

Splitting the Baby in Two:
How to Solve Solomon's Dilemma when
Agents are Boundedly Rational*

Giovanni Ponti

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Correspondence to: Giovanni Ponti - Departament de Fundaments del Anàlisis Econòmic - Universitat de Alicante - 03071 Alicante - SPAIN. Voice: ++ 34 96590 3619 - Fax: ++34 96590 3898 - E-Mail: giuba@merlin.fae.ua.es.

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Abstract

We study the *dynamic implementation* of the first-best for *King Solomon's Dilemma*, on the assumption that boundedly rational players find their way to equilibrium using monotonic evolutionary dynamics, and also with best-reply dynamics. We find that, although the mechanisms proposed by the literature are dynamically implementable with best-reply dynamics, the same does not hold when monotonic dynamics are considered. To solve this problem, we propose an alternative mechanism, whose game-form is still implementable in the traditional sense. However, it is also dynamically implementable, as every interior path of the adjustment processes we consider converges to the first-best, which is also asymptotically stable.

JEL CLASSIFICATION: C72, D78, D83.

KEYWORDS: Solomon's Dilemma, Implementation Theory, Evolutionary Dynamics.

1 Introduction

King Solomon is called to resolve a dispute between two women, “Anna” and “Betta”, who both claim to be the mother of a baby. In economic terms, an indivisible prize is to be allocated between two individuals who holds different evaluations. The identity of the *true mother* (i.e. the agent with the highest evaluation) is common knowledge between Anna and Betta, but unknown to Solomon, whose objective is to rightfully resolve the dispute at no cost for the true mother.

To solve this dilemma, Glazer and Ma [9] propose the simple mechanism sketched in Figure 1, to be used when there are only two feasible evaluations, \underline{v} and \bar{v} , with $\underline{v} < \bar{v}$.

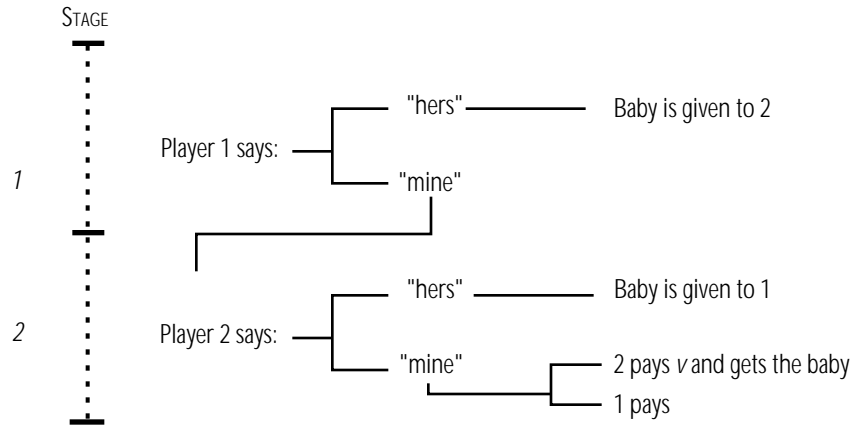


FIGURE 1
Glazer and Ma’s mechanism (M_1)

Denote by $v_i \in \{\underline{v}, \bar{v}\}; i = A, B$, i ’s evaluation. According to the game-form of Figure 1 (labeled as “ M_1 ” in what follows), Betta and Anna sequentially announce the identity of the true mother. The mechanism is designed in such a way that a statement in which Anna (Betta) attributes the baby to Betta (Anna) is never challenged. If both claim to be the true mother, then the baby is given to the player who has been chosen to speak last, subject to a lump-sum transfer to Solomon whose utility value, v , lies somewhere in between the two evaluations (i. e. $\underline{v} < v < \bar{v}$).¹

It is not difficult to show that M_1 implements the first-best when subgame perfection is employed as equilibrium notion. To see why, assume (without loss of generality) that Betta is the true mother (i.e. $v_A = \underline{v} < \bar{v} = v_B$) and evaluate

¹Under this circumstance, also the “looser” pays Solomon a fine, $\delta > 0$.

the optimal course of action for player 2.² Since $v_B > v > v_A$, only Betta (given she has been chosen to speak last) has an incentive to claim (only her evaluation exceeds the transfer). This, in turn, implies that the unique subgame-perfect equilibrium of the game induced by the mechanism requires Anna to attribute the baby to Betta. This outcome does not depend on the order in which the two mothers are called to speak: if Anna is player 2, she has no direct incentive to claim ($v_A < v$); if Anna is player 1, she is better off by not claiming to avoid the fine (since Betta, given $v_B > v$, will claim in return).

However, the game induced by this mechanism has many other “social inefficient” Nash equilibria. In particular, there is a component (i.e. a closed and connected set) of Nash equilibria in which Betta, conditional on being selected to speak first, gives up the baby under the (“incredible”) threat that Anna will challenge in return (leaving Betta without the baby and with a fine to pay).³ Thus, to ensure that the first-best is achieved by way of M_1 , we need to assume that both mothers are rational (in the sense that they would never use a dominated action), and they know that their opponent is also rational.

