STOCHASTIC OLG MODELS, MARKET STRUCTURE AND OPTIMALITY*

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ABSTRACT

For a general class of pure exchange OLG economies under uncertainty, we provide a complete characterization of the efficiency properties of competitive equilibria when markets are only sequentially complete and the criterion of efficiency is conditional Pareto optimality. We also consider a particular case in which markets fail to be even sequentially complete and provide a characterization when the criterion of efficiency is weakened to ex post Pareto optimality.

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1. INTRODUCTION

We consider a general class of pure exchange, two-period overlapping generations (OLG) economies under uncertainty and characterize the optimality properties of competitive equilibria under different market structures and with respect to alternative criteria of optimality.

Following Cass' [6] characterization of efficient growth paths in terms of the properties of their supporting competitive prices (later generalized in Benveniste [3]), Balasko and Shell [2] and Okuno and Zilcha [20] provided similar results for the competitive equilibria of standard, deterministic OLG economies.[1] These authors considered pure exchange economies under certainty with a fixed number of goods available in every period, a fixed number of two-period lived agents born in every period, and a complete set of markets at the beginning of history where all agents are free to trade.[2] They showed that (i) if aggregate endowments are uniformly bounded above and preferences are locally nonsatiated, strictly convex and the curvature of the indifference sets is bounded below uniformly across agents, then divergence of the series obtained in the "Cass criterion" (where the terms appearing in the series are the reciprocals of the norms of the price vectors) implies that the associated competitive allocation is Pareto optimal; (ii) if preferences are strictly monotonic, the curvature of the indifference sets is bounded above uniformly across agents and the equilibrium allocation is interior, then convergence of the same series implies that the competitive allocation is not Pareto optimal. The argument exploits a pair of *curvature conditions* for each agent; these are inequalities which relate changes in the composition of expenditure across the two periods of the agent's life, relative to the composition at the equilibrium level of consumption, to changes in the agent's utility level. The interaction between the curvature conditions and the feasibility conditions lead to the characterization result. This result has been subsequently extended, by Burke [5] and Geanakoplos and Polemarchakis [12] among others, in particular to economies where the number of agents born, and the number of commodities available, vary over time.

Our focus is on OLG economies with uncertainty. No special assumptions are made on the structure of the uncertainty; it is simply assumed to be represented by a date-event tree where each node has a finite number of successors. As a consequence, the number of (contingent) commodities at each date increases over time, and will typically tend to infinity. If we were to maintain the assumption that agents are free to trade on a complete set of markets at the initial period of the economy, so that agents evaluate their consumption plans in terms of their ex ante utility (i.e. before any realization of the uncertainty), then the only complication that we would face is the fact that the dimension of the commodity space increases over time (and, typically, tends to infinity). In this set-up, agents' preferences are strictly monotone in all commodities available at a given date and the characterization of those complete market competitive equilibrium allocations which are ex ante Pareto optimal is easily obtained from the results proved in [5] and [12] for general deterministic OLG economies; it is given by the divergence of a series whose terms are the reciprocals of the norms of the vector of all prices at a date.

[1] See also Bose [4].

[2] As is well known, the equilibrium allocations with this market structure are the same as the equilibria with a sequential structure of markets—a system of spot markets and one asset, possibly in nonzero net supply (e.g. fiat money), allowing transfers of income across periods.

Under uncertainty we cannot find a sequential structure of markets where agents trade only after they are born and which supports the same equilibrium allocations as when agents have unrestricted access to a complete set of markets at the initial date; this is in contrast with the case of certainty. When trading is sequential, agents will in fact be unable to insure against the realization of the uncertainty at the time of their birth so that markets are necessarily incomplete. The "most complete" market structure which is compatible with the demographic structure of the economy and the sequential nature of trading is called *sequentially complete markets* where, in each period, and for each realization of the uncertainty, spot markets exist for every commodity and there is also a complete set of one-period contingent claims whose payoff is contingent on all possible events next period, conditional on the current event. By trading in these markets in the two periods of his life, each agent can fully insure against the realization of the uncertainty, but *only* conditionally on the event at his birth.

When markets are only sequentially complete, typically, competitive equilibria will not be ex ante Pareto optimal. A more appropriate notion for evaluating the efficiency properties of allocations when trades take place sequentially is the notion of *conditional Pareto optimality* (CPO), first proposed by Muench [19].[3] According to this criterion, agents' welfare is evaluated by conditioning their utility on the event at the date of their birth. Agents are thus distinguished not only according to their type and their date of birth but also according to the event at that date, and an agent's preferences are defined over a subset of all the commodities available in the two periods of that agent's lifetime since the commodities which enter the utility function of an agent born at date t are the date t and $t+1$ commodities whose availability is contingent on the occurrence of the particular event at the agent's birth.

The problem of evaluating a possible reallocation of resources in terms of the CPO criterion, where the reallocation is relative to a competitive equilibrium obtained with sequentially complete markets, cannot be reduced to a one-dimensional problem in which only the change in the total expenditure at a date matters; this is in contrast to the case of the deterministic OLG model (or, more generally, to the complete markets case). With uncertainty, a reallocation of resources induces sequences of transfers (of income or "value") along distinct paths in the date-event tree with the feasibility condition satisfied at each date-event. When markets are only sequentially complete, agents of the same type born at the same date but in different events are considered to be distinct so that nonzero levels of transfers conditional on more than one event at a given date lead to an extremely rich set of redistributional possibilities since they affect different agents. Moreover, as each agent cares about only some of the commodities available at a date, the curvature conditions that one obtains are inequalities which consider changes in expenditure in the event at which the agent is born and the changes in expenditure in the set of immediate successor events. Whether or not a reallocation is improving depends on the interaction, across a collection of paths, among the transfers that the reallocation generates; in particular, the changes in expenditure at different events at the same date, and not just the value of the change at a given date, need to be considered. Hence the issue of characterizing those equilibrium allocations, obtained with sequentially complete

[3]See Dutta and Polemarchakis [11] for a discussion of the various criteria for optimality in OLG economies under uncertainty.

markets, which are efficient relative to the CPO criterion is far more complex than the analogous problem in the deterministic case (and its generalizations as presented in [5] and [12]) and those earlier results cannot be easily extended to the economies considered in this paper.

One of the main results of this paper is the derivation of a *necessary and sufficient condition for the conditional Pareto optimality* of competitive equilibria with sequentially complete markets under general assumptions on the economy; this is established in Theorems 1 and 2. The condition we obtain requires us to consider, for each given collection of paths (or more precisely, for each "sub-tree"), the behaviour of all the price series associated with these paths. The terms appearing in each price series are the reciprocals of the norms of the vectors of prices at a node, multiplied by the values of a weight function defined for each node in the sub-tree. As we argued above, the consideration of possible redistributions of resources requires us to examine the interaction among transfers defined across a collection of paths; hence we need to evaluate the relative "importance" of various nodes at each date. This is indeed the role of the weight function whose values can be interpreted as a conditional probability of reaching a node from the root of the sub-tree.

The form of the condition that we obtain is rather different from the one derived for the deterministic OLG model (or, equivalently, for the case of complete markets) though it reduces to that condition in the special case in which there is no uncertainty. At a more abstract level, our results indicate how a characterization of efficiency can be obtained in the more general case in which agents care for only a subset of the set of commodities available at each point in time.[4]

The necessary and sufficient condition presented in Theorems 1 and 2 is not easy to verify and to work with. Hence, in Theorem 3 and Corollary 1 we provide alternative sufficient conditions for CPO which are stronger but easier to verify. These conditions require the consideration, again for each given collection of paths, of a single series whose terms are the reciprocal of the sum of the norms of the vectors of prices across nodes at a given date.

When we consider a restricted class of economies, and, in particular, a more specific uncertainty structure, our characterization result takes a simpler form. For the special case of one good OLG economies with stationary Markov uncertainty, we show (in Theorem 4) that the sufficient condition derived in our Corollary 1 is also necessary for stationary equilibria with sequentially complete markets to be CPO, and reduces to a very simple condition on the value of the dominant root of the matrix of contingent claim prices.

When markets are not sequentially complete, i.e. when agents are unable to insure against all sources of uncertainty affecting them after their birth, competitive equilibria are typically not even CPO. A general investigation of the efficiency properties of competitive equilibria in this case goes beyond the scope of this paper. Here we will only examine the case of a simple asset structure with a single one-period asset, a bond with a constant real return, and examine its efficiency properties with respect to a notion of optimality which is weaker than CPO, *ex post Pareto optimality* (EPPO). According to this criterion,

[4]Deterministic OLG economies in which agents live for more than two periods provide another example where the composition of transfers at a date across agents born at different dates needs to be considered. The reformulation of such an economy as an economy with two period lifetimes results in agents caring about only a subset of commodities available in each period in the transformed economy.

agents' welfare is evaluated by conditioning not only on the state when young but also on the state realized in the second period of their lives. Agents are then distinguished by the realization of the uncertainty over their entire lifetime. In this situation efficiency concerns only the allocation of commodities at each given date-event, and over time, but not across states (i.e. not the allocation of risk).

