

SHORT-TERM OPTIONS WITH STOCHASTIC VOLATILITY: ESTIMATION AND EMPIRICAL PERFORMANCE*

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A B S T R A C T

This paper examines the stochastic volatility model suggested by Heston (1993). We employ a time-series approach to estimate the model and we discuss the potential effects of time-varying skewness and kurtosis on the performance of the model. In particular, it is found that the model tends to overprice out-of-the-money calls and underprice in-the-money calls. It is also found that the daily volatility risk premium presents a quite volatile behavior over time; however, our evidence suggests that the volatility risk premium has a negligible impact on the pricing performance of Heston's model.

Keywords: Stochastic, Volatility, Skewness, Kurtosis, Pricing.

1. INTRODUCTION

Given the Black-Scholes (1973) (BS henceforth) assumptions, all option prices on the same underlying security with the same expiration date but with different exercise prices should have the same implied volatility. However, the well known smiles and smirks suggest that the BS formula exhibits systematic biases in pricing options. There have been various attempts to deal with this apparent failure of the BS valuation model. In principle, as explained by Das and Sundaram (1999), the existence of the smile effect may be attributed to the presence of excess kurtosis in the conditional return distributions of the underlying assets. It is clear that excess kurtosis makes extreme observations more likely than in the BS case. This increases the value of out-of-the-money and in-the-money options relative to at-the-money options, creating the smile. Meanwhile, if the pattern shown by data contains a clear asymmetry in the shape of the smile, i.e. a smirk pattern, then this may be due to the presence of skewness in the distribution which has the effect of accentuating just one side of the smile.

Given this evidence, extensions to the BS model that exhibit excess kurtosis and skewness have been proposed in recent years along two lines of research: Jump-diffusion models under a Poisson-driven jump process, and the stochastic volatility framework are the key developments in the theoretical option pricing literature¹.

A recent and important attempt to summarize alternative option pricing models is carried out by Bakshi, Cao and Chen (1997). They are able to derive a closed-form jump-diffusion model that includes previously studied models. It allows not only stochastic volatility, but also stochastic interest rates and stochastic jumps. Moreover, following Bates (1996), they use a cross-sectional framework to implement their model, and analyze the performance and hedging behavior of the nested option pricing models. Das and Sundaram (1999) also examine the extent to which these models are able to capture the observed anomalies discussed in literature. Bakshi, Cao and Chen (1997) find that both stochastic volatility and jumps are important for pricing. However, they suggest that recognizing stochastic volatility alone produces the best hedging performance. On the other hand, Das and Sundaram (1999) argue that, generally speaking, stochastic volatility models yield better

¹ Alternatively, Corrado and Su (1996), and Backus, Foresi, Li and Wu (1997) adapt a Gram-Charlier series expansion of the normal density function to obtain skewness and kurtosis adjustment terms for the BS formula. Eberlein, Keller and Prause (1998) introduce the hyperbolic density to account for excess kurtosis and skewness, and they are even able to obtain a closed option pricing formula under this assumption. Rosenberg (1998) suggests the so called flexible density function methodology to estimate risk-neutral densities implied by option pricing data.

pricing results than jumps, although none of them is able to explain all patterns of kurtosis, skewness and volatility smiles found in empirical pricing literature.

Stochastic volatility option valuation start with the bivariate diffusion processes of Hull and White (1987), Scott (1987) and Wiggins (1987). In these models the volatility risk premium is not rewarded and thus removed from the valuation equation. Further, the correlation between the volatility and the stock return is zero in Hull and White (1987) and Scott (1987). Heston (1993) provides a closed-form solution for a European call option without imposing the restrictions of zero correlation and zero price of volatility risk by using Fourier inversion methods.

The key objective of the paper is to analyze the empirical performance of the stochastic volatility model proposed by Heston (1993) relative to the BS framework for options on the Spanish IBEX-35 stock exchange index. At the same time, the empirical and theoretical behavior of the parameters characterizing the diffusion process assumed for the instantaneous variance is studied. Their behavior is discussed relative to the appropriate skewness and kurtosis underlying in Heston's model. This provides us with insights clarifying the reasons behind the poor performance found for the stochastic volatility option pricing model.

Since the Spanish option market shows a very limited number of exercise prices traded simultaneously and where liquidity is also less generalized than in the US market, then there are serious consequences for the empirical implementation of the cross-sectional estimation of implied parameters proposed by Bates (1996) and Bakshi, Cao and Chen (1997), that is; there are just not enough prices available in order to estimate jointly all parameters embedded in Heston's model. This forces us to turn to estimate the parameters in two separate steps. First, we estimate the parameters of the stochastic volatility, which are required inputs of Heston's model, by employing the indirect inference estimation technique of Gouriéroux, Monfort and Renault (1993) on a time-series of the underlying hourly return. Second, the price of the volatility risk and instantaneous variance are backed out by minimizing the sum of squared pricing errors between the option model and market prices as in Bakshi, Cao and Chen (1997). Besides this practical argument, it should also be pointed out that the cross-sectional approach may easily ignore relevant information in the original series that may not be embedded in the option prices.

The remainder of the paper is organized as follows: Section 2 contains a brief summary of the Spanish options market features and the options data. The theoretical model,

i.e. Heston's model, employed in this paper appears in Section 3. The empirical results regarding the time series estimation of the stochastic volatility parameters, the theoretical discussion on the relationship between these parameters and the appropriate skewness and kurtosis in Heston's framework appears in Section 4. The implied volatility graphs of the daily implied variance, the estimation and testing of the volatility risk premium through option prices is shown in Section 5. In Section 6, the out-of-sample pricing performance is carried out. Finally, in Section 7, we conclude with a summary.

2. THE SPANISH IBEX-35 INDEX OPTIONS

2.1. *Market description*

The Spanish IBEX-35 index is a value-weighted index comprising the 35 most liquid Spanish stocks traded in the continuous auction market system. The official derivative market for risky assets, which is known as MEFF, trades a futures contract on the IBEX-35, the equivalent option contract for calls and puts, and individual option contracts for blue-chip stocks. Trading in the derivative market started in 1992. The market has experienced tremendous growth from the very beginning. Relative to the volume traded in the Spanish continuous market, trading in MEFF represented 40% of the regular continuous market in 1992, 156% in 1994, and 170% in 1995. The number of all traded contracts in MEFF relative to the contracts traded in the CBOE reached 20% in 1995.

The IBEX-35 option contract is a cash settled European option with trading during the three nearest consecutive months and the other three months of the March-June-September-December cycle. The expiration day is the third Friday of the contract month. Trading occurs from 10:30 to 17:15. During the sample period covered by this research, the contract size is 100 Spanish pesetas times the IBEX-35 index, and prices are quoted in full points, with a minimum price change of one index point or 100 pesetas². The exercise prices are given in 50 index point intervals.

It is important to point out that liquidity is concentrated in the nearest expiration contract. Thus, during 1995 and 1996 almost 90% of crossing transactions occurred in this type of contracts. Finally, it should be noticed that options and futures contracts are directly associated. The futures contract has exactly the same contract specifications as the IBEX-35

² This has recently been changed to 1,000 pesetas.

options. This will allow us to employ the futures price rather than the spot price in our empirical exercise. In fact, this is what is usually done by practitioners.

2.2. *The data*

For this paper, our database is comprised of all call options on the IBEX-35 index traded daily on MEFM during the period January 3, 1996 through April 30, 1996. Given the concentration in liquidity, our daily set of observations includes only calls with the nearest expiration day. Moreover, we eliminate all transactions taking place during the last week before expiration (to avoid the expiration-related price effects).

As usual in this type of research, our primary concern is the use of simultaneous prices for the options and the underlying security. The data, which are based on all reported transactions during each day throughout the sample period, do not allow us to observe simultaneously enough options with the same time-to-expiration on exactly the same underlying security price but with different exercise prices. In order to avoid large variations in the underlying security price, we restrict our attention to the 45-minute window from 16:00 to 16:45. It turns out that, on average and during our sample period, almost 25% of crossing transactions occur during this interval. Moreover, care was also taken to eliminate the potential problems with artificial trading that are most likely to occur at the end of the day. Thus, all trades after 16:45 were eliminated so that we avoid data which may reflect trades to influence market maker margin requirements. At the same time, using data from the same period each day avoids the possibility of intraday effects in the IBEX-35 index options market.

These exclusionary criteria yield a final daily sample of 768 observations. Table 1 describes the sample properties of the call option prices employed in this work. Average prices, average relative bid-ask spread and the number of available calls are reported for each moneyness category. Moneyness is defined as the ratio of the exercise price to the futures price. A call option is said to be deep out-of-the-money if the ratio K/F belongs to the interval $(1.03, 1.08)$; out-of-the-money if $1.03 > K/F \geq 1.01$; at-the-money when $1.01 > K/F \geq 0.99$; in-the-money when $0.99 > K/F \geq 0.97$; and deep-in-the-money if $0.97 > K/F \geq 0.90$. As we already discussed, there are 768 call option observations, with OTM, ATM and ITM options respectively representing 51%, 32% and 17%. The average call price ranges from 12.88 pesetas for deep OTM options to 185.42 pesetas for deep ITM options. The average relative

bid-ask spread goes exactly in the opposite direction to the average price. In particular, it ranges from 0.39 for deep OTM options to 0.10 for deep ITM calls.

