THE THREE MUSKETEERS: FOUR CLASSICAL SOLUTIONS TO BANKRUPTCY PROBLEMS

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A B S T R A C T

This paper provides a comparative analysis of some classical solutions to bankruptcy problems from an axiomatic viewpoint. These rules are the constrained equal-awards rule, the constrained equal-losses rule, the proportional rule and the Talmud rule. The purpose of this study is to facilitate the understanding of their di¤erences and to clarify the type of situations in which each of these rules is better.

Key-words: Bankruptcy problems, proportional solution, equal-awards solution, equal-losses solution, Talmud solution.

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1 INTRODUCTION

A bankruptcy problem is a distribution problem involving the allocation of a given amount of a single (perfectly divisible) good among a group of agents, when this amount is insufficient to satisfy all their demands. This type of problem arises in many real life situations. The canonical example is that of a bankrupt firm that is to be liquidated; namely, a situation in which the creditors' entitlements exceed the worth of the firm. Another familiar example refers to the division of an estate among several heirs when the estate falls short of the deceased's commitments. A different case is that in which, in a fixprice setting, the demand for a given commodity exceeds the available supply. The collection of a given amount of taxes in a community can also be given this form.

The available quantity of the good to be divided is usually called the estate. The agents are also referred to as creditors, whereas the term claims is meant to describe the agents' entitlements, demands or needs, depending upon the problem at hand. A solution to a bankruptcy problem is to be interpreted as the application of an allocation rule that gives a sensible distribution of the estate as a function of agents' claims. Therefore, we are interested in the analysis of rules that can be applied to a family of problems, rather than in the solvability of a particular problem. Note that, within this context, one can also think of allocation rules as rationing schemes that distribute the existing shortages. Indeed, this dual interpretation will play a significant role in the sequel and will prove extremely useful in simplifying the discussion.

Alternative rules typically represent different ways of applying a fairness criterion to the resolution of bankruptcy problems. The analysis of their structural properties permits one to select a particular rule by choosing the set of these properties that this rule satisfies. This venue becomes more fruitful the closer we get to the following recommendations:

(1) Each property is intuitive and represents a single and clear ethical principle.

(2) We can identify each rule as the only one satisfying a distinctive set of properties (that is, a collection of these properties characterizes the rule); moreover all these properties are logically independent.

(3) This set of distinctive properties is small whereas alternative rules share most of the properties (in order to clearly identify their ethical differences).

Structural properties express invariance of the solutions with respect to changes in the parameters, and are usually motivated by particular concerns. They are intended to ensure that the solution has some desirable features or

to prevent some inconveniences. Hence it is not surprising that a particular rule can be characterized by diﬀerent sets of independent axioms. Each characterization provides an insight on the type of problems for which a rule is satisfactory. The reader is referred to Thomson (1998) for a discussion of the axiomatic method.

The resolution of bankruptcy-like situations is a major practical issue and has a long history as a conceptual problem [see the references provided in Rabinovitch (1973), O'Neill (1982), Aumann & Maschler (1985), Young (1994, ch. 4)]. Modern economic analysis has addressed this class of problems from two main perspectives. The ﬁrst one is the game theoretic approach, where a bankruptcy problem is formulated as a TU game [see for instance O'Neill (1982), Aumann & Maschler (1985), Curiel, Maschler & Tijs (1988), Dagan & Volij (1993)]. The second one is the axiomatic method, where alternative solutions are characterized in terms of intuitive properties that express diﬀerent value judgements [e.g. Young (1987), Dagan (1996), Herrero, Maschler & Villar (1999), Herrero & Villar (1998)]. The reader is referred to Thomson (1995) for a survey of this literature.

This paper provides a comparative analysis of three main rules to solve bankruptcy problems from an axiomatic viewpoint. These are:
(i) The proportional rule, that divides the estate proportionally to the agents' claims.
(ii) The constrained equal-awards rule, that divides equally the estate among the agents under the condition that nobody gets more than her claim.
(iii) The constrained equal-losses rule, that divides equally the diﬀerence between the aggregate claim and the estate, provided no agent ends up with a negative transfer.
The proportional rule satisﬁes a number of appealing properties and, when compared with other rules, it has much to recommend itself. The idea of equality underlies another well-known rule: the constrained equal-awards rule. It makes awards as equal as possible, subject to the condition that no creditor receives more than her claim. A dual formulation of equality, focusing on the losses creditors incur as opposed to what they receive, underlies the constrained equal-losses rule. It proposes a distribution of the estate in which agents' losses are as equal as possible, subject to the condition that no one ends up with a negative award.
The choice of these three solutions is by no means arbitrary. First because they are among the most common methods of solving practical problems. Second for their long tradition in history. And last but not least, because they are almost the only sensible ones within the family of solutions that

treat equally equal claims. As Moulin (1997, p. 3) puts it: "One unambiguous conclusion emerges from the axiomatic analysis of rationing methods: ... [these three rules] stand out by virtue of their multifarious axiomatic properties".

As the Three Musketeers were four so are our three rules. The Talmud rule will play here the role of D'Artagnan. This is an appealing allocation rule that amounts to solve bankruptcy problems by combining the principles that inform the three rules above. It is worth stressing that even though all these rules are far from new, their axiomatization turns out to be quite recent.

So, if the rules are old and their characterization has already been done, what is new here? On the one hand, there is a methodological contribution since the discussion is based on the use of the duality relationship both for rules and properties. This approach greatly simpli...es the analysis dispensing with the need of proving some results and making the proofs rather elementary. On the other hand, there is an analytical contribution in the sense that we provide new properties and new characterizations of these rules. The combination of these two features brings about an extremely easy comparative analysis that helps clarifying the class of real life problems for which each of these rules might be better. Following the recommendations given above, we concentrate on characterizations that permit an easy comparison of these rules (in particular, all rules can be compared in terms of a single di¤erential property).

The paper is organized as follows. We start by formally introducing the family of bankruptcy problems and the three basic rules. Then we present several appealing properties for bankruptcy rules and o¤er di¤erent characterizations of the three selected rules. The Talmud rule is discussed next. A ...nal section commenting on the relationship between these rules closes the paper.

