A COMPUTATIONAL APPROACH TO THE FUNDAMENTAL THEOREM OF ASSET PRICING IN A SINGLE-PERIOD MARKET

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A B S T R A C T

In this paper we provide a new approach to the Fundamental Theorem of Asset Pricing. The proof of this result is usually based on Projection (Separation) Theorems and is far more intuitive. Our approach follow the relation between the projection problem an equivalent least squares problem. More precisely, we will use and iterative procedure in order to obtain solutions of a bounded least square problem. This solutions will give, under some conditions, either the state price vector or the arbitrage opportunity of the problem under consideration.

Keywords: Asset Pricing; Arbitrage; Mathematical Finance

1 Introduction

The basic idea of the whole pricing philosophy in finance consists in the construction of a linear functional π which strictly separates the arbitrage opportunities obtained by trading strategies. Moreover, this functional gives the state–price vector necessary for pricing contingent claims in a financial market. In a more formal way, if we denote by M the linear subspace all trading strategies and by $\mathbb{K}$ the non–negative cone, then there not are arbitrage opportunities if and only if $M \cap \mathbb{K} = \{\mathbf{0}\}$. This result is know as the *Fundamental Theorem of Asset Pricing.* We remark that the existence of state price vectors follows by *the Separate Hyperplane Theorem [7]*–a version of the Hahn–Banach Theorem (see [5]). However, the Hahn–Banach theorem not is a useful tool in order to construct the state price vector. Despite its practical importance, a computational approach to the calibration of the state price vector has received little attention in the Mathematical Finance theory in discrete time (see [8, Theorem 1.2], [2, Theorem 1.4.1] and [6, 1.16]). We remark that the single period model is important because it provides much of the intuition that is necessary for more general models of financial markets and because it is possible to combine single periods results to prove the *Fundamental Theorem of Asset Pricing for the Multi–Period Model* (see [2, Proposition 4.2.3] or [8, Section 2.3 and Theorem 3.1]).

The aim of this paper is to give a constructive proof of this fundamental result. To see this we will reduce the problem to solve a bounded least square problem, a minimization problem. In this sense this strategy has some similarities to the problem solved by Avellaneda [1] in order to obtain a risk–neutral probability that minimizes the relative entropy with respect to a given prior distribution. From this approach we will obtain, from the construction of the state price vectors, how are the conditions in order to have non–arbitrage opportunities and exact replication of market portfolios.

The paper is organized as follows. In the next section we will introduce some definitions and we state the main result of this paper. In Section 3 we will prove it.

2 Definition and statement of results

In this paper we shall consider a single period market, that is, we have two indices, namely $t = 0$ which is the current time, and $t = \Delta t$, which is the terminal date for all economic activities under consideration.

The financial market contains N traded financial assets, whose prices at time $t = 0$ are denoted by

$$\mathbf{S}_0 = \begin{pmatrix} S_0^1 & S_0^2 & \cdots & S_0^N \end{pmatrix}' \geq \mathbf{0},$$

here $'$ denotes the transpose of a matrix or vector. At time Δt, the owner of financial asset number i receives a random payment depending on the state of the world. We model this randomness by introducing a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_i) > 0$ for all $i \in \{1, 2, \ldots, k\}$.

We note that the random payment arising from financial asset i is a $\mathbb{R}^k$–vector

$$\left(S^i_{\Delta t}(\omega_1), S^i_{\Delta t}(\omega_2), \ldots, S^i_{\Delta t}(\omega_k)\right)' \geq \mathbf{0}.$$

At time $t = 0$ the agents can buy and sell financial assets. The portfolio position of an individual agent is given by a trading strategy, which is a vector

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)' \in \mathbb{R}^N.$$

Here θ_i denotes the quantity if the i–th asset bought at time $t = 0$, which may negative, if the agent has a short position, as well positive, if he has a long position.