In Theorem 5 we provide necessary and sufficient conditions for the ex post Pareto optimality of competitive equilibria when a bond is the only asset. The conditions we obtain are again on collections of series of weighted reciprocals of the norms of price vectors, as in Theorems 1 and 2. A crucial difference, however, is the fact that the weighting factor is now univocally determined by the average of the agents' Lagrange multipliers (thus, to evaluate the efficiency of competitive equilibrium allocations we need to know not only the supporting prices but also the Lagrange multipliers of all agents).

There has been some earlier work on the efficiency properties, with respect to the CPO criterion, of one commodity OLG economies with stationary uncertainty (Peled [21], Manuelli [17], Zilcha [24], etc.). We discuss in detail the relation of our work to this literature and show that all these earlier results can be obtained as special cases of our results.

The rest of the paper is structured as follows. Section 2 presents the model and notation. In Section 3 we present a characterization of equilibrium allocations that are CPO when markets are sequentially complete, while in Section 4 we focus our attention on stationary equilibria. In Section 5 we introduce a (particular) incomplete asset structure and we characterize the efficiency properties of equilibria obtained with that asset structure with respect to the weaker EPPO criterion of optimality. All proofs are collected in Section 6.

2. THE MODEL

We consider a general, two period, pure exchange overlapping generations (OLG) economy under uncertainty. The economy evolves in discrete time with uncertainty described by an abstract date-event tree as in Chapter 7 of Debreu [9] or Radner [22].

In this section (and in most of the paper) we will consider the case in which markets are sequentially complete so that agents are able to insure against all risks that arise after they are born but are unable to insure against the risk of being born in a particular event. This situation is usually described by assuming that at every date-event, there exists a complete set of spot markets and one period forward markets, contingent on all the possible realizations of uncertainty at the next date given the current date-event. Here we consider the following equivalent structure of markets: at the initial date, there exists a complete set of Arrow-Debreu contingent commodity markets, but each agent is allowed to trade only on the markets for delivery contingent on the event at his birth and on the events that can occur in the next period given the event at birth. It is easy to show that the agent's opportunities for trade are the same in the two market structures.

We turn to a formal description of the model and the notation used.

Time is discrete and dates are denoted by $t = 1, 2, 3, \cdots$.

Uncertainty is described by a *date-event tree* which is defined by (i) σ_0, the *root* of the tree, (ii) finite sets Σ_t, for $t \geq 1$, and (iii) a set of functions $f^0 : \Sigma_1 \to \sigma_0$ and $f^t : \Sigma_{t+1} \to \Sigma_t$, for $t \geq 1$, where each function is surjective. Define $\Sigma := \cup_{t\geq 1}\Sigma_t$ and $\Gamma := \Sigma \cup \{\sigma_0\}$; we will abuse notation and use Γ to denote a generic date-event tree. Elements of Γ are called *nodes* (to be thought of as the "date-events" or simply "events"), and a generic node is denoted by σ. Σ_t is the set of all nodes at date t; thus if $\sigma \in \Sigma_t$ we say that σ is a node at date t (sometimes the notation σ_t will be used to stress the fact that we are referring to a node at date t), and the function $f^{t-1}(\sigma)$ identifies the node at $t-1$ which is the unique predecessor of σ.[5]

Given a node $\sigma \in \Sigma$, $t(\sigma)$ denotes the value of t at which $\sigma \in \Sigma_t$; the set of immediate successor nodes of σ will be denoted by σ^+ (thus, if $\sigma \in \Sigma_t$, σ^+ is the set of nodes, at date $t+1$, such that σ is their immediate predecessor). Similarly, the unique predecessor of a node σ, given by $f^{t(\sigma)-1}(\sigma)$, will also be referred to as σ_{-1}. The total number of nodes at t, $\#\Sigma_t$, will be denoted by S_t; $S(\sigma) := \#\sigma^+$ is then the number of immediate successors of σ (or the branching number of σ).[6]

A *path* is defined by an infinite sequence of nodes $\{\sigma_t\}_{t\geq 1}$ such that, for all $t \geq 1$, $\sigma_t = f^t(\sigma_{t+1})$, and will be denoted by σ^∞.

L commodities are available for consumption at each node $\sigma \in \Sigma$.

At each node $\sigma \in \Sigma$ a generation of H agents is born. Each agent lives at two dates, $t(\sigma)$ and $t(\sigma)+1$. The fact that agents are unable to trade in markets offering insurance against the event at their birth is captured by requiring that the consumption plan of

[5]This definition of a date-event tree leads directly to another standard definition in which the ordered pairs $(f^t(\sigma_{t+1}), \sigma_{t+1})$ are called *arcs* and induce a partial order on Γ with the properties that (i) each node traces its origin, by the partial order of precedence, to σ_0, and (ii) each node, except σ_0, has exactly one predecessor which is also an element of Γ.

[6]Since all the functions $f^t(\cdot)$ are surjective, $S(\sigma) \geq 1$ for all nodes. We treat this restriction (i.e. the economy "never ends") as part of the definition of a tree. Given that our interest is in optimality, this is without loss of generality.

an agent specify the level of consumption in the event at birth and in its immediate successor nodes, and that his preferences be defined over such plans. So, an agent can be distinguished according to his type h *and* the node identifying the event at his birth; consequently, a member of generation σ of type h is denoted by (σ, h).

In addition, there is a set of H one-period lived agents who enter the economy at each node $\sigma \in \Sigma_1$ at date 1; they constitute the generation of the "initial old", and are indexed by (σ, h, o), where $\sigma \in \Sigma_1$.

Each agent (σ, h) is described by a consumption set, $X_{\sigma,h}$, an endowment vector, $\omega(\sigma, h)$, and a utility function, $u_{\sigma,h}(\cdot)$ (for the initial old, $X_{\sigma,h,o}$, $\omega(\sigma; h, o)$, and $u_{\sigma,h,o}(\cdot)$). A consumption plan for agent (σ, h) will be denoted by $x(\sigma, h)$ ($x(\sigma; h, o)$ for the initial old).[7]

For all $(\sigma, h) \in \Sigma \times H$, the elements of the endowment vector $\omega(\sigma, h)$ of agent (σ, h), describing the endowment at birth and in all successor nodes, will be written as follows: $(\omega(\sigma; \sigma, h), (\omega(\sigma'; \sigma, h))_{\sigma' \in \hat{\sigma+}})$. Similarly, the elements of the consumption vector $x(\sigma, h)$ are $(x(\sigma; \sigma, h), (x(\sigma'; \sigma, h))_{\sigma' \in \hat{\sigma+}})$.

Denoting by $\omega(\sigma)$ the total endowment at node σ, we have then:

$\omega(\sigma) := \sum_{h \in H} \omega(\sigma; \sigma, h) + \sum_{h \in H} \omega(\sigma; h, o)$ for $\sigma \in \Sigma_1$,

$\omega(\sigma) := \sum_{h \in H} \omega(\sigma; \sigma, h) + \sum_{h \in H} \omega(\sigma; \sigma_{-1}, h)$ for $\sigma \in \cup_{t \geq 2} \Sigma_t$.

Agents' preferences and endowments are assumed to satisfy the following standard conditions:

ASSUMPTION 1:

(i) $1 \leq L < \infty$, $1 \leq H < \infty$, and $1 \leq S(\sigma) \leq \bar{S} < \infty$ for all $\sigma \in \Gamma$.

(iia) For all $(\sigma, h) \in \Sigma_1 \times H$, $X_{\sigma,h,o} = R^L_+$, $\omega(\sigma; h, o) \in R^L_+$,
$u_{\sigma,h,o} : X_{\sigma,h,o} \to R$ is C^2, strictly monotone, and differentiably strictly quasi-concave.

(iib) For all $(\sigma, h) \in \Sigma \times H$, $X_{\sigma,h} = R^{L(1+S(\sigma))}_+$, $\omega(\sigma, h) \in R^{L(1+S(\sigma))}_+$,
$u_{\sigma,h} : X_{\sigma,h} \to R$ is C^2, strictly monotone, and differentiably strictly quasi-concave.

(iii) For all $\sigma \in \Sigma$, $\omega(\sigma) \in R^L_{++}$.

DEFINITION 1: A *feasible allocation* x is given by an array $((x(\sigma; h, o))_{(\sigma,h) \in \Sigma_1 \times H}, (x(\sigma, h))_{(\sigma,h) \in \Sigma \times H})$ such that $x(\sigma; h, o) \in X_{\sigma,h,o}$ for all $(\sigma, h) \in \Sigma_1 \times H$, $x(\sigma, h) \in X_{\sigma,h}$ for all $(\sigma, h) \in \Sigma \times H$, and

$\sum_{h \in H} x(\sigma; \sigma, h) + \sum_{h \in H} x(\sigma; h, o) \leq \omega(\sigma)$ for all $\sigma \in \Sigma_1$,

$\sum_{h \in H} x(\sigma; \sigma, h) + \sum_{h \in H} x(\sigma; \sigma_{-1}, h) \leq \omega(\sigma)$ for all $\sigma \in \cup_{t \geq 2} \Sigma_t$.