2   THREE OLD BANKRUPTCY RULES

2.1   Preliminaries

We aim at modelling the situation faced by an arbitrator that has to allocate a given amount of a perfectly divisible commodity among a group of claimants, when the available amount is not enough to satisfy all the admissible demands. The data of the problem are the amount of the good to be distributed, the number of agents, and their claims. The arbitrator has to

apply some ethical and procedural criteria to solve the problem, criteria that may depend on the nature of the problem under consideration.[1]

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents that represents a collectivity (also referred to as the society). For notational simplicity we take $|N| = n$. A bankruptcy problem for N is a pair $(E; c)$, where $E \in \mathbb{R}_+$ represents the estate (the amount of the good to be distributed), and $c \in \mathbb{R}_+^n$ is a vector of claims (demands, needs, rights, etc.). The ith component of c, denoted by c_i, represents the ith agent's individual claim, for all $i \in N$. Since our analysis focuses on bankruptcy problems it is assumed along the text that $\sum_{i \in N} c_i \geq E$ and also that $E > 0$: The family of all those problems is denoted by B^N:

A rule for B^N is a mapping F that associates with every $(E; c) \in B^N$ a unique point $F(E; c) \in \mathbb{R}^n$ such that:

(i) $0 \leq F(E; c) \leq c$:

(ii) $\sum_{i \in N} F_i(E; c) = E$:

The point $F(E; c)$ is to be interpreted as a desirable way of dividing E among the creditors in N: Requirement (i) is that each creditor receives an award that is non-negative and bounded above by her claim. Requirement (ii) is that the entire estate is to be allocated. These two requirements imply that $F(E; c) = c$ whenever $E = \sum_{i \in N} c_i$: It is implicitly assumed that F is homogeneous of degree one in $(E; c)$, meaning that E, c and $F(\cdot)$ are measured in the same units.

A more general model refers to the case of a variable population. Let $\mathbb{N}$ denote the set of all potential agents (a set with an infinite number of members), and let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$: For any $N \in \mathcal{N}$, we denote by n the cardinal of N, as before. Now a bankruptcy problem is a triple $(N; E; c)$, where $N \in \mathcal{N}$ stands for the particular set of agents involved in this problem, $E \in \mathbb{R}_+$ is the estate, and $c \in \mathbb{R}_+^n$ is the vector of claims, with $\sum_{i \in N} c_i \geq E$: We shall denote by $B = \bigcup_{N \in \mathcal{N}} B^N$ the family of all such bankruptcy problems with variable population. In this context a rule is a mapping F that associates with every $(N; E; c) \in B$ a unique point $F(N; E; c) \in \mathbb{R}^n$, such that: (i) $0 \leq F(N; E; c) \leq c$ and (ii) $\sum_{i \in N} F_i(N; E; c) = E$:

To simplify notation, for any given problem $(E; c) \in B^N$, we call C the

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[1] We focus our analysis on those problems in which each individual has a claim to some quantity of a common asset, but the claims are not against specific portions of the asset. Moreover, our approach fits better the "involuntary claims" case, as incentives are not an issue. See the discussion in Young (1994, ch. 4).

aggregate claim and L the aggregate loss. That is to say,

$$C = \sum_{i \in N} c_i; \qquad\qquad L = C - E$$

Note that solving a bankruptcy problem may be interpreted in two different ways. One is that of distributing what it is, namely E; as function of the claims vector c: The other is that of allocating what it is missing, namely L; with respect to the claims. This dual interpretation of the problem can be used to derive new rules and properties. Let us make precise these ideas.

Both in the ...xed population and in the variable population framework, for any given rule F we can consider another rule associated to it by means of a duality procedure. This duality procedure can be easily explained in the following way: Suppose that, when solving a problem, we start by temporarily awarding every agent her claim. Since this is not feasible, now we apply rule F to the problem of allocating losses. By this operation we obtain a new rule, the dual rule of the initially used. Formally,

Dual rule of F, $F^{\ast}$ (Aumann and Maschler, 1985): For all $N \in \mathcal{N}$ and all $(E; c) \in B^N$, $F^{\ast}(E; c) = c - F(L; c)$:

The rules F and $F^{\ast}$ are related in a simple way: $F^{\ast}$ divides what is available in the same way as F divides what is missing. Note that, for all $(E; c) \in B^N$; it follows that $C > L \geq 0$ so that $(L; c) \in B^N$. Moreover, $0 \leq F(L; c) \leq c$ and $\sum_{i \in N} F_i(L; c) = L$; so that $0 \leq F^{\ast}(E; c) \leq c$ and $\sum_{i \in N} F_i^{\ast}(E; c) = E$; that is, $F^{\ast}$ is well de...ned.

It is worth observing that duality is an idempotent operation, that is, $(F^{\ast})^{\ast} = F$: The notion of duality is naturally extended to the properties a solution satis...es. Formally,

Dual properties: Given two properties P, $P^{\ast}$, we say that $P^{\ast}$ is the dual property of P if and only if for very rule F that satis...es P its dual rule $F^{\ast}$ satis...es $P^{\ast}$:

The following result is an immediate consequence of the de...nitions given above:

Theorem 0. If a rule F is characterized by a set of independent properties $\Gamma = \{\odot_1; \odot_2; ....; \odot_k\}$; then the dual rule $F^{\ast}$ is characterized by the corresponding set of dual properties $\Gamma^{\ast} = \{\odot_1^{\ast}; \odot_2^{\ast}; ....; \odot_k^{\ast}\}$: Moreover, the properties in $\Gamma^{\ast}$ are also independent.

Proof. Suppose that a rule F is characterized by a set of independent properties Γ: By de...nition, the dual rule $F^{\ast}$ satis...es the dual properties

$\acute{A}_1^\alpha, \acute{A}_2^\alpha, ..., \acute{A}_k^\alpha$: Suppose that F^α is not characterized by the set $\mathcal{A}^\alpha$: This implies the existence of another rule G that satis...es all properties in $\mathcal{A}^\alpha$: But is such a case, G^α must satisfy all properties in $\mathcal{A}$ which is only possible if $G^\alpha = F$: This is a contradiction since the duality relation is idempotent.

Now suppose that the properties in $\mathcal{A}^\alpha$ are not independent. This implies that there is a property $\acute{A}_j^\alpha$ that can be eliminated without a¤ecting the characterization of F^α; that is, F^α ( ) $\mathcal{A}^\alpha \ _{i} \ \{\acute{A}_j^\alpha\}$. But in this case it follows that F ( ) $\mathcal{A} \ _{i} \ \{\acute{A}_j\}$; against the assumption. ∎

This preliminary theorem will allow us to simplify substantially the ensuing discussion, since many of the rules and properties analyzed exhibit the duality relationship.

2.2   The three musketeers

The time is ripe to present our ...rst three rules. Each of these rules applies the idea of "equality" focusing on a particular reference variable (ratios, awards, losses).