The dynamics of this model using the trading strategy $\boldsymbol{\theta}$ are as follows:

1. At time $t = 0$ the agent invests the amount

$$\mathbf{S}'_0 \boldsymbol{\theta} = \theta_1 S^1_0 + \theta_2 S^2_0 + \cdots + \theta_N S^N_0,$$

2. and at time $t = \Delta t$ the agent receives a random payment $\mathbf{P}$ that we can represent by using a matrix as follows, let

$$\mathbf{S}_{\Delta t} = \begin{pmatrix} S^1_{\Delta t}(\omega_1) & S^1_{\Delta t}(\omega_2) & \cdots & S^1_{\Delta t}(\omega_k) \\ S^2_{\Delta t}(\omega_1) & S^2_{\Delta t}(\omega_2) & \cdots & S^2_{\Delta t}(\omega_k) \\ \vdots & \vdots & \ddots & \vdots \\ S^N_{\Delta t}(\omega_1) & S^N_{\Delta t}(\omega_2) & \cdots & S^N_{\Delta t}(\omega_k) \end{pmatrix},$$

then

$$\mathbf{P} = \mathbf{S}'_{\Delta t} \boldsymbol{\theta}.$$

We remark that each component of vector $\mathbf{P}$ represents the payment received depending on the realized stated of the world ω.

Then we can define an *arbitrage opportunity* as a vector $\boldsymbol{\theta} \in \mathbb{R}^N$ such that one of the following two conditions hold.

(Arb1) $\mathbf{S}'_0 \boldsymbol{\theta} = 0$ and $\mathbf{P} = \mathbf{S}'_{\Delta t} \boldsymbol{\theta} \geq \mathbf{0}$, with $\mathbf{S}'_{\Delta t} \boldsymbol{\theta} \neq \mathbf{0}$.

(Arb2) $\mathbf{S}'_0 \boldsymbol{\theta} < 0$ and $\mathbf{P} = \mathbf{S}'_{\Delta t} \boldsymbol{\theta} \geq \mathbf{0}$.

Note that in the case of and arbitrage opportunity satisfying (Arb 1) the agent's net investment at time $t = 0$ is zero, and there exists a $\omega \in \Omega$ such that

$$\sum_{i=1}^{N} S^i_{\Delta t}(\omega)\, \theta_i > 0,$$

that is, there exists non–zero probability to obtain a "free lunch". In the case of condition (Arb 2), we have that $\mathbf{S}_0'\boldsymbol{\theta} < 0$, that is, the agent borrows money for consumption at time $t = 0$, and he doesn't has to repay anything in the time Δt.

By using the well–know result called, *the separated hyperplane theorem* (see [7]) that is a version of the *Hahn–Banach Theorem* (see [8]) it follows the following result (see [2], [5], [6] or [8]).

Theorem 1 *There is no arbitrage opportunity if and only if there exists $\Psi > \mathbf{0}$ such that*

$$\mathbf{S}_{\Delta t}\Psi = \mathbf{S}_0. \tag{1}$$

We will say that a vector $\Psi > \mathbf{0}$ satisfying (1) is a state price vector. Moreover, we can state that *the separated hyperplane theorem implies the existence of state price vector in the proof Theorem 1.* The main goal of this paper is to construct either the state price vector if non arbitrage opportunities exist or an arbitrage opportunity if there no are state price vectors. To see this we will use an algorithm due to Dax [3] that it was used in order to give an elementary and comprehensive proof of Farkas' Lemma. More precisely, Dax's Algorithm provides a solution $\mathbf{y}^*$ of the following bounded least square problem

$$\begin{aligned} &\min \|A\mathbf{y} - \mathbf{b}\|^2 \\ &\text{subject to } \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{2}$$

(see Appendix A). As we will see, this algorithm will provides the necessary tools to obtain as solution of it either the state price vector or the arbitrage oportunity.

Now, we can give some preliminary definitions and results about basic Linear Algebra. Let A be a $m \times n$–matrix then we define the column space of A, that we denote by $\operatorname{col} A$, as

$$\operatorname{col} A = \operatorname{span}\{A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n\},$$

where $\mathbf{e}_i$ denotes the i–th column of the $n \times n$ unit matrix. In particular, if we set

$$\mathbf{S}^i = \mathbf{S}_{\Delta t}\mathbf{e}_i$$

for $i = 1, 2, \ldots, k$, then

$$\operatorname{col} \mathbf{S}_{\Delta t} = \operatorname{span}\left\{\mathbf{S}^1, \mathbf{S}^2, \ldots, \mathbf{S}^k\right\}.$$

In a similar way as above we define de row space of A, denoted by $\operatorname{row} A$, by

$$\operatorname{row} A = \operatorname{col} A'.$$

Let

$$\mathrm{nul} A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\},$$

and for a vectorial subspace $E \subset \mathbb{R}^n$, we will denote by $E^{\perp}$ the orthogonal complement of E, that is,

$$E^{\perp} = \{\mathbf{x} : \mathbf{x}'\mathbf{y} = \mathbf{0} \text{ for all } \mathbf{y} \in E\}.$$

