: D63

1 INTRODUCTION

1.1 Multidimensional inequality: the welfare evaluation of social goods

This paper deals with the evaluation of welfare and inequality in a distributive problem consisting of the allocation of a bundle of goods among a group of people. We aim at devising an indicator that measures group welfare as a function of the amounts and the distribution of the available goods. The term *multidimensional* refers to the presence of several goods whose distribution is jointly evaluated. The term *inequality* conveys the idea that equal division is the social optimum when agents are "homogeneous" in some appropriate sense (they have equal claims with respect to the goods that are being distributed). Therefore, the ensuing analysis is applicable to those distribution problems involving commodities that *should* be distributed equally among homogeneous agents. These commodities will be called here **social goods,** for the sake of simplicity in exposition.

Many distributive problems of this nature refer to the allocation of commodities which have to do with the equality of opportunity of economic agents (e.g. basic schooling, primary health services, social security benefits). These goods typically represent the material counterpart of some basic rights that define the entitlements of the citizens in a given society. Note that in some cases the consumption of social goods is to be interpreted as *accessibility* rather than real consumption, as it happens with public goods. A different case is that in which one has to evaluate the allocation of a single good in a society made of heterogeneous "types". The classical example is the analysis of income distribution in a society made of agents who have exercised different degrees of effort. In this context agents are homogeneous within each type but types may not be comparable. The multidmensional approach can be invoked here by interpreting the consumption of the good by different types as the consumption of different goods. An example where both several goods and different types may be present is that in which we want to analyze the distribution of wealth, when it consists of several components and we want to keep track of the distributional differences between these components (e.g. the evaluation the joint distribution of personal income, access to public facilities, and services provided by durable goods possessed by the agents).

The notion of social goods is reminiscent of the Rawlsian concept of *primary goods,* that he proposes as a reference for his theory of justice (goods that are deemed essential for the survival and self-respect of agents, such as rights, liberties, powers, income and opportunities). Yet, there are some relevant differences between these two concepts, which mostly derive from the fact that the scope of our analysis is much narrower than Rawls', as we focus on pure distribution problems. In particular, the substitutability between social goods is permitted (even though in our formulation it will become more and more difficult as the amounts of these goods decrease). Moreover, agents can be "heterogeneous" in which case the equal distribution is not compelling anymore (it might be the case that not all agents in the society are entitled to some of the goods considered). Indeed, our concerns are closer to the ideas of some economists who have tried to disentangle variables such as opportunities, rights or needs from utilities. Among them let us mention Kolm's notion of fundamental preferences, Sen's capability approach and Roemer's formulation of equality of opportunity. The reader is referred to the works of Rawls (1971), (1982), Kolm (1972), Sen (1985), Roemer (1998), and the discussion in Fleurbeay (1996), Herrero (1996) Roemer (1996, ch. 5), and Peragine (2000).

1.2 French dressing

The French Revolution consecrated in an admirable compact slogan some of the basic principles that inform the ethical system of our modern societies: *"Liberté, Egalité, Fraternité"*. These three words involve an extremely reach system of ethical references which can be recognized behind many social, legal and political institutions. Here we shall apply these ideas to the economic analysis of inequality and welfare, identifying these three abstract principles with the more specific value judgments of *responsibility, equal treatment of equals,* and *solidarity.*[1] These value judgements facilitate the analysis of distributive problems taking into account not only the evaluation of the observed outcomes, but also the distribution of opportunities that conditions the agents' choices. The starting point is the assumption that individual outcomes are partly dependent on individual decisions and partly dependent on the agents' exogenous circumstances. That is, we recognize the individual agents' autonomy of choice but also the existence of conditioning variables

[1] See Moulin (1995, ch. 1) for a different economic interpretation of these principles.

that are external to the agents' decisions.

Responsibility is the counterpart of "autonomous choice". In the context of distribution problems it means that individuals are accountable for their autonomous choices but not for their circumstances. As a consequence, those outcome differences that derive from personal choices made by the agents who share the same circumstances are perfectly admissible, whereas those outcome differences that arise from people's different opportunities will be deemed socially unfair.

Equal treatment of equals is a general principle that establishes that agents with identical characteristics should be treated equally. The notion of "identical characteristics" is to be interpreted relative to the problem under consideration. Here they correspond to the combination of the agents' external circumstances and those personal choices that are related to the distribution problem considered (e.g. labour decisions when evaluating the income distribution). Observe that this property says neither that everybody is equal nor that perfect equality *per se* is the social desideratum. What it says is that we shall consider unfair those outcome differences that are due to the circumstances for which the agents are not accountable (e.g. parental characteristics), or to features unrelated to the distribution problem, such as names, religion, political ideas, etc. We call "homogeneous" agents those who have the same relevant characteristics.

Solidarity is the political response to the lack of equality. It is an expression of the society's will to reduce the social and economic differences. Therefore, it recognizes the right of the less favoured agents to be "compensated" for some adverse circumstances that do not derive from their autonomous choices. This compensation may refer to some social benefits, the existence of a progressive tax system, the implementation of a positive discrimination policy, etc. In our setting this will translate into giving priority to the less favoured in the social evaluation function.

Taking these three value judgements as our philosophical blanket, the analysis is carried out according to two methodological principles:

(a) *Social welfare is defined directly in terms of the allocation of the goods and not as a function of agents' utilities.* This is an explicit departure from welfarism because agents' utilities are not the leading variables in the welfare evaluation.

