EQUILIBRIUM UNIQUENESS IN OLIGOPOLY GAMES WITH STRATEGIC COMPLEMENTS*

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ABSTRACT

I show the uniqueness of equilibrium for a class of oligopoly models with strategic complements. Product differentiation models are considered in which the contraction mapping theorem cannot necessarily be applied.

JEL Classification: D43

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1 Introduction

This note analyzes symmetric supermodular games, which are of importance in the analysis of imperfectly competitive product markets in which products are strategic complements. It is often easily checked how many symmetric equilibria the game has. Supermodularity then provides a convenient way to characterize all equilibria. In case there is a single symmetric equilibrium candidate, dominance solvability follows. The latter result also holds in games in which there is local interaction. Particular models of product differentiation are discussed.

2 Oligopoly games with strategic complements: definitions

With $\Gamma = \{N, (S_i, \pi_i, i \in N)\}$ I denote a normal form oligopoly game where $N = \{1, \ldots, n\}$ is the finite set of firms, S_i the complete strategy set of firm i, partially ordered by $\geq_i$, and π_i its profit function. To simplify, each firm controls only one strategic variable, say price, which will be called $p_i \in S_i$. For my purpose it will be sufficient to consider compact intervals as strategy sets. Denote $p = (p_1, \ldots, p_n)$ which is also written as (p_i, p_{-i}).[1]

The profit function π_i has *increasing differences* if $\pi_i(p'_i, p'_{-i}) - \pi_i(p''_i, p'_{-i}) \geq \pi_i(p'_i, p''_{-i}) - \pi_i(p''_i, p''_{-i})$ for $p'_i \geq p''_i$ and $p'_{-i} \geq p''_{-i}$. This captures the notion of strategic complementarity: an increase in the variables which are not controlled by firm i increases its incremental profits. In the differentiable case this simply reads $\partial^2 \pi_i / \partial p_i \partial p_j \geq 0$, $j \neq i$.

Consider an extended notion of strategic complementarity (see Milgrom and Shannon, 1994). The function π_i satisfies the *single crossing property* in (p_i, p_{-i}) if for $p'_i \geq p''_i$ and $p'_{-i} \geq p''_{-i}$

$$\pi_i(p'_i, p''_{-i}) \geq \pi_i(p''_i, p''_{-i}) \text{ implies } \pi_i(p'_i, p'_{-i}) \geq \pi_i(p''_i, p'_{-i}).$$

This can be interpreted as a profit gain due to an increase in firm i's price to remain a gain after an increase of the competitors' prices.

A game Γ is called *ordinal (cardinal) supermodular* if all π_i satisfy the single-crossing property (have increasing differences) and, since strategies are continuous variables, if π_i is upper semi-continuous in p_i for given p_{-i} and continuous in p_{-i} for given p_i.

[1] Since strategies are one-dimensional and S_i complete, the set $S = \times_{i \in N} S_i$ is a complete lattice. In this case the definition of cardinal supermodularity of a function and increasing differences are equivalent. One can generalize the set-up by making the strategy set of firm i dependent upon the actions of the other firms; important is only that S is a complete lattice. Also, the analysis is easily extended to allow for a vector of strategic variables (see Milgrom and Shannon, 1994).

3 Symmetric oligopoly: results and application

In this section I consider symmetric, supermodular games. A game is called *symmetric* if, apart from labeling, profits are not affectd by any permutation of firms' indices, i.e. in particular $\pi_i(p) = \pi_j(p_1, \ldots, p_{i-1}, p_j, p_{i+1}, \ldots, p_{j-1}, p_i, p_{j+1}, \ldots, p_n)$ for any i, j and $p \in S$.

Proposition 1 (follows from Amir, 1996, p.145). *Any symmetric ordinal supermodular game has only symmetric pure strategy Nash equilibria.*

Proposition 1 implies that there exists a unique equilibrium and that the game is dominance solvable whenever one can show that there exists only a single symmetric equilibrium candidate.[2] This is stated as a corollary.

Corollary. *Any symmetric ordinal supermodular game with only one symmetric equilibrium candidate has a unique pure strategy Nash equilibrium and is dominance solvable.*

Note that one does not have to show that this candidate is an equilibrium because this is implied by the result above together with a version of Tarski's fixed point theorem (Tarski, 1955). Hence, in applications it is sufficient to show that the first-order conditions of payoff maximization have only one symmetric solution (see Application 2).