It is well–known (see [9]) that

$$E \cap E^{\perp} = \{\mathbf{0}\}$$

and for all $\mathbf{x} \in \mathbb{R}^n$ there exist $\mathbf{x}_1 \in E$ and $\mathbf{x}_2 \in E^{\perp}$ such that

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2.$$

Moreover,

$$(\mathrm{nul}\, A)^{\perp} = \mathrm{row}\, A = \mathrm{col}\, A'.$$

Finally, set $\mathbb{K} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$, $\overset{\circ}{\mathbb{K}} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$ and for $\mathbf{x} \in \mathbb{R}^n$, let $\mathcal{Z}(\mathbf{x}) = \{i : x_i = 0\}$.

Theorem 2 *Let Ψ^* be a solution of*

$$\begin{aligned} &\min \left\|\mathbf{S}_{\Delta t}\Psi - \mathbf{S}_0\right\|^2 \\ &\textit{subject to } \Psi \geq \mathbf{0}. \end{aligned} \tag{3}$$

and take $\boldsymbol{\theta}^* = \mathbf{S}_{\Delta t}\Psi^* - \mathbf{S}_0$. If $\boldsymbol{\theta}^* \neq \mathbf{0}$ then $\boldsymbol{\theta}^*$ satisfies (Arb 2). Otherwise, if $\boldsymbol{\theta}^* = \mathbf{0}$ then one and only one of the following statement hold:

1. *If $\Psi^* > \mathbf{0}$ then there no are arbitrage opportunities.*
2. *If $\Psi^* \geq \mathbf{0}$ and*

   $$\mathrm{span}\left\{\mathbf{S}^i : i \in \mathcal{Z}(\Psi^*)\right\} \subset \mathrm{span}\left\{\mathbf{S}^i : i \notin \mathcal{Z}(\Psi^*)\right\},$$

   then there exist $\delta > 0$ and a continuous path of state price vectors Ψ^*_{ε}, where $\varepsilon \in (0, \delta]$, and such that

   $$\lim_{\varepsilon \to 0} \Psi^*_{\varepsilon} = \Psi^*.$$

   Moreover, there no are arbitrage opportunities.
3. *If $\Psi^* \geq \mathbf{0}$ and*

   $$\mathrm{span}\left\{\mathbf{S}^i : i \in \mathcal{Z}(\Psi^*)\right\} \not\subseteq \mathrm{span}\left\{\mathbf{S}^i : i \notin \mathcal{Z}(\Psi^*)\right\},$$

then there are arbitrage opportunities satisfying (Arb1) and there no exists a state price vector. Moreover, let $\mathbf{y}^*$ *be a solution of*

$$\begin{aligned} &\min \left\| [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\, \mathbf{y} + \mathbf{e}' \left[\mathbf{S}_{\Delta t}, -\mathbf{S}_0\right]' \right\|^2 \\ &\textit{subject to } \mathbf{y} \geq \mathbf{0}, \end{aligned} \tag{4}$$

where $\mathbf{e} = (1, 1, \ldots, 1)'$ *, then*

$$\boldsymbol{\theta}^* = [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\, \mathbf{y}^* + \mathbf{e}' \left[\mathbf{S}_{\Delta t}, -\mathbf{S}_0\right]'$$

is an arbitrage opportunity.

From the above theorem we obtain the following result.

Corollary 3 *Theorem 2 implies Theorem 1.*

Note that in both cases, either the existence of a state price vector or the existence of an arbitrage opportunity, we only need to solve a bounded least square problem. To solve this problem and for completeness we give, in Appendix A, an algorithm due to Dax [4].

3 Proof of the main result

This section is devoted to the proof of Theorem 2. Assume that $\Psi^* \geq \mathbf{0}$ is a solution of (3). If $\boldsymbol{\theta}^* \neq \mathbf{0}$, then we will use the following useful lemma (see [3, Lemma 2]).