(b) *The social marginal worth attached to the consumption of a given good depends on the good considered and the agent who consumes it.* This principle is related to Sen's (1976) "personalized goods approach" and allows us to evaluate allocations by means of a system of shadow prices which weight commodities differently, depending on the agent who consumes them.

This methodological approach has been successfully applied to the welfare analysis of one-dimensional distribution problems, following the ideas of Sen (1976), (1979) [see for instance Osmani (1982), Chakravarty & Dutta (1990), Herrero &Villar (1992), Ruiz-Castillo (2000)]. Our contribution here extends the works in Herrero & Villar (1989) and Tomás & Villar (1993) on the evaluation of income distributions allowing for a multidimensional variable, refining the axiomatization of the welfare measure, and providing new fields of application.

1.3 Outline of the paper

Section 2 contains the reference model. We aim at identifying a welfare index for the evaluation of the distribution of n social goods in a given society. For each social good there is a subset of the society, made of those agents who are entitled to this good, that we assume perfectly homogeneous. That is, all agents within each group share the same characteristics and have no differential claims with respect to the good that is being distributed (because all of them have contributed equally to its production or simply because the amount available of this good is independent on the agents' actions). Note that this formulation permits one to treat the case of a society made of a single group and n social goods, the case of a single good being allocated among the agents that belong to n different types, or a mixture of both. Be as it may, within context the principle of responsibility does not apply whereas equal treatment implies that equal division is the social optimum. The principle of solidarity will enter the picture in the welfare evaluation of the resulting distribution.

The main result in this section consists of the axiomatization of a *cardinal welfare measure.* For that we start by restricting the choice of welfare indices to the family of homogeneous functions. This allows to evaluate allocations as weighted sums of individual consumption vectors. These weights describe the individuals' social marginal worth in the evaluation function. Therefore, our value judgements on social welfare can be naturally expressed in terms of the weighting system.

There are two key axioms that lead to the evaluation formula. The first one, *independence*, says that agents' weights in social welfare only depend on their individual relative consumption. This can be regarded as an expression of the principle of equal treatment of equals. The second one, *progressivity*, is an instance of the principle of solidarity. It establishes that the ith agent's social marginal worth, with respect to a given commodity, is negatively correlated and inversely proportional to her share in total consumption. That is, we are going to give progressively more weight in social welfare to those agents with smaller shares in total consumption.

Combining these two axioms with a suitable scaling system permits one to measure social welfare as the sum of n partial indices, each of which is a function of the amount consumed of a particular good and of its dispersion, measured by Theil's first inequality index. Let us point out that the choice of the "scaling system" is a relevant part of the model when different commodities are involved.

Section 3 provides three different applications of this evaluation formula. One refers to the case of a society consisting of different subsocieties and aims at getting estimates of the between and within welfare loss due to the unequal distribution of the goods. Other deals with the welfare evaluation of income and opportunities. Following the ideas developed in Roemer (1998) we show how our formula can be used in the analysis of the equality of opportunity. Here we shall apply the notion of responsibility to asses the relevant inequality as opposed to the observed outcome inequality. The last one deals with the welfare evaluation of the provision of local public goods. Here again agents are heterogenous even though they are so in a different way which requires an alternative treatment.

The paper concludes with a few final comments in Section 4.

2 THE REFERENCE MODEL

We consider here the evaluation of social welfare in a distribution problem with n social goods in a society consisting of a (finite) set of agents. For each social good there is a well defined subset of homogenous agents who are *entitled* to the consumption of that good. Since agents are homogenous within groups, equal division is the social optimum.

We shall assume from the outset that:

(i) Both the class of social goods and the subsets of the population that are entitled to each of these goods are given *a priori,* as the outcome of a social agreement.

(ii) Social goods are measurable by some real numbers that describe their availability (physical units) and each social good has associated with it a *reference price* (or a well defined unitary cost) which is kept fixed during the analysis. We choose commodity units so that each reference price is equal to one.

These two preliminary assumptions are very convenient, but far from innocuous. They put clear limits on the nature of the ensuing analysis and introduce some aggregation principles that are left unexplained. The first one amounts to saying that we shall not discuss here the key question of which goods should enter the evaluation function and which agents are entitled to these goods. In some cases this might be far more important than the actual formula of measurement. The second one implies that we take as given (through an exogenous reference price vector) the weights with which different goods enter social welfare. Here again the choice of the reference price vector conditions the significance of the evaluation formula. We discuss further this point the final section.

Consider a society $M = \{1, 2, ..., m\}$ and a distribution problem involving n social goods, which are measured in some given units. For each social good j there is a group $M(j) \subset M$, of agents who are entitled to the consumption of this good. There are m_j agents in group $M(j)$, $j = 1, 2, ..., n$, in the understanding that each agent may be part of several or all of these groups, depending on the problem under consideration. In particular, when all agents are entitled to the consumption of all social goods, then $M(j) = M$ for all j. When goods j and k correspond to the same commodity, which is interpreted as two different goods because it is consumed by two different types of agents, we have $M(j) \cap M(k) = \emptyset$.

Let $a = \sum_{j=1}^{n} m_j$. A point $\mathbf{x} \in \mathbb{R}^a_{++}$ is an **allocation**. That is, x_{ij} describes the amount of the jth good consumed by agent $i \in M(j)$, that we take to be strictly positive. For every $j = 1, 2, ..., n$, call $X_j = \sum_{i \in M(j)} x_{ij}$ —that is, X_j is the aggregate amount of commodity j in the distribution $\mathbf{x}$. When commodity units are chosen so that all prices are unity, as it is assumed here, x_{ij} can be interpreted both as a quantity or as a value.