The implied volatility for each of our 768 options is estimated next. Note that we take as the underlying asset the average of the bid and ask price quotation given for each futures contract associated with each option during the 45-minute interval³. Recall that we are allowed to use futures prices given that the expiration day of the futures and options contracts systematically coincides during the expiration date cycle. Moreover, note that dividends are already taken into account by the futures price. To proxy for riskless interest rates, we use the daily series of annualized repo T-bill rates with either one week, two weeks or three weeks to maturity. One of these three interest rates will be employed depending upon how close the option is to the expiration day. Finally, as discussed by French (1984), volatility appears to be a phenomenon that is basically related to trading days. However, interest rates are paid by the calendar day. Thus, in order to estimate the implied volatility of each option in our sample, we employ Black's (1976) option pricing formula adjusted by two time measures to reflect both trading days and calendar days until expiration. These implied volatilities will be used later as the basis for comparison with Heston's implied instantaneous volatilities.

3. HESTON'S STOCHASTIC VOLATILITY OPTION PRICING MODEL

Heston (1993) obtains a closed-form solution for the price of a European call option on an asset with stochastic volatility. Heston works with Fourier transforms of conditional probabilities that the option expires in-the-money. The characterization of these probabilities is achieved through their characteristic function.

The stochastic volatility model proposed by Heston generalizes Geometric Brownian Motion by allowing the volatility of the return process itself to evolve stochastically over time in a square root mean-reverting fashion:

³ It might be that lack of liquidity in the futures market is responsible for the lack of variation in the price of the underlying asset during the 45-minute window. However, this is not the case. In fact, the futures market is at least as liquid as the spot market in terms of comparable measures of trading volume.

$$\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_{1t} \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{2t} \\
dW_{1t}dW_{2t} &= \rho dt
\end{aligned} \tag{1}$$

where μ is the instantaneous expected rate of return of the underlying asset, V_t is the instantaneous stochastic variance, θ is the long-term mean of the variance, κ governs the rate at which the variance converges to this mean, and σ represents the volatility of the variance process. The parameters of the variance process, $\theta, \kappa,$ and σ are all strictly positive constants. W_{1t} and W_{2t} are each a standard Brownian motion allowed to be instantaneously correlated. Thus increases (decreases) in volatility could be related to the level of the underlying asset.

The risk-neutral probability measure incorporates the market price of volatility, denoted as λ , to distinguish the objective probability measure from the risk-neutral one. The volatility risk premium is assumed to be proportional to the instantaneous variance, λV_t , and its sign arises from the (sign of) correlation between the Brownian processes assumed for the instantaneous variance and the (aggregate) consumption. The model is given by:

$$\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_{1t}^* \\
dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t}dW_{2t}^* \\
dW_{1t}^*dW_{2t}^* &= \rho dt
\end{aligned} \tag{2}$$

where $\kappa^* = \kappa + \lambda$, $\theta^* = \kappa\theta/(\kappa + \lambda)$.

Let $c(S, v, t)$ be the value of a European call option where $S \equiv S_t$ and $v \equiv V_t$ to abbreviate. Heston's formula is given by:

$$c(S, v, t) = S_t P_1 - K e^{-r(T-t)} P_2 \tag{3}$$

where, P_1 and P_2 are two risk-neutralized probabilities having the same interpretation as in the standard BS expression.

In the application below, we use future options so that the actual version of the formula we employ is given by:

$$c(F, v, t) = e^{-r(T-t)}(F_t P_1 - K P_2) \quad (4)$$

where F is the future price on the underlying spot price, and

$$P_j(x, v, T-t; \ln[K]) = \text{Prob}(x_T \geq \ln[K] | x_t = x, v_t = v) \quad (5)$$

where, $x \equiv \ln[F_t]$, $j = 1$ or 2 (the probability of the event $\{x \geq \ln[K]\}$ depends on whether we chose the futures for $j = 1$, or the riskless rate for $j = 2$), and P_j is, therefore, the conditional probability that the option expires in-the-money. These probabilities depend on the vector of parameters $(\kappa, \theta, \lambda, \sigma, \rho)$ given by the process assumed by Heston under the original probability. It is important to note that the expressions for these probabilities are slightly different from the original values given by Heston (1993) since we are using futures. The actual formulae employed in this paper are provided in Appendix A.

In the empirical implementation of the model, we could always go to option prices and implicitly infer the parameters under the risk neutral probabilities. Given the limited number of daily option prices available to the 45-minute window from 16:00 to 16:45, the cross-sectional approach is just not possible if we wish to estimate all the parameters implied in Heston's model for calculating daily call prices. Our estimation approach consists of a two step procedure. First, we employ the original asset data (a time series of the spot IBEX-35 return) to estimate the parameters from the true process, i.e. equation (1), by adjusting the discretization biases throughout the indirect inference estimation methodology which is discussed in the following section. Second, we estimate the volatility risk premium and daily instantaneous variance from option prices. Definitively, this methodology needs two different sources of data for estimating the parameters, both time series data and cross section data.

4. INDIRECT ESTIMATION

4.1. *Parameter estimation of the stochastic variance process*

To estimate the process under the original probability measure given by (1), we have to assume that available data are discrete-time observations of a continuous time process. If we apply regular econometric methods to discrete-time approximations, we would have a serious estimation -discretization- bias in our results. In order to avoid this bias, we employ the indirect estimation proposed by Gouriéroux, Monfort and Renault (1993). The procedure consists of two steps. First of all, by maximum likelihood techniques, we estimate an appropriate auxiliary model. Secondly, the estimates of the auxiliary model are compared with estimators based on simulations of the path of the continuous time process given by (1). More specifically, we have to introduce a discrete time analogue of (1) corresponding to a small time unit τ , such that $1/\tau$ is an integer. This is done by Euler approximation. Then for a given value of the parameters, we simulate the process, and obtain simulated values for the observation dates by merely selecting the values corresponding to integer indexes. This yields an accurate simulation of V as long as τ is sufficiently small. In Appendix B, we precisely describe the steps necessary for the indirect estimation procedure.

With respect to the empirical results, we first estimate the in-sample period from January 2, 1994 to January 2, 1996 with continuously compounded hourly returns from the Spanish IBEX-35 index. The calculation of returns is based on the last recorded logarithmic index levels over consecutive hourly intervals. It is well known that the first hourly return incorporates adjustments to the information which has arrived overnight, and therefore presents a higher average return variability than any other hourly return. This basically implies that this first return is not an hourly return, and we consequently delete it from the estimation. Our final sample length consists of 2,450 hourly observations.

The results for the in-sample period are reported in Table 2. We consider four alternative combinations of simulations and frequencies and the Euler discretization specification⁴. The results contained in the first column are obtained simulating the process once ($N = 1$), and 2,450x10 data points since the frequency used is equal to $\tau = 1/10$. In the second column of Table 1, we simulate (η_1, η_2) 10 times ($N = 10$), and we generate 10 series of size 2,450x10 given that the frequency is again $\tau = 1/10$. Hence, our final estimates are

⁴ Nowman (1997) discretization was also carried out but the results were similar to Euler discretization.

based on the following minimization: $\min_{\Omega} \left(\hat{\Psi} - 1/10 \sum_{i=1}^{10} \tilde{\Psi}_i \right)$. In the third column, the process

is simulated once ($N = 1$), but the calibration of the Nagarch (1,1) model is performed with more data. In particular, the length of the vector used in the estimation is $2,450 \times 10$. Given that the frequency is $\tau = 1/10$, we have a total of $2,450 \times 10 \times 10$ data points. Note, of course, that once we go back to the Nagarch (1,1) model using simulated data of equal frequency as the real data, we have $2,450 \times 10$ observations. This is the methodology employed in the rolling procedure for the out-of-sample estimation which will be used in testing Heston's option pricing model under Euler discretization. Finally, the results reported in the fourth column simulate the process once ($N = 1$), but the frequency is established at $\tau = 1/50$.

The estimations contained in Table 2 suggest that, independently of the alternative procedure employed, the results tend to be quite similar. There seems to be some minor difference in the estimator of the volatility of the variance process, and the estimator of the correlation coefficient between the shocks when we use the frequency at $\tau = 1/50$. In any case, independently of the procedure employed, the estimate of the correlation coefficient is, surprisingly, positive and close to zero. Given the asymmetry coefficient found in the Nagarch (1,1) structure, we would have expected a negative correlation coefficient to reflect the asymmetry generally observed in literature to reflect that agents seem to react more to bad news than to good news⁵.

For the out-of-sample estimation, the same process is estimated 80 times using systematically 2,450 past observations. This rolling procedure of the indirect inference is necessary to yield estimates of the parameters involved in Heston's expression for each day between January 3, 1996 and April 30, 1996. Thus, the daily changing estimates of these parameters are used as inputs in Heston's option pricing formula. Figure 1 presents the evolution of the parameters associated with the stochastic variance process throughout the out-of-sample period. As we can observe from the figure, the long-term volatility, $\sqrt{\theta}$, the rate of convergence of the instantaneous variance to the long-term average, κ , and the volatility of the variance process, σ , remain quite stable over the out-of-sample period. However, the coefficient of correlation between the shocks for the stock and the variance, ρ ,

⁵ Duan (1997) shows that both the GJ (1,1) and Nagarch (1,1) models converge to the same limiting diffusion process for the stochastic volatility, i.e. the mean-reverting Geometric Brownian motion. León and Sentana (1998) shows that the relationship between the parameters of this Brownian process and the ones corresponding to the Nagarch (1,1) model is exactly given by: $\rho = \gamma\sqrt{2}/\sqrt{1+2\gamma^2}$. Then, it must be the case that $\text{sign}(\rho) = \text{sign}(\gamma)$. Of course, this limiting result does not hold for the square-root mean reverting process.

increases continuously. As before, it is interesting to observe that the correlation remains positive throughout the out-of-sample period.