Whether this is to be considered as a demanding assumption is, essentially, an empirical matter. In this respect, there is already substantial experimental evidence that casts doubts on the use of standard game-theoretic equilibrium notions to describe how people play games in the lab. What we learn from experiments is that subjects often *fail* to play the equilibrium, especially if the equilibrium notion is fairly refined (as is the case of subgame perfection).⁴ Better results are observed when subjects can acquire some experience through repeated play.⁵ However, for some games, players may still fail to play an equilibrium, even with experience. As for the specific allocation problem studied here, Elbittar and Kagel [5] have done experiments with Moore’s [13] and Perry and Reny’s [16] mechanisms to solve Solomon’s Dilemma.⁶ Although their study appears inconclusive in isolating a unique behavioral pattern from the various experimental sessions, what stands clear from the experiments is that “...the emergence of non-expected responses in the experimental sessions makes both mechanisms fail in their predictions...” (p. 44).

Prompted by these experimental findings, we approach Solomon’s dilemma taking *bounded rationality* into account. The underlying theory is based upon an alternative definition of implementation. Among the variables which specify

²In what follows, we shall always associate Betta with the *role* of the agent with the higher evaluation (i.e. the “true mother”). As for players’ *positions* are concerned, we adopt the convention of labeling as player k , $k = 1, 2$, the one who is required to move at stage k .

³This threat is to be considered “incredible” in the spirit of subgame-perfection insofar, by claiming the baby, Anna would choose a dominated action.

⁴See McKelvey and Palfrey [12] and Binmore *et al.* [2].

⁵See Güth *et al.* [10] and Cooper *et al.* [4].

⁶Moore’s [13] mechanism is a modification of GMM. Perry and Reny’s [16] mechanism employs the iterated elimination of weakly dominated strategies and is supposed to be used in presence of incomplete information of the agents’ evaluations. As shown in Ponti [17], failure of monotonic dynamic implementation can be proved also for these mechanisms.

the “environment” in which the mechanism is supposed to operate, this definition includes the learning protocols agents may use, as well as initial conditions of the learning process. According to this alternative approach, a social choice rule will be said to be *dynamically implemented* by a mechanism if, for all possible environments (i.e. preferences, adjustment processes, initial conditions), the limiting set of outcomes *i*) coincides with the the first-best and *ii*) is also *asymptotically stable*, that is, robust to arbitrarily small perturbations.⁷

As for the dynamic implementation of the mechanism of M_1 , Proposition 1 shows that if the learning dynamics satisfy Nachbar’s [14] *monotonicity* condition, then many equilibria in the inefficient Nash equilibrium component can be limit points of the adjustment process. In other words, Betta’s incredible threat can be *sustainable* if, in the first repetitions of the game, it is exercised sufficiently often.⁸ However, as Proposition 1 shows, both mechanisms pass the *stability test*, that is, for initial conditions starting sufficiently close to the first-best, both mechanisms display the desired stability properties. We also study the dynamic implementation problem under best-reply dynamics (Matsui [11]), a limiting case of monotonic dynamic by which only strategies that are a best response to the current mixed strategy profile grow. This choice of dynamics allows us to understand the effects of increasing levels of responsiveness to past payoffs of the players (which could be interpreted as a proxy for “sophistication”) on the performance of the mechanism. In this respect, we find (proposition 2) that under best-reply dynamics the mechanism of figure 1 dynamically implements the first-best. Rather than accepting such a drastic restriction in the domain of admissible learning environments, we look for an alternative mechanism which may solve this problem.

There is a caveat here. Implicit in our evolutionary approach there is the restriction of the set of admissible preference relations to those preferences that can be represented as VNM utility functions. This restriction is standard in many fields of economics (game theory included), although it is an unusual assumption for implementation theory. This is because mechanisms usually employ pure strategy equilibria and need no assumptions on the agents’ preferences under risk. The reason why we use VNM utility functions is essentially technical, that is, we need to specify the payoff functions for mixed strategies, as dynamics are defined over the mixed strategy space. However, given this restriction, why should not Solomon exploit the fact that agents’ behavior must be consistent with VNM axiomatization to achieve his goal?

Consider the game-form of figure 2.

⁷Cabrales and Ponti (forthcoming) discuss in detail the rationale behind this alternative approach.

⁸The sustainability of incredible threats under evolutionary dynamics was first pointed out, by way of simulation, by Gale *al.* [7] for the *Ultimatum Game*.