[7] The fact that the number of commodities available at each node and the number of agents born at each node are date-event independent is without loss of generality; our results can be easily extended to more general specifications so long as the total number of commodities, and agents born, before a given date is finite, and this for all dates. Our specification prevents the notation from getting more cluttered.

Applying the notion of Pareto efficiency to the economy described above, where agents are distinguished by the event at their birth, yields the criterion of conditional Pareto Optimality, first proposed by Muench [19]:

DEFINITION 2 (CPO): Let x be a feasible allocation. x is *conditionally Pareto optimal (CPO)* if there does not exist another feasible allocation $\widehat{x}$ such that
(i) for all $(\sigma, h) \in \Sigma_1 \times H$, $u_{\sigma,h,o}(\widehat{x}(\sigma; h, o)) \geq u_{\sigma,h,o}(x(\sigma; h, o))$,
for all $(\sigma, h) \in \Sigma \times H$, $u_{\sigma,h}(\widehat{x}(\sigma, h)) \geq u_{\sigma,h}(x(\sigma, h))$,
(ii) either for some $(\sigma', h') \in \Sigma_1 \times H$, $u_{\sigma' h',o}(\widehat{x}(\sigma'; h', o)) > u_{\sigma',h',o}(x(\sigma'; h', o))$,
or for some $(\sigma', h') \in \Sigma \times H$, $u_{\sigma',h'}(\widehat{x}(\sigma', h')) > u_{\sigma',h'}(x(\sigma', h'))$.

In addition to allocating commodities optimally at each node, a CPO allocation requires that the risk carried in the second period of the agents' lives be allocated optimally.

We will define now competitive equilibria for the economy we described. A complete set of contingent commodity markets is available at the initial date. Let $p(\sigma_t)$ be the vector of prices, quoted at the initial date, for contingent delivery of the L commodities at the node σ_t at date t. A price system is then defined by a non-negative sequence $\{p_t\}_{t\geq 1}$ where $p_t = ((p(\sigma))_{\sigma\in\Sigma_t}) \in R_+^{S_t L}$. Prices are normalized as follows: $p_l(\sigma) = 1$ where $l = 1$ and $\sigma \in \Sigma_1$.

DEFINITION 3 (S-CE):[8] (x^*, p^*) is a *competitive equilibrium with sequentially complete markets (S-CE)* if x^* is a feasible allocation, and,
(i) for all $(\sigma, h) \in \Sigma_1 \times H$,
$p^*(\sigma) \cdot x^*(\sigma; h, o) \leq p^*(\sigma) \cdot \omega(\sigma; h, o)$,
$u_{\sigma,h,o}(x(\sigma; h, o)) > u_{\sigma,h,o}(x^*(\sigma; h, o)) \quad \Rightarrow \quad p^*(\sigma) \cdot x(\sigma; h, o) > p^*(\sigma) \cdot \omega(\sigma; h, o)$,
(ii) for all $(\sigma, h) \in \Sigma \times H$,
$(p^*(\sigma), (p^*(\sigma'))_{\sigma'\in\sigma^+}) \cdot x^*(\sigma, h) \leq (p^*(\sigma), (p^*(\sigma'))_{\sigma'\in\sigma^+}) \cdot \omega(\sigma, h)$,
$u_{\sigma,h}(x(\sigma, h)) > u_{\sigma,h}(x^*(\sigma, h)) \quad \Rightarrow$
$(p^*(\sigma), (p^*(\sigma'))_{\sigma'\in\sigma^+}) \cdot x(\sigma, h) > (p^*(\sigma), (p^*(\sigma'))_{\sigma\in\sigma^+}) \cdot \omega(\sigma, h)$.

REMARK 1: The date-event tree structure, together with Assumption 1, implies that the set of nodes is countable; hence, so is the set of agents. The economy defined satisfies also the other assumptions of Corollary 1 in Geanakoplos and Polemarchakis [12] so that existence of S-CE is guaranteed. Furthermore, our assumption of strict monotonicity of preferences implies that at an S-CE allocation or at a CPO allocation, the feasibility condition holds as an equality at all nodes.

[8] It is easy to show that if, at each node $\sigma \in \Sigma$, there are L spot commodity markets and $LS(\sigma)$ markets for delivery contingent on every possible realization $\sigma' \in \sigma^+$ (i.e. at every successor node), then the specification of the agents' budget constraints is the same as in Definition 3; as a consequence, the set of equilibrium allocations is also the same.

3. SEQUENTIALLY COMPLETE MARKETS AND CPO

In this section we examine the efficiency properties of competitive equilibria with sequentially complete markets with respect to the CPO criterion. We derive a necessary and sufficient condition on equilibrium prices under which the equilibrium allocation is CPO.

As a preliminary step to the results we derive an implication of the curvature conditions imposed on preferences by Assumption 1.

DEFINITION 4:[9] Let $x(\sigma,h)$ solve the utility maximization problem of agent $(\sigma,h) \in \Sigma \times H$ at prices $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ such that $\| p(\sigma) \| > 0$, $\| p(\sigma') \| > 0$ for all $\sigma' \in \sigma\hat{+}$. For $k > 0$, let

$$\begin{aligned}\underline{\mathrm{P}}_{\sigma,h}(k) := \{\rho \in R \mid & \text{ for all } \widehat{x}(\sigma,h) \in X_{\sigma,h} \text{ satisfying (i) } \| \widehat{x}(\sigma,h) - x(\sigma,h) \| \leq k \text{ and} \\ & \text{(ii) } u_{\sigma,h}(\widehat{x}(\sigma,h)) \geq u_{\sigma,h}(x(\sigma,h)) \quad \text{we have} \\ & \textstyle\sum_{\sigma'\in\sigma\hat{+}} \delta_2(\sigma',\sigma,h) \geq -\delta_1(\sigma,h) + \rho\frac{(\delta_1(\sigma,h))\hat{2}}{\|p(\sigma)\|}\}\end{aligned}$$

where $\delta_1(\sigma,h) := p(\sigma)\cdot[\widehat{x}(\sigma;\sigma,h) - x(\sigma;\sigma,h)]$, $\delta_2(\sigma',\sigma,h) := p(\sigma')\cdot[\widehat{x}(\sigma';\sigma,h) - x(\sigma';\sigma,h)]$, for $\sigma' \in \sigma^+$.

(a) Given $k > 0$, $\underline{\rho}_{\sigma,h}(k)$ is the *lower curvature coefficient* of agent (σ,h) at $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ where $\underline{\rho}_{\sigma,h}(k) := \sup\{\rho \in \underline{\mathrm{P}}_{\sigma,h}(k)\}$ if $\underline{\mathrm{P}}_{\sigma,h}(k) \neq \emptyset$ and $\underline{\rho}_{\sigma,h}(k) := -\infty$ if $\underline{\mathrm{P}}_{\sigma,h}(k) = \emptyset$.[10]

(b) The preferences of agent (σ,h) satisfy the *non-vanishing Gaussian curvature condition at* $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ if there exists $\underline{k}_{\sigma,h} > 0$ for which $\underline{\rho}_{\sigma,h}(\underline{k}_{\sigma,h}) > 0$.

DEFINITION 5: Let $x(\sigma,h)$ solve the utility maximization problem of agent $(\sigma,h) \in \Sigma \times H$ at prices $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ such that $\| p(\sigma) \| > 0$, $\| p(\sigma') \| > 0$ for all $\sigma' \in \sigma\hat{+}$. For $k > 0$, let

$$\begin{aligned}\bar{\mathrm{P}}_{\sigma,h}(k) := \{\rho \in R \mid & \text{ for all } \widehat{x}(\sigma,h) \in X_{\sigma,h} \text{ satisfying (i) } \| \widehat{x}(\sigma,h) - x(\sigma,h) \| \leq k \\ & \text{(ii) } p(\sigma)\cdot[\widehat{x}(\sigma;\sigma,h) - x(\sigma;\sigma,h)] < 0 \text{ and} \\ & \text{(iii) } \textstyle\sum_{\sigma'\in\sigma^+} \delta_2(\sigma',\sigma,h) \geq -\delta_1(\sigma,h) + \rho\frac{(\delta_1(\sigma,h))\hat{2}}{\|p(\sigma)\|} \quad \text{we have} \\ & u_{\sigma,h}(\widehat{x}(\sigma,h)) \geq u_{\sigma,h}(x(\sigma,h))\}\end{aligned}$$

where $\delta_1(\sigma,h) := p(\sigma)\cdot[\widehat{x}(\sigma;\sigma,h) - x(\sigma;\sigma,h)]$, $\delta_2(\sigma',\sigma,h) := p(\sigma')\cdot[\widehat{x}(\sigma';\sigma,h) - x(\sigma';\sigma,h)]$, for $\sigma' \in \sigma^+$.