4.2. Skewness, kurtosis and the parameters of the stochastic volatility process

The time-varying behavior of the correlation coefficient found above may have a serious impact on the capacity of Heston's model to explain option pricing data. It suggests that, in this case, we may have a similar problem than the one we have when assuming constant volatility in the BS context. If correlation between prices and volatility changes continuously over time, skewness of the underlying asset may also exhibit time-varying behavior. This is clearly a potential and relevant problem for models with stochastic volatility.

On the other hand, according to the estimates shown in Figure 1, it seems that the behavior of the volatility of volatility is rather stable over time. This suggests that accounting for changing kurtosis of the underlying asset may not be as crucial as taking into consideration changing skewness.

Das and Sundaram (1999) obtain closed-form expressions for conditional and unconditional skewness and kurtosis under Heston's stochastic volatility model. Their expressions, for a given frequency of $\Delta t = 1$ can be seen in Appendix C.

Given our rolling estimates of ρ and σ from January 1996 to April 1996, we can daily estimate (C1) and (C2) from Appendix C, so that we may observe how the characteristics of the distribution of the underlying asset -skewness and kurtosis- change with both the correlation coefficient and the volatility of volatility.

Figure 2 depicts the conditional and unconditional skewness over the sample period. As we can easily observe, their behavior closely follows the pattern of the correlation coefficient of Figure 1. As expected, skewness mainly arises from the correlation between changing prices and stochastic volatility of the underlying asset. The problem is that, of course, Heston's model assumes a constant unconditional skewness over time⁶.

⁶ Given the similarity between the behavior of unconditional and conditional skewness, the impact of the instantaneous stochastic variance on the conditional skewness must be negligible relative to the effect of the correlation coefficient.

In León and Rubio (2000), it is shown that, all else being constant, the relationship between skewness and the correlation coefficient is positive; i.e. $\partial \text{SKEW} / \partial \rho > 0$ for both the conditional and unconditional cases. This explains the behavior of skewness in Figure 2. Therefore, if (as is in fact the case) the correlation coefficient tends to be rather unstable, an option pricing model with a stochastic differential equation for correlation would be welcomed⁷. Given this evidence, we should not expect to find a good performance of the stochastic volatility model when we compare observed market prices with theoretical prices. We will come back to this issue later in the paper.

Figure 3 contains the same evidence for the kurtosis case. In principle, the impact of time-varying kurtosis on the misspecification of Heston's model seems to be much less severe than the influence of the correlation coefficient. Its pattern over time is much more stable than ρ . This is related to the behavior of the volatility of volatility, σ , in Figure 1. This is the parameter that allows for kurtosis in the stochastic volatility option pricing model, and it does not seem to change much over time. Once again, in León and Rubio (2000), it is shown that the relationship between kurtosis and σ is positive, i.e. $\partial \text{KURT} / \partial \sigma > 0$, for both the conditional and unconditional cases.

As a summary, time-varying skewness may be the key issue to analyze if we want to understand the failure of the tests we report below. Its consequences may be much more serious than the potential effects of changing kurtosis. However, this point will be clarified in Section 6.

5. ESTIMATING THE IMPLIED VARIANCE AND THE VOLATILITY RISK PREMIUM

Volatilities from the BS model and both the instantaneous variance and the volatility risk premium from Heston model are estimated every day from option prices, specifically all available call options transacted over the 45-minute interval from 16:00 to 16:45 during the period January 3, 1996 to April 30, 1996.

⁷ See Nandi (1998) for an exhaustive discussion of the importance of the correlation coefficient

5.1. Estimation procedure

Note that, once we have estimated for each day the parameters of the variance process, κ, θ, σ , and ρ , using the rolling indirect inference procedure, we still need to estimate the volatility risk premium, λ , and the instantaneous variance, V_t , before we can actually price a given call option under Heston's model. Therefore, it seems reasonable to expect that we may use cross-sectional data to implicitly infer the parameters that minimize the sum of squared errors (SSE) in a given day of the sample.

Given the set of parameters of the indirect estimation procedure obtained for a particular day t in the sample, $\hat{\Omega}_t = (\hat{\kappa}_t, \hat{\theta}_t, \hat{\sigma}_t, \hat{\rho}_t)$, and for each option, i ($i = 1, \dots, n$) and each day t , we define the pricing error as:

$$e_{it}(V_t, \lambda; \hat{\Omega}_t) = \hat{c}_{it}(K_i) - c_{it}(K_i) \quad (6)$$

where $\hat{c}_{it}(K_i)$ is the theoretical price of call i in day t , and $c_{it}(K_i)$ is the corresponding observed market price. We then want to find the instantaneous variance, V_t , and the risk premium parameter, λ , to solve:

$$SSE_t \equiv \min_{\{V_t, \lambda\}} \sum_{i=1}^n \left[e_{it}(V_t, \lambda; \hat{\Omega}_t) \right]^2 \quad (7)$$

Direct inspection of the quadratic form in (7) shows that it is highly valley shaped and, therefore, it is extremely flat along a direction corresponding to a nontrivial combination of λ and V_t . As a consequence, derivative based minimization methods are expected to perform poorly since the numerically computed gradient is very unstable in a neighborhood of a minimum. This is confirmed by some experiments that we performed with the Newton-Raphson method. In order to avoid the above problem, a derivative-free minimization algorithm is called for. Since the function in (7) only depends on two parameters, the downhill simplex method of Nelder and Mead (1965) seems to be a natural candidate. We have robustified the method by choosing randomly several starting triplets and in those cases of convergence to different local minima their minimum was selected as the global solution. So, we obtain daily estimates of both λ and V_t from January 3, 1996 to April 30, 1996; i.e. a

sample of 80 days. Finally, a series for daily implied volatilities estimated in the corresponding BS's versions of (6) and (7) is also obtained.

5.2. *The volatility risk premium*

In Figure 4, we can see that the estimated daily values of λ further suggest that the volatility risk premium varies over time in the IBEX-35 futures option market. Notice that this time-varying behavior of volatility risk premium is not consistent with Heston's model since λ must be a constant value as we can see in equation (2). The estimated values of λ ranges from -0.252 to 0.333 . The mean (median) for the time-series of λ is -0.021 (-0.027) and the standard deviation is 0.113 . The skewness and the excess kurtosis are 0.527 and 0.747 respectively. The p-value of the Jarque-Bera test (normality test) for the time-series of λ is 0.085 , so the normal distribution is rejected at the 10% significance level. The null hypothesis of $\lambda = 0$ was tested under three different methods. First, assuming that λ values were drawn from a normal distribution, a Student t test was performed, so the Student t ratio was -1.58 (p-value= 0.1183).

Second, a nonparametric test for the population median of λ values, specifically the sign test⁸, was performed whose p-value was 0.1544 .

Third, we also test $H_0 : \lambda = 0$ by obtaining accurate confidence intervals for λ based on the "bootstrap-t" confidence interval (see Efron and Tibshirani, 1993). By selecting 10,000 independent bootstrap samples, each consisting of 80 data values drawn with replacement from the series of λ , then the 90%, 95% and 99% bootstrap-t confidence intervals for the mean are respectively: $[-0.0426; 0.0021]$, $[-0.0467; 0.007]$ and $[-0.0551; 0.0175]$. Summing up, we conclude that the evidence does not support rejection of the null hypothesis, i.e. the volatility risk premium is zero. The finding of a zero volatility risk premium for Ibex-35 index options is opposite to those of Guo (1998), Sarwar and Krehbiel (2000) using currency options, Lin, Strong and Xu (1999) using FTSE 100 index options. Nevertheless, our result is consistent with Bakshi, Cao and Chen (1997) who find that the maximum likelihood estimates of κ and θ are statistically indistinguishable from their respective S&P 500 option-implied counterparts, i.e. κ^* and θ^* .

⁸ Since the empirical distribution is skewed, the sign test is more appropriate than the popular nonparametric Wilcoxon signed-rank test because the last one assumes that the underlying distribution is symmetric (see Rohatgi, 1984).

Since the λ parameter in Heston's model is a constant value and given that we can impose that $\lambda = 0$ for the volatility risk premium embedded in Heston option prices on the Spanish Ibex-35 index, according to the conclusions from the above tests, then we can solve equations (6) and (7) and obtain a daily estimate of instantaneous variance from January 3, 1996 to April 30, 1996. We also repeat the above procedure for two alternative λ values, specifically $\lambda = 0.50$ and $\lambda = -0.50$, in order to analyze the sensibility of the estimated instantaneous variance under different magnitudes of λ which are in accordance with our sample of λ values. Definitively, in Section 6 we will test the out-of-sample pricing performance for Heston's model under the three candidate values of λ and compare each with BS's performance.

5.3. *Implied volatility graphs*

Figure 5 contains the evolution for several series of implied volatilities from January 3, 1996 to April 30, 1996 for each version of Heston's model, i.e. for $\lambda = -0.5, 0, 0.5$ and for a different daily λ value, and also, of course, for the BS formula. It can be easily appreciated that Heston's volatility, independently of the volatility risk premium assumed, tends to be higher than BS's volatility. *A priori*, this may have serious implications for pricing. Also, note that the four implied volatility series under Heston's model are very similar and for $\lambda = -0.5, 0, 0.5$, in general, the larger the required volatility risk premium imposed, the (very slightly) larger the instantaneous volatility.

To obtain a general picture of the potential misspecifications of the option pricing models employed in this research, we report the average pattern of implied volatilities across degree of moneyness. The results are shown in Figure 6.