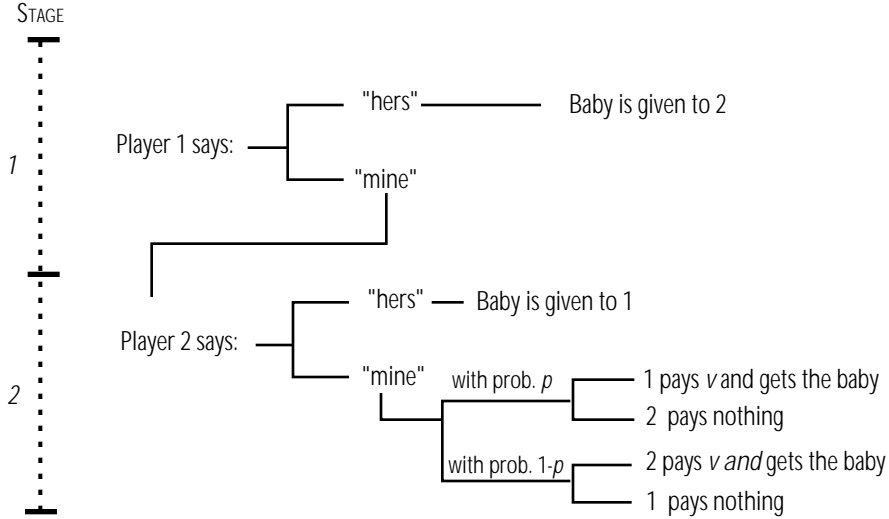


FIGURE 2
Our proposed solution (M_2)

According to the biblical story, Solomon was able to solve the dilemma by threatening to “split the baby in two”. By adopting M_2 , Solomon can still achieve his goal by introducing a lottery in which splitting occurs only “in expected terms”, with no risk of blood being spilled.⁹ Since the game induced by M_2 is strictly dominance solvable, it is also dynamically implementable with monotonic dynamics.

The remainder of the paper is arranged as follows. The implementation problem with monotonic dynamics and two feasible evaluations is solved in section 2, while section 3 deals with best-reply dynamics. Section 4 extends the above result to the N -player case, while section 5 concludes, followed by an appendix which collects the proofs.

2 Monotonic Dynamics

For a given normalform game $\mathcal{G} = \{I, S_i, u_i\}$, denote by $x_i = \{x_i^{s_i}\}$, $s_i \in S_i$, a generic mixed strategy for player i . We formalize players’ behavior in terms of the mixed strategy profile $x(t) \equiv (x_A(t), x_B(t)) \in \Delta$ played at each point in time, with Δ^0 denoting the relative interior of the state space Δ , i.e. the set of completely mixed strategy profiles. The evolution of $x(t)$ is given by the following system of continuous-time differential equations:

⁹Although non standard, the use of lotteries is not new in the implementation literature (see, e.g., [8], [1] and [16]).

$$\dot{x}_i^{s_i} = f_i^{s_i}(x(t)); i = A, B, \quad (1)$$

with $\sum_{s_i \in S_i} f_i^{s_i}(x(t)) = 0$.

As we explained in the introduction, we shall use asymptotic stability with respect to the interior as a sufficient condition (together with global interior convergence) for the first-best outcome to be *dynamically implementable*.¹⁰ As for the relation between the evolutionary dynamics and game payoffs, this section focuses on solutions of (1) which satisfy the following condition.

Assumption 1 (Monotonic Dynamics.) *A selection dynamic (1) is called payoff monotonic if*

- i) growth rates $g_i^s(x(t)) \equiv \frac{\dot{x}_i^s(t)}{x_i^s(t)}$ are Lipschitz continuous in an open domain X containing Δ ;
- ii) for all $s_i, s'_i \in S_i$ and all $x_{-i} \in \Delta_{-i}$,

$$\text{sign}[g_i^{s_i}(x(t)) - g_i^{s'_i}(x(t))] = \text{sign}[u_i(s_i, x_{-i}(t)) - u_i(s'_i, x_{-i}(t))], \quad (2)$$

where $u_i(\cdot, \cdot)$ denotes player i 's (VNM) payoff function.

Firstly introduced by Nachbar [14], condition (2) is commonly used in the evolutionary literature to capture the essence of a *selective* process. Given the mixed strategy profile played at each point in time, strategies with higher expected payoff grow faster than poorly performing ones.

For the dynamic analysis of the two mechanisms presented in the introduction, we shall focus on the case in which $\bar{v} = v^B$ (i.e. the true mother is Betta), and denote by $G(\Gamma)$ the game induced by the mechanism of figure 1(2).

For both games, a pure strategy $s_i \in S_i$ is an ordered pair $(h, k); h, k \in \{A, B\}$ of messages concerning the identity of the true mother. Each message is to be used in one of the two subgames, $G_1(\Gamma_1)$, in which Anna speaks first, and $G_2(\Gamma_2)$, in which Betta speaks first. The evolutionary properties of the two games will be derived using a two-population model (one population of Annas and one population of Bettas). The implicit assumption is that agents from each population are randomly paired; which action is to be used depends on the order of statements, which is randomly determined (by tossing a “fair” coin) by the matching technology.¹¹

Proposition 1 *If repeated play evolves according to monotonic dynamics, then the first-best i) fails to be dynamically implemented by the mechanism of figure 1, although ii) it can be dynamically implemented by the mechanism of figure 2.*

¹⁰By interior asymptotic stability, every trajectory starting arbitrarily close stays sufficiently close and eventually converges to the solution. By interior global convergence, every interior trajectory converges to the first-best. For formal definitions, see Weibull [21].