Lemma 4 *Let* $\Psi^* \in \mathbb{R}^k$. *Then* Ψ^* *holds (3) if and only if* Ψ^* *and* $\boldsymbol{\theta}^*$ *satisfy*

$$\Psi^* \geq \mathbf{0}, \mathbf{S}_{\Delta t}\, \boldsymbol{\theta}^* \geq \mathbf{0} \textit{ and } (\Psi^*)' \, \mathbf{S}'_{\Delta t} \boldsymbol{\theta}^* = 0. \tag{5}$$

By using the above lemma, we have that

$$\begin{aligned} \mathbf{S}'_0 \boldsymbol{\theta}^* &= (\mathbf{S}_{\Delta t} \Psi^* - \boldsymbol{\theta}^*)^T \boldsymbol{\theta}^* \\ &= (\Psi^*)' \, \mathbf{S}'_{\Delta t} \boldsymbol{\theta}^* - (\boldsymbol{\theta}^*)' \, \boldsymbol{\theta}^* \\ &= -\left\| \boldsymbol{\theta}^* \right\|^2 < 0. \end{aligned}$$

Since, $\mathbf{S}'_{\Delta t}\, \boldsymbol{\theta}^* \geq \mathbf{0}$, we obtain that $\boldsymbol{\theta}^*$ is an arbitrage opportunity satisfying (Arb 2).

Now, assume that $\boldsymbol{\theta}^* = \mathbf{0}$. Then we will consider the following two situations. First, assume that $\Psi^* > \mathbf{0}$. In this case Ψ^* is a state price vector. We claim that there no are arbitrage opportunities. Otherwise, there exists $\boldsymbol{\theta} \in \mathbb{R}^N$ satisfying that

$$\mathbf{S}'_0 \boldsymbol{\theta} = (\mathbf{S}_{\Delta t} \Psi^*)' \, \boldsymbol{\theta} = (\Psi^*)' \, \mathbf{S}'_{\Delta t} \boldsymbol{\theta} > 0,$$

a contradiction and the claim follows. Thus statement 1 holds.

Now, assume that $\Psi^* \geq \mathbf{0}$. In the case we can write

$$\Psi^* = \left(0, \ldots, 0, \psi^*_{s+1}, \ldots, \psi^*_k\right)'$$

without loss of generality. We recall that $\mathbf{S}^i = \mathbf{S}_{\Delta t}\mathbf{e}_i$. From now one we will denote by $\mathbf{S}_{\Delta t,2}$ the matrix given by

$$\left[\mathbf{S}^{s+1}\mathbf{S}^{s+2}\ldots\mathbf{S}^k\right].$$

Now, assume that there exists $1 \leq l \leq k-s$ be such that

$$\operatorname{col}\mathbf{S}_{\Delta t,2} = \operatorname{span}\left\{\mathbf{S}^{k-l+1}, \mathbf{S}^{k-l+2}, \ldots, \mathbf{S}^k\right\}$$

and where $\left\{\mathbf{S}^{k-l+1}, \mathbf{S}^{k-l+2}, \ldots, \mathbf{S}^k\right\}$ is a set of lineally independents vectors. Then

$$\mathbf{S}^i = \sum_{j=k-l+1}^{k} \lambda_{i,j}\mathbf{S}^j, \tag{6}$$

where $\lambda_{ij} \in \mathbb{R}$, for $i = 1, 2, \ldots, k-l$ and $j = k-l+1, \ldots, k$. Moreover,

$$\mathbf{S}_0 = \mathbf{S}'_{\Delta t}\Psi^* = \sum_{j=s+1}^{k} \psi^*_j\mathbf{S}^j. \tag{7}$$

Finally, we take $\mathbf{S}_{\Delta t,1} = \left[\mathbf{S}^1\mathbf{S}^2\cdots\mathbf{S}^s\right]$. Note that $\mathbf{S}_{\Delta t} = \left[\mathbf{S}_{\Delta t,1}\mathbf{S}_{\Delta t,2}\right]$. The next lemma will be useful to prove statement 2.

Lemma 5 *If* $\operatorname{col}\mathbf{S}_{\Delta t,1} \subset \operatorname{col}\mathbf{S}_{\Delta t,2}$, *then there exist* $\delta > 0$ *and a continuous path of state price vectors* Ψ^*_ε, *where* $\varepsilon \in (0, \delta]$, *and such that*

$$\lim_{\varepsilon \to 0} \Psi^*_\varepsilon = \Psi^*.$$

P roof. o prove the claim we need to find $\Psi = (\psi_1, \ldots, \psi_k)' > \mathbf{0}$ such that

$$\mathbf{S}_0 = \sum_{j=s+1}^{k} \psi^*_j\mathbf{S}^j = \sum_{j=1}^{k} \psi_j\mathbf{S}^j. \tag{8}$$