In the BS case, we back out the implied volatility of each call option and for each day of the above period using the procedure discussed in sub-section 2.2. Then, the equally-weighted implied volatility for each moneyness category and each day in the sample period is calculated. There is a U-shaped pattern with a hump in the middle in Figure 6. This suggests that the BS model tends to underprice deep OTM and deep ITM calls. This is the typical smile pattern of the Spanish option market analyzed by Peña, Rubio, and Serna (1999). Any reasonable alternative model to BS must be able to properly price deep OTM and deep ITM call options. Of course, Heston's stochastic volatility model, given by equations (4) and (5), is a potential and particularly interesting candidate.

For $\lambda = -0.5, 0, 0.5$, we may analyze the pattern of implied volatilities across alternative degrees of moneyness. The evidence reported in this regard in Figure 6 suggests a rather asymmetric smile in Heston's model independently of the volatility risk premium imposed. It seems that Heston's approach tends to underprice (deep) ITM calls and overprice (deep) OTM calls.

We now turn to formal tests of the alternative option pricing specifications considered in this paper.

6. OUT-OF-SAMPLE PRICING PERFORMANCE

In order to test the out-of-sample pricing performance for each model analyzed in this work, we employ two years of rolling data to estimate, by indirect inference, the parameters of the stochastic variance process assumed in (1). Given these estimates, and a chosen volatility risk premium, λ , we use all call options available in our 45-minute window to compute for each day from January 2, 1996 to April 29, 1996 the instantaneous variance that minimized the squared error between the theoretical value and the market price of the call options according to (6) and (7). We then compute the theoretical price of each option using the previous day's instantaneous variance and the corresponding parameters of the stochastic volatility process. For the BS case, the previous day's implied volatility that minimized the squared error between the theoretical value and the market price of the options is used to obtain the theoretical BS price of each option in the sample.

In this way, we have 768 pricing errors for each of the calls available from January 3, 1996 to April 30, 1996, and for each of the models analyzed. These pricing errors are the basis for our analysis. Table 3 reports two measures of performance for the alternative model specifications. Panel A contains the absolute pricing error which is the sample average of the absolute difference between the model price and the market price for each call in the testing sample period. This statistic is reported for each moneyness category and for all calls in the sample. In Panel B, the reported percentage pricing error is the sample average of the theoretical price minus the market price, divided by the market price. Again, this statistic is calculated for each moneyness category and for all calls in the sample.

Overall, in absolute terms, Heston's option pricing model tends to value slightly better than BS. The absolute pricing error over all calls is approximately 2.8 pesetas for Heston's model independently of the volatility risk premium assumed, and 3.2 pesetas for the BS model. It is quite important to notice that the level of the volatility risk premium does not seem to have any influence on the performance of Heston's stochastic volatility model. The pricing errors obtained under alternative λ values are practically identical. There might be some evidence in favor of a positive risk premium, but we can safely conclude that option prices do not seem to be sensitive to the volatility risk premium. This is an important empirical result. Kapadia (1998) shows that the expected value of the delta-hedged gain, under a stochastic volatility model where volatility risk is not priced, is equal to zero. This is exactly the same result that holds under BS. Consequently, if we may assume that volatility risk is not priced, the analysis of the dynamic hedging performance of both models is clearly facilitated⁹.

It should be pointed out that the overall slightly better performance of Heston's model is not maintained throughout all moneyness categories. In particular, Heston's model tends to value ATM and ITM calls better than BS. However, the opposite result holds for OTM calls. This is also the case when we analyze the percentage pricing error in Panel B. Heston's model, regardless of the volatility risk premium imposed, tends to overvalue OTM calls and, at the same time, the model undervalues ITM calls. However, the percentage pricing for ATM calls is practically zero. In fact, ATM calls are clearly more consistent with a stochastic volatility model than with the well known lognormal assumption. BS, on the other hand, tends to undervalue deep OTM and deep ITM calls. This is consistent with a U-shaped volatility smile. In Heston's case, the evidence points towards a sneer rather than a regular smile.

6.1. *The statistical significance of the out-of-sample performance*

Overall, at least over the sample period studied and regardless of λ , Heston's model seems to overvalue, while BS tends to undervalue call options. Unfortunately, however, the simple statistics reported above do not help in making inferences in terms of the statistical significance of improvement when we contrast one model versus another.

⁹ On the other hand, if volatility risk is priced, the average delta-hedged gain is proportional to the magnitude and sign of the volatility risk premium. Comparisons of the dynamic hedging performance between Heston's model and BS may be much more complicated than the analysis carried out by Bakshi, Cao, and Chen (1997).

In this paper, the statistical significance of performance for out-of-sample pricing errors is assessed by analyzing the proportion of theoretical prices lying outside their corresponding bid-ask spread boundaries¹⁰. The following Z-statistic for the difference between two proportions is employed in the tests. The statistic is given by:

$$Z = \frac{p_1 - p_2}{\sqrt{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2}}$$

where p_1 is always the proportion of BS prices outside the bid-ask boundaries, and p_2 is the equivalent proportion for alternative Heston's specifications. n_1 and n_2 are sample sizes corresponding to these proportions. The statistic is asymptotically distributed as a standardized normal variable.

The empirical results are reported in Table 4. Given that we are also interested in knowing whether a given theoretical valuation model undervalues or overvalues market prices, the Z-statistic is also calculated to obtain the proportion for which the theoretical model yields a price below the bid quote, and the proportion for which the model gives a price above the ask quote. If a theoretical model tends to undervalue market prices, it would yield a higher proportion of prices below the bid quote. If, on the other hand, the model tends to overvalue market prices, it would have a higher proportion of prices above the ask quote.

When we consider all call options together, the p-value for statistical improvement of Heston's model over the BS formula is equal to 0.068. The proportion of BS prices lying outside the bid-ask boundaries is 48%, while the proportion of Heston's model, regardless of the volatility risk premium, is about 42%. As we see, there is a slight improvement, but we might interpret the results as rather disappointing. It is not clear at all that, given the costs of implementation, Heston's approach is worthwhile in terms of practical applications.

It is also the case that 35% of BS prices are below the bid quote. Given that Heston's model yields approximately 18% of prices below the bid, we can conclude that BS, on average, significantly undervalues market prices relative to Heston's approach. At the same time, Heston's prices are 24% of total observations above the ask quote. We can also conclude that Heston's model tends to significantly overvalue market prices relative to the BS formula. Thus, mispricing associated with BS is basically related to the tendency of the model to yield prices below market values. However, Heston's mispricing is a consequence of the model's tendency to offer prices above their corresponding market values.

¹⁰ See Corrado and Su (1996).

When we classify all call options by moneyness, similar conclusions are obtained. A call option is said to be OTM if $K/F > 1$, and a call is classified as ITM when $K/F < 1$. In general, within a given category of moneyness, neither model is statistically superior to the other. However, once again, BS mispricing comes from the tendency of the model to undervalue either OTM or ITM calls relative to Heston's model (and relative to market prices). The opposite is true for Heston's formula relative to BS (and market prices). It should be pointed out that in the latter case, the main problem arises when we value OTM with Heston's formula. There seems to be a strong tendency in Heston's model to overvalue this type of options. This was also reflected in Table 3.

It was mentioned above that the percentage pricing error of Heston's model for ATM calls turns out to be quite small. It may be the case that Heston's formula works particularly well for ATM options. Calls are said to be ATM when $1.01 > K/F \geq 0.99$. The exercise described above is repeated for this type of options. Overall, as expected, Heston's model tends to yield lower proportions of prices lying outside the bid-ask spread than in previously analyzed cases. However, the p-value for the difference relative to BS proportions is just 0.078. As before, BS tends to yield a statistically significant higher proportion of prices lower than the bid quote relative to Heston's prices, and Heston's model presents a statistically significant higher proportion of prices above the ask quote relative to BS prices. Now, however, Heston's formula has similar proportions above the ask or below the bid. Regarding ATM calls, Heston's misspecification cannot be explained by either overvaluation or undervaluation of call prices.

6.2. *The structure of pricing errors*

Given the poor empirical performance of both BS formula and Heston's stochastic volatility model, a further analysis trying to understand the structure of pricing errors of these models would seem to be called for. Following the evidence reported by Peña, Rubio, and Serna (1999) and Bakshi, Cao, and Chen (1997), we use a simple regression framework to study the relation between the percentage pricing errors and factors that are either option-specific or market dependent. We first take as given an option pricing model, and let e_{it} be the i -th call option's percentage pricing error on day t . Finally, we run the following regression for the whole sample period:

$$e_{it} = \alpha + \beta_1 X_{it} + \beta_2 \tau_{it} + \beta_3 SP_t + \beta_4 VOL_t + \beta_5 TERM_t + \beta_6 SKEW_t + \beta_7 KURT_t + \omega_{it}$$

(8)

where X is the moneyness of the i th call at time t as defined by the ratio between the exercise price (K) and the futures price (F); τ_{it} is the annualized time to expiration of the i th call on day t ; SP is the average relative bid-ask spread of all calls and puts transacted between 16:00 and 16:45 on date t ; VOL_t is the annualized standard deviation of the IBEX-35 index returns computed from 1-minute intradaily returns; $TERM_t$ is the yield differential between the annualized ten-year government bond and the annualized one-month repo Treasury bill; $SKEW_t$ is the conditional skewness, and $KURT_t$ the conditional kurtosis. They are given by expression (C1)¹¹ from Appendix C.

Table 5 contains the regression results based on the entire sample period and 768 call options, and where the standard error for each coefficient estimate is adjusted by the White (1980) heteroskedasticity-consistent estimator.