¹¹In other words, we use the “symmetrization procedure” of the strategy space proposed by Selten [19]. Clearly, Selten’s symmetrization acts only on strategies, not on payoffs, as players hold different evaluations concerning the event they are given the prize.

Proof. In the Appendix. ■

3 Best-Reply dynamics

In this section, we shall consider an alternative scenario.

Assumption 2 (Best-Reply Dynamics.) *A selection dynamic (1) is called best-reply dynamics if $x(t)$ evolves as follows:*

$$\dot{x} = BR(x) - x, \tag{3}$$

with $BR(x)$ denoting the mixed strategy best-reply correspondence $BR : \Delta \mapsto \Delta$.

This alternative dynamic defines a (continuous-time) version of the classic *best-reply* dynamics, often proposed as an alternative learning model to the evolutionary dynamics studied hereto. We can give two interpretations to (3). Following Matsui [11], we can use (3) to approximate the evolution of an infinite population of players who occasionally update their strategy, selecting a best reply to the current population state $x(t)$. Alternatively, (3) can be regarded as the continuous-time limit (up to a reparametrization of time) of the well known *fictitious play* dynamic.¹² This dynamic accounts for the evolution of players' *beliefs*, when these beliefs follow the empirical frequencies with which each pure strategy profile has been played (and perfectly observed) in the past, and agents select, at each point in time, a pure strategy among those which maximize their expected payoff, given their current beliefs.

Proposition 2 *If repeated play evolves according to the best-reply dynamic, then the first-best can be dynamically implemented by both mechanisms of figures 1 and 2.*

Proof. In the Appendix. ■

As for best-reply dynamics, we have shown that every interior solution converges to the unique equilibrium whose outcome is what Solomon wants. This is so because every interior solution of (3) has the property that weakly dominated strategies in which players lie about their own evaluation in stage 2 are played with vanishing weight. In other words, under best-reply dynamics, these incredible threats *are not sustainable*, and this implies convergence to the (subgame-perfect) solution.

¹²Firstly introduced by Brown [3] as an algorithm to compute Nash equilibria, fictitious play has been recently re-interpreted as a learning model in the by Fudenberg and Kreps [6].

4 Some extensions and negative results

All the results hereto depend on the assumption that there are only two players and two possible evaluations (i.e. Solomon knows what is the highest evaluation). As it turns out, the latter restriction can be easily relaxed when the implementation problem is to be solved dynamically, while the former is crucial to obtain the result.

4.1 Multiple evaluations

Assume Anna's and Betta's evaluations v^A and v^B ($v^A < v^B$) are drawn from the finite set $V \equiv \{v_k\}_{k=1}^K; v_k > 0$.

Moore's [13] proposed solution to Solomon's Dilemma for the case of (finite) multiple evaluations is sketched in figure 3.

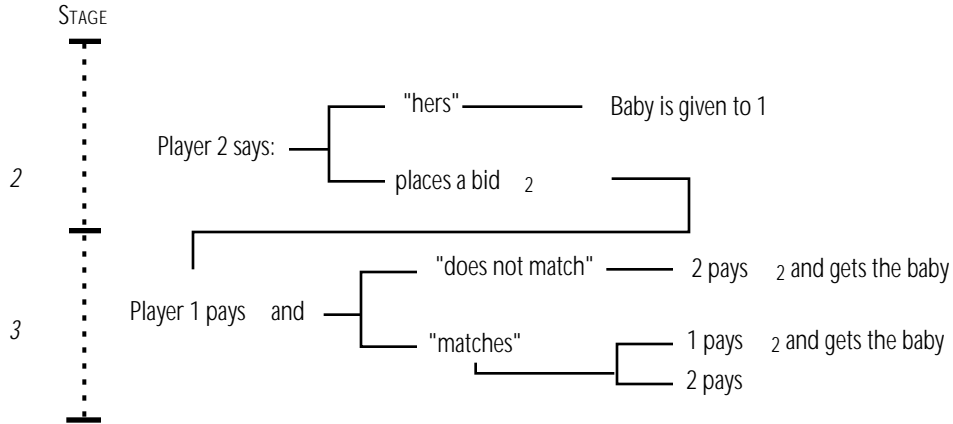


Figure 3
Multiple evaluations: Moore's [13] mechanism.

Stage 1 is identical to that of figure 1. If player 2 does not give up the prize in stage 2, she has to post the price $\pi_2 \in V$ she is willing to pay to get it. Then, player 1 (after having paid a fixed fine, δ) has to decide whether she wants to match the bid. If she matches, she pays π_2 and gets the prize.¹³ Otherwise, the prize is given to player 2.