The ...rst is the proportional rule, probably the best known and most widely used solution concept. This rule distributes awards proportionally to claims. Hence it equalizes the ratios between claims and awards. It is formally de...ned as follows:

Proportional rule, P: For all $N \ 2 \ \mathcal{N}$; all $(E;c) \ 2 \ B^N$; and all $i \ 2 \ N$,

$$P(E;c) = \ _{,}c$$

Note that the very de...nition of a rule implies that $_{,} = \frac{E}{C} \ 2 \ (0;1]$:

The second is the constrained equal-awards rule. The underlying idea is that every agent should receive the same amount as long as this does not exceed her claim. Hence it equalizes the awards, under the restriction given by the de...nition of a solution. As it is made explicit in Aumann and Maschler (1985), "this rule has been adopted as law by most major codi...ers, including Maimonides (in his Laws for Lending and Borrowing)". Formally,

Constrained equal-awards rule, CEA: For all $N \ 2 \ \mathcal{N}$; all $(E;c) \ 2 \ B^N$; and all $i \ 2 \ N$;
$$CEA_i(E;c) = \min\{c_i; \ _{,}\}$$

The de...nition of a rule implies that ¸ solves $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Our third rule proposes to distribute equally the di¤erence between the estate and the aggregate claims. Namely, to give each agent the amount $c_i - \frac{L}{n}$: Yet, as this amount might be negative, the solution applies this principle with one proviso: no agent ends up with a negative transfer. Hence the constrained equal-loss solution equalizes the rationing experienced by the agents, as long as this is compatible with feasibility. >From a geometric viewpoint this way of solving the problem amounts to selecting that point in the feasible set which is closest to the vector of claims (according to the Euclidean distance). Aumann and Maschler (1985) point out that this rule also appears in Maimonides, dealing with auctions and looking at the losses the seller may experience when bidders renege (in his Laws of Appraisal).

Constrained equal-losses rule, CEL: For all $N \in \mathcal{N}$; all $(E; c) \in B^N$; and all $i \in N$;
$$CEL_i(E; c) = \max\{0, c_i - \lambda\}$$

The de...nition of a rule implies that ¸ solves $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

The constrained equal-awards rule corresponds to the uniform rule in the case of distribution problems with single-peaked preferences, when the task is smaller than the supply of e¤ort. In the context of taxation this rule is known as the "head tax". The principle underlying the constrained equal-losses rule, the equal-loss principle, has been applied to other distribution problems, such as cost-sharing, taxation or axiomatic bargaining [see Young (1987), (1988), Chun (1988b), Herrero and Marco (1993)]. In the context of taxation it is known as the "leveling tax".

The reader can easily check that the constrained equal-awards rule and the constrained equal-losses rule are dual from each other, whereas the proportional rule is dual of itself. That is,
$$CEL = CEA^{\ast}$$
$$P^{\ast} = P$$

3   FOUR COMMON PROPERTIES AND A JOINT CHARACTERIZATON

Let us now introduce four basic properties that these three solutions do satisfy. The ...rst, equal treatment of equals, has a clear ethical content.

The remaining three, composition, path independence and consistency, can be regarded as procedural requirements. They prevent the solution of a problem to be dependent on the agenda or unstable with respect to subgroup renegotiations.

3.1 Equal treatment of equals

This is probably the most basic equity requirement: agents with identical claims should be treated identically. Hence, we exclude di¤erentiating between agents on the basis of their names, gender, religion, political ideas, etc. Formally:

Equal treatment of equals: For all N 2 N; all $(E; c)$ 2 B^N; and all $i; j$ 2 N; $c_i = c_j$ implies $F_i(b) = F_j(b)$:

Equal treatment of equals is an instance of impartiality. It establishes that all agents with the same claims will receive the same amount.

Mind that this property says neither that everybody is equal nor that one should treat all agents equally. It says that when two agents are indistinguishable, with regard to the problem under consideration, they should be treated equally. The very de…nition of a bankruptcy problem as a pair $(E; c)$ implies discarding any information about individuals other than their claims. Therefore, in this case equal treatment of equals collapses to "agents with equal claims receive the same". But this need not be the case in a more general setting (e.g. Moulin, 1997).

3.2 Composition and path independence

In order to motivate the next property think of the case of a bankrupt …rm endowed with two types of assets (buildings and machinery, say). Suppose that these two assets are sold to di¤erent buyers in di¤erent dates. After selling the …rst asset the amount obtained is distributed among the …rm's creditors according to some pre-established rule. When the second asset is sold its price is distributed again among the creditors, reducing their claims by the amount already obtained. It seems reasonable to require that the …nal allocation of the …rm's worth should not depend on the order in which these two assets are sold, provided the total price is the same. In particular one would require to get the same outcome whether the …rm is liquidated one shot or it is done by parts (assuming again that the total price does not change). The property of composition makes precise this requirement:

Composition (Young, 1988): For all $N \in \mathcal{N}$; all $(E; c) \in \mathcal{B}^N$; and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$;

$$F(E; c) = F(E_1; c) + F[E_2; c - F(E_1; c)]$$

This property says that the problem $(E; c)$ can also be solved as the sum of two partial problems. The first corresponds to a problem with the initial claims c and a fraction E_1 of the estate. The second is that problem made out of the outstanding claims $c - F(E_1; c)$ and the reminder estate, $E - \sum_{i \in N} F_i(E_1; c)$. When a solution satisfies composition, solving a problem in stages does not change agents' final awards.

A slightly different interpretation of this property is the following [cf. Moulin (1997)]: Suppose that, when faced with a set of creditors whose claims are given by a vector c, a conservative arbitrator makes an estimation of the budget equal to E_1 and recommends an allocation $F(E_1; c)$: Later the arbitrator discovers the budget is actually larger than estimated so that there is still a portion E_2 to be distributed, with respect to the outstanding claims $[c - F(E_1; c)]$: Composition requires that this allocation of the resources $(E_1 + E_2)$ in two steps yields precisely the same result as the direct allocation of $(E_1 + E_2)$ according to the initial claims, c:

Think again of the situation faced by an arbitrator that makes a tentative division of the estate based on an estimate of its value. Contrary to the former case now suppose that once the tentative division is done, it turns out that the actual value of the estate falls short of what was assumed. Then, two options are open: either the tentative division is cancelled altogether and the actual problem is solved, or the rule is applied to the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. Path independence says that both procedures should yield the same outcome. Formally:

Path Independence (Moulin, 1987): For all $N \in \mathcal{N}$; all $(E; c) \in \mathcal{B}^N$, and all $E' > E$; we have $F(E; c) = F[E; F(E'; c)]$

Path independence is a property that applies when after solving a problem $(E'; c)$ it turns out that the actual worth of the estate E falls short of what was expected. It requires in this case that the solution of the real problem $(E; c)$ be the same as that of the problem with estate E and a vector of claims $c' = F(E'; c)$:

It is easy to see that if a rule satis...es either composition or path-independence it is monotonic with respect to the estate. That is, for any two problems $(E; c); (E^0; c) \in B^N; E \leq E^0$ implies $F_i(E; c) \leq F_i(E^0; c);$ for all $i \in N:$ Moreover:

- ² Claim 1. If solution F satis...es either composition or path independence then $F_i(E; c)$ is continuous in $E;$ for all $i \in N:$

- ² Claim 2. Composition and path independence are dual properties.