Then

$$\begin{aligned}
\sum_{i=1}^{k} \psi_i\mathbf{S}^i &= \sum_{i=1}^{s} \psi_i\mathbf{S}^i + \sum_{t=s+1}^{k-l} \psi_t\mathbf{S}^t + \sum_{j=k-l+1}^{k} \psi_j\mathbf{S}^j \\
&= \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i\lambda_{i,j}\right)\mathbf{S}^j + \sum_{t=s+1}^{k-l} \psi_t\mathbf{S}^t + \sum_{j=k-l+1}^{k} \psi_j\mathbf{S}^j \\
&= \sum_{t=s+1}^{k-l} \psi_t\mathbf{S}^t + \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i\lambda_{i,j} - \psi_j\right)\mathbf{S}^j,
\end{aligned}$$

by using (8) we obtain that,

$$\sum_{j=s+1}^{k} \psi_j^* \mathbf{S}^j = \sum_{t=s+1}^{k-l} \psi_t \mathbf{S}^t + \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} - \psi_j \right) \mathbf{S}^j.$$

Finally, Ψ must be hold that $\psi_t = \psi_t^*$ for $t = s+1, \ldots, k-l$ and

$$\left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} \right) + \psi_j = \psi_j^*$$

for $j = k-l+1, \ldots, s$. If we take $\psi_i = \varepsilon > 0$ for $i = 1, 2, \ldots, s$, then

$$\varepsilon \left(\sum_{i=1}^{s} \lambda_{i,j} \right) + \psi_j = \psi_j^*.$$

Thus, we only need to choose $\varepsilon > 0$ satisfying that

$$\psi_j = \psi_j^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,j} \right) > 0 \tag{9}$$

for all $j = k-l+1, \ldots, s$. To see this we consider the set

$$\mathcal{K} = \left\{ j \in \{k-l+1, \ldots, s\} : \sum_{i=1}^{s} \lambda_{i,j} > 0 \right\},$$

and we will take

$$\delta = \min \left\{ \frac{\psi_j^*}{\sum_{i=1}^{s} \lambda_{i,j}} : j \in \mathcal{K} \right\} > 0.$$

Then for all $\varepsilon \leq \delta$ (9) holds for $j = k-l+1, \ldots, s$. We conclude the proof of claim by considering Ψ_ε^* as

$$\left(\varepsilon, \ldots, \varepsilon, \psi_{s+1}^*, \ldots, \psi_{k-l}^*, \psi_{k-l+1}^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,k-l+1} \right), \ldots, \psi_k^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,k} \right) \right)'.$$

Clearly Ψ_ε^* is a state price vector for all $\varepsilon \in (0, \delta]$ satisfying that $\lim_{\varepsilon \to 0} \Psi_\varepsilon^* = \Psi^*$.
■

We note that by the above lemma and the proof of statement 1 if $\Psi^* \geq \mathbf{0}$ and

$$\operatorname{span} \left\{ \mathbf{S}^i : i \in \mathcal{Z}(\Psi^*) \right\} = \operatorname{col} \mathbf{S}_{\Delta t},$$

then there no are arbitrage opportunities. Thus, statement 2 follows.

Finally, in order to prove statement 3, assume that $\operatorname{col}\mathbf{S}_{\Delta t,1} \nsubseteq \operatorname{col}\mathbf{S}_{\Delta t,2}$. Then we choose from the set $\operatorname{col}\mathbf{S}_{\Delta t,1}$ the columns belonging to the subspace $\operatorname{col}\mathbf{S}_{\Delta t,1} \cap \operatorname{col}\mathbf{S}_{\Delta t,2}$. Then we add these subset to the set of columns of $\mathbf{S}_{\Delta t,2}$. By this procedure we construct two new matrices, namely, $\mathbf{S}^*_{\Delta t,1}$ and $\mathbf{S}^*_{\Delta t,2}$ such that $\mathbf{S}_{\Delta t} = \left[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}\right]$ and where $\operatorname{col}\mathbf{S}^*_{\Delta t,1} \cap \operatorname{col}\mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$. Moreover, from the same argument used to prove Lemma 5, we can construct a vector $\Psi_2^* > \mathbf{0}$ satisfying that

$$\mathbf{S}^*_{\Delta t,2}\Psi_2^* = \mathbf{S}_0,$$

that is,

$$\left[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}\right]\begin{bmatrix}\mathbf{0}\\ \Psi_2^*\end{bmatrix} = \mathbf{S}_0.$$

The following lemma gives a characterization of the arbitrage opportunities in this context.