It is not difficult to show that also the mechanism of figure 3, although subgame-perfect implementable, *fails* to be dynamically implementable under monotonic dynamics. This is because player 1 (regardless of whether she holds the highest evaluation) can always play a (Nash equilibrium) strategy by which she claims the prize at stage 1, and matches whatever bid at stage 3. If this (weakly dominated) strategy is played with sufficiently high probability, nothing can prevent the system to converge to a Nash equilibrium in which player 2 gives

¹³In this case, also player 2 has to pay Solomon the fine, δ .

up the prize, no matter how “credible” is the threat she is facing. However, this feature of the mechanism does not affect its dynamic implementation, if we restrict the class of learning environments to best-reply dynamics. This is because, if initial conditions are in the relative interior of the mixed strategy space, weakly dominated “threats” will be played with vanishing probability, so as to force Anna to give up the prize in the stage which precedes the bidding phase.

It is important to note that this difficulty is not peculiar of Moore’s [13] mechanism, but intrinsic to the fact that Solomon does not know what the highest evaluation is. In this case, the mechanism must employ a *bidding stage* in which players make statements about their own evaluation. If this bidding stage is arranged *after* (or is simultaneous to) the phase in which players may give up the prize, then:

- a)* if loosing the auction yields a punishment, like in [13], then there exist reachable Nash equilibria by which Anna bids over Betta’s evaluation and Betta gives up the prize to avoid the fine;¹⁴
- b)* if loosing the auction does not yield a punishment, then Anna has no incentive to withdraw, since bidding their true evaluation is a weakly dominant strategy for both players.

Things are similar if the bidding stage is arranged *before* the stage in which players can attribute the prize to their opponent (as is for the mechanism proposed by Perry and Reny [16]). In this case, the first-best would be achieved when Anna wins the auction and then withdraws. However:

- a)* if loosing the auction (when the winner does not exercise her option to withdraw) yields a punishment, then there exist reachable Nash equilibria in which Anna bids over Betta’s evaluation and Betta bids even higher and then withdraws;
- b)* if loosing the auction does not yield a punishment, then Anna has an incentive to bid her true value and to stick on it (as this strategy weakly dominates any strategy in the support of the first-best outcome).

Can a lottery solve this problem, as in the two-evaluation case? The answer is no. Since Solomon does not know what the highest evaluation is, he cannot construct a lottery with a strictly positive (negative) expected value for Betta (Anna).¹⁵

We conclude this section by observing that, to construct our model, we had to assume two populations (one population of Annas and one population of Bettas). To make sense of the model, even when agents may differ in their private evaluation, there must exist two disjoint sets of feasible evaluations,

¹⁴It can be proved that this feature does not depend on how this bidding stage is arranged.

¹⁵The same result holds if the lottery depends upon the bidding stage. If the discriminatory value is determined as (an ϵ above) the lowest bid (like in a second price auction), there is no incentive for either player to withdraw; if this discriminatory price is determined by (an ϵ below) the highest bid (like in a first price auction), the mechanism suffers the same problems as Moore’s [13].

whose ranges do not overlap. Incidentally, this is the only information Solomon needs to know to set up a lottery which, as in the two-evaluation case, has a strictly positive (negative) expected value for the high (low) evaluation player.

4.2 Two evaluations, multiple players

Let us assume that there are N populations of agents (1 population of Bettas and $N - 1$ populations of Annas), repeatedly engaged in the $N - player$ version of Solomon’s dilemma. In this case, there is an easy way to implement the first-best dynamically, by way of the following mechanism. Each player is arranged to simultaneously announce whether to claim the prize or not. If a player does not claim, her payoff is zero, unless no other player claims, in which case a fine $\delta > 0$ is levied to all. If some player claims, a lottery is drawn which assign the prize among the claimants (each of which has some positive probability of winning), in exchange to a side payment equal to v . To see why this mechanism can implement the first best dynamically with both monotonic and best-reply dynamics, notice that the underlying game is strictly dominance solvable. This is because claiming is a strictly dominant strategy for Betta; if Betta claims the prize, then not claiming is strictly optimal for all the $N - 1$ players in Anna’s position. In consequence, standard results can be used to show that the first-best can be dynamically implemented in both the learning environments we consider.

5 Conclusion

In the recent years, there has been an impressive progress in the theory of implementation. We are now in possess of a wide variety of instruments, by which, “...with enough ingenuity, the planner can implement ‘anything’ ”¹⁶.

We can identify (at least) two sources of ingenuity:

- agents are assumed to *play by the book*, i.e. to follow the rules of the mechanism in any possible circumstance;
- they are also assumed to play the equilibrium prescribed by the social rule.