(a) Given $k > 0$, $\bar{\rho}_{\sigma,h}(k)$ is the *upper curvature coefficient* of agent (σ,h) at $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ where $\bar{\rho}_{\sigma,h}(k) := \inf\{\rho \in \bar{\mathrm{P}}_{\sigma,h}(k)\}$ if $\bar{\mathrm{P}}_{\sigma,h}(k) \neq \emptyset$ and $\bar{\rho}_{\sigma,h}(k) := \infty$ if $\bar{\mathrm{P}}_{\sigma,h}(k) = \emptyset$.

(b) The preferences of agent (σ,h) satisfy the *bounded Gaussian curvature condition at* $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$ if there exists $\bar{k}_{\sigma,h} > 0$ for which $\infty > \bar{\rho}_{\sigma,h}(\bar{k}_{\sigma,h}) \geq 0$.

The fact that in Definitions 4 and 5, $x(\sigma,h)$ is assumed to solve the utility maximization problem of agent $(\sigma,h) \in \Sigma \times H$ at prices $(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$, together with local nonsatiation of preferences, implies that the agent's preferred set is contained in the half-space defined by the budget constraint (i.e. $x(\sigma,h)$ minimizes expenditure) and

[9] Here, as well as in the rest of the paper, $\|\cdot\|$ will denote the Euclidean norm unless otherwise noted.

[10] The coefficient $\underline{\rho}_{\sigma,h}$ depends on $x(\sigma,h)$ but this dependence has been suppressed in order to simplify the notation (hence the qualifier at "$(p(\sigma),(p(\sigma'))_{\sigma'\in\sigma^+})$").

hence $\underline{\rho}_{\sigma,h}(k) \geq 0$ and $\bar{\rho}_{\sigma,h}(k) \geq 0$ for all $k > 0$; (b) in Definition 4 requires that the inequality be strict, while (b) in Definition 5 imposes an upper bound. Lemma 1 shows that Assumption 1 guarantees both of these properties for interior maxima.

LEMMA 1: *Let (x^*, p^*) be an S-CE. If $x^*(\sigma, h) \in R_{++}^{L(1+S(\sigma))}$ then, under Assumption 1, the preferences of the agent $(\sigma, h) \in \Sigma \times H$ satisfy both i) the non-vanishing Gaussian curvature condition, and ii) the bounded Gaussian curvature condition, at prices $(p^*(\sigma), (p^*(\sigma'))_{\sigma' \in \sigma^+})$. In fact, for all $k > 0$ we have $\underline{\rho}_{\sigma,h}(k) > 0$ and for all $k > 0$ we have $\infty > \bar{\rho}_{\sigma,h}(k) \geq 0$.*[11]

Part i) of Lemma 1 says that, if the alternative consumption plan $\widehat{x}(\sigma, h)$ improves with respect to the optimal choice $x^*(\sigma, h)$ of agent (σ, h) at prices p^*, then the difference in value of the alternative plan must obey a quadratic relation; it requires that preferences be locally nonsatiated and that they satisfy a differentiable form of strict quasiconcavity (both of which are imposed in Assumption 1). Part ii) of Lemma 1 provides us with a condition on the income transfers along the lifetime of an agent which ensures that we have an improvement in the agent's welfare; this result requires a minimum degree of substitutability among goods in the agent's preferences (ensured by the strict monotonicity and the smoothness of the utility functions imposed in Assumption 1).

Notice that the terms appearing in the two inequalities which define $\underline{\mathrm{P}}_{\sigma,h}(k)$ and $\bar{\mathrm{P}}_{\sigma,h}(k)$ in Definitions 4 and 5 are the norm of the price vector and the change in the expenditure in the two periods of an agent's life, where both are indexed by the events in which they occur. This, as pointed out in the introduction, constitutes a crucial difference relative to the case of certainty (or of complete markets) as it implies that transfers to and from agents must be identified not only by the date but also by the event in which they take place since, for the agent, transfers received at an arbitrary pair of events at the same date are not substitutable unless both events succeed the event in which he is born.

In order to state the main results, we need some additional notation.

Given $\hat{\sigma} \in \Gamma$, we define a *sub-tree* (of the tree Γ) *with root* $\hat{\sigma} \in \Gamma$, denoted by $\Gamma_{\hat{\sigma}}$, as a collection of nodes such that (i) $\Gamma_{\hat{\sigma}} \subset \Gamma$, (ii) $\Gamma_{\hat{\sigma}}$ is itself a tree with $\hat{\sigma}$ as its root. Hence the tree itself, Γ, is a sub-tree; so is any path, σ^∞.

Given a sub-tree $\Gamma_{\hat{\sigma}}$, a *weight function* is a function $\lambda_{\Gamma_{\hat{\sigma}}} : \Gamma_{\hat{\sigma}} \to [0, 1]$ such that $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} \lambda_{\Gamma_{\hat{\sigma}}}(\sigma') = 1$ for all $\sigma \in \Gamma_{\hat{\sigma}}$.

This function associates to the immediate successors (in the sub-tree $\Gamma_{\hat{\sigma}}$) of each given node nonnegative weights such that the weights sum to one. Hence, the weights $\lambda_{\Gamma_{\hat{\sigma}}}(\cdot)$ can be interpreted as a subjective conditional probability of reaching, from each given node, the different immediate successor nodes in the specified sub-tree.

Given a pair $(\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})$, the *induced weight function*, denoted $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}} : \Gamma_{\hat{\sigma}} \times \lambda_{\Gamma_{\hat{\sigma}}} \to [0, 1]$, is defined by $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\hat{\sigma}) = 1$, $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) = \lambda_{\Gamma_{\hat{\sigma}}}(\sigma) \cdot \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{-1})$ for $\sigma_{-1} \in \Gamma_{\hat{\sigma}}$. It associates to each node σ the product of the values $\lambda_{\Gamma_{\hat{\sigma}}}(\cdot)$ along the chain of nodes from $\hat{\sigma}$ (the root of the sub-tree) to the given node σ.

[11] Clearly, in Lemma 1 and the other results to follow, the condition that the equilibrium allocation is interior for all the agents can be replaced by a standard assumption on the boundary behaviour of preferences.

Since, for $t \geq t(\hat{\sigma})$, $\sum_{\sigma \in \Sigma_t \cap \Gamma_{\hat{\sigma}}} \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) = 1$,[12] the function $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}$ can also be interpreted as the subjective probability of reaching the different nodes $\sigma \in \Sigma_t \cap \Gamma_{\hat{\sigma}}$ conditional on having reached $\hat{\sigma}$, the root of the sub-tree.

Given a path σ^∞, let σ_t^∞ denote the t-th coordinate of the path. Given a sub-tree $\Gamma_{\hat{\sigma}}$, define a *path in the sub-tree* $\Gamma_{\hat{\sigma}}$, denoted by $\sigma^\infty(\Gamma_{\hat{\sigma}})$, as a path with the property that for $t \geq t(\hat{\sigma})$ all the nodes are elements of the sub-tree, i.e. $\sigma^\infty(\Gamma_{\hat{\sigma}}) \subset \{\sigma_1^\infty, \cdots, \sigma_{t(\hat{\sigma})-1}^\infty\} \cup \Gamma_{\hat{\sigma}}$.

Notice that there is an obvious way of associating a node to the t-th coordinate of a specified path; in what follows we shall use σ_t and σ_t^∞ interchangeably when referring to a node.[13]

The following results identify necessary and sufficient conditions for an S-CE to be conditionally Pareto optimal.

THEOREM 1: *(Sufficiency) Let (x^*, p^*) be an S-CE and suppose Assumption 1 holds.*[14] *Assume that $x^*(\sigma, h) \in R_{++}^{L(1+S(\sigma))}$ for every $(\sigma, h) \in \Sigma \times H$, and that there are numbers $K > 0$ and $\underline{\rho} > 0$ such that for all nodes $\sigma \in \Sigma$*
(i) $\omega_l(\sigma) \leq 2KH$ for all $l = 1, \cdots, L$
(ii) for all $h \in H$, $\underline{\rho} \leq \underline{\rho}_{\sigma,h}(2KHL(1+\bar{S}))$.[15]
If the equilibrium allocation is not conditionally Pareto optimal then there exists a pair, given by a sub-tree and a weight function $(\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})$, and an $A < \infty$ such that, for every path $\sigma^\infty(\Gamma_{\hat{\sigma}})$ in the sub-tree,

$$A(\ \sigma^\infty(\Gamma_{\hat{\sigma}}); (\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})) := \sum_{t=t(\hat{\sigma})}^{\infty} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^\infty)}{\parallel p^*(\ \sigma_t^\infty)\ \parallel} \leq A.$$

THEOREM 2: *(Necessity) Let $(x\hat{*}, p\hat{*})$ be an S-CE and suppose Assumption 1 holds. If there exist numbers $\varepsilon > 0$, $\bar{\rho} > 0$, $\bar{k} > 0$, $A < \infty$, and a pair, given by a sub-tree and a weight function $(\ \Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})$, such that at every node $\sigma \in\ \Gamma_{\hat{\sigma}}$, there exists an agent $h_\sigma \in H$ for whom*

[12] This follows easily since $\sum_{\sigma \in \Sigma_t \cap \Gamma_{\hat{\sigma}}} \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) = \sum_{\sigma \in \Sigma_{t-1} \cap \Gamma_{\hat{\sigma}}} [\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) \cdot \sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} \lambda_{\Gamma_{\hat{\sigma}}}(\sigma')] = \sum_{\sigma \in \Sigma_{t-1} \cap \Gamma_{\hat{\sigma}}} \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) = \cdots = 1$, where we repeatedly use the fact that $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} \lambda_{\Gamma_{\hat{\sigma}}}(\sigma') = 1$ for all $\sigma \in \Gamma_{\hat{\sigma}}$.