The explanatory variables employed in the regressions tend to be statistically significant. However, there are important differences between the percentage errors associated with either BS or Heston.

In particular, a key point of the results is the statistical significance of the coefficient estimates related to skewness and kurtosis, when we consider Heston's model under any of the three volatility risk premia used in the analysis. This is a very important result. As we argued in sub-section 4.2, the assumption of constant correlation between stochastic variance and price changes, and even the assumption of constant volatility of variance under Heston's model do not seem to be the appropriate assumptions to adequately explain the behavior of option prices even allowing for stochastic volatility.

Note that, on the other hand, the coefficient associated with kurtosis is not statistically significant in the BS case. However, as in Heston's case, the skewness bias is also relevant to explain the BS pricing errors.

Table 5 also shows that the annualized standard deviation of the IBEX-35 index slightly explains the percentage pricing errors independently of the model employed in the estimation. On average, pricing errors tend to be lower the higher the volatility of the index. However, the statistical significance of the coefficients is very weak.

¹¹ The same regression was run using one day lagged values for the market dependent variables. Very similar results were found. The actual regressions employ excess kurtosis as an explanatory variable.

Traditional biases are not corrected for any of the models. The bias associated with moneyness has the opposite sign in both types of models. As expected, given previous results, Heston tends to price OTM options worse than ITM calls. However, on average, the opposite result is found for BS. Moreover, the bias related to time to expiration seems to be relevant for both types of models. On average, the percentage pricing errors are larger the longer the time to expiration.

The influence on percentage pricing errors of both the yield differential between interest rates and the average spread for all calls and puts transacted over the 45 minute window are, interestingly enough, different for Heston and BS expressions. In the BS case, the higher the long-term yield relative to the short-term rate, the lower the percentage pricing error. However, this result disappears when we allow for stochastic volatility under any of Heston's specifications.

Let us analyze the spread variable. Regressions of a similar type were run including the relative bid-ask spread at date t of the call i . This is, contrary to the results reported in Table 5, a contract-specific variable. Surprisingly, the estimated coefficients are never significant regardless of the model considered. By doing this, we are really incorporating a transaction cost variable directly associated with the liquidity of each individual option. Again, this variable does not seem to be significant in explaining percentage pricing errors. On the other hand, however, when we include the average spread over all call and put options for a particular day t , the estimated coefficient becomes positive and significant in Heston's case. This aggregate variable may indicate the average consensus about the uncertainty of trading throughout the option market. It may be understood as the average adverse selection confronting market makers in trading options on the Spanish index. As we see from Table 5, the larger the average spread -larger average adverse selection among traders- the higher the percentage pricing error in Heston's stochastic volatility models. The impact of this type of uncertainty does not seem to be relevant in explaining the percentage pricing errors of BS.

In short, the pricing errors from Heston's framework have some moneyness, maturity, average (aggregate) bid-ask spread, skewness and kurtosis related biases. On the other hand, the BS case presents some moneyness, maturity, yield differential and skewness associated biases. Neither model seems to capture appropriately the underlying distribution characteristics of the underlying asset. Further research is clearly justified.

7. CONCLUSIONS

This paper introduces a two-step procedure, based on both time series and cross section data, for estimating all the parameters that we need to compute Heston call price. This estimation approach should be particularly useful in *thin* markets where a single cross-sectional estimation approach may be difficult to implement, for instance Spanish options data on the Ibex-35 stock index. Moreover, to employ just a cross-sectional procedure may ignore relevant information that may be included in the original series but not in the option prices.

The empirical results, however, are quite disappointing. On average, over all call options available in our sample, Heston's model improves the (poor) performance of BS just marginally. It is clear that this extremely limited improvement cannot justify the implementation costs involved in the estimation of Heston's approach¹². The overall rejection of Heston's model coincides with recent findings by Bakshi, Cao and Chen (1997) and Chernov and Ghysels (1998) for options written on the S&P 500 index.

We are quite convinced that the ultimate reasons behind the performance failure of Heston's model are closely related to the time-varying skewness and kurtosis found in the data. In particular, the assumption of a constant correlation coefficient between returns and stochastic volatility should be relaxed if we really want to have a richer model. Unfortunately, of course, the complexities needed to price options seem to increase without bounds. It may be the case that simple nonparametric methodologies are able to incorporate the missing (realistic) factors in our option pricing models.

It is also found that the daily volatility risk premium presents a quite volatile behavior over time. However, our evidence suggests that the volatility risk premium has a negligible impact on the pricing performance of Heston's model.

A potentially interesting area of research might be related to endogenously incorporating liquidity costs in option pricing models with either stochastic volatility, stochastic jumps or both. Once again, this approach may be extremely demanding from a theoretical point of view but it would probably be welcomed.

¹² Hedging performance of alternative models is not analyzed in this paper. It is possible that the hedging improvement under Heston's stochastic volatility model might be clearly superior to BS.

APPENDIX A

Heston's stochastic volatility model using futures options

Given that we use futures options, the actual version of the formula we employ is given by:

$$c(F, v, t) = e^{-r(T-t)} (F_t P_1 - K P_2)$$

where F is the future price on the underlying spot price, K is the exercise price and the probabilities are given by:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left(\frac{e^{-i\phi \ln[K]} f_j}{i\phi} \right) d\phi ; j = 1, 2$$

where $\operatorname{Re}(y)$ is the real part of the function y ; i is the imaginary number $i = \sqrt{-1}$, and

$$f_j(x, v, T-t; \phi) = \exp[C(T-t; \phi) + D(T-t; \phi)v + i\phi x]$$

where¹³,

$$C(T-t; \phi) = \frac{a}{\sigma^2} \left\{ (b_j - \rho\sigma\phi i + d)(T-t) - 2 \ln \left[\frac{1 - g e^{d(T-t)}}{1 - g} \right] \right\}$$

$$D(T-t; \phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2} \left[\frac{1 - e^{d(T-t)}}{1 - g e^{d(T-t)}} \right]$$

¹³ As we have already pointed out, the expression for $C(T-t; \phi)$ below is slightly different than the original value given by Heston (1993) since we are using futures.

$$g = \frac{b_j - \rho\sigma\phi_i + d}{b_j - \rho\sigma\phi_i - d}$$

$$d = \sqrt{(\rho\sigma\phi_i - b_j)^2 - \sigma^2(2u_j\phi_i - \phi^2)}$$

$$a = \kappa\theta$$

$$b_1 = \kappa + \lambda - \rho\sigma$$

$$b_2 = \kappa + \lambda$$

$$u_1 = 1/2; u_2 = -1/2$$

verifying that $C(0) = D(0) = 0$, and where $C(T-t;\phi)$ and $D(T-t;\phi)$ (and therefore the probabilities P_j ; $j = 1, 2$) depend on the vector of parameters, $(\kappa, \theta, \lambda, \sigma, \rho)$ given by the processes assumed by Heston under the original probability.

APPENDIX B

Indirect estimation procedure

1. Let $x_t \equiv \ln S_t$, then equation (1) becomes

$$dx_t = (\mu - V_t/2)dt + V_t^{1/2}dW_{1t} \quad (B1)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma V_t^{1/2}dW_{2t} \quad (B2)$$

$$dW_{1t}dW_{2t} = \rho dt$$

The above expressions are used to obtain the set of estimators $\Omega \equiv (\mu, \kappa, \theta, \sigma, \rho)$.

2. Let us consider next the Euler discretization¹⁴ of both (B1) and (B2) with frequency τ :

$$x_t = x_{t-\tau} + \mu\tau - \frac{V_{t-\tau}}{2}\tau + V_{t-\tau}^{1/2}\tau^{1/2}\eta_{1t} \quad (B3)$$

$$V_t = \kappa\theta\tau + (1 - \kappa\tau)V_{t-\tau} + \varepsilon_t \quad (B4)$$

where

$$\varepsilon_t = \sigma\tau^{1/2}V_{t-\tau}^{1/2}[\rho\eta_{1t} + (1-\rho^2)^{1/2}\eta_{2t}]$$

$$(\eta_{1t}, \eta_{2t})' \stackrel{iid}{\approx} N(0, I).$$

3. Our auxiliary model is given by the Nagarch (1,1) model¹⁵:

$$R_t = \mu + \xi_t; \xi_t = h_t^{1/2}\varepsilon_t; \varepsilon_t \stackrel{iid}{\approx} N(0,1) \quad (B5)$$

$$h_t = \omega + \beta h_{t-1} + \alpha \left(\xi_{t-1} + \gamma h_{t-1}^{1/2} \right)^2$$

where $R_t \equiv x_t - x_{t-1}$ and γ represents the relation between the shocks and the conditional variance. The set of parameters to be estimated by maximum likelihood is given by $\Psi \equiv (\mu, \omega, \alpha, \beta, \gamma)$.

4. Now we discuss the steps for the indirect estimation itself:

¹⁴ Bakshi, Cao and Chen (2000) also use the Euler discretization for the method of simulated moments which is employed to estimate the structural parameters of the continuous stochastic volatility (SV) process of the underlying asset. They also use a square root process for the SV model..

¹⁵ See León and Mora (1999) for the behavior of alternative specifications within the GARCH family in the Spanish stock market.

[4.1] Estimation of the auxiliary Nagarch (1,1) model with the observed data ($\tau = 1$)

$$\Rightarrow \hat{\Psi} \equiv (\hat{\mu}, \hat{\omega}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}).$$

[4.2] Let Ω_0 be a initial value for Ω .