The key point of our analysis is the identification of a welfare criterion that enables the evaluation of allocations $\mathbf{x} \in \mathbb{R}^a_{++}$. That is, we look for a **social evaluation function** $V : \mathbb{R}^a_{++} \to \mathbb{R}$ which permits one to perform welfare assessments of the overall allocation of social goods. The properties of this evaluation function will reflect the value judgments involved. We shall restrict the search of this evaluation function to the family of cardinal (and smooth) measures. To be precise, let $\mathbb{V}$ denote the family of *evaluation functions* $V : \mathbb{R}^a_{++} \rightarrow \mathbb{R}$ *that are homogeneous of degree one and twice differentiable.*

The homogeneity property is equivalent to the existence of a complete, continuous and homothetic social preference preordering on the set of allocations $\mathbb{R}^a_{++}$. It introduces a cardinal element in the evaluation, as $V(\lambda \mathbf{x}) = \lambda V(\mathbf{x})$ for every $\lambda > 0$. The differentiability property is an operational requirement that will facilitate our reasoning.

For any given $V \in \mathbb{V}$ Euler's theorem implies that the evaluation of a given allocation $\mathbf{x} \in \mathbb{R}^a_{++}$ can be expressed as $V(\mathbf{x}) = \nabla V(\mathbf{x})\mathbf{x}$, where $\nabla V(\mathbf{x})$ stands for the vector of partial derivatives of V, and $\nabla V(\mathbf{x})\mathbf{x}$ denotes the scalar product. Hence, calling $f_{ij}(\mathbf{x}) = \partial V(\mathbf{x})/\partial x_{ij}$ for each $\mathbf{x}$ in $\mathbb{R}^a_{++}$ we have:

$$V(\mathbf{x}) = \sum_{j=1}^{n} \left(\sum_{i \in M(j)} f_{ij}(\mathbf{x}) x_{ij} \right) \qquad [1]$$

This expression says that the social welfare of allocation $\mathbf{x}$ can be measured as a weighted sum of individual consumption levels, where the coefficient $f_{ij}(\mathbf{x})$ describes the **social marginal worth** of individual i as a consumer of good j. Note that taking V in $\mathbb{V}$ implies that these weights are homogeneous of degree zero. That is, each agent's social marginal worth depends on the distribution of social goods but not on their levels.

Now we shall establish some assumptions on this evaluation function $V \in \mathbb{V}$. These axioms express our value judgments, in terms of properties of the weighting system $f_{ij}(\mathbf{x})$.

Axiom 1 (Independence) *For all* $i \in M(j)$, *all* $j = 1, 2, ..., n$, *all* $\mathbf{x} \in \mathbb{R}^a_{++}$, $f_{ij}(\mathbf{x}) = f_j(x_{ij}, X_j)$.

Axiom 2 (Progressivity) *For all* $i \in M(j)$, *all* $j = 1, 2, ..., n$, *all* $\mathbf{x} \in \mathbb{R}^a_{++}$,

$$\frac{\partial f_{ij}}{\partial x_{ij}} = \frac{-1}{x_{ij}}$$

Axiom 3 (Scale) $x_{ij} = X_j/m_j$ *for all* $i \in M(j)$, *all* $j = 1, 2, ..., n$, *implies* $f_{ij}(\mathbf{x}) = 1$.

Axiom 1 translates into this context the notion of *decentralizability* commonly used in the literature on cost/surplus sharing problems [e.g. Moulin (1988, ch. 6)]. It says that the social marginal worth of agent i as a consumer of good j only depends on her own consumption and the total amount available of this good. This is mostly an operational axiom which permits one to evaluate individual agents' weights with a minimal information. It also entails the property of "equal treatment of equals", as any two agents that receive the same amount of a given commodity are treated alike.

Axiom 2 tells us about the sign and the size of the change in the social marginal worth of the ith agent due to a change in her own consumption, other things equal. On the one hand it says that f_{ij} decreases when x_{ij} increases. This corresponds to the principle of *minimal equity*, introduced by Sen (1973, p. 18), and constitutes a basic value judgment: we are going to give more weight in social welfare to those agents with smaller consumption. On the other hand, axiom 2 also establishes that the change in f_{ij} due to a change in x_{ij} is inversely proportional to x_{ij}. Therefore, the ith agent's social marginal worth as a consumer of the jth good changes more the smaller her share. Note that Progressivity implies the "principle of Dalton", which postulates that a transfer from rich to poor that does not change their ranking increases social welfare.

Finally, axiom 3 establishes that when equal division prevails we take agent i's social marginal worth equal to unity. It follows from axioms 1 and 3 and equation [1] that the contribution to social welfare of an amount X_j of good j which is equally distributed is precisely X_j.

Let us recall here the definition of Theil's first inequality index.[2] Let $\mathbf{y} \in \mathbb{R}^m_{++}$ stand for a distribution of a given one-dimensional variable, and call $z_i = y_i/\mu$ (where μ stands for the average). Theil's first inequality index is given by:

$$T(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} z_i \ln z_i$$

When $\mathbf{y}$ is a vector of personal incomes $T(\mathbf{y})$ can be interpreted as a measure of the distance between population shares and income shares [see Theil (1967)]. It is easy to see that $0 \leq T(\mathbf{y}) \leq \ln m$.