[13] This notational convention will be used throughout without further mention. It implies that if we are given a pair $(\Gamma_{\hat{\sigma}},\ \lambda_{\Gamma_{\hat{\sigma}}})$, then we are also given the values of the weight function on the coordinates $t \geq t(\hat{\sigma})$ of each path in the sub-tree $\Gamma_{\hat{\sigma}}$.

[14] The result also holds if the restrictions on preferences, imposed as Assumption 1, and the interiority condition on the equilibrium allocation, are replaced by the assumption that, at the S-CE being considered, the preferences of all the agents being improved satisfy the nonvanishing Gaussian curvature condition. The assumptions we made are only sufficient conditions for the curvature conditions to hold; that is the content of Lemma 1. Similar considerations apply to the other results presented in the paper. In particular, Theorem 1 holds even if indifference sets have unbounded curvature ("kinks") while Theorem 2 below holds even if they have zero curvature ("flat" segments).

[15] Definition 4 specifies the way in which the term $\underline{\rho}_{\sigma,h}(k)$, for $k = 2KHL(1+\bar{S}))$, is obtained from the solution to the optimization problem faced by agent (σ, h) at prices $(p^*(\sigma), (p^*(\sigma'))_{\sigma' \in \sigma^+})$; similarly for $\bar{\rho}_{\sigma,h}(\bar{k})$ in Theorem 2. Lemma 1 shows that, under the assumptions of the theorems, the two curvature conditions hold for each agent.

(i) $x^*(\sigma, h_\sigma) \in R_{++}^{L(1+S(\sigma))}$ *and* $\hat{x*}(\sigma; \sigma, h_\sigma) \geq \varepsilon \dot{1}_{L\times 1}$
(ii) $\bar{\rho}_{\sigma,h_\sigma}(\bar{k}) \leq \bar{\rho}$,
and for every path $\sigma^\infty(\Gamma_{\hat{\sigma}})$ *in the sub-tree,*

$$A(\ \sigma^\infty(\Gamma_{\hat{\sigma}}); (\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})) := \sum_{t=t(\hat{\sigma})}^{\infty} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^\infty)}{\parallel p^*(\ \sigma_t^\infty)\ \parallel} \leq A,$$

then the equilibrium allocation is not conditionally Pareto optimal.

Clearly, the characterization provided by the two theorems is tight.

The sufficient condition for CPO obtained in Theorem 1 says that if for every pair $(\ \Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})$, given by a sub-tree and a weight function defined on it, and every real number B, there exists a path in the sub-tree, $\sigma^\infty(\Gamma_{\hat{\sigma}})$, for which $A(\ \sigma^\infty(\Gamma_{\hat{\sigma}}); (\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})) > B$, then the S-CE allocation is CPO. Here $A(\ \sigma^\infty(\Gamma_{\hat{\sigma}}); (\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}}))$ is a series whose t-th element is given by the term $\frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^\infty)}{\|p^*(\ \sigma_t^\infty)\|}$, i.e. by the reciprocal of the norm of the price vector at a node, where the node is given by the t-th coordinate of the path $\sigma^\infty(\Gamma_{\hat{\sigma}})$, weighted by the associated value of $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\cdot)$.

The structure of the proof of Theorem 1 is the following. The first step consists in showing that if there exists a CPO improvement then necessarily there exists a sub-tree (which could be just a path) with the property that the per capita transfer (of income or "value") to the old agents at each node in the sub-tree is strictly positive. We then construct a weight function by associating to each node σ the proportion of the total per capita transfer which is received at that node by the old agents, i.e. the per capita transfer to the old agents at that node divided by the total per capita transfer received by those agents at different nodes when old. But then, by the definition of the weight function, the total per capita transfer received when old can also be written as the per capita transfer received at *one* node when old divided by the weight assigned to that node. This allows us to isolate the relation between the transfer at a node and the (weighted) transfer at *only one* successor node. Now, by invoking the feasibility condition at each node, and iteratively applying the nonvanishing Gaussian curvature condition, we are able to show that along every path, the sequence of these per capita transfers multiplied by the induced weight increases according to a quadratic function. Given the uniform bound on endowments, the admissibility of the improvement requires that the norm of the price vector divided by the value of the induced weight function grow at an appropriate rate. Hence we find that the existence of a CPO improvement requires that the series $\sum_{t=t(\tilde{\sigma})}^{\infty} \frac{\widehat{\lambda}_{\Gamma_{\tilde{\sigma}}}(\sigma_t^\infty)}{\|p^*(\ \sigma_t^\infty)\|}$ converges, and this must happen on every path in the sub-tree.

Turning to necessity, the condition in Theorem 2 says that a CPO improvement can be constructed if we can find a sub-tree (which could simply be a path) with the property that along every path in this sub-tree, the price norm at a date-event does not go to zero faster than the induced weight function at the date-event. This condition ensures the existence of transfer sequences with the property that the transfer that an agent makes when young (in the event at his birth) is more than compensated (in utility terms) by the transfer that he receives when old (in possibly more than one event).

The argument of the proof of Theorem 2 is as follows. First a collection of transfers (of income) to agents along the sub-tree is constructed. Next we show that there exists a corresponding collection of vectors of commodity transfers, supporting these income transfers, which give rise to vectors which are in the consumption set of each agent and are feasible, i.e. compatible with the endowment of the economy. Finally, we show that the proposed collection of commodity transfers generates a Pareto improvement by verifying that the associated collection of transfers satisfy the bounded Gaussian curvature condition. The proportions according to which the total transfer received by an agent when old should be distributed over each of the possible events when old are determined according to the values of the function $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}$.

The logic of the proof of both the theorems is similar to the argument usually given in the case of deterministic OLG economies (in particular, Benveniste [3] and Balasko and Shell [2]); not surprisingly, the necessary and sufficient conditions established in the two theorems are in the form of criteria like the one obtained by Cass (in fact, in the special case in which there is no uncertainty, they reduce to the standard Cass criterion).

The truly novel features of our results are the fact that transfers are defined on non-trivial sub-trees, and the role played by the weight function. We see that the difficulty in the proofs, relative to the arguments given in the case of the deterministic OLG model, comes from the fact that in our set-up the preferences of each agent are defined on a subset of the set of commodities available at each point in time; moreover, the number of commodities and agents increases over time. In both the theorems we have to allow for the possibility that transfers are made at more than one event at a given date; this leads naturally to the consideration of sub-trees whose presence is a manifestation of the rich redistributional possibilities in the present model. In addition, the Gaussian curvature conditions relate one node at date t to one or more nodes at date $t+1$ (but typically a strict subset of the set of all nodes at date $t+1$). So, in order to evaluate whether a sequence of transfers is feasible and improving we have to distinguish them according to the event in which they take place in addition to the date of the transfer. Using the weight function, we are able to disaggregate transfers across the different immediate successor nodes of a given node; hence, we can isolate the pattern of transfers across pairs of successive nodes so that the admissibility of the transfers, which is defined in terms of paths, can be verified directly along all the paths in a sub-tree. We can interpret the weight as the importance given to the transition from a given node to one of its immediate successors, i.e. as a subjective conditional probability.[16]

The sufficient condition for CPO obtained in Theorem 1 is rather cumbersome to verify because one has to check that for every pair given by a sub-tree and a weight function $(\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})$, and every real number B, there exists a path in the sub-tree, $\sigma^{\infty}(\Gamma_{\hat{\sigma}})$, for which $A(\sigma^{\infty}(\Gamma_{\hat{\sigma}}); (\Gamma_{\hat{\sigma}}, \lambda_{\Gamma_{\hat{\sigma}}})) > B$. We now provide another sufficient condition for CPO, Theorem 3, that is stronger but that is somewhat easier to verify.

For any $\bar{\sigma} \in \Gamma_{\sigma}$, we denote by $\Gamma(\bar{\sigma}, \Gamma_{\sigma})$ the sub-tree that has $\bar{\sigma}$ as its root and includes

[16] The system of subjective conditional probabilities defined by the weight function need not bear any relationship with the system of local "subjective" probabilities on the date-event tree implicit in the specification of agents' preferences.

all the nodes that are successors of $\bar{\sigma}$ and are elements of Γ_σ.