Although it has been developed for a very specific allocation problem (and, therefore, lacks of the generality typical of the implementation literature), our result move in the direction of a substantial relaxation of the second set of assumptions. If Solomon is “patient” enough, so as to accept the fact that it might take some time to achieve his goal, he may exploit the dynamic features of the equilibrating process to make sure that the first-best will be eventually attained.

¹⁶See Sjöström [20].

Whether the learning environments we explore capture the essence of real-life bounded rationality is, essentially, an empirical matter. In this respect, there is already some experimental evidence on the specific allocation problem we study in this paper which substantially supports the thesis that traditional implementation techniques may poorly perform in laboratory, as they do not adequately consider the effects of disequilibrium play. Elbittar and Kagel [5] have done experiments with Moore's [13] and Perry and Reny's [16] mechanisms. Although their study appears inconclusive in isolating a unique behavioral pattern from the various experimental sessions, what stands clear from the experiments is that "...the emergence of non-expected responses in the experimental sessions makes both mechanisms fail in their predictions..." (p. 44). The extent to which our proposed solutions may correct these deficiencies is left to future research.

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6 Appendix

6.1 Some useful results

The formal analysis developed in this paper is based on results, some of those are proved elsewhere. The aim of this section is to provide the reader with a quick reference of these theoretical findings.

Lemma 1 *Let $\mathcal{G} \equiv \{\{A, B\}, S_i, u_i\}$ denote a two-player normalform game. If there exists some Nash equilibrium $s^* \equiv (s_i^*, s_{-i}^*)$ such that i) s_i^* is a weakly dominant strategy for player i and ii) s_{-i}^* is a strict best response to s_i^* , then*

1. s^* is contained in a Nash equilibrium component which is interior asymptotically stable under any monotonic dynamic (2).
2. s^* is interior asymptotically stable under best-reply dynamics (3).

Proof.

1. See Oechssler and Schlag [15], Proposition 1.

2. Since s_i^* is a weakly dominant strategy, it must be $\dot{x}_i^{s_i^*} = 1 - x_i^{s_i^*}$ for any interior solution of (3). Moreover, since s_{-i}^* is a strict best response to s_i^* , there must exist some $\epsilon > 0$ such that s_{-i}^* is the unique best response for any x_i such that $x_i^{s_i^*} \geq 1 - \epsilon$. This also implies that there exists a closed neighborhood $B(\epsilon)$ of s^* such that:

i) trajectories starting from $B(\epsilon) \cap \Delta^0$ do not leave $B(\epsilon) \cap \Delta^0$ (i.e. s^* is stable);
ii) trajectories starting from $B(\epsilon) \cap \Delta^0$ converge to s^* (i.e. s^* is interior attracting).

Since these two requirements characterize interior asymptotic stability, the result follows. ■

Definition 1 (τ -dominance) *Let $x(x(0), t)$ be an interior solution of (1). A pure strategy s_i is said to be strictly τ -dominated by some pure strategy s_i' ($s_i <_\tau s_i'$ hereafter) if we can identify a time τ and a non-empty compact set C_{-i} for which:*

$$x_{-i}(t) \in C_{-i}, \forall t > \tau; \quad (4)$$

$$u_i(s_i, x_{-i}) < u_i(s_i', x_{-i}), \forall x_{-i} \in C_{-i}. \quad (5)$$

Moreover, s_i is weakly τ -dominated by s_i' ($s_i \leq_\tau s_i'$ hereafter) if (4) holds and (5) is replaced by the following conditions:

$$u_i(s_i, x_{-i}) \leq u_i(s_i', x_{-i}), \forall x_{-i} \in C_{-i}, \quad (6)$$

$$u_i(s_i, x_{-i}) < u_i(s_i', x_{-i}), \forall x_{-i} \in C_{-i}^0, \quad (7)$$

where, by analogy, $C_{-i}^0 = C_{-i} \cap \Delta^0$.

Corollary 1 *If s_i is strictly (weakly) dominated by s'_i , then $s_i <_\tau s'_i$ ($s_i \leq_\tau s'_i$).*

Proof. Fix $\tau = 0$ and $C_{-i} = \Delta_{-i}$. ■

Let $\omega_i(x(0))$ be the ω -limit set for player i of an interior solution $x(x(0), t)$, i.e. $\omega_i(x(0)) = \{\hat{x}_i : x_i(x(0), t) \rightarrow \hat{x}_i \text{ for some sequence } \langle t_m \rangle_{m=1}^\infty\}$.

Lemma 2 *Let $x(x(0), t)$ be an interior solution of a monotonic dynamic $\dot{x} = f(x)$. If $s_i \leq_\tau s'_i$ then*

1. $\frac{d}{dt} \frac{x_i^{s'_i}}{x_i^{s_i}} < 0$ for any $t > \tau$.
2. $\lim_{t \rightarrow \infty} \frac{x_i^{s'_i}}{x_i^{s_i}} = L_i^{(s_i, s'_i)} \geq 0$.
3. If $L_i^{(s_i, s'_i)} > 0$ then $u_i(s_i, x_{-i}) = u_i(s'_i, x_{-i}), \forall x_{-i} \in \omega_{-i}(x(0))$.