[2] For a discussion of this inequality index the reader is referred to Blackorby & Donaldson (1978), Bourguignon (1979), Cowell & Kuga (1981) and Foster (1983).

For $\mathbf{x} \in \mathbb{R}^a_{++}$, let $T_j(\mathbf{x})$ denote the value of Theil's inequality index relative to the distribution of the jth social good. The following result is obtained:

Theorem 1 *A social evaluation function $V \in \mathbb{V}$ satisfies the axioms of independence, progressivity and scale if and only if, for every $\mathbf{x} \in \mathbb{R}^a_{++}$ we have*

$$V(\mathbf{x}) = \sum_{j=1}^{n} X_j \left[1 - T_j(\mathbf{x})\right]$$

Proof.

(i) We know that the evaluation function $V \in \mathbb{V}$ can be written as $V(\mathbf{x}) = \sum_{j=1}^{n} \sum_{i \in M(j)} f_{ij}(\mathbf{x}) x_{ij}$. The axioms of independence and progressivity imply that:

$$\frac{\partial f_{ij}(\mathbf{x})}{\partial x_{ij}} = \frac{\partial f_j(x_{ij}, X_j)}{\partial x_{ij}} = \frac{-1}{x_{ij}} \qquad [2]$$

Moreover, as f_{ij} is homogeneous of degree zero in $\mathbf{x}$, it follows that $f_j(\lambda_j x_{ij}, \lambda_j X_j) = f_j(x_{ij}, X_j)$. Therefore, letting $\lambda_j = m_j / X_j$ we can define an auxiliary function $\gamma_j : \mathbb{R}_{++} \to \mathbb{R}$ as follows:

$$f_{ij}(\frac{1}{\mu_j}\mathbf{x}) = f_j(\frac{x_{ij}}{\mu_j}, m_j) = \gamma_j(s_{ij})$$

where $\mu_j = X_j / m_j$ is the average consumption of the jth social good and $s_{ij} = x_{ij}/\mu_j$ represents the ith agent's share in this average. Now we can write:

$$\frac{\partial f_j}{\partial x_{ij}} = \frac{d\gamma_j}{ds_{ij}} \frac{1}{\mu_j} = \frac{-1}{x_{ij}}$$

which gives us:

$$\frac{d\gamma_j(s_{ij})}{ds_{ij}} = \frac{-1}{s_{ij}}$$

Solving this differential equation we obtain:

$$\gamma_j(s_{ij}) = \beta_j - \ln s_{ij} \qquad [3]$$

In view of axiom 3 (scale) we deduce: $\gamma_j(1) = \beta_j = 1$.

Therefore,

$$\begin{aligned} V(\mathbf{x}) &= \sum_{j=1}^{n}\left[\sum_{i\in M(j)}(1-\ln s_{ij})\,x_{ij}\right]=\sum_{j=1}^{n}\left[X_j-\sum_{i\in M(j)}x_{ij}\ln s_{ij}\right] \\ &= \sum_{j=1}^{n}\left[X_j-\mu_j\sum_{i\in M(j)}s_{ij}\ln s_{ij}\right]=\sum_{j=1}^{n}X_j\left[1-\frac{1}{m_j}\sum_{i\in M(j)}s_{ij}\ln s_{ij}\right] \\ &= \sum_{j=1}^{n}X_j\left[1-T_j(\mathbf{x})\right] \end{aligned}$$

(ii) The converse implication is trivially obtained when we let $f_{ij}(\mathbf{x}) = 1-\ln\frac{x_{ij}}{\mu_j}$, which is the weighting system in $V(\mathbf{x})$. ■

Theorem 1 says that choosing an evaluation function in the set $\mathbb{V}$ that satisfies axioms 1 to 3 is equivalent to measuring social welfare as a weighted sum of the amounts of goods available, deflated by a term that expresses the distance with respect to the egalitarian distribution, measured by Theil's inequality index.

Note that each term $X_j\left[1-T_j(\mathbf{x})\right]$ corresponds to the **egalitarian equivalent** amount of social good j, in the sense of Atkinson-Kolm-Sen. That is, the amount of commodity j that equally distributed among the icumbent agents would make the society as well-off as with the real amount available, when inequality is measured by Theil's inequality index. Consequently, by letting X_j^* to denote the egalitarian equivalent amount of j, Theorem 1 boils down to:

$$V(\mathbf{x})=\sum_{j=1}^{n}X_j^*$$

That is, $V(\mathbf{x})$ is nothing else than the sum of the egalitarian equivalent amounts of the n goods.

Observe that the homogeneity of V ensures a one to one correspondence between welfare measures and inequality measures [see Blackorby & Donaldson (1978)]. Also note that, as a consequence of axiom 3 (scale) and the nature of the inequality index, we take the aggregate worth of social goods as a suitable measure of social welfare if and only if all goods are equally distributed.