THEOREM 3: *(Sufficiency) Let (x^*, p^*) be an S-CE and suppose Assumption 1 holds. Assume that $x^*(\sigma, h) \in R_{++}^{L(1+S(\sigma))}$ for every $(\sigma, h) \in \Sigma \times H$, and that there are numbers $K > 0$ and $\underline{\rho} > 0$ such that for all nodes $\sigma \in \Sigma$*
(i) $\omega_l(\sigma) \leq 2KH$ for all $l = 1, \cdots, L$
(ii) for all $h \in H$, $\underline{\rho} \leq \underline{\rho}_{\sigma,h}(2KHL(1+\bar{S}))$.
The equilibrium allocation is conditionally Pareto optimal if for every sub-tree Γ_σ there exists a node $\bar{\sigma} \in \Gamma_\sigma$, for which

$$A(\bar{\sigma}, \Gamma_\sigma) := \sum_{t=t(\bar{\sigma})}^{\infty} \frac{1}{\sum_{\hat{\sigma} \in \Sigma_t \cap \Gamma(\bar{\sigma}, \Gamma_\sigma)} \| p^*(\hat{\sigma}) \|}$$

diverges.

The sufficient condition for CPO stated in Theorem 3 requires, for every sub-tree Γ_σ, the existence of a node $\bar{\sigma}$ for which the series defined by $A(\bar{\sigma}, \Gamma_\sigma)$ diverges. The terms of the series $A(\bar{\sigma}, \Gamma_\sigma)$ are constructed by taking the reciprocal of the sum of the norm of prices at all nodes at a given date which are successors of $\bar{\sigma}$, and are elements of the sub-tree Γ_σ.

As in the proof of Theorem 1, the first step in the proof of Theorem 3 consists in showing that if there exists a CPO improvement then necessarily there exists a sub-tree $\Gamma_{\tilde{\sigma}}$ (which could be just a path) with the property that the per capita transfer to the old agents at each node σ is strictly positive. For each date we consider the sum of these transfers across nodes in the sub-tree at that date; due to the non-vanishing Gaussian curvature condition the sequence of these terms increases according to a quadratic function. Consider the sequence we obtain starting from the date of the root of the sub-tree, $t(\tilde{\sigma})$. Given the uniform bound on endowments, feasibility requires that the sum of price norms across all nodes in the sub-tree at a given point in time increases sufficiently fast, so that the existence of a CPO improvement requires that the series

$$\sum_{t=t(\tilde{\sigma})}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t \cap \Gamma_{\tilde{\sigma}}} \| p^*(\sigma) \|} \tag{1}$$

converges, for the sub-tree $\Gamma_{\tilde{\sigma}}$. Since the same argument can be made starting from *any* other node in the specified sub-tree, we find that the existence of an improvement implies that a whole family of series must converge. This directly leads to the sufficient condition for CPO stated in Theorem 3 that, for every sub-tree, at least one of the series should diverge.[17]

To understand the difference between the sufficient conditions in Theorems 1 and 3, and the fact that the latter are stronger, notice that in Theorem 1 we define weights

[17] Of course, the divergence of the series in (1) for *every* sub-tree Γ_σ, is also a sufficient condition for CPO. Note, however, that the convergence of the series in (1), obtained when we start summing from the root of the sub-tree, does not imply the convergence of the series we get when we start summing from some other node in the sub-tree. Hence, the sufficient condition given in Theorem 3 is tighter than requiring the divergence of the series in (1) for every sub-tree.

which allow us to disaggregate the total transfer at a date, obtained from the curvature conditions, in terms of the transfer at a node, and hence to identify the pattern of the transfers along paths in a sub-tree. In Theorem 3, on the other hand, we aggregate the transfers of agents born at different date-events into a single term by taking the sum. Still, the condition in Theorem 3 also deals with the limiting behaviour of a whole family of series; note, however, that unlike in Theorem 1, there is no claim that the family of series is uniformly bounded.

As a corollary to Theorem 3 we obtain an even simpler, but stronger, sufficient condition for CPO; Example 1 below illustrates the difference between the two conditions. The condition in the corollary rules out the existence of a special type of CPO improvement, one which requires positive average transfers at every node in the date-event tree Γ. But it is easy to show that if the series in Corollary 1 diverges then the series in (1) also diverges for every sub-tree; since the latter is a sufficient condition for CPO, as noted in Footnote 17, the condition in Corollary 1 also implies the nonexistence of *any* improvement.

COROLLARY 1: *Under the assumptions of Theorem 1, the equilibrium allocation* $\hat{x*}$ *is conditionally Pareto optimal if, for the corresponding price sequence* $\hat{p*}$*, the series*

$$\sum_{t=1}^{\hat{\infty}} \frac{1}{\sum_{\sigma \in \Sigma_t} \parallel \hat{p*}(\sigma) \parallel}$$

diverges.

Since sub-trees include paths as special cases, our previous results also characterize the conditions under which improvements can be achieved by transfers along only one path. By Theorem 1, if an improvement exists on only a path then, the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|\hat{p*}(\sigma_t^\infty)\|}$ must converge along this path. Conversely, Theorem 2 shows that if the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|\hat{p*}(\sigma_t^\infty)\|}$ converges on some path then the allocation is not CPO; in this case, an improvement can be constructed via transfers which are different from zero only along the specified path. Clearly, the divergence of the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|\hat{p*}(\sigma_t^\infty)\|}$ along every path is not sufficient to ensure that the allocation is CPO.

REMARK 2: Here we examine the relationship between existing results for the general deterministic OLG model (Burke [5] and Geanakoplos and Polemarchakis [12]) and our characterization of CPO. We will argue that these results allow us to derive the necessary and sufficient conditions for ex ante Pareto optimality when markets are complete, since in that case the economy is isomorphic to a deterministic OLG economy in which the number of commodities varies with time, but do not have any direct implication as far as CPO is concerned.

With complete markets, an agent is able to trade in the full menu of contingent commodity contracts so that he *can* insure against the uncertainty both at the date of his birth and in the second period of his life; consequently, his consumption plan specifies his consumption at the date of his birth, for all possible realizations of the event at birth, and at the subsequent date, again for all possible realizations of the uncertainty. Hence an agent can be distinguished simply by the date of his birth, t, and his type, h, and is identified by the pair (t, h). His consumption set is given by $X_{t,h} := R_+^{L(S_t+S_{t+1})}$, with

generic element $x(t,h)$, his endowment vector is $\omega(t,h) \in X_{t,h}$, and his utility function $u_{t,h} : X_{t,h} \to R$ which is strictly increasing in every coordinate. The agent's budget restriction takes the form $(p_t, p_{t+1})\dot{x}(t,h) \leq (p_t, p_{t+1})\dot{\omega}(t,h)$. In all other respects, equilibria with complete markets are as in Definition 3. Let (p^{**}, x^{**}) denote a competitive equilibrium with complete markets.

The "natural" criterion for evaluating the efficiency properties of competitive equilibria when markets are complete is *ex ante Pareto optimality* which evaluates the welfare of every agent before any realization of the uncertainty; at an ex ante Pareto optimum, risk is allocated optimally in both periods of the agents' lives. Evidently, ex ante Pareto optimality implies CPO.

As in Lemma 1 (i), it can be shown that the agent's preferences satisfy the following non-vanishing Gaussian curvature condition if an assumption analogous to Assumption 1 is made: for $x^{**}(t,h) \in R_{++}^{L(S_t+S_{t+1})}$, and $k^{**} = 4KHL$, there exists $\underline{\rho}_{t,h}(k^{**}) > 0$ such that for $\| \hat{x}(t,h) - x^{**}(t,h) \| \leq k^{**}$,

$$u_{t,h}(\hat{x}(t,h)) \geq u_{t,h}(x^{**}(t,h)) \qquad \Rightarrow$$

$$\Rightarrow \qquad \sum_{\sigma \in \Sigma_t} \sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma, h) \geq - \sum_{\sigma \in \Sigma_t} \delta_1(\sigma, h) + \rho \frac{(\sum_{\sigma \in \Sigma_t} \delta_1(\sigma, h))^2}{\| p_t^{**} \|}$$

for all $0 \leq \rho \leq \underline{\rho}_{t,h}(k^{**})$. Similarly, one obtains a bounded Gaussian curvature condition.

¿From [5] and [12] it follows that, under conditions analogous to those of Theorems 1 and 2 a competitive equilibrium allocation with complete markets is ex ante Pareto optimal if and only if the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|p_t^{**}\|}$ diverges, i.e. the competitive prices satisfy the classical Cass criterion with a variable number of commodities.[18]

The condition for ex ante Pareto optimality cannot be compared to the condition for CPO derived in Theorems 1 and 2 since the results refer to different market structures and prices (p^* and p^{**}) obtained with different equilibrium concepts.