Proof. See Ponti [18], Proposition 4.1. ■

Lemma 3 *Let $x(x(0), t)$ be an interior solution of a monotonic dynamic $\dot{x} = f(x)$. If $s_i <_\tau s'_i$ then $\hat{x}_i^{s_i} \equiv \lim_{t \rightarrow \infty} x_i^{s_i} = 0$.*

Proof. See Ponti [18], Proposition 4.2. ■

Lemma 4 *Let $x(x(0), t)$ be an interior solution of the best-reply dynamics (3). If $s_i \leq_\tau s'_i$ ($s_i <_\tau s'_i$), then $\lim_{t \rightarrow \infty} x_i^{s_i} = 0$.*

Proof. If $s_i \leq_\tau s'_i$ (and, a fortiori, if $s_i <_\tau s'_i$), then (3) implies

$$\dot{x}_i^s(t) = -x_i^s(t); t > \tau,$$

for any interior solution. This is because, since $s_i \leq_\tau s'_i$ ($s_i <_\tau s'_i$), s_i cannot be a best-reply to any $x_{-i} \in C_{-i}$. This, in turn, implies $\hat{x}_i^{s_i} = \lim_{t \rightarrow \infty} x_i^{s_i}(\tau)e^{-t} = 0$. ■

6.2 The Proofs

We are now in the position of proving propositions 1-2.

Proof of proposition 1. Denote by Θ the “inefficient” Nash equilibrium component of game G :

$$\Theta = \{(x_A, x_B) \in \Delta : (\{\alpha s_A^{(B,A)} + (1 - \alpha) s_A^{(B,B)}\}, \{\beta s_B^{(A,A)} + (1 - \beta) s_B^{(B,A)}\})\},$$

with $\alpha \in [\frac{v^B}{1+v^B}, 1]$ and $\beta \in [0, 1]$. All strategy profiles in Θ are outcome equivalent to the Nash equilibrium in pure strategies $(s_A^{(B,A)}, s_B^{(B,A)})$ by which the first-best is attained in game G_1 and the inefficient equilibrium is attained in G_2 . To prove the proposition, it is sufficient to show that Θ is reachable from a non-zero measure set of initial conditions. We characterize this set as follows:

$$\Omega \equiv \{(x_A, x_B) \in \Delta^0 : x_A^{(B,A)} > 1 - \epsilon_A, x_B^{(B,A)} > 1 - \epsilon_B; 0 < \epsilon_i \leq \eta, i = A, B\}.$$

with η sufficiently small. To show that any trajectory starting from $\Delta^0 \cap \Omega$ converges to Θ , notice that, $\forall x \in \Delta^0$ such that $x_A^{(B,A)} > \frac{v+v^B}{v+v^B+\delta}$, it must be $g_B^{(B,A)}(x) > 0$ (which implies $\dot{x}_B^{(B,A)} > 0$), since $s_B^{(B,A)}$ is the unique best response. This, in turn, implies that, provided $x_A^{(B,A)}(t)$ stays arbitrarily high, $\hat{x}_B^{(B,A)}(t)$ stays arbitrarily high.

By the same token, $\forall x \in \Delta^0$ such that $x_B^{(B,A)} > \frac{2v^A}{2v^A+\delta}$, it must be $g_A^{(B,B)}(x) > 0$. Moreover, for any sequence $\langle x_m \rangle_{m=1}^\infty$ such that $\lim_{m \rightarrow \infty} x_m = (s_A^{(B,A)}, s_B^{(B,A)})$ it must be $\lim_{m \rightarrow \infty} g_A^{(B,B)}(x_m) > 0$ since the difference in growth rates between strategy $s_A^{(B,B)}$ and strategies $s_A^{(A,A)}$ and $s_A^{(A,B)}$ is bounded away from zero (by monotonicity) when $x_B^{(B,A)}$ is sufficiently high. This, in turn, implies $\lim_{m \rightarrow \infty} g_A^{(B,A)}(x_m) > 0$, since (by Lipschitz continuity), $|(g_A^{(B,B)}(x_m) - g_A^{(B,A)}(x_m))| \rightarrow 0$ as $m \rightarrow \infty$. Therefore, there exists a neighborhood $\bar{B}(\epsilon_B)$ of $(s_A^{(B,A)}, s_B^{(B,A)})$ such that $g_A^{(B,A)}(x) > 0, \forall x \in \bar{B}(\epsilon_B)$. In other words, provided $x_B^{(B,A)}(t)$ stays arbitrarily high, $x_A^{(B,A)}(t)$ stays arbitrarily high. This concludes the proof.