Lemma 6 *Let*

$$E = \left\{\mathbf{X} : \mathbf{X} = \left(\mathbf{S}^*_{\Delta t,1}\right)'\boldsymbol{\theta} \text{ for some } \boldsymbol{\theta} \in \operatorname{nul}\left(\mathbf{S}^*_{\Delta t,2}\right)'\right\}.$$

Then $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ if and only if there are arbitrage opportunities satisfying (Arb 1)

P roof. ote that if $\mathbf{X} = \left(\mathbf{S}^*_{\Delta t,1}\right)'\boldsymbol{\theta} \in E \cap \mathbb{K}$ and $E \cap \mathbb{K} \neq \{\mathbf{0}\}$, then

$$\mathbf{S}'_{\Delta t}\boldsymbol{\theta} = \begin{bmatrix}\left(\mathbf{S}^*_{\Delta t,1}\right)'\\ \left(\mathbf{S}^*_{\Delta t,2}\right)'\end{bmatrix}\boldsymbol{\theta} \geq \mathbf{0}.$$

Since

$$\left(\operatorname{nul}\left(\mathbf{S}^*_{\Delta t,2}\right)'\right)^{\perp} = \operatorname{col}\mathbf{S}^*_{\Delta t,2}$$

and $\mathbf{S}_0 \in \operatorname{col}\mathbf{S}^*_{\Delta t,2}$, then

$$\mathbf{S}'_0\boldsymbol{\theta} = \mathbf{0}.$$

Thus $\boldsymbol{\theta}$ is an arbitrage opportunity satisfying (Arb 1). Conversely, if $\boldsymbol{\theta}$ is an arbitrage opportunity satisfying (Arb 1) then

$$\mathbf{S}'_0\boldsymbol{\theta} = \begin{bmatrix}\mathbf{0} & (\Psi_2^*)'\end{bmatrix}\begin{bmatrix}\left(\mathbf{S}^*_{\Delta t,1}\right)'\\ \left(\mathbf{S}^*_{\Delta t,2}\right)'\end{bmatrix}\boldsymbol{\theta} = \mathbf{0}$$

and

$$\begin{bmatrix}\left(\mathbf{S}^*_{\Delta t,1}\right)'\\ \left(\mathbf{S}^*_{\Delta t,2}\right)'\end{bmatrix}\boldsymbol{\theta} \geq \mathbf{0}.$$

Thus

$$\left(\mathbf{S}^*_{\Delta t,1}\right)' \boldsymbol{\theta} \geq \mathbf{0} \text{ and } \left(\mathbf{S}^*_{\Delta t,2}\right)' \boldsymbol{\theta} = \mathbf{0},$$

and in consequence $\left(\mathbf{S}^*_{\Delta t,1}\right)' \boldsymbol{\theta} \in E \cap \mathbb{K}$ and the lemma follows. ■

Lemma 7 $E^{\perp} = \operatorname{nul} \mathbf{S}^*_{\Delta t,1}$.

P roof. et $\mathbf{Y} \in E^{\perp}$. Then $\mathbf{Y}'\mathbf{X} = 0$ for all $\mathbf{X} \in E$. Thus, $\mathbf{Y}' \left(\mathbf{S}^*_{\Delta t,1}\right)' \boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in$ nul $\left(\mathbf{S}^*_{\Delta t,2}\right)'$, that is, $\left(\mathbf{S}^*_{\Delta t,1}\mathbf{Y}\right)' \boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in$ nul $\left(\mathbf{S}^*_{\Delta t,2}\right)'$. In consequence $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} \in \left(\text{nul } \left(\mathbf{S}^*_{\Delta t,2}\right)'\right)^{\perp} = \operatorname{col} \mathbf{S}^*_{\Delta t,2}$. Since $\operatorname{col} \mathbf{S}^*_{\Delta t,1} \cap \operatorname{col} \mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$, we have that $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} = \mathbf{0}$ and $\mathbf{Y} \in \operatorname{nul} \mathbf{S}^*_{\Delta t,1}$. On the other hand if $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} = \mathbf{0}$ then $\mathbf{Y}' \left(\mathbf{S}^*_{\Delta t,1}\right)' \boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in$ nul $\left(\mathbf{S}^*_{\Delta t,2}\right)'$ and $\mathbf{Y} \in E^{\perp}$ and the lemma follows. ■

Now, we give a characterization of the existence of a state price vector.