3.3 Consistency

This property refers to the case of a variable population. Consistency is a powerful property that links the solution to a problem for a given society N with the solutions of the problems corresponding to its sub-societies. To formally de...ne this notion, let S be a proper subset of N and suppose that, after solving a problem $(N; E; c)$ by means of the rule $F;$ the members of group S reconsider the allocation of what they got, $\sum_{i \in S} F_i(N; E; c):$ Let $[S; \sum_{i \in S} F_i(N; E; c); c_S]$ be the associated reduced problem, where $c_S = (c_i)_{i \in S}$. The rule F is consistent if applied to any of its reduced problems it gives the incumbent agents the same amounts they obtained in the original problem. Formally:

Consistency: For all $N \in \mathcal{N};$ all $S \subseteq N;$ all $(N; E; c) \in B;$ and all $i \in S;$ we have: $F_i(N; E; c) = F_i[S; \sum_{i \in S} F_i(N; E; c); c_S]:$

Consistency is a procedural requirement with two relevant implications:

(i) Once an allocation has been agreed upon, no group of agents is willing to re-apply the rule in the reduced problem that appears when the other agents leave bringing with them their allotted shares. Hence, what is good for the large group is also good for the smaller ones.

(ii) If the agreement on how to solve a two-person problem can be consistently extended to any number of them, then that extension is unique. Hence, what is good for the smallest possible group is good for larger ones.[2]

The ...rst implication provides us with a stability feature: consistency prevents subgroups of agents to renegotiate once there is a solution proposed for the society. The second one helps in assessing the value judgements

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[2]The requirement of agreement on the shares from small groups to large groups is usually refered to as "converse consistency". In general, consistency and its converse are independent properties, but in the case of bankruptcy problems if a rule satis...es consistency it also satis...es converse consistency. See Chun (1998).

of alternative solutions, as what is "fair" is usually easier to check and to understand in the two-person case.

Remark 1 It may happen that no consistent extension of a particular solution exists [see Dagan, Serrano and Volij (1997)]. The uniqueness in the procedure of consistently extending solutions to bankruptcy problems was ...rst noticed by Aumann and Maschler (1985) for the consistent extension of the contested garment rule.

> ² Claim 3. The three rules, P; CEA; and CEL satisfy equal treatment of equals, consistency, composition, and path-independence.

3.4   A joint characterization

We have just mentioned that the proportional rule, the constrained equal-awards rule, and the constrained equal-losses rule all satisfy the properties of equal treatment of equals, composition, path-independence and consistency. The following result tells us that these rules are actually the only ones satisfying all these requirements.

Theorem 1 (Moulin, 1997, Corol. to Th. 2) There are three and only three rules on B satisfying simultaneously equal treatment of equals, composition, path-independence, and consistency: The proportional rule, the constrained equal-awards rule, and the constrained equal-losses rule.

This result provides additional support to the choice of the three bankruptcy rules discussed so far. One way of checking the strength of this theorem is by asking ourselves what property to drop in order to buy other solutions. On the one hand, equal treatment of equals seems di¢ cult to object unless we consider a wider family of problems in which agents have other relevant differences, to be included in the information that describes the problem. Composition, path-independence and consistency can be regarded as procedural requirements. The ...rst two ensure coherence with respect to subdivisions of the estate; the third one ensures coherence with respect to all reduced problems, when some of the agents leave taking with them their allotted shares. If either composition or path-independence fails, then the outcome of the resolution becomes dependent on the agenda. That is to say, it varies according to the way in which the problem is subdivided into partial problems. The lack of consistency implies that the resolution of the problem proposed by the rule may be altered by renegotiations within population subgroups.

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Once we know the features that these rules have in common we have to ...nd those properties that permit one to separate them. It will be shown that the equal awards rule obtains when we give priority in the distribution to those agents with smaller claims and the equal losses rule obtains when priority is given to those agents with larger claims. The proportional solution remains in the mid-way, as it gives priority in the distribution neither to large nor small claims.

4.1 Exemption and exclusion

These two properties refer to the behaviour of a rule when agents' claims are very asymmetric. They embody two opposite principles of claims enforce-ability that express clear cut values on how allocation rules should perform in extreme situations.[3] They will help us choosing among di¤erent rules, depending upon the problem at hand.

The ...rst property considered, exemption, establishes that when the re-sources to be divided are large enough relative to the agents' claims, only those individuals with larger claims are to be rationed. This is an instance of the general principle of progressivity, according to which those agents with smaller claims are given priority in the distribution and can actually be ex-empted from being rationed. More precisely, exemption says that when the claim of an individual is smaller than equal division, the rule should grant her the full claim. In the context of taxation problems[4] this property establishes that those individuals whose income is below the average tax burden should be exempted.

Formally:

Exemption: For all N 2 N and for all $(E; c)$ 2 B^N, if $c_i \cdot \frac{E}{n}$ then $F_i(E; c) = c_i$:

The rationale of this principle is twofold. On the one hand, it re‡ects the idea that the small claimants cannot be held responsible for the shortage. Indeed, would all agents have claims smaller than $E=n$ there would be no bankruptcy at all. On the other hand, it can be associated with the idea that

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[3] John Stuart Mill (1859) argued that the strength of our moral values is to be judged in extreme situations.

[4] The problem $(E; c)$ 2 R^n corresponds to a tax problem when E is interpreted as a given amount of taxes to be collected in a society N; and c is interpreted as the vector of agents' gross incomes.

small claimants correspond to relatively poorer people for whom the claims represent a larger fraction of their wealth.

It is worth noting that this principle is applied by Law in some real-life bankruptcy problems, as it is the case of ...nancial intermediaries, where the debts of clients with small savings are honored ...rst. The exemption of small incomes in a tax system is also an obvious instance of this principle.