[4.3] We simulate $(\eta_{1t}, \eta_{2t})'$ with $\tau = 1/10$, so $(\eta_{1t}, \eta_{2t})'$ is a $(T/\tau) \times 1$ dimensional vector, i.e.

$$\Theta_\tau \equiv \left\{ (\eta_{1t}, \eta_{2t})' : t = 1, 2, \dots, T/\tau \right\}.$$

[4.4] Given [4.2] and [4.3], simulate with Euler discretization equation (B4) to generate V_t , and then equation (B3), where the set of simulated values from (B3) is denoted by:

$$X(\Omega_0, \Theta_\tau) \equiv \{ \tilde{x}_t ; t = 0, \tau, 2\tau, \dots, T/\tau \}$$

[4.5] In order to work with the same frequency as the real data, take from $X(\Omega_0, \Theta_\tau)$ the following values:

$$\Gamma(\Omega_0, \Theta_\tau) \equiv \{ \tilde{x}_0, \tilde{x}_{10\tau}, \tilde{x}_{20\tau}, \dots, \tilde{x}_T \}$$

[4.6] Now, we go back to the auxiliary model and with the data from [4.5], we have a new set of R_t in (B5). We estimate the Nagarch model with the simulated data at real frequency and get $\tilde{\Psi}(\Omega_0, \Theta_\tau)$ which is the vector of ML estimators of the Nagarch model with $\tau = 1/10$ for $\Gamma(\Omega_0, \Theta_\tau)$.

[4.7] We have the same number of parameters in Ω and Ψ , so that we, in fact, minimize a distance with I as the weighting matrix¹⁶. Then, if

$$\hat{\Psi} = \tilde{\Psi}(\Omega_0, \Theta_\tau) \Rightarrow \text{indirect estimators are } \Omega_0 \Rightarrow \text{END}$$

[4.8] If $\hat{\Psi} \neq \tilde{\Psi}(\Omega_0, \Theta_\tau) \Rightarrow \text{GO TO [4.2]}$. So, let Ω_1 be another new value for Ω , continue the process and stop if $\hat{\Psi} = \tilde{\Psi}(\Omega_1, \Theta_\tau)$; otherwise come back to [4.2] and repeat the process until convergence.

¹⁶ Since the model is exactly identified, the results are unaffected by the choice of the weighting matrix.

APPENDIX C
(Un)conditional skewness and kurtosis

a) The conditional case:

$$\text{SKEW} = \left(\frac{3\rho\sigma e^{\kappa/2}}{\sqrt{\kappa}} \right) \left[\frac{\theta(2 - 2e^{\kappa} + \kappa + \kappa e^{\kappa}) - \nu(1 + \kappa - e^{\kappa})}{\left[\theta(1 - e^{\kappa} + \kappa e^{\kappa}) + \nu(e^{\kappa} - 1) \right]^{3/2}} \right] \quad (\text{C1})$$

$$\text{KURT} = 3 \left[1 + \sigma^2 \left(\frac{\theta A_1 - \nu A_2}{B} \right) \right]$$

where

$$A_1 = \left(1 + 4e^{\kappa} - 5e^{2\kappa} + 4\kappa e^{\kappa} + 2\kappa e^{2\kappa} \right) + 4\rho^2 \left(6e^{\kappa} - 6e^{2\kappa} + 4\kappa e^{\kappa} + 2\kappa e^{2\kappa} + \kappa^2 e^{\kappa} \right)$$

$$A_2 = 2 \left(1 - e^{2\kappa} + 2\kappa e^{\kappa} \right) + 8\rho^2 \left(2e^{\kappa} - 2e^{2\kappa} + 2\kappa e^{\kappa} + \kappa^2 e^{\kappa} \right)$$

$$B = 2\kappa \left[\theta(1 - e^{\kappa} + \kappa e^{\kappa}) + \nu(e^{\kappa} - 1) \right]^2$$

and where $\nu \equiv V_t$.

b) The unconditional case:

$$\text{SKEW} = 3 \left(\frac{\rho\sigma}{\sqrt{\kappa\theta}} \right) \left[\frac{1 - e^{\kappa} + \kappa e^{\kappa}}{\kappa^{3/2} e^{\kappa}} \right] \quad (\text{C2})$$

$$\text{KURT} = 3 \left[1 + \frac{\sigma^2}{\kappa\theta\kappa^2 e^{\kappa}} \left(1 - e^{\kappa} + \kappa e^{\kappa} + 4\rho^2 \left[2 - 2e^{\kappa} + \kappa + \kappa e^{\kappa} \right] \right) \right]$$

TABLE 1.
SAMPLE CHARACTERISTICS OF IBEX-35 FUTURES OPTIONS

Average prices, average relative bid-ask spread and the number of available calls are reported for each moneyness category. All call options transacted over the 45 minute interval from 16:00 to 16:45 are employed from January 2, 1996 to April 30, 1996. K is the exercise price and F denotes the futures price of the IBEX-35 index. Moneyness is defined as the ratio of the exercise price to the futures price. OTM, ATM, and ITM are out-of-the-money, at-the-money, and in-the-money options respectively.

	Moneyness (K/F)	Average Price	Average Bid-Ask Spread	Number of Observations
DEEP OTM CALLS:	1.03-1.08	12.88	0.3903	116
OTM CALLS:	1.01-1.03	28.68	0.2205	273
ATM CALLS:	0.99-1.01	57.20	0.1491	245
ITM CALLS:	0.97-0.99	98.54	0.1273	108
DEEP ITM CALLS:	0.90-0.97	185.42	0.0987	26
ALL CALLS:	-	50.52	0.2015	768

TABLE 2.
INDIRECT INFERENCE IN-SAMPLE ESTIMATION
January 1994-December 1995

The parameters of the following processes are estimated using the indirect inference technique:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_{1t} \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2t} \\ dW_{1t} dW_{2t} &= \rho dt \end{aligned}$$

where μ is the instantaneous expected rate of return of the underlying asset, V_t is the instantaneous stochastic variance, θ is the long-term mean of the variance, κ governs the rate at which the variance converges to this mean, σ represents the volatility of the variance process, and ρ is the instantaneous correlation. The auxiliary model employed in the estimation is the following Nagarch(1,1) model^{1/}:

$$\begin{aligned} R_t &= \mu + \xi_t; \xi_t = h_t^{1/2} \varepsilon_t; \varepsilon_t \stackrel{\text{iid}}{\sim} N(0,1) \\ h_t &= \omega + \beta h_{t-1} + \alpha \left(\xi_{t-1} + \gamma h_{t-1}^{1/2} \right)^2 \end{aligned}$$

where R_t is the hourly rate of return of the IBEX-35 index, and γ is the asymmetric parameter of the Nagarch. Euler discretization technique is employed in the estimation. Moreover, alternative frequencies and simulations are also used.

	N = 1; $\tau = 1/10^{2/}$ Length = 2,450	N = 10; $\tau = 1/10^{3/}$ Length = 2,450	N = 1 $\tau = 1/10^{4/}$ Length = 2,450x10	N = 1; $\tau = 1/50^{5/}$ Length = 2,450
$\hat{\mu}^{6/}$	0.619	0.696	0.702	0.868
$\sqrt{\hat{\theta}}^{7/}$	12.09	12.77	11.92	11.73
$\hat{\kappa}$	0.029	0.034	0.034	0.025
$\hat{\sigma}^{8/}$	1.631	1.864	1.857	1.542
$\hat{\rho}$	0.044	0.038	0.020	0.087

1/ The estimates of the Nagarch(1,1) parameters are:

$$\hat{\mu} = 0.00412; \hat{\omega} = 0.00301; \hat{\alpha} = 0.07680; \hat{\beta} = 0.89025; \hat{\gamma} = -0.1225$$

2/ The process is simulated once ($N = 1$), so that we employ 2,450x10 data points since $\tau = 1/10$, where 2,450 is the number of hourly returns available from January 1994 to December 1995.

3/ The process is simulated 10 times ($N = 10$), so that we generate 10 series of size 2,450x10 since, as in 2/, $\tau = 1/10$.

4/ The process is simulated once ($N = 1$), but now we employ 2,450x10x10 data points since $\tau = 1/10$. Thus, the calibration of the Nagarch is done with more data: 2,450x10.

5/ The process is simulated once ($N = 1$), so that we employ 2,450x50 data points since $\tau = 1/50$

6/ This is annualized and is given in percentage terms.

7/ This represents the standard deviation of the long-term variance. It is annualized and is given in percentage terms.

8/ This is annualized and is given in percentage terms.

TABLE 3.
OUT-OF-SAMPLE PRICING ERROR FOR ALTERNATIVE OPTION PRICING MODELS
ABSOLUTE PRICING ERROR AND PERCENTAGE PRICING ERROR

Two years of rolling daily data are employed to estimate by indirect inference the parameters of the stochastic volatility process under Heston's model. Given these estimates, and a chosen volatility risk premium, λ , we use all call options transacted over the 45 minute interval from 16:00 to 16:45 to compute for each day from January 2, 1996 to April 29, 1996, the instantaneous variance that minimized the squared error between the theoretical value and the market price of the options. We then compute the theoretical price of each option using the previous day's instantaneous variance and the corresponding parameters of the stochastic volatility process. For the Black-Scholes case, the previous day's implied volatility that minimized the squared error between the theoretical value and the market price of the options is used to obtain the theoretical price of each option in the sample. The reported absolute pricing error is the sample average of the absolute difference between the model price and the market price for each call in a given moneyness category. The reported percentage pricing error is the sample average of the theoretical price minus the market price, divided by the market price. K is the exercise price and F denotes the futures price of the IBEX-35 index. Moneyness is defined as the ratio of the exercise price to the futures price. OTM, ATM, and ITM are out-of-the-money, at-the-money, and in-the-money options respectively.