Lemma 8 $\operatorname{nul} \mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$ *if and only if there exists a state price vector.*

P roof. f $\operatorname{nul} \mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$ then there exists $\Psi_1 > 0$ be such that $\mathbf{S}^*_{\Delta t,1}\Psi_1 = 0$. Thus,

$$\left[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}\right] \begin{bmatrix} \Psi_1 \\ \Psi_2^* \end{bmatrix} = \mathbf{S}_0,$$

and the claim follows. Conversely, let $\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 \end{bmatrix}'$ a state price vector. Then

$$\left[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}\right] \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \mathbf{S}_0,$$

that is,

$$\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{S}_0 - \mathbf{S}^*_{\Delta t,2}\Psi_2 \in \operatorname{col} \mathbf{S}^*_{\Delta t,2}.$$

By the fact that $\operatorname{col} \mathbf{S}^*_{\Delta t,1} \cap \operatorname{col} \mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$, we have that

$$\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{0}.$$

Thus $\Psi_1 \in \operatorname{nul} \mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}}$ and the lemma follows. ■

From the separate hyperplane Theorem and the Riesz's Lemma it follows the following result.

Lemma 9 *Let F be a subspace of $\mathbb{R}^n$ be such that $F^{\perp} \neq \{\mathbf{0}\}$. If $F \cap \mathbb{K} = \{\mathbf{0}\}$ then there exists $\mathbf{y}^* \in F^{\perp}$ such that $\mathbf{x}'\mathbf{y}^* > 0$ for all $x \in \mathbb{K}$ that is $F^{\perp} \cap \mathbb{K} \neq \{\mathbf{0}\}$.*

Now, assume that $\operatorname{nul}\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$. Then there exists $\Psi_1 > \mathbf{0}$ satisfying that $\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{0}$ a contradiction because $\mathbf{S}^*_{\Delta t,1} \geq \mathbf{0}$. In consequence there no are state price vectors. Thus $\operatorname{nul}\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} = \emptyset$, that is, $E^{\perp} \cap \overset{\circ}{\mathbb{K}} = \emptyset$. It is not difficult to see that if $E^{\perp} \cap \partial\mathbb{K} \neq \emptyset$, where $\partial\mathbb{K}$ denotes the topological boundary of $\mathbb{K}$, then $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ and from Lemma 6 there are arbitrage opportunities satisfying (Arb 1). In consequence statement 3 follows. On the other hand, if we assume that $E^{\perp} \cap \partial\mathbb{K} = \emptyset$, that is, $E^{\perp} \cap \mathbb{K} = \{\mathbf{0}\}$, by using Lemma 9, $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ and the first part of statement 3 follows.

To prove the second statement we need the following two lemmas. The first one follows in an easy way.

Lemma 10 *There exists a state price vector if and only there exists $\mathbf{y}^* > \mathbf{0}$ such that*

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\,\mathbf{y}^* = \mathbf{0}.$$

By using Stiemkes's Theorem (see [3]) we have the following result.

Lemma 11 *Either the system*

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\,\mathbf{y}^* = \mathbf{0} \text{ and } \mathbf{y}^* > \mathbf{0} \tag{10}$$

has a solution, or the system

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\,\boldsymbol{\theta} \geq \mathbf{0}, \;\; [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\,\boldsymbol{\theta} \neq \mathbf{0} \tag{11}$$

has a solution $\boldsymbol{\theta}$, but never both.

We remark that (11) implies that $\boldsymbol{\theta}$ is an arbitrage opportunity. An equivalent way to write (11) is

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\,\boldsymbol{\theta} \geq \mathbf{0} \text{ and } -\mathbf{e}'\,[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\,\boldsymbol{\theta} < 0, \tag{12}$$

where $\mathbf{e} = (1, 1, \ldots, 1)'$. Recall that from all said above we have the existence of arbitrage opportunities. Thus (12) has a solution. From [3, Theorem 1.1], it follows the existence of a solution $\mathbf{y}^*$ of

$$\begin{aligned} &\min \left\| [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\,\mathbf{y} + \mathbf{e}'\,[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]' \right\|^2 \\ &\text{subject to } \mathbf{y} \geq \mathbf{0}, \end{aligned} \tag{13}$$

such that

$$\boldsymbol{\theta}^* = [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\,\mathbf{y}^* + \mathbf{e}'\,[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'$$

solves (12). In consequence $\boldsymbol{\theta}^*$ is an arbitrage opportunity. Finally, to obtain a solution of (13) we can use the Dax's Algorithm given in Appendix A.