Equation [3] in the proof of Theorem 1 can help us to understand better this welfare measure. If one plugs the value $s_{ij} = e$ into this equation one finds that $\gamma_j(e) = 0$. That is, when a single agent gets more than three times the average, it enters with a negative weight in the social evaluation function. This is easily seen in the following figure that plots the semilogarithmic weighting function γ_j:

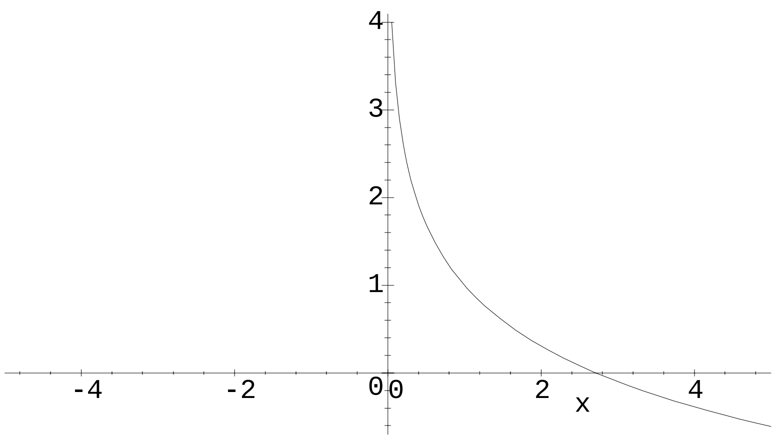

Figure 1

Note that for $T_j(\mathbf{x}) > 1$ the term $X_j\,[1 - T_j(\mathbf{x})]$ becomes negative, so that increasing the amount of the good available, other things equal, decreases the social welfare. Even though one does not find empirically such values, this theoretical consequence illustrates well the extent of the inequality aversion associated with the progressivity axiom. In other words, this welfare measure establish an inequality threshold, $T(\mathbf{x}) = 1$, above which getting more is not socially better if inequality is not reduced [see also the final section].

Remark 1 *This welfare index satisfies the principle of population replications, in the following sense: If we take two identical populations and consider the aggregate welfare associated with the society consisting of the union of them both, one should find that it is twice as much as the single case.*

3 FROM THEORY TO APPLICATIONS

We shall consider here three different problems for which our welfare measure becomes a useful tool of analysis. The first of these problems refers to the analysis of a society made of k different sub-societies, the object being the evaluation of the welfare loss that is due to the unequal distribution of social goods *between* and *within* the sub-societies. The second corresponds to the case in which we analyze the allocation of a single good among a society made of different types. It refers to the welfare evaluation of income and opportunities in a society in which agents are heterogeneous and exert diverse degrees of effort. Here we shall interpret the income of each degree of effort as a different social good. The last problem considered deals with the welfare evaluation of n different goods. We focus in this case on the provision of local public goods in a Federal State, identifying the agents with the constituent states.

3.1 The several-groups case

Our first application refers to the analysis of a society made of k different sub-societies, under the maintained assumption of homogeneous agents. The standard example is that of the regions within a country or the states in a federation. This is a classical topic in the study of inequality measurement, that is usually addressed under the heading of "decomposability" [see, for instance, Shorrocks (1984)].

Consider a society made of k sub-societies. To fix ideas let us consider the case of a state consisting of k regions. Let m_r denote the population of region r $(r = 1, 2, ..., k)$. We shall assume, for the sake of simplicity in exposition, that all agents within each region are entitled to the consumption of all goods. Therefore, the distribution of the jth social good *within* region r can be described by a vector $\mathbf{x}(j, r) \in \mathbb{R}^{m_r}_{++}$, whereas $\mathbf{x}(r) = [\mathbf{x}(1, r), ..., \mathbf{x}(n, r)] \in \mathbb{R}^{nm_r}_{++}$ stands for the vector that describes the distribution of the n social goods within region r.

Under axioms 1 to 3 we can measure the welfare of region r associated with a distribution $\mathbf{x}(r)$ as follows:

$$V[\mathbf{x}(r)] = \sum_{j=1}^{n} X_j^r \left[1 - T[\mathbf{x}(j, r)]\right] \qquad r = 1, 2, ..., k$$

where X_j^r is the total amount of good j that is available in region r and $T[\mathbf{x}(j,r)]$ denotes Theil's inequality measure in region r with respect to good j.

Similarly, let $\widetilde{\mathbf{x}} = [\mathbf{x}(1), \mathbf{x}(2), ..., \mathbf{x}(k)] \in \mathbb{R}^{nm}_{++}$, with $m = \sum_{r=1}^{k} m_r$, denote the whole country's distribution of the n goods. Now for each $j = 1, 2, ..., n$, let $X_j = \sum_{r=1}^{k} X_j^r$ and let $T_j(\widetilde{\mathbf{x}})$ denote the inequality index in the whole population, relative to the jth social good. Applying our welfare measure country-wise, we obtain:

$$V(\widetilde{\mathbf{x}}) = \sum_{j=1}^{n} X_j \left[1 - T_j(\widetilde{\mathbf{x}})\right]$$

Making use of the decomposability properties of the Theil index we can obtain a precise description of the terms that reflect the different inequality elements. In particular, for each $j = 1, 2, ..., n$ the corresponding Theil's inequality index can be decomposed as follows:

$$T_j(\widetilde{\mathbf{x}}) = \frac{1}{m} \sum_{r=1}^{k} \sum_{i=1}^{m_j} \frac{x_{ij}^r}{\mu_j} \ln \frac{x_{ij}^r}{\mu_j} = \frac{1}{X_j} \sum_{r=1}^{k} X_j^r \left(T_j \left[\mathbf{x}(j,r)\right] + \ln \frac{\mu_j^r}{\mu_j} \right)$$