The following property, exclusion, conveys the opposite message: those agents with very low claims are to be disregarded. To make this precise, let $(E; c) \in B$ be a bankruptcy problem. The number $\frac{L}{n}$ is the average rationing experienced by the agents in problem $(E; c)$: We say that the ith agent's claim is residual if it is smaller than the per capita loss. Namely, if $c_i \cdot \frac{L}{n}$. Exclusion says that residual claims are ignored. Formally:

Exclusion: For all $N \in \mathcal{N}$ and for all $(E; c) \in B^N$, $F_i(E; c) = 0$ whenever $c_i \cdot \frac{L}{n}$:

This property simply says that when the claim of agent i is so small that it does not reach the average loss, then she gets nothing. In other words, the solution to the problem $(E; c)$ coincides with that of the problem $F(E; b)$ that results from substituting c by the vector b given by $b_i = \max\{0; c_i \mathbin{\text{\textexclamdown}} \frac{L}{n}\}$; $i \in N$:

The principle behind this property is also used in many real-life situations. An illustration can be obtained from many public health systems, in which major diseases are fully covered whereas minor a¤ections are excluded (e.g. heart attacks versus headaches).

The CEA rule satis...es exemption and fails to satisfy exclusion. The CEL rule, on the contrary, satis...es exclusion and fails to satisfy exemption. The proportional rule fails to satisfy both properties. Moreover,

> ² Claim 4. Exemption and exclusion are dual properties. Moreover, there is no rule that can satisfy these two properties simultaneously.

The following results tell us the bite of these properties:

Theorem 2 The constrained equal-awards rule is the only rule in B satisfying equal treatment of equals, path-independence, consistency and exemption.

Proof. It is easy to see that the CEA rule satis...es these properties. Let us prove the converse.

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Since consistency and converse consistency are equivalent in bankruptcy problems, it is enough to prove the result for the two-person case.

Let $(E; c) \in B^2$: If $c_1 = c_2$; then equal treatment of equals implies $F_1(E; c) = F_2(E; c) = \frac{E}{2}$ which is the CEA for this case. Let $c_1 \neq c_2$; without loss of generality let $c_1 < c_2$: There are two cases to be considered.

Take ...rst the case in which $\frac{E}{2} \geq c_1$: By exemption, $F_1(E; c) = c_1$; $F_2(E; c) = E - c_1$; which is the CEA solution in this case.

Now suppose that $\frac{E}{2} < c_1$: Let $E^0 = 2c_1 > E$: By exemption if follows that $F_1(E^0; c) = F_2(E^0; c) = c_1$: Path independence implies that $F(E; c) = F[E; F(E^0; c)]$: Then, equal treatment implies $F(E; c) = \left(\frac{E}{2}; \frac{E}{2} \right)$; which is the CEA solution for this case. ■

Theorem 3 The constrained equal-losses rule is the only rule in B^N satisfying equal treatment of equals, composition, consistency and exclusion.

Proof. We know that CEL and CEA are dual rules. Moreover, composition is the dual property of path independence (Claim 2), exclusion is the dual property of exemption (Claim 4), and equal treatment of equals and composition are dual of themselves. Therefore, the result follows from Theorem 0. ■

Remark 2 For two-person bankruptcy problems the notions of exemption and exclusion turn out to be equivalent to those of sustainability and preeminence, respectively, introduced in Herrero & Villar (1998). In the n-person case, however, exemption and exclusion are weaker properties.

4.2   Independence of claims truncation and composition from minimal rights

The following properties also represent alternative principles of claims enforceability. One focuses on the case in which there is some agent whose individual claim exceeds the available estate. It proposes to scale down this unfeasible claim to reality. The other introduces the notion of undisputed amounts, as those which are left to the agent when all other got their claims fully honored. It says that the solution cannot grant an agent less than her undisputed amount. Let us be more precise about these ideas.

Consider a bankruptcy problem in which the claim of some individual agent is larger than the estate. How a rule should treat her demand? One of the principles that appears in the Talmud says that one should not consider

any claim that is greater than the estate. That is, replacing c_i by E if $c_i > E$ should not a¤ect the recommendation. Formally,

Independence of claims truncation (Dagan, 1996): For all $N \in \mathcal{N}$ and for all $(E; c) \in B^N$; $F(E; c) = F(E; c^T)$; where $c_i^T = \min\{E; c_i\}$ for all $i \in N$:

This property establishes that if an individual claim exceeds the total to be allocated, the excess claim should be considered irrelevant. The rationale behind is that "one cannot claim more than there is; thus the excess of a claim above the estate is irrelevant. A rule is independent of claims truncation if it allocates the estate taking into account only the relevant claims." [Cf. Dagan (1996, p.53)].

In order to present the next property let us start by introducing the notion of minimal rights. For a given problem $(E; c) \in B$ de…ne the agents' minimal rights as:

$$m_i(E; c) = \max\{0; E_i - \sum_{j \neq i} c_j\}$$

The number $m_i(E; c)$ represents the amount of the estate that is left to the ith agent when the claims of all other agents are fully honored, provided this amount is nonnegative. And it is taken to be zero otherwise. Let $m(E; c)$ denote the vector in R^n_+ whose components are the minimal rights $m_i(E; c)$; $i \in N$:

The next property says that a rule should honor agents' minimal rights before any further step is taken. Hence it asks the rule to allocate …rst the amounts corresponding to these minimal rights and then solving the remaining problem. Formally,

Composition from minimal rights: For all $N \in \mathcal{N}$ and for all $(E; c) \in B^N$, $F(E; c) = m(E; c) + F[E_i - \sum_{i \in N} m_i(E; c); c_i - m(E; c)]$:

Composition from minimal rights is a particular form of composition that says the following: the solution of any problem $(E; c) \in B^N$ coincides with the outcome of a process in which minimal rights $m(E; c)$ are allocated …rst, and the rule is applied to the problem consisting of the remaining estate $E_i - \sum_{i \in N} m_i(E; c)$ and the outstanding claims $c_i - m(E; c)$:

It is easy to see that CEA satis…es independence of claims truncation and fails to satisfy composition from minimal rights. CEL satis…es composition from minimal rights and fails to satisfy independence of claims truncation. P fails to satisfy both. Moreover,

² Claim 5. Independence of claims truncation and composition from minimal rights are dual properties.

The following results are obtained:

Theorem 4 (Dagan, 1996, Prop.1) The constrained equal-awards rule is the only rule in B satisfying equal treatment of equals, composition, and independence of claims truncation.

>From this result, together with Claims 2, 5 and Theorem 0, it follows that:

Theorem 5 (Herrero, 2000) The constrained equal-losses rule is the only rule in B satisfying equal treatment of equals, path independence, and composition from minimal rights.

4.3 Self-duality

A rule is called self-dual when it coincides with its dual, that is, when $F(E;c) = c_i \ F(L;c)$. Formally,

Self-duality (Aumann and Maschler, 1985): For all $N \ 2 \ N$ and for all $(E;c) \ 2 \ B^N$; $F(E;c) = F^\alpha(E;c)$:

Self-duality is a property that introduces a principle of symmetry in the behaviour of the solution with respect to awards and losses. It says that the same principle is to be applied if we think of $(E;c)$ either as a distribution problem or as a rationing scheme. This property is also inspired in the Talmud and establishes that dividing "what is there" and dividing "what is not there" should be treated in a symmetric way.