If there exists more than one symmetric equilibrium, the set of serially undominated strategies is a subset of a ray from the origin. The equilibrium refinement of coalition proofness selects the equilibrium which Pareto dominates the others. In two applications I apply the Corollary to models of product differentiation.

Application 1. Here I point out the difference between the uniqueness result of Proposition 1 and the contraction mapping theorem which is usually applied (see e.g. Friedman, 1977). For this consider a differentiable version of the game Γ which is log-supermodular. Each firm chooses its price p_i to maximize profits $\pi_i(p) = (p_i - c)X_i(p)$, where X_i the demand function firm i is facing. Possible specifications are for instance the symmetric CES demand model by Dixit and Stiglitz (1977) or the symmetric multinomial logit model (see e.g. Anderson, de Palma, and Thisse, 1992). Logarithmic profit functions in these models have increasing differences in logarithmic prices $x_i \equiv \log p_i$, i.e. $\partial^2 \log \pi_i(p)/\partial x_i \partial x_j \geq 0$, $j \neq i$. This implies that the game is ordinal supermodular and that the best response is nondecreasing.

[2]Milgrom and Roberts (1990, p. 1266) already noted for cardinal supermodular games that the existence of a single symmetric equilibrium implies the existence of a unique equilibrium.

Furthermore, it is easy to show that there exists only one symmetric equilibrium candidate. Uniqueness of equilibrium then follows from Proposition 1.

Alternatively, uniqueness of equilibrium is established with the help of the contraction mapping theorem. The contraction mapping property is satisfied (in the transformed game) if $\sum_{j=1}^{n} |\partial r_i(x_{-j})/\partial x_j| < 1$ where r_i is the best response which, for simplicity, is assumed to be a function. This property holds e.g. in the multinomial logit as shown by Anderson, de Palma, and Thisse (1992, pp. 221). Instead of showing the best response property directly consider the transformed game using logs of profits, strategies, and strategy sets (see Milgrom and Roberts, section 4, example 2). Uniqueness in such a transformed game holds if the following dominant diagonal property holds, $|\partial^2 \log \pi_i(p)/\partial x_i^2| > \sum_{j \neq i} |\partial^2 \log \pi_i(p)/\partial x_i \partial x_j|$. This property requires π_i to be log-concave in x_i. Also, the sum of cross derivatives has to be dominated by the second derivative of logarithmic profits in absolute value.

In general, these properties cannot be derived from the single crossing property: it imposes a restriction on the sign of the cross derivatives which implies that best responses are nondecreasing in the competitors' actions and does not imply any restrictions on the absolute value of these functions. The dominant diagonal property above is satisfied for the CES and the logit. Together with supermodularity it implies that there exists a unique equilibrium in the game Γ. However, as implied by Proposition 1, neither this property nor the more general contraction mapping property is needed to show uniqueness in symmetric supermodular games with a unique symmetric equilibrium candidate.

Application 2. I modify the previous model to consider profit functions which possibly do not satisfy any dominant diagonal property and which are even not necessarily quasiconcave (so that Kakutani's fixed point theorem cannot be applied to show existence of equilibrium). Assume that firms have profit functions

$$\pi_i = (p_i - c)\left(X_i^m(p_i) + X_i^c(p)\right)$$

where (i) $(p_i - c)X_i^c(p)$ is ordinal supermodular and (ii) $\lim_{p_i \to \infty}(p_i - c)X_i^m(p_i) = 0$. The first property relates to application 1: if demand $X_i^c(p)$ is derived from the CES or the logit model, this property holds. The second property simply states that the part of profits which is independent of the prices of the competitor vanishes for price turning to infinity and is sufficient for compact strategy spaces. In addition, the assumption of upper semi-continuity of profit functions (see above) is made; it is implied by the continuity of demand functions. Since I do not make any shape assumption on X_i^m payoffs do not necessarily satisfy the dominant diagonal property nor are they necessarily quasi-concave in own price.

The model can be interpreted as follows. Each firm has a subset of consumers who are uninformed of the existence of competing products and hence it can act as a monopolist for this group of consumers. However, firms cannot price-discriminate between informed and uninformed consumers. The informed consumers choose between all the products in the market and this gives rise to demand functions X_i^c which for instance are derived from the CES or logit model (see above).