We now argue that the results in [5] and [12] do not have any direct implication as far as a characterization of CPO is concerned; this is because those results utilize the fact that preferences are strictly monotonic in all the commodities available at a given date, a condition which cannot be satisfied when agents are distinguished by the event at their birth (as required by the CPO criterion). In fact, if the curvature coefficients, described in Definitions 4 and 5, satisfy the uniform bounds specified in Theorems 1 and 2 then divergence of the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|p_t^*\|}$ is neither a necessary condition nor a sufficient condition for CPO with sequentially complete markets: as Example 1 below shows, an allocation could be CPO even though the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|p_t^*\|}$ converges, while Example 2 shows that a CPO improvement may exist even though the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{\|p_t^*\|}$ diverges. Furthermore, if the curvature coefficients are as in Theorems 1 and 2 then divergence of the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{S_t \cdot \|p_t^*\|}$ is a sufficient condition for CPO but not a necessary condition; this criterion obtains from [12] when the total number of agents born at date t is HS_t while the total number of commodities is LS_t, as in our set-up (where agents are distinguished according to the event at their birth). Divergence of the series $\sum_{t=1}^{\hat{\infty}} \frac{1}{S_t \cdot \|p_t^*\|}$ necessarily

[18] It is worth emphasizing that in this case for sufficiency we need $\omega_l(\sigma) S_{t(\sigma)}$, rather than $\omega_l(\sigma)$, to be uniformly bounded for all $l \in L$ and for all $\sigma \in \Sigma$.

implies that the series $\sum_{t=1}\hat{\infty}\frac{1}{\sum_{\sigma\in\Sigma_t}\|\hat{p*}(\sigma)\|}$ diverges,[19] which, by Corollary 1 of this paper, is a sufficient condition for CPO; but Example 1 below shows how an allocation could be CPO even though $\sum_{t=1}\hat{\infty}\frac{1}{S_t\cdot\|\hat{p_t*}\|}$ converges.

The following example shows that equilibrium prices could satisfy the criterion in Theorem 3 even though both the series $\sum_{t=1}\hat{\infty}\frac{1}{\|\hat{p_t*}\|}$ and the series $\sum_{t=1}\hat{\infty}\frac{1}{S_t\cdot\|\hat{p_t*}\|}$ converge. Moreover, it shows that the condition in Theorem 3 is different from the condition in Corollary 1 since the series $\sum_{t=1}^{\infty}\frac{1}{\sum_{\sigma\in\Sigma_t}\|p^*(\sigma)\|}$ converges; this follows from the inequality $\sum_{t=1}^{\infty}\frac{1}{\sum_{\sigma\in\Sigma_t}\|p^*(\sigma)\|}\leq\sum_{t=1}\hat{\infty}\frac{1}{\|\hat{p_t*}\|}$.

EXAMPLE 1: Consider an economy where $H=1$, $L=1$, $S(\sigma)=2$ for all $\sigma\in\Gamma$; also, $\Sigma_1=\{\sigma_a,\sigma_b\}$. Let prices be as follows:

$p^*(\sigma)=1$ if $t(\sigma)=2,4,6,\cdots$ and σ succeeds σ_a
or if $t(\sigma)=1,3,5,\cdots$ and σ succeeds σ_b
$p^*(\sigma)=(1/3)^{t(\sigma)}$ otherwise

so that at even dates all the nodes in the "top half" of the tree are assigned price one, at odd dates all the nodes in the "bottom half" of the tree are assigned price one, and at all other nodes prices decline at a geometric rate; so we have a perverse form of nonstationarity in the prices. It is easy to specify preferences and endowments so that the prices at an S-CE take this form.

Given *any* node $\bar{\sigma}$, consider the sub-tree $\Gamma_{\bar{\sigma}}$ that has $\bar{\sigma}$ as its root and includes every successor of $\bar{\sigma}$ in Γ. The sub-tree has $2^{t-t(\bar{\sigma})}$ nodes at date t and

$$\sum_{t=t(\bar{\sigma})}^{T}\frac{1}{\sum_{\sigma\in\Sigma_t\cap\Gamma_{\bar{\sigma}}}\parallel p^*(\sigma)\parallel}=\sum_{t\geq t(\bar{\sigma})\text{ and }t\text{ even}}^{T}\frac{1}{2^{t-t(\bar{\sigma})}}+\sum_{t\geq n\text{ and }t\text{ odd}}^{T}\frac{1}{2^{t-t(\bar{\sigma})}\cdot(1/3)^t}.$$

As $T\to\infty$, this sum diverges since the second term on the right diverges. So the criterion in Theorem 3 is satisfied, indicating that the equilibrium allocation is CPO (if the additional conditions specified in the theorem hold).

However, the series

$$\sum_{t=1}\hat{\infty}\frac{1}{\parallel\hat{p_t*}\parallel}=\sum_{t=1}\hat{\infty}\frac{1}{\left(2^{t-1}(1+(1/3)^{2t})\right)^{1/2}}<\sum_{t=1}\hat{\infty}\frac{1}{2^{(t-1)/2}}$$

converges. Clearly, the series $\sum_{t=1}\hat{\infty}\frac{1}{S_t\cdot\|\hat{p_t*}\|}$ also converges since $S_t=2^t$.

The next example shows how equilibrium prices could satisfy the condition that $\sum_{t=1}\hat{\infty}\frac{1}{\|\hat{p_t*}\|}$ diverges even though the allocation is not CPO.

EXAMPLE 2: Consider an economy in which $H=1$, $L=1$, $S(\sigma)=2$ for all $\sigma\in\Gamma$; $p^*(\sigma)=(1/2)^{t/2}$ for all $\sigma\in\Sigma$. Since the series $\sum_{t=1}\hat{\infty}\frac{1}{\|\hat{p*}(\sigma_t^{\infty})\|}$ converges along every path, by Theorem 2 the allocation is not CPO (if the additional conditions specified in the theorem hold).

[19] Since $\parallel p_t^*\parallel>\max_{\sigma\in\Sigma_t}\parallel p^*(\sigma)\parallel$ we have $S_t\cdot\parallel p_t^*\parallel>S_t\cdot\max_{\sigma\in\Sigma_t}\parallel p^*(\sigma)\parallel>\sum_{\sigma\in\Sigma_t}\parallel p^*(\sigma)\parallel$.

Also

$$\sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t} \parallel p^*(\sigma) \parallel} = \sum_{t=1}^{\infty} \frac{1}{2^t \cdot \frac{1}{2^{t/2}}} = \sum_{t=1}^{\infty} \frac{1}{2^{t/2}}$$

converges; in fact, for $\bar{\sigma}$ an arbitrary node, $\sum_{t=t(\bar{\sigma})}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t \cap \Gamma_{\bar{\sigma}}} \|p^*(\sigma)\|}$ converges where $\Gamma_{\bar{\sigma}}$ is the sub-tree that has $\bar{\sigma}$ as its root and includes every successor of $\bar{\sigma}$ in Γ.

Furthermore,

$$\sum\nolimits_{t=1}\hat{\infty} \frac{1}{\parallel p_t\hat{*} \parallel} = \sum\nolimits_{t=1}\hat{\infty} \frac{1}{\left(2^t((1/2)^t)\right)^{1/2}} = \sum\nolimits_{t=1}\hat{\infty} \frac{1}{1}$$

diverges.

The last example shows that the condition that there is no improvement along only one path is weaker than CPO.

EXAMPLE 3: Consider an economy in which $H = 1$, $L = 1$, $S(\sigma) = 2$ for all $\sigma \in \Gamma$; $p^*(\sigma) = 1$ for all $\sigma \in \Sigma$. Clearly, the series $\sum_{t=1}\hat{\infty} \frac{1}{\|p\hat{*}(\sigma_t^{\infty})\|}$ diverges along every path. However, the allocation is not CPO. To see this, consider the weight function $\lambda(\sigma) = 1/2$ for all $\sigma \in \Sigma$; it follows that the series defined in Theorem 2 are uniformly bounded across paths so that a CPO improvement exists (if the additional conditions specified in the theorem hold).

REMARK 3: Zilcha [24] provides a characterization of the efficiency properties of allocations in the framework of a one good OLG model with uncertainty and production according to a different efficiency criterion. He provides necessary and sufficient conditions both for allocations to be SOSD-efficient and FOSD-efficient, i.e. to be not dominated, for any agent, in terms of second order stochastic dominance (respectively, first order stochastic dominance), conditional upon the event at birth. Note that CPO is a stronger welfare criterion than SOSD-efficiency and, a fortiori of FOSD-efficiency.[20]

Peled [21] on the other hand provided a sufficient condition for a competitive equilibrium to be CPO in an economy with one commodity and one agent per generation, stationary Markov uncertainty, and money as the only asset. It is well known that with $H = 1$ a market structure with money as the only asset is equivalent to having sequentially complete markets; so we can convert Peled's formulation into our set-up. We can then show that (see Section 6 for the details) if Peled's sufficient condition for CPO holds then, for some $\epsilon > 0$, the inequality $\frac{1}{\sum_{\sigma \in \Sigma_t} p^*(\sigma)} \geq \epsilon$ holds for all t. Note that this condition

[20] A precise study of the relationship between the results in [24] and this paper is difficult here since the structure of the economy considered in [24] is rather different from the one examined here, and so is the efficiency notion. The precise relationship between our characterization of the conditions for CPO and the conditions for SOSD-efficiency derived in [24] is given in Chattopadhyay [8], where the current analysis is applied to the case in which uncertainty is described by a filtered probability space instead of a date-event structure. Results analogous to Theorems 1 and 2 of this paper are shown to hold with the weights replaced by a sequence of densities; the criterion in [24] is obtained when the densities are restricted to take the value one everywhere. This suggests that SOSD efficiency is much weaker than CPO since the path sums could diverge under the criterion in [24] even though a subset of the sums are uniformly bounded for densities which are not identically equal to one, indicating that CPO improvements exist.

implies the validity of the condition given in Corollary 1 above; the converse is not true though (consider for example the case where $\sum_{\sigma\in\Sigma_t} p^*(\sigma) = t$). Hence Peled's condition is stronger than the condition in Corollary 1 (which we saw was in turn stronger than the conditions provided in Theorems 1 and 3).