ii). We begin by noting that $s_A^{(B,B)}$ ($s_A^{(A,B)}$) weakly dominates $s_A^{(B,A)}$ ($s_A^{(A,A)}$). Also notice that Betta has a weakly dominant strategy, namely $s_B^{(B,B)}$. In particular, $s_B^{(B,B)}$ strictly dominates $s_B^{(A,A)}$ and $s_B^{(B,A)}$ and weakly dominates $s_B^{(A,B)}$. Then, lemma 2.2 implies convergence of Betta's mixed strategy, with $\hat{x}_B^{(A,A)} = \hat{x}_B^{(B,A)} = 0$ by lemma 3. Only two cases need be discussed:

CASE 1. $\hat{x}_B^{(A,B)} = 0$ (i.e. $\hat{x}_B^{(A,B)} = 1$). This would imply $s'_A <_\tau s_A^{(B,B)}$, for all $s'_A \neq s_A^{(B,B)}$, which in turn implies, by lemma 3, $\hat{x}_A^{(B,B)} = 1$.

CASE 2. $\hat{x}_B^{(A,B)} > 0$ (i.e. $\hat{x}_B^{(A,B)} = 1 - \hat{x}_B^{(B,B)}$). This would imply, by lemma 2.2-3, $\hat{x}_A^{(A,A)} = \hat{x}_A^{(A,A)} = 0$ and $\hat{x}_A^{(B,A)} = 1 - \hat{x}_A^{(B,B)}$.

Given that both cases imply convergence of Anna's mixed strategy (and, therefore, convergence to a Nash equilibrium), and that every Nash equilibrium of game Γ is outcome equivalent to the first-best, global interior convergence follows. As for interior asymptotic stability, the result follows directly from lemma 1.1. ■

Proof of proposition 2.

2.1 Game G.

2.1.1 Convergence Weak domination of the corresponding strategies implies, by lemma 4,

$$\hat{x}_A^{(A,A)} = \hat{x}_A^{(B,A)} = \hat{x}_B^{(A,A)} = \hat{x}_B^{(A,B)} = 0.$$

This, in turn, implies strict τ -domination of strategies $s_A^{(A,B)}$ and $s_B^{(B,A)}$ which, in turn, implies convergence to the first-best. More explicitly, since $s_B^{(A,A)}$ and $s_B^{(A,B)}$ are weakly dominated (by $s_B^{(B,A)}$ and $s_B^{(B,B)}$ resp.), lemma 4 implies that there exists some $\tau_A \geq 0$ such that

$$x_B^{(A,A)}(t) + x_B^{(A,B)}(t) \leq \frac{\delta}{\delta + 2v^A}, \forall t \geq \tau_A.$$

In other words, given any interior initial condition, Anna has a *strictly τ -dominant strategy*, namely $s_A^{(B,B)}$. We can evaluate τ_A explicitly:

$$\begin{aligned} \tau_A &= 0 \quad \text{if } x_B^{(A,A)}(0) + x_B^{(A,B)}(0) \leq \frac{\delta}{\delta + 2v^A}, \\ &= \ln(\delta) - \ln((x_B^{(A,A)}(0) + x_B^{(A,B)}(0))(\delta + 2v^A)) \quad \text{otherwise.} \end{aligned} \quad (8)$$

By virtue of (8), $\tau_A < \infty$; by lemma 4, $\hat{x}_A^{(B,B)} = 1$. Thus, there exists some $\tau_B \geq 0$ such that $x_A^{(B,B)}(t) > \frac{\delta+v}{\delta+v+v^B}, \forall t > \tau_B$ (i.e. $s_B^{(B,B)}$ also strictly τ -dominates every other strategy in Betta's support). Fix $\bar{\tau} = \max[\tau_A, \tau_B]$. Since $BR(x(t)) = (s_A^{(B,B)}, s_A^{(B,B)})$ for any $t \geq \bar{\tau}$, any interior solution of (3) is characterized by the following system of differential equations:

$$\dot{x}_i^{(B,B)}(t) = 1 - x_i^{(B,B)}(t), i = A, B; \quad (9)$$

for $t \geq \bar{\tau}$. Thus, $\hat{x}_i^{(B,B)} = 1, i = A, B$. This concludes the proof for game G.

2.1.2. Asymptotic Stability. Since we just defined a neighborhood of $(s_A^{(B,B)}, s_A^{(B,B)})$ within which every trajectory of (2) converges monotonically to $(s_A^{(B,B)}, s_A^{(B,B)})$, the result follows.

2.2. Game Γ .

2.2.1. Convergence As we already noticed in the proof of proposition ??, in game Γ Betta has a weakly dominant strategy, namely $s_B^{(B,B)}$. This already implies, by lemma 4, $\hat{x}_B^{(B,B)} = 1$ for any interior solution of (3). By analogy with the proof for game G, $\hat{x}_B^{(B,B)} = 1$ also implies $\hat{x}_A^{(B,B)} = 1$, since $s_A^{(B,B)}$ strictly τ -dominates every other strategy in Anna's support.

2.2.2. Asymptotic Stability. The result follows directly from lemma 1.2. ■