Note that, according to our de...nition of a rule, it follows that $c = F(L + E;c)$ so that self-duality can be rewritten as

$$F(L + E;c) = F(L;c) + F(E;c)$$

which may be regarded as an instance of additivity.

The following result is obtained:

Theorem 6 (Young, 1988) The proportional rule is the only rule in B satisfying equal treatment of equals, composition and self-duality.

Remark 3 Young's original proof includes the requirement of continuity.[5] Yet it is easy to see that the only continuity which is actually needed is continuity with respect to the estate and this follows from composition (Claim 1).

By using the duality relation, the following result is also obtained:

Theorem 7 The proportional rule is the only rule in B satisfying equal treatment of equals, path-independence and self-duality.

5   AND D'ARTAGNAN

The Talmud rule is the fourth of our three rules. It is a procedure that has been designed in order to accommodate the solutions given in the Talmud to a number of practical distribution problems. Auman & Maschler (1985) propose this rule as the consistent extension of the so called contested garment rule, a solution concept de...ned for two-person bankruptcy problems.

The contested garment rule can be described as follows:

Contested garment rule, G: For all $(E; c) \in B^2$; and all $i \in N$;

$$G_i(E; c) = m_i(E; c) + \frac{1}{2}[E_i \ M(E; c)]$$

where $M(E; c) = m_1(E; c) + m_2(E; c)$:

Therefore, G concedes to both agents their minimal rights and then divides equally the reminder.

The contested garment rule satis...es equal treatment of equals, self-duality, independence of claims truncation and composition from minimal rights. From this and the theorems in section 4 it follows that it fails to satisfy composition and path-independence. It is also easy to see that it satis...es neither exemption nor exclusion, which also serves the purpose of separating these properties from those of composition from minimal rights and independence of claims truncation.

The following characterization result is obtained:

Theorem 8 (Dagan, 1996) The contested garment rule is the only two-person rule satisfying self-duality and independence of claims truncation.

---

[5] This property says the following: For all $N \in N$; all $(E; c) \in B^N$; and all $(E_q; c_q) \in B^N$ such that $\lim_{q! \ 1} E_q = E$; and $\lim_{q! \ 1} c_q = c$; then $\lim_{q! \ 1} F(E_q; c_q) = F(E; c)$.

The following result is a direct implication of Theorem 8 and the duality relation stated in Claim 4:

Theorem 9 The contested garment rule is the only two-person rule satisfying self-duality and composition from minimal rights.

Remark 4 Dagan (1996) also shows that the contested garment rule is the only rule in B^2 that satis...es equal treatment of equals, independence of claims truncation and composition from minimal rights.

Therefore, the contested garment rule appears as a way of reaching an agreement between the principles of composition from minimal rights and independence of claims truncation. Indeed, this rule can also be regarded as a compromise between the CEA and the CEL rules. To see this simply note that an elementary calculation permits one to rewrite this rule as follows:

$$G(E,c) = \begin{cases} CEA(E; \frac{1}{2}c) & E \cdot \frac{1}{2}C \\ \frac{1}{2}c + CEL(E - \frac{1}{2}C); c & E > \frac{1}{2}C \end{cases}$$

>From this point of view the contested garment rule behaves as the constrained equal awards rule for values of the estate not exceeding the minimum claim of the agents, and it behaves as the constrained equal-losses rule for values of the estate above the maximum claim.

Aumann and Maschler (1985) introduced the Talmud rule as the consistent extension of the contested garment rule.

Talmud rule, T (Aumann and Maschler, 1985): For all $N \in \mathbb{N}$, all $(E,c) \in B^N$, and all $i \in N$,

$$T_i(E,c) = \begin{cases} \min\{\frac{1}{2}c_i, \lambda\} & \text{if } E \cdot \frac{1}{2}C \\ \max\{\frac{1}{2}c_i, c_i - \mu\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} T_i(E,c) = E$.

Apart from its justi...cation as the consistent extension of the contested garment rule, the rationale of the Talmud rule is based in the psychological principle of "more than half is like the whole, whereas less than a half is like nothing". Thus, it seems natural to look at the size of the awards when they are below half of the claim, and to look at the size of the losses above half of the claim. This, together with a principle of equal treatment, in which all agents are at the same side of the half-way psychological watershed amounts to construct the Talmud rule.

As a consequence, we have the following result:

Theorem 10 The Talmud rule is the only rule in B satisfying consistency, self-duality and composition from minimal rights.

Because of self-duality, and the dual relationship between composition from minimal rights and independence of claims truncation, the following alternative characterization of the Talmud solution is obtained:

Theorem 11 The Talmud rule is the only rule in B satisfying consistency, self-duality and independence of claims truncation.

6 FINAL REMARKS

We have presented a number of results that characterize four classical solutions to bankruptcy problems: the proportional rule, the constrained equal awards rule, the constrained equal losses rule, and the Talmud rule. Each rule may be regarded as implementing a speci…c notion of fairness, whose application may depend on the type of problem considered. One can say, roughly speaking, that the …rst three rules are similar in that they apply an egalitarian principle, and di¤er in the variable they aim at equalizing (ratios, awards or losses). The Talmud rule applies a protective criterion that ensures that each individual agent su¤ers a rationing that is "of the same sort" of that experienced by the whole society.

Theorem 1 provides us with a joint characterization of the proportional rule, the constrained equal awards rule, and the constrained equal losses rule. This result emphasizes the common features of these three rules and gives support to their choice as the leading candidates to the resolution of bankruptcy problems. The characterization of each of these rules is taken up in Theorems 2 to 6. A relevant feature of these characterizations is that they allow us to compare the three rules in terms of a single di¤erential property. This facilitates the selection among these rules depending on the nature of the bankruptcy problem considered. The last theorems serve the purpose of characterizing the Talmud rule and also to show the relationship between this and the former rules (indeed the Talmud rule happens to satisfy some of the properties that permit the separation of the other rules).

We have shown that choosing one of the rules that satisfy simultaneously equal treatment of equals, composition, path independence and consistency, amounts to choosing one among three clear-cut principles: exemption, exclusion or self-duality (theorems 2, 3 and 6 or 7). Or, alternatively, independence

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of claims truncation, composition from minimal rights or self-duality (theorems 4, 5 and 6 or 7). The Talmud rule exhibits the surprising feature of satisfying simultaneously three of the properties that permits one to achieve the separation among the constrained equal awards rule, the constrained equal losses rule and the proportional rule. That is to say, the Talmud rule satis...es independence of claims truncation, composition from minimal rights and self-duality (theorems 10 and 11).