4. STATIONARY EQUILIBRIA

In this section we restrict attention to stationary equilibria, characterized by the fact that for each agent the equilibrium consumption level only depends on the realization of the uncertainty during his lifetime, not on past realizations nor on the date of his birth. The market structure is unchanged, characterized by sequentially complete markets.

We will assume here that the uncertainty in the environment is generated by the realizations of a time homogeneous Markov process with fixed finite state space $\mathcal{S}$ (where $\#\mathcal{S} = S$). The structure of the date-event tree induced by all possible realizations of the Markov process from an initial date $t = 0$ is as follows. The initial node σ_0 is an element of $\mathcal{S}$; set $\Sigma_1 := \{\sigma_0\} \times \mathcal{S}$, and iteratively $\Sigma_t := \Sigma_{t-1} \times \mathcal{S}$ for $t = 2, 3, \cdots$. In this framework a node σ_t can be identified with the string $(\sigma_0, s_1, s_2, \cdots, s_t)$, where $s_\tau \in \mathcal{S}$ denotes the realization of the Markov process at date τ, $\tau = 1, \cdots, t$ (σ_0 is then the realization at the initial date).[21] The date event tree Γ_{σ_0} thus induced has at every node the same number of successors, $\#S(\sigma) = S$ for all σ, and $\#\Sigma_t = S^t$.

For stationary equilibria to exist additional conditions are needed. We will assume that the economy is also stationary, i.e. that the endowments and utility functions of each agent only depend on the realizations of the Markov process during his lifetime, not on time nor on past realizations: $\omega(\sigma_t, h) = \omega(s_t, h) = (\omega(s_t; s_t, h), (\omega(s_t, s; s_t, h))_{s\in\mathcal{S}})$, and $u_{\sigma_t,h} = u_{s_t,h}$, for all h, σ_t, s_t, t such that $\sigma_t = (\sigma_0, s_1, s_2, \cdots, s_t)$. In addition we will assume that a single commodity is available at each date, i.e. $L = 1$.

Under the above stationarity conditions of the environment, each agent can simply be identified by the triple $(s_t, t, h) \in \mathcal{S} \times \{1, 2, \cdots\} \times H$, describing the state and the date at his birth as well as the agent's type (or $(\sigma_0, 0, h)$ for the initial old).

Formally, we say a competitive equilibrium with sequentially complete markets (x^*, p^*) is *stationary* if, in addition to the properties stated in Definition 3, the following condition also holds: $x^*(\sigma_t, h) = x^*(s_t, h)$ for all $(\sigma_t, h) \in \Sigma \times H$ (i.e. the consumption allocation of each agent only depends on the state at the date of his birth and the states at the next date). Under the above conditions, and Assumption 1, stationary equilibria with S-CE always exist.[22]

To characterize efficiency we will use the fact that the system $\{p_t\}_{t\geq 1}$ of prices of all contingent commodities at the initial date implicitly defines a system of spot prices for the commodities and one-period contingent prices at all nodes (given the equivalence already noticed in Section 2). Since $L = 1$, spot prices can be ignored here; the vector $q(\sigma_t)$ of the prices at a generic node σ_t of one period claims, contingent on all possible realizations of the uncertainty at the next date, conditional on σ_t, is then given by $((p(\sigma')/p(\sigma_t))_{\sigma'\in\sigma_t^+})$. It

[21] In the present set-up, s_t identifies the arc leading from node $(\sigma_0, s_1, s_2, \cdots, s_{t-1})$ to node $(\sigma_0, s_1, s_2, \cdots, s_t)$. It is a general property of date-event trees that nodes can also be identified by the sequence of arcs leading to them from the root of the tree—see Footnote 5 in Section 2.

[22] See e.g. Gottardi [13]. When $L > 1$, as shown by Spear [23], stationary equilibria do not exist, generically (in endowments and utility functions).

is easy to see that at a stationary equilibrium $q(\sigma_t)$ only depends on s_t; hence the system of one-period contingent prices is simply described by a matrix Q, of dimension $S \times S$, with generic element $q_{ss'}$, describing the price quoted in state s for a claim promising the delivery of one unit of the good at the next date in state s'.

Let $\lambda(Q)$ denote the dominant root of the matrix Q. The next result shows that a complete characterization of the efficiency properties of stationary equilibria can be given in terms of the value of $\lambda(Q)$:

THEOREM 4: *Let (x^*, p^*) be a stationary competitive equilibrium (with sequentially complete markets) and Q^* be the matrix of the associated one-period contingent claim prices. Under Assumption 1, if the environment is stationary, $L = 1$, and $x^*(s;h) \in R^{1+S}_{++}$ (i.e. the equilibrium allocation is interior for each agent), we have:*
(i) the equilibrium allocation x^* is conditionally Pareto optimal if $\lambda(Q^*) \leq 1$;
(ii) if $\lambda(Q^*) > 1$, x^* is not conditionally Pareto optimal; moreover, it is Pareto dominated (in the CPO sense) by another stationary allocation.[23]

The result is to a large extent an implication of the following:

LEMMA 2: *Let p^* be a system of prices at a stationary competitive equilibrium (with sequentially complete markets) and Q^* be the matrix of the associated one period contingent claim prices. Then we have:*
(i) $\sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t} p^*(\sigma)} < \infty \Longrightarrow$ there exists a weight function $\lambda_{\Gamma_{\sigma_0}}$ and a scalar $A < \infty$ such that, for every path $\sigma^\infty(\Gamma_{\sigma_0})$,

$$A(\ \sigma^\infty(\Gamma_{\sigma_0}); (\Gamma_{\sigma_0}, \lambda_{\Gamma_{\sigma_0}})) := \sum_{t=1}^{\infty} \frac{\widehat{\lambda}_{\Gamma_{\sigma_0}}(\sigma_t^\infty)}{p^*(\ \sigma_t^\infty)} \leq A.$$

(ii) $\lambda(Q^*) > 1 \Longleftrightarrow \sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t} p^*(\sigma)} < \infty$.

The result in part (i) of Lemma 2 says that, for the special case of the stationary equilibria of a stationary economy, the sufficient condition for CPO obtained in Corollary 1 is also a necessary condition for CPO (its violation implies in fact the validity of the necessary condition for CPO of Theorem 2). Part (ii) of Lemma 2 then establishes the equivalence between this condition—the convergence of the series appearing in Corollary 1, whose terms are given by the reciprocals of the sum of the prices $p^*(\sigma)$ at all nodes at a given date—and the condition that the dominant root of the matrix of the associated one-period contingent prices $q^*_{ss'}$ is greater than 1. Combining these two results with Corollary 1 and Theorem 2 we obtain the characterization provided in Theorem 4: the necessary and sufficient condition for stationary equilibria to be CPO is $\lambda(Q^*) \leq 1$. The

[23] In Theorem 4 we no longer need to impose uniform upper and lower bounds on the curvature coefficients as in Theorems 1 and 2. The stationarity of the environment and of the equilibrium considered ensure in fact that the set of equilibrium consumption levels is described by a finite collection of vectors, so that under Assumption 1 uniform bounds always hold.

theorem actually establishes a slightly stronger result as it shows that if $\lambda(Q^*) > 1$ a CPO improvement exists even if we limit our attention to stationary allocations.

REMARK 4: Aiyagari and Peled [1] examine a stationary stochastic OLG economy, similar to the one of this section. By studying the solutions of a finite dimensional planner's problem for such an economy, they show that an interior stationary equilibrium allocation is optimal, according to the CPO criterion, in the class of all *stationary* allocations if and only if $\lambda(Q) \leq 1$. The result in Theorem 4 is significantly stronger as it shows that if the same condition ($\lambda(Q) \leq 1$) holds, stationary equilibria are optimal, in the CPO sense, in the class of all allocations, i.e. even if we allow for nonstationary reallocations.

Manuelli [17] considers the case of a stationary stochastic OLG economy where uncertainty is described by a stationary Markov process with infinite support, described by a compact set. Under the additional condition that there is a single agent per generation, i.e. $H = 1$, he shows that the necessary and sufficient