Let me impose symmetry (possibly after a positive monotone rescaling of prices and marginal costs and, independently, profits). Symmetry here means that $X_i^m(p_i) = X_j^m(p_i)$ and $X_i^c(p) = X_j^c(p_1, \ldots, p_{i-1}, p_j, p_{i+1}, \ldots, p_{j-1}, p_i, p_{j+1}, \ldots, p_n)$. It follows from Proposition 1 that there do not exist asymmetric equilibria. It depends on the shape of the demand function whether there exists more than one symmetric equilibrium. For specific functional forms of X^m and X^c it is easy to check (at least numerically) whether there exists exactly one symmetric solution to the first-order conditions of profit maximization: the problem reduces to finding the zeros of a function in one variable.

4 Locally symmetric oligopoly: result and application

The first result applies only to symmetric games. In applications with local interaction (such as in many address models of product differentiation or models with local spillovers) the actions of the other firms do not symmetrically enter the profit function. A game is called *locally symmetric* if for any pair of firms' indices (a, b) there exists a permutation of firms' indices such that for all prices the profits of any firm i become the profits of some firm j after permutation and in particular firm a's profits become firm b's profits. In the previous section this was supposed to hold for any permutation. Local symmetry allows firms not to interact directly but says that the firms' local environments are the same when abstracting from their identity.

Proposition 2. *Any locally symmetric, ordinal supermodular game with a single symmetric equilibrium candidate has a unique pure strategy Nash equilibrium and is dominance solvable.*

Proof. Suppose that there exists an asymmetric equilibrium. Denote $p^- \in S_i$ the minimal price by any firm in the set of asymmetric equilibrium price vectors P and p^+ the maximal price. Formally, p^- is chosen such that $\nexists(\widetilde{p}, \ldots, \widetilde{p}) > (p^-, \ldots, p^-) : (\widetilde{p}, \ldots, \widetilde{p}) \leq p' \, \forall p' \in P$, analogously for p^+. Clearly, $p^- < p^+$. Suppose that p^- is at position i in the asymmetric equilibrium p^*. By definition, there exists for each position j a permutations of strategies such that p^- is at position j in this vector and $\pi_i(p) =$

$\pi_j(p'_1, \ldots, p'_n)$. Any such permutation is an equilibrium. Analogously, for p^+. Since there exists a minimal and a maximal element in the equilibrium set (see Theorem 12 in Milgrom and Shannon, 1994), there exist symmetric equilibrium vectors $p^1 \leq (p^-, \ldots, p^-)$ and $p^2 \geq (p^+, \ldots, p^+)$. Since, by assumption, there exists only one symmetric equilibrium candidate these two vectors have to coincide and P is empty. This equilibrium candidate has to be an equilibrium because there does not exist any other equilibrium and because, by Tarski's fixed point theorem (for correspondences), the existence of an equilibrium is guaranteed. Furthermore, the game is dominance solvable because all serially undominated strategies lie between the minimal and the maximal elements of this set, and these elements coincide. ■

Application 3. Consider the following model of localized competition. To the circle model of horizontal product differentiation in the specification of Economides (1989) I add consumers which are not informed about the existence of more than one good. Firms set prices to maximize profits and profit functions are written as $\pi_i = (p_i - c)(X^m(p_i) + X_i^c(p))$. Firms have monopoly power over consumers which are uninformed about competitors. Any consumer is equally likely to know any of the available products so that the resulting demand functions X^m are symmetric by assumption and are assumed to satisfy the same property as in example 2.

X_i^c is determined as follows. Consumers are uniformly distributed on the a circle with circumference 1 and incur transportation costs which are quadratic in distance between consumer and firm location. The population of informed consumers is of mass 1 and all informed consumers are assumed to buy one unit in the market. The n firms are located equidistantly on the circle. Consequently, demand functions are $X_i^c = (p_{i-1} - 2p_i + p_{i+1})n/2 + 1/n$ when all firms are active (compare Economides, 1989). The model is an example of a locally symmetric game. Profit functions have increasing differences. It follows from Proposition 2 that there exists a unique price equilibrium whenever there exists only one symmetric solution to the first-order conditions of profit maximization, which is easily checked.

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