Note that the principles of exemption and independence from claims truncation, on the one hand, and exclusion and composition from minimal rights, on the other, are pair-wise logically independent. Indeed there is no rule that can satisfy exemption and exclusion simultaneously, whereas the properties of independence of claims truncation and composition from minimal rights are satis...ed by the Talmud rule.

All these results illuminate on the kind of problems for which each solution might be better. The constrained equal awards rule seems appropriate for those problems in which individuals are the primary concern, whereas their claims only represent maximal aspirations. The principles of exemption and independence of claims truncation express this notion in two di¤erent formats. In one case it is assumed that those claims that are relatively small are to be fully honored. In the other case that all those claims larger than the estate are indistinguishable. Here individuals go ...rst. As a consequence, agents with smaller claims obtain a relatively higher satisfaction of their demands.

The constrained equal-losses rule is a sensible rationing scheme for those problems in which claims represent real entities of an absolute nature (e.g. unalienable rights or vital needs, to take two extreme cases). The principles of exclusion and composition from minimal rights convey this message. Exclusion says that in case of great need those who are only entitled to minor shares are to be disregarded. Composition from minimal rights takes as a starting point a distribution that cannot be objected unless we are ready to violate agents' undisputed claims. Therefore, now claims go ...rst. As a consequence, agents with larger claims are given priority in the distribution.

The proportional rule lies somewhere in between since it gives priority neither to smaller nor larger claims. Hence claims and agents are treated on an equal foot. The distinctive feature of self-duality suggests that this is the most natural distribution rule when we think of bankruptcy as a subfamily of distribution problems in which E can exceed of fall short of C; as self-dual rules allocate awards and losses in the same manner.

The Talmud rule combines the features of these three solutions by taking into account the size of the estate with respect to the aggregate claim.

Nobody gets more than half of her claim if the estate is less than half of the aggregate claim. And nobody gets less than half of her claim if the amount to be distributed exceeds one half of the total demand. This can be regarded as implementing a protective criterion according to which nobody losses too much when there is a small discrepancy and nobody gains too much when there is a large de...cit. Composition from minimal rights ensures the ...rst part of this protective criterion, and independence from claims truncation the second one. Then, self-duality introduces the symmetry between both parts.

The following table summarizes the results in former sections. "Y" means that the rule satis...es the property and "N" that it does not. "Y(*)" (resp. "Y(+)") means that this property, together with the other with (*) (resp. with (+)) in the column, characterizes the rule.

Properties / Rules	CEA	CEL	P	T
Equal treatment	Y(*)(+)	Y(*)(+)	Y(*)(+)	Y
Composition	Y(*)(+)	Y	Y(*)	N
Path-Independence	Y	Y(*)(+)	Y(+)	N
Consistency	Y(*)	Y(*)	Y	Y(*)(+)
Exemption	Y(*)	N	N	N
Exclusion	N	Y(*)	N	N
Ind.Claims Trunc.	Y(+)	N	N	Y(+)
Comp.Min.Rights	N	Y(+)	N	Y(*)
Self-duality	N	N	Y(*)(+)	Y(*)(+)

A P P E N D I X : Proofs of the claims

$\square$ Claim 1. If a solution F satis...es either composition or path independence then $F_i(E; c)$ is continuous in E; for all $i \in N$:

Proof.
Let us prove the result for the case in which F satis...es composition. Let $(E; c) \in B^N$, and suppose that $F_i(\cdot; c)$ is not continuous with respect to its ...rst argument for some $i \in N$. That is to say,

$$a = \lim_{" \to 0} F_i(E \mathbin{\text{¡}} "; c) \ne \lim_{" \to 0} F_i(E + "; c) = b$$

Since F is monotonic, $a < b$ and $a \le F_i(E; c) \le b$: Let $F_i(E; c) = °$:
(a) Assume that $a \le ° < b$: For all $j \in N$; let $c_j^0 = c_j \mathbin{\text{¡}} F_j(E; c)$ and let $± > 0$ be such that $F_i(±; c^0) < b \mathbin{\text{¡}} °$. By composition, $b < F_i(E + ±; c) = F_i(E; c) + F_i(±; c^0) < ° + b \mathbin{\text{¡}} °$; which is a contradiction.
(b) In the case $a < ° \le b$; a similar argument applies. $\blacksquare$

$\square$ Claim 2. Composition and path independence are dual properties.

Proof.
Let $(E; c) \in B^N$; $E_1; E_2 \in R_+$ with $E_1 + E_2 = E$: By de...nition, $F^{¤}(E; c) = c \mathbin{\text{¡}} F(L; c) = c \mathbin{\text{¡}} F(C \mathbin{\text{¡}} E_1 \mathbin{\text{¡}} E_2; c)$: Now call $L_1 = C \mathbin{\text{¡}} E_1$; $z = F(L_1; c)$, $Z = \sum_{i=1}^{n} z_i$: When F satis...es path independence we have $F(L; c) = F(L; z) = F(Z \mathbin{\text{¡}} E_2; z) = z \mathbin{\text{¡}} F^{¤}(E_2; z)$: Therefore,

$$
\begin{aligned}
F^{¤}(E; c) &= c \mathbin{\text{¡}} z + F^{¤}(E_2; z) \\
&= c \mathbin{\text{¡}} F(L_1; c) + F^{¤}(E_2; z) \\
&= F^{¤}(E_1; c) + F^{¤}(E_2; z)
\end{aligned}
$$

That is, $F^{¤}$ satis...es composition.
Similarly, $F(E; c) = c \mathbin{\text{¡}} F^{¤}(L; c)$: Let $y = F^{¤}(L_1; c)$ and suppose that $F^{¤}$ satis...es composition, that is,

$$
\begin{aligned}
F(E; c) &= c \mathbin{\text{¡}} F^{¤}(L; y) = c \mathbin{\text{¡}} z + F(E_2; z) \\
&= F(E_1; c) + F(E_2; z)
\end{aligned}
$$

That is, F satis...es path independence. $\blacksquare$

² Claim 3. The three rules, P; CEA; and CEL satisfy equal treatment of equals, consistency, composition, and path-independence.

Proof.

Consistency and equal treatment are immediate. It is also very easy to see that the proportional rule satis...es composition and path independence. Let us prove that the CEL rule satis...es composition and path independence (the proof for the CEA rule goes along the same lines).

(i) Consider ...rst the property of composition.