Therefore, we can express our welfare index in the following form:

$$\begin{aligned} V(\widetilde{\mathbf{x}}) &= \sum_{j=1}^{n} \sum_{r=1}^{k} X_j^r \left[1 - \frac{1}{X_j} \sum_{r=1}^{k} X_j^r \left(T_j \left[\mathbf{x}(j,r)\right] + \ln \frac{\mu_j^r}{\mu_j} \right) \right] \\ &= \sum_{j=1}^{n} \sum_{r=1}^{k} X_j^r \left(1 - T_j \left[\mathbf{x}(j,r)\right] - \ln \frac{\mu_j^r}{\mu_j} \right) \qquad [5] \end{aligned}$$

This formula says that the contribution to the aggregate welfare of group r with respect to good j, is given by $X_j^r \left(1 - T_j \left[\mathbf{x}(j,r)\right] - \ln \frac{\mu_j^r}{\mu_j}\right)$. That is, as the total amount of good j in group r adjusted by two different components. The first one, $X_j^r T_j \left[\mathbf{x}(j,r)\right]$, measures the welfare loss due to the unequal distribution of j within r, irrespective of what happens in other regions. The second one, $X_j^r \ln \frac{\mu_j^r}{\mu_j}$ measures the relative share of this region with respect to the whole country.

Expression [5] can be rewritten as:

$$V(\widetilde{\mathbf{x}}) = \sum_{r=1}^{k} V[\mathbf{x}(r)] - \sum_{r=1}^{k}\sum_{j=1}^{n} X_j^r \ln \frac{\mu_j^r}{\mu_j} \qquad [6]$$

which says that the aggregate welfare of the country is given by the sum of the welfare levels of its constituent regions, minus a term that captures the welfare loss due to the inequality between the regions (a weighted sum of the logarithm of the ratios of the regional means with respect to the global mean).

3.2 The welfare evaluation of income and opportunities

Consider a society consisting of m agents and let $\mathbf{y} = (y_1, y_2, ..., y_m)$ stand for its income distribution vector. This context can be regarded as a particular case of our model for $n = 1$. If we let $Y = \sum_{i=1}^{m} y_i$, the welfare evaluation of $\mathbf{y}$ induced by axioms 1 to 3 is simply given by:

$$V(\mathbf{y}) = Y\,[1 - T(\mathbf{y})]$$

Here the egalitarian equivalent income, $Y^* = Y\,[1 - T(\mathbf{y})]$, gives us the aggregate income that equally distributed would make the society as well-off as with the real amount available.

Note, however, that the welfare loss due to the income inequality which is observed empirically derives from two sources which are very different from a welfare viewpoint. One refers to the income differences that result from the diverse *effort* decisions taken by people with the same characteristics (different choices on labour and leisure between people in the same social category, say). Another is that corresponding to the differences in *opportunity*, that is, the differences between people that perform the same degree of effort but have different exogenous characteristics.

Even though the concepts of effort and opportunity have fuzzy boundaries, they refer to relevant categories that matter for the ethical evaluation of income distribution. Effort has to do with responsibility whereas opportunity refers to the agents' external circumstances. Those differences derived from different choices made by people who share the same circumstances, are not relevant for the welfare analysis, whereas those differences that result from the unequal distribution of opportunities are those to be worried about. The bottom line is that a *fair* society should compensate agents for differences in opportunities and not for differences derived from personal decisions [see however Mariotti (2001)].

Following the ideas developed by Roemer (1998), Peragine (2000) and Ruiz-Castillo (2000) provide models in which the evaluation of opportunities is formulated in a way which seems well suited for empirical purposes. We shall adapt these ideas to our model and see that the possibility of evaluating several goods simultaneously greatly simplifies the analysis..

Suppose that this society, made of m agents, can be partitioned into k groups, $r = 1, 2, ..., k$, called *types*. Each type corresponds to a population subgroup with the same *circumstances*. Here we implicitly assume that there is a social agreement about which are the set of circumstances that provide equal opportunity for the economic agents. Therefore, being in the same group means having the same opportunities and all income differences within a group correspond to differences in people's effort decisions. Moreover, income differences between people of different types are not directly comparable, as they involve both differences in effort and differences in opportunity.

Now assume that effort is a single-valued variable which is positively correlated with income. According to Roemer's (1998) persuasive discourse, we shall consider that the distribution of effort is type-dependent (i.e. the effort distribution function is a characteristic of the type). Following Peragine (2000) we define a *tranche* as a subset of the population who have exercised a comparable *degree* of effort. This can be measured in terms of the quantiles of the effort distribution within types. Income differences in a tranche essentially reflect the different opportunities that people enjoy (provided the partition in quantiles and the space of characteristics that define the types are sufficiently fine). This is the relevant inequality we are concerned with.

Let us be more precise on these ideas. Consider n quantiles that are indexed by j. Then, the tranche $j = 1, 2, ..., n$ is the income vector

$$\mathbf{y}(j) = \bigcup_{r=1}^{k} \mathbf{y}(j, r)$$

where $\mathbf{y}(j, r)$ is the income vector of those agents in the jth quantile of group r. Letting $m(j) = \sum_{r=1}^{k} m(j, r)$, where $m(j, r)$ denotes the population in the jth quantile of group r, we can take $\mathbf{y}(j)$ as a point in $\mathbb{R}_{++}^{m(j)}$, for $j = 1, 2, ..., n$.