A The Dax's Algorithm

In this Appendix we will introduce a simple iterative algorithm due to Dax [3] in order to establish the existence of a point $\mathbf{y}^*$ that solves (2). It is possible to show (see [4]) that the algorithm ends in a finite number of iterations. Assume that

$$A = \left[\mathbf{A}^1 \mathbf{A}^2 \cdots \mathbf{A}^k\right],$$

where $\mathbf{A}^i = A\mathbf{e}_i$. We proceed by its i–th iteration, $i = 1, 2, \ldots$ that consists of the following two steps.

- Step 1

Let $\mathbf{y}_i = (y_1, y_2, \ldots, y_k)' \geq \mathbf{0}$ denote the current estimate of the solution beginning of the i–th iteration. Define

$$\mathbf{r}_i = A\mathbf{y}_i - \mathbf{b}.$$

If Cardinal $\mathcal{Z}(\mathbf{y}_i)^c = 0$, where $\mathcal{Z}(\mathbf{y}_i)^c = \{j : y_j > 0\}$, or $\mathbf{r}_i = \mathbf{0}$ then skip to Step 2. Otherwise, let A_i, the matrix whose columns are $\mathbf{A}^l$, with $l \in \mathcal{Z}(\mathbf{y}_i)^c$. For simplicity we assume that

$$A_i = \left[\mathbf{A}^{s+1} \cdots \mathbf{A}^k\right], \mathcal{Z}(\mathbf{y}_n) = \{1, 2, \ldots, s\} \text{ y } \mathcal{Z}(\mathbf{y}_n)^c = \{s+1, \ldots, k\}.$$

Let the vector $\mathbf{w}_i = (w_{s+1}, w_{s+2}, \ldots, w_k)'$ solve the unconstrained least squares problem

$$\min \|A_i \mathbf{w}_i - \mathbf{r}_i\|^2.$$

We note $\mathbf{0}$ solves this problem if and only if $A_i'\mathbf{r}_i = \mathbf{0}$. In this case skip to Step 2. Otherwise, define a nonzero search direction $\mathbf{u}_i = (u_1, u_2, \ldots, u_k)'$ by the following rule

$$u_l = 0 \text{ for } l = 1, \ldots, s \text{ and } u_l = w_l \text{ for } l = s+1, \ldots, k.$$

The next point is defined as

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \nu_i \mathbf{u}_i$$

where $\nu_i > 0$ is the largest number in the interval $[0, 1]$ that keeps the point $\mathbf{y}_i + \nu_i \mathbf{u}_i$ feasible. In others words, ν_i is the smallest number in the set $\{1\} \cup \{-y_i/u_i : u_i < 0\}$.

- Step 2.

In this step we have $A_i'\mathbf{r}_i = \mathbf{0}$ which means that $\mathbf{y}_i$ solves the problem

$$\min \|A\mathbf{y} - \mathbf{b}\|^2 \tag{14}$$

$$\text{subject to } y_l = 0 \text{ for } l \in \mathcal{Z}(\mathbf{y}) \tag{15}$$

$$\text{and } y_l \geq 0 \, \text{para } l \in \mathcal{Z}(\mathbf{y})^c . \tag{16}$$

In this case $\mathbf{y}_n$ is called a *dead point*. To test whether or not $\mathbf{y}_i$ is optimal, we compute an index j such that

$$\left(\mathbf{A}^j\right)' \mathbf{r}_i = \min \left\{ \left(\mathbf{A}^l\right)' \mathbf{r}_i : l \in \mathcal{Z}(\mathbf{y}) \right\}.$$

If $\left(\mathbf{A}^j\right)' \mathbf{r}_i \geq 0$ then $\mathbf{y}_i$ and $\mathbf{r}_i$ satisfy (5). From Lemma 4 we have that $\mathbf{y}_i$ solves (2) and the algorithm ends in this case. Otherwise, the next point is defined as

$$\mathbf{y}_{i+1} = \mathbf{y}_i - \left(\frac{\left(\mathbf{A}^j\right)' \mathbf{r}_i}{\left(\mathbf{A}^j\right)' \mathbf{A}^j} \right) \mathbf{e}_j .$$

Note that

$$- \left(\frac{\left(\mathbf{A}^j\right)' \mathbf{r}_n}{\left(\mathbf{A}^j\right)' \mathbf{A}^j} \right) > 0,$$

and this point solves the problem

$$\min f(\lambda) = \left\| A\left(\mathbf{y}_n + \lambda \mathbf{e}_j\right) - \mathbf{b} \right\|^2 .$$

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