Let $(E;c)$ 2 B^N; E_1; E_2 2 R_+ with $E_1 + E_2 = E$; and consider the associated problems $(E_1;c)$; $(E_2;c^0)$ where for all i 2 N; $c_i^0 = c_i$ ¡ $CEL_i(E_1;c)$: We have to show that $CEL(E;c) = CEL(E_1;c) + CEL(E_2;c^0)$:

Clearly, $C > E_1$: Moreover, $C^0 = C$ ¡ $E_1 > E_2$: Thus, there exist $¸$; $¸^0 > 0$ such that for all i 2 N;

$$CEL_i(E_1;c) = \max f0; c_i \, ¡ \, ¸g;$$
$$CEL_i(E_2;c^0) = \max f0; c_i^0 \, ¡ \, ¸^0 g:$$

Note that for all i 2 N; $c_i^0 \cdot ¸$ because, by de...nition, $c_i^0 = c_i$ if $¸ ¸ c_i$ and $c_i^0 = ¸$ if $¸ < c_i$: Therefore, $¸^0 \cdot ¸$: Next we show that, for all i 2 N; $CEL_i(E_1;c) + CEL_i(E_2;c^0) = \max f0; \, c_i \, ¡ \, ¸^0 g$:

Let i 2 N be such that $CEL_i(E_1;c) = 0$: Since $c_i^0 = c_i$; then $CEL_i(E_1;c) + CEL_i(E_2;c^0) = \max f0; \, c_i \, ¡ \, ¸^0 g$: Now let i 2 N be such that $CEL_i(E_1;c) > 0$: Note that $CEL_i(E_1;c) = c_i$ ¡ $¸$; and $c_i^0 = ¸$: Thus, $CEL_i(E_2;c^0) = \max f0; ¸ ¡ ¸^0 g = ¸ ¡ ¸^0$; because $¸ ¸ ¸^0$: Hence, $CEL_i(E_1;c) + CEL_i(E_2;c^0) = c_i$ ¡ $¸^0$: As c_i ¡ $¸ ¸ 0$ and $¸ ¸ ¸^0$; it follows that c_i ¡ $¸^0 ¸ 0$; and thus $CEL_i(E_1;c) + CEL_i(E_2;c^0) = \max f0; \, c_i \, ¡ \, ¸^0 g$:

(ii) Let us consider the case of path independence. We have to show that for any problem $(E;c)$ 2 B^N; and all E^0 2 R_+ with $E^0 > E$; we have $CEL(E;c) = CEL(E;c^0)$; where $c^0 = CEL(E^0;c)$:

By de...nition,

$$c_i^0 = \max f0; \, c_i \, ¡ \, ¸g$$
$$CEL_i(E;c) = \max f0; \, c_i \, ¡ \, ¸^0 g$$
$$CEL_i(E;c^0) = \max f0; \, c_i^0 \, ¡ \, ¸^1 g$$

with $¸^1 < ¸ < ¸^0$:

When $c_i^0 = 0$ we have $c_i \cdot ¸ < ¸^0$; so that $CEL_i(E;c^0) = 0 = CEL_i(E;c)$. Take now the case in which $c_i^0 = c_i$ ¡ $¸ > 0$. Then, $c_i^0 > c_i$ ¡ $¸^0$: Let N_+ ½ N denote the set of indices for which $c_i^0 > 0$: Since $\sum_{i=1}^{n} CEL_i(E;c) =$

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$\sum_{i=1}^{n} CEL_i(E; c^0)$; and $c_i^0 = 0$ if and only if $CEL_i(E; c^0) = CEL_i(E; c) = 0$; it follows that

$$\sum_{i \in N_+} (c_i - \gamma^0) = \sum_{i \in N_+} (c_i^0 - 1) = \sum_{i \in N_+} (c_i - \gamma - 1)$$

which is only possible if $\gamma^0 = \gamma + 1$: Therefore,

$$CEL_i(E; c) = \begin{cases} 0; & \text{if } c_i^0 = CEL_i(E; c^0) = 0 \\ c_i - \gamma^0 = c_i^0 - 1 = CEL_i(E; c^0); & \text{otherwise} \end{cases}$$

■

² Claim 4. Exemption and exclusion are dual properties. Moreover, there is no rule that can satisfy these two properties simultaneously.

Proof.
Let $(E; c) \in B^N$ and let F be a rule that satis…es exclusion. By de…nition we have $F^\ast(E; c) = c - F(L; c)$: We know that $F_i(L; c) = 0$ whenever $c_i \cdot \frac{c_i - L}{n} = \frac{E}{n}$, which implies:

$$F_i^\ast(E; c) = c_i$$

That is, $F^\ast$ satis…es exemption. Similarly, suppose that $F^\ast$ satis…es exemption. Then, $F(E; c) = c - F^\ast(L; c)$: When $c_i \cdot \frac{L}{n}$ we have $F_i^\ast(L; c) = c_i$; that is to say, $F_i(E; c) = 0$: Therefore, F satis…es exclusion.

To see that there is no rule that can satisfy simultaneously these two properties, take the case $n = 2$ with $C = 2E$: Without loss of generality suppose that $c_1 \cdot c_2$: Since $c_1 \cdot \frac{E}{2}$; exemption implies that $F_1(E; c) = c_1$: Moreover, as $\frac{L}{2} = \frac{E}{2} \cdot c_1$; exclusion implies that $F_1(E; c) = 0$: But this is incompatible with the former conclusion.. ■

² Claim 5. Independence of claims truncation and composition from minimal rights are dual properties.

Proof.
Let $(E; c) \in B^N$ and let F be a rule that satis…es composition from minimal rights. By de…nition we have:

$$
\begin{aligned}
F^\ast(E; c) &= c - F(L; c) \\
&= c - m(L; c) - F[L - M(L; c); c - m(L; c)]
\end{aligned}
$$

Let $c^0 = c - m(L; c)$; so that $C^0 = C - M(L; c)$; and let $L^0 = C^0 - E$: Then, $F^\ast(E; c) = c^0 - F(L^0; c^0) = F^\ast(E; c^0)$:

Now observe that:

$$c_k^0 \;=\; c_k - m_k(L;c) = c_k - \max\{0; L - \sum_{j \in N\setminus\{k\}} c_j\}$$

$$= c_k - \max\{0; c_k - E\} = \min\{c_k; E\}$$

That is, F^{*} satis...es independence of claims truncation.

Similarly, assume that F^{*} satis...es independence of claims truncation. Then, $F(E;c) = c - F^{*}(L;c) = c - F^{*}(L;c^0)$; where:

$$c_k^0 \;=\; \min\{c_k; L\} = c_k - \max\{0; c_k - C + E\}$$

$$= c_k - m_k(E;c)$$

Therefore,

$$F(E;c) \;=\; c - c^0 + F(C^0 - C + E; c^0)$$

$$= m(E;c) + F[E - M(E;c); c - m(E;c)]$$

That is, F satis...es composition from minimal rights. ∎

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