Now let us identify the income of each tranche as a different social good (so we have as many social goods as quantiles). This amounts to saying that an income distribution satisfies the principle of equality of opportunity whenever all people who exercise the same degree of effort receive the same income. Therefore, letting $Y_j = \sum_{i=1}^{m(j)} y_{ij}$ and $T[\mathbf{y}(j)]$ denote the aggregate income and Theil's inequality index of tranche j, respectively, we can evaluate the overall income distribution $\mathbf{y}$ as follows:

$$V(\mathbf{y}) = \sum_{j=1}^{n} Y_j \left[1 - T[\mathbf{y}(j)]\right]$$

Here the term

$$\sum_{j=1}^{n} Y_j T[\mathbf{y}(j)]$$

gives us the welfare loss that is due to the unequal opportunities in this society.

As in the approach developed in Ruiz-Castillo (2000, III.3) this way of measuring the aggregate welfare loss implies that the greater the degree of effort, the greater the weight received by the inequality of opportunities in the social welfare loss.

Remark 2 *It is worth noting the difference between the "population subgroups" in the former sub-section and the "types" in this one. Population subgroups are defined with respect to a variable which is not correlated with the goods whose distribution is being considered. Types, on the contrary, are defined precisely with respect to the distribution problem under study. Moreover, the consumption in population subgroups is directly comparable (homogeneous agents) whereas the consumption of the same good by two different types is considered as two different commodities, because agents are heterogeneous.*

3.3 The provision of local public goods

A different application refers to the social evaluation of local public goods. This is an important problem when we consider an economy consisting of m different regions which are responsible for the provision of a number of public services (health, education, transport, unemployment benefits, etc.). One can think of a Federal State that tries to give all citizens the right to enjoy these basic services, no matter where they choose to live.

Now our economic agents are not the individuals but the "regions" (States, Lander, Comunidades Autónomas, etc.) that constitute a Federal State. The social goods correspond to a number of public goods that are provided at regional level, such as education, health care, transport facilities, public infrastructures, etc. The basic value judgement behind our evaluation formula is that all citizens have the right to enjoy the same basic services within the region in which they live. Therefore, we have m regions (agents) and n different local public goods (social goods). These goods are public within the regions and private between them. We implicitly assume that these local public goods are subject to congestion, so that the number of users affects the quality of the service.

Needless to say that in this case agents may be widely different with respect to their size and characteristics, and that the homogenization problem comes to the forefront.[3] How can we transform regions that may differ in many relevant aspects into homogeneous entities? We propose an approach based on the following idea: A consumption vector $\mathbf{x} \in \mathbb{R}^n_{++}$ does not represent the same real consumption for two regions when there are significant differences in aspects such as *size and characteristics* of the population. Size matters for the quality of the service, as mentioned above. The relevant characteristics may vary from one service to another and usually refer to three types of variables: demographic (population pyramid, birth and mortality rates, population density, etc.), socio-economic (unemployment levels, growth rate, private wealth, capital stock accumulated, etc.), and environmental variables (e.g. climate, orography, pollution). Therefore, we shall assume that when the ith region consumes a vector $\mathbf{x}_i \in \mathbb{R}^n_{++}$, it gets an equivalent consumption vector $\mathbf{g}_i = (g_{i1}, g_{i2}, ..., g_{in})$. To make this operational, we can think of each g_{ij} as a function

$$g_{ij} = \Phi_j(x_{ij}, \nu_i, \theta_{ij})$$

where Φ_j is the transformation that describes how x_{ij} becomes real consumption, depending on the characteristics of the region, that are summarized in the variables ν_i, the population size of region i, and θ_{ij}, the *index of relative need* of service j in region i.[4]

[3]There are standard procedures to deal with this problem by means of equivalence scales, as analyzed for instance in Deaton & Muellbauer (1980), Ruiz-Castillo (1995). Note however that utilities are not the arguments of our evaluation function, so that the approach based on cost functions is straneous to our model.

[4]To obtain this relative index one usually makes use of factor analysis to identify the

Note that Φ_j is a sort of "production function" that transforms "raw commodities" into comparable consumption vectors. Function Φ_j is increasing in x_{ij} and decreasing both in ν_i and θ_{ij}. A simple formulation that reflects these features and has a natural interpretation is the following:

$$g_{ij} = \frac{x_{ij}}{\nu_{ij}^e}$$

where $\nu_{ij}^e = \nu_i(1+\theta_{ij})$ is the *equivalent population.* That is, ν_{ij}^e is the outcome of re-scaling the actual population of the ith region ν_i, by a "needs rate" θ_{ij} that varies from region to region and from one public good to another. Therefore, we can say that g_{ij} is simply the per capita consumption in terms of the equivalent population.

Our evaluation formula becomes in this case:

$$V(\mathbf{x}) = \sum_{j=1}^{n} G_j \left[1 - \alpha_j T_j(\mathbf{g})\right]$$

where

$$\begin{aligned} G_j &= \sum_{i=1}^{m} \frac{x_{ij}}{\nu_{ij}^e}, \qquad j = 1, 2, ..., n \\ \mathbf{g} &= \left(\frac{x_{11}}{\nu_{11}^e}, ..., \frac{x_{1n}}{\nu_{1n}^e}, ..., \frac{x_{m1}}{\nu_{m1}^e}, ..., \frac{x_{mn}}{\nu_{mn}^e} \right) \end{aligned}$$