PANEL A: ABSOLUTE PRICING ERROR					
	Moneyness (K/F)	Black-Scholes (Ptas.)	Heston (Ptas.)		
			($\lambda=0$)	($\lambda=0.5$)	($\lambda=-0.5$)

DEEP OTM CALLS:	1.03-1.08	1.829	2.033	2.030	2.038
OTM CALLS:	1.01-1.03	2.641	2.815	2.819	2.774
ATM CALLS:	0.99-1.01	3.847	2.874	2.854	2.896
ITM CALLS:	0.97-0.99	4.690	3.593	3.586	3.600
DEEP ITM CALLS:	0.90-0.97	4.357	3.809	3.795	3.823
ALL CALLS:	-	3.249	2.859	2.852	2.853

PANEL B: PERCENTAGE PRICING ERROR					
	Moneyness (K/F)	Black-Scholes (%)	Heston (%)		
			($\lambda=0$)	($\lambda=0.5$)	($\lambda=-0.5$)

TABLE 4.
NONPARAMETRIC TESTING FOR ALTERNATIVE OPTION PRICING MODELS

The statistical significance of performance for out-of-sample pricing errors is assessed by analyzing the proportion of theoretical prices lying outside their corresponding bid-ask spread boundaries. The following Z-statistic for the difference between two proportions given by:

$$Z = \frac{p_1 - p_2}{\sqrt{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2}}$$

is employed in the tests, where p_1 is always the proportion of Black-Scholes prices outside the bid-ask boundaries, and p_2 is the equivalent proportion for alternative Heston's model specifications. n_1 and n_2 are sample sizes corresponding to these proportions. The statistic is asymptotically distributed as a standardized normal variable. All call options transacted over the 45 minute interval from 16:00 to 16:45 from January 3, 1996 to April 30, 1996 are used in the tests below. K is the exercise price and F denotes the futures price of the IBEX-35 index. Moneyness is defined as the ratio of the exercise price to the futures price.

Categories	BS	Heston ($\lambda=0$)	Z-stat. (p.value)	Heston ($\lambda=0.5$)	Z-stat. (p-value)	Heston ($\lambda=-0.5$)	Z-stat. (p-value)
<u>ALL OPTIONS:</u>							
p(Bid> c_{MODEL} >Ask)	0.4761	0.4249	1.825 (0.068)	0.4249	1.825 (0.068)	0.4233	1.882 (0.060)
p(c_{MODEL} <Bid)	0.3487	0.1837	6.730 (0.000)	0.1821	6.804 (0.000)	0.1901	6.434 (0.000)
p(c_{MODEL} >Ask)	0.1274 (0.000)	0.2412	-5.253 (0.000)	0.2428	-5.319 (0.000)	0.2332	-4.919
<u>OTM OPTIONS (K/F>1):</u>							
p(Bid> c_{MODEL} >Ask)	0.4821	0.4347	1.421 (0.155)	0.4347	1.421 (0.155)	0.4324	1.421 (0.155)
p(c_{MODEL} <Bid)	0.3184	0.1351	6.694 (0.000)	0.1329	6.791 (0.000)	0.1419	6.399 (0.000)
p(c_{MODEL} >Ask)	0.1637	0.2995	-4.864 (0.000)	0.3018	-4.940 (0.000)	0.2905	-4.566 (0.000)
<u>ITM OPTIONS (K/F<1):</u>							
p(Bid> c_{MODEL} >Ask)	0.4615	0.4011	1.166 (0.244)	0.4011	1.166 (0.244)	0.4011	1.1656 (0.244)
p(c_{MODEL} <Bid)	0.4231	0.3022	2.418 (0.015)	0.3022	2.418 (0.015)	0.3077	2.303 (0.021)
p(c_{MODEL} >Ask)	0.0385	0.0989	-2.294 (0.022)	0.0980	-2.294 (0.022)	0.0934	-2.123 (0.034)
<u>ATM OPTIONS (1.01>K/F≥0.99):</u>							
p(Bid> c_{MODEL} >Ask)	0.4583	0.3750	1.762 (0.078)	0.3749	1.763 (0.078)	0.3735	1.765 (0.077)
p(c_{MODEL} <Bid)	0.3981	0.1806	5.134 (0.000)	0.1800	5.395 (0.000)	0.1893	5.189 (0.000)
p(c_{MODEL} >Ask)	0.0602	0.1944	-4.272 (0.000)	0.1987	-4.341 (0.000)	0.1915	-4.231 (0.000)

TABLE 5.
PERCENTAGE PRICING ERRORS AND EXPLANATORY VARIABLES

For a given option pricing model, the following regression is employed to explain the percentage pricing errors of all call options transacted over the 45 minute interval from 16:00 to 16:45 from January 3, 1996 to April 30, 1996:

$$e_{it} = \alpha + \beta_1 X_{it} + \beta_2 \tau_{it} + \beta_3 SP_t + \beta_4 VOL_t + \beta_5 TERM_t + \beta_6 SKEW_t + \beta_7 KURT_t + \omega_{it}$$

where e_{it} is the percentage pricing error of the i th call on date t ; X is the moneyness of the i th call at time t as defined by the ratio between the strike price (K) and the futures price (F); τ_{it} is the annualized time to expiration of the i th call on day t ; SP is the average relative bid-ask spread of all calls and puts transacted between 16:00 and 16:45 on date t ; VOL_t is the annualized standard deviation of the IBEX-35 index returns computed from 1-minute intradaily returns; $TERM_t$ is the yield differential between the annualized ten-year government bond and the annualized one-month repo Treasury bill; $SKEW_t$ is the conditional skewness and $KURT_t$ is the conditional (excess) kurtosis. The t -statistic reported in parenthesis is based on White's heteroskedasticity consistent estimator of standard errors. A total of 768 call options are employed in the regressions.

Coefficient	Black-Scholes	Heston ($\lambda=0$)	Heston ($\lambda=0.5$)	Heston ($\lambda=-0.5$)
Constant	0.683 (2.17)	-0.563 (-1.79)	-0.590 (-1.87)	-0.550 (-1.77)
Moneyness (X)	-0.746 (-2.20)	0.715 (2.02)	0.737 (2.08)	0.694 (1.98)
Time to expiration (τ)	0.025 (2.67)	0.027 (2.84)	0.026 (2.77)	0.027 (2.84)
Spread (SP)	0.225 (1.48)	0.404 (2.69)	0.409 (2.72)	0.406 (2.71)
Volatility (VOL)	-0.998 (-1.69)	-0.976 (-1.70)	-0.986 (-1.72)	-0.941 (-1.64)
Term structure (TERM)	-0.129 (-4.65)	-0.008 (-0.26)	-0.007 (-0.24)	-0.009 (-0.30)
Skewness (SKEW)	1.545 (2.65)	1.238 (2.15)	1.227 (2.13)	1.223 (2.13)
Kurtosis (KURT)	0.217 (0.77)	-0.827 (-2.38)	-0.805 (-2.31)	-0.808 (-2.33)

FIGURE1
INDIRECT INFERENCE (January 96-April 96)

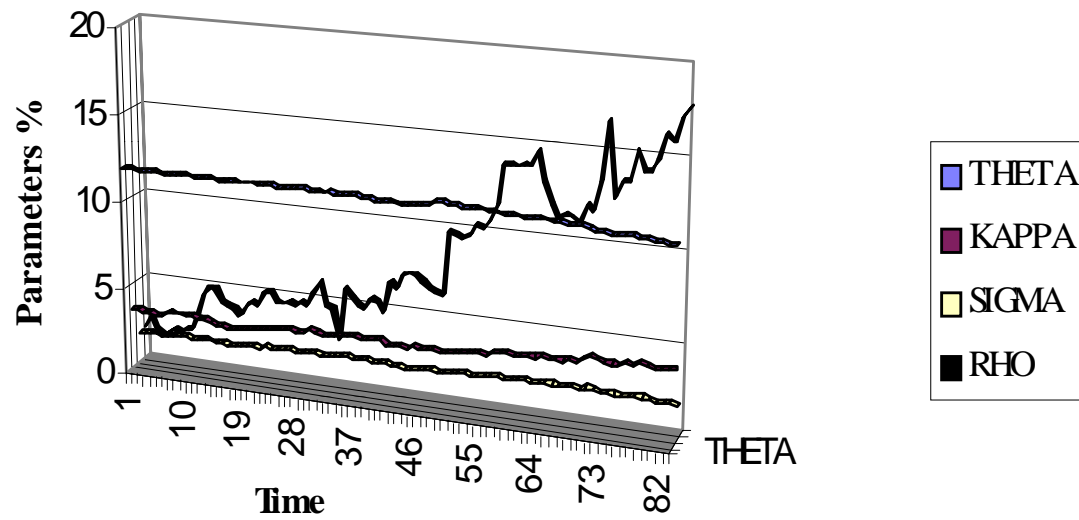


FIGURE 2
TIME-VARYING SKEWNESS (January 96-April 96)