To understand the implications of this welfare measure let us consider the following problem. Let $\mathbf{q} \in \mathbb{R}_{++}^n$ denote a given vector of goods that the central authority plans to distribute among the m regions. The planner's problem is the following:

$$\left. \begin{array}{l} \max \;\; \sum_{j=1}^{n} G_j[1 - T_j(\mathbf{g})] \\ s.t. : \sum_{i=1}^{m} g_{ij}\nu_{ij}^e \leq q_j \quad \forall \;\; j = 1, 2, ..., n \end{array} \right\}$$

The optimal solution is given by a vector $\mathbf{g}^*$ such that

$$g_{ij}^* = \frac{q_j}{\sum_{i=1}^{m} \nu^e{}_{ij}}$$

weights of the different factors that may affect the access of the population to a given service. We shall ignore here all the difficulties that have to be solved in order to get an operational index of this type. See Bosch & Escribano (1988) for a detailed explanation of these techniques and an empirical aplication to the Spanish case.

for all $i = 1, 2, ..., m$, all $j = 1, 2, ..., n$. Therefore,

$$x^*_{ij} = q_j \frac{\nu^e{}_{ij}}{\sum_{i=1}^m \nu^e{}_{ij}}$$

That is, the optimal allocation of the jth good corresponds to the proportional distribution of the available amount with respect to the equivalent population.

4 FINAL COMMENTS

We have characterized a welfare measure V which permits one to evaluate the allocation of a given bundle of n goods as the sum of n terms. The jth term of this sum corresponds to the amount available of good j deflated by Theil's inequality index. Let us conclude by comenting on two aspects of this modelization: the choice of commodity units and the axiom of progressivity.

We have assumed throughout the paper that commodity units have been normalized with respect to some reference price vector, so that all prices are equal to one and each term x_{ij} can be interpreted either as a physical amount or as a value. The use of a reference price vector makes the evaluation function independent of changes in the units of measurement.[5] Yet it has some consequences which should be taken into account.

The welfare implication of this modelling choice is easily understood if one takes the case of perfect equality and measures commodities in value terms, with the aforementioned reference price vector. So let $\mathbf{p} \in \mathbb{R}^n_{++}$ stand for our reference price vector, let $x_{ij} = p_j q_{ij}$, where q_{ij} is the physical amount of social good j and p_j denotes its price, and let $Q_j = \sum_{i \in M(j)} q_{ij}$. When $q_{ij} = Q_j / m_j$ for all $j = 1, 2, ..., n$, our evaluation formula becomes:

$$V(\mathbf{x}) = \sum_{j=1}^n p_j Q_j$$

which says that we accept relative prices as the relative weights of social goods for group welfare, under perfect equality.

[5] The choice of commodity units affects the values that function V takes on. In particular, for $n > 1$, it might be that $V(\mathbf{x}) > V(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^a_{++}$, whereas $V(\mathbf{x}') < V(\mathbf{y}')$, where $\mathbf{x}', \mathbf{y}'$ are the same commodity vectors measured in different (physical) units.

When the reference price vector corresponds to market prices one may interpret this choice as a way of measuring social welfare which is respectful with agents' *unanimous* judgments. Namely, all agents agree on the marginal worth of that commodity (as market prices correspond precisely to the ratio of agents' marginal utilities), and all agents enjoy the same consumption. Yet, this reference price vector can be criticized because market prices partly reflect the initial distribution of endowments, via the income and substitution effects between social goods and other commodities. To avoid this problem one might take a different reference price vector, such as the equilibrium price vector associated to the egalitarian distribution of all resources. Needless to say that the existence of a suitable system of prices for social goods may be far from obvious in many real-life situations. The determination of shadow prices will typically be a relevant part of the application of this formula to any empirical research.

The progressivity axiom postulates that the change in f_{ij} due to a change in x_{ij} is inversely proportional to x_{ij}, with a coefficient of proportionality equal to -1. As a consequence, agents enter the evaluation function with negative weights when they enjoy a consumption of more than three times the average. Moreover, an increase in aggregate consumption can produce reductions in social welfare in cases of extreme inequality. This implications show a high inequality aversion, which may be critiziced in some applications. To gain flexibility on this respect, one can introduce a more general formulation of the progressivity axiom, as follows:

Axiom 2' (Generalized Progressivity) *For all* $i = 1, 2, ..., m$, *all* $j = 1, 2, ..., n$, *some* $\alpha_j > 0$,

$$\frac{\partial f_{ij}}{\partial x_{ij}} = \frac{-\alpha_j}{x_{ij}}$$

It follows immediately from Theorem 1 that a social evaluation function $V \in \mathbb{V}$ satisfies the axioms of independence, generalized progressivity and scale if and only if, for every $\mathbf{x} \in \mathbb{R}^{mn}_{++}$ we have

$$V(\mathbf{x}) = \sum_{j=1}^{n} X_j \left[1 - \alpha_j T_j(\mathbf{x})\right]$$

This result gives us a family of functions that depends on the parameters α_j, for $j = 1, 2, ..., m$. The choice of these parameters expresses our value judgements with respect to the impact of the inequality measure on the welfare index. Therefore, we can interpret the coefficient α_j as a measure of our "inequality aversion" with respect to the jth primary good, because it establishes the weight that we put on inequality to transform amounts of goods into *welfare*. For instance, if we let $\alpha_j = \frac{1}{\ln m_j}$ we ensure that $[1 - \alpha_j T_j(\mathbf{x})] \geq 0$, for all possible values of the inequality index, which may be a desirable property in some cases (even though it implies making the impact of inequality on welfare dependent on the number of agents).

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