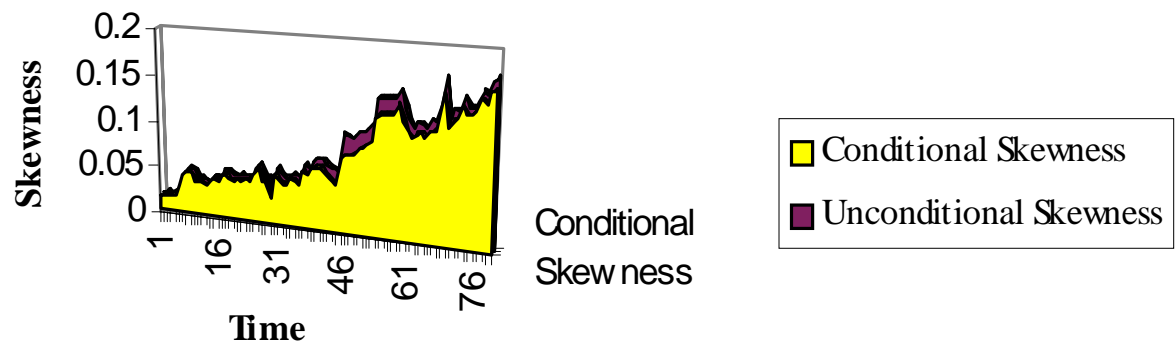


FIGURE 3
TIME-VARYING KURTOSIS (January 96-April 96)

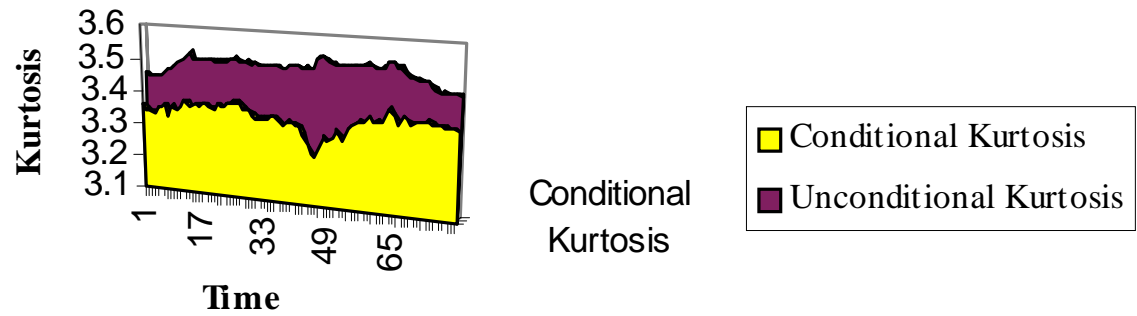


FIGURE 4
DAILY VOLATILITY RISK PREMIUM (January 96-April 96)

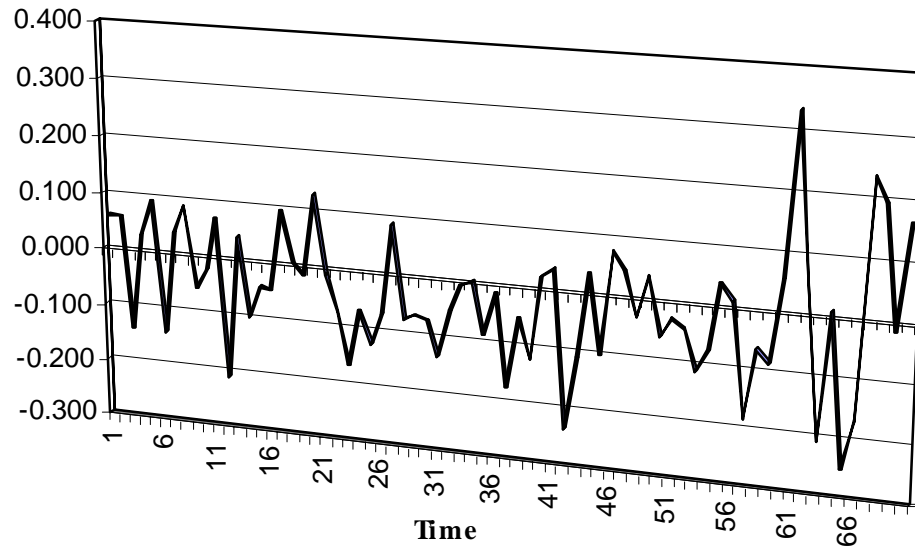


FIGURE 5
DAILY IMPLIED VOLATILITIES (January 96-April 96)

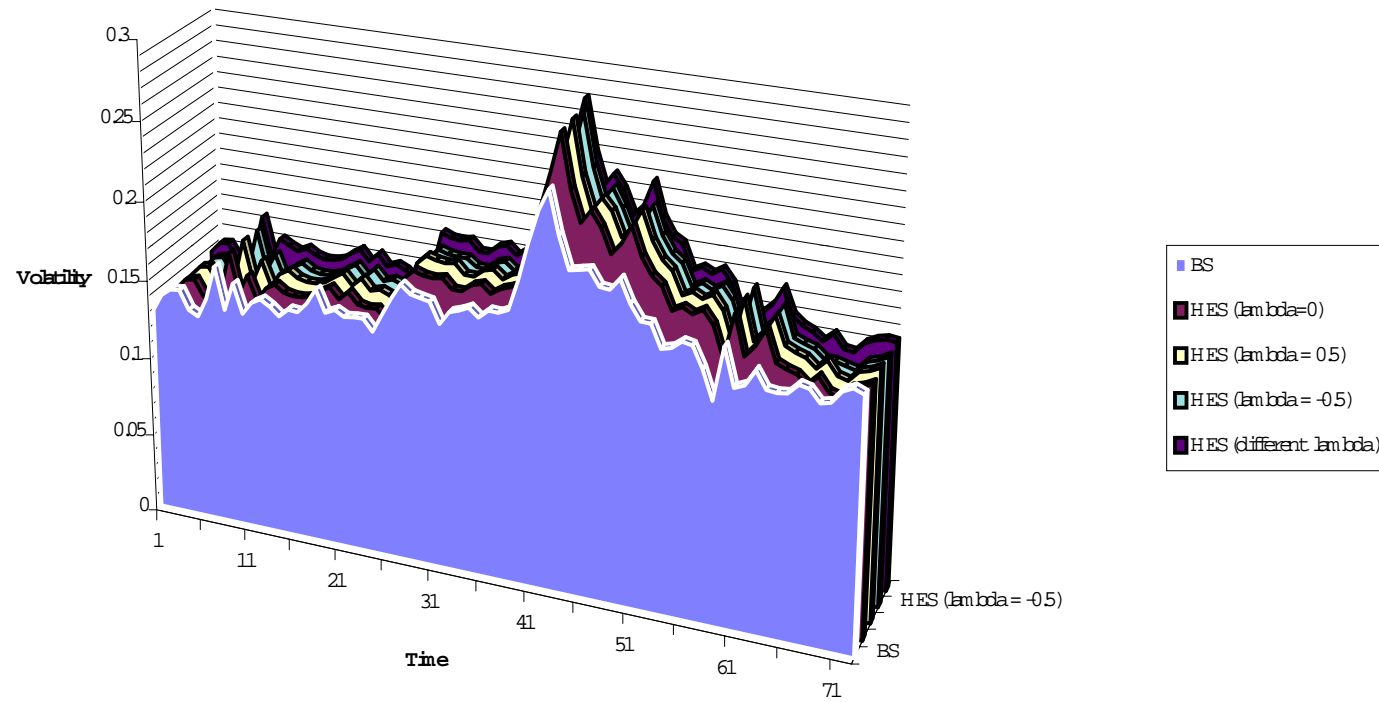
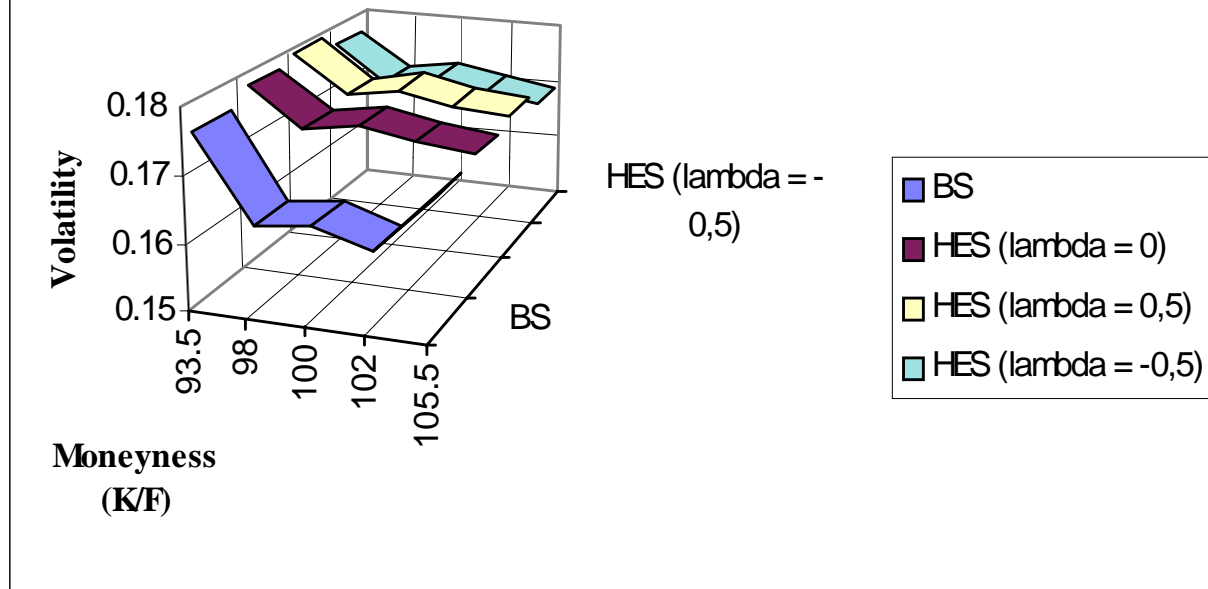


FIGURE 6
SMILES (January 96-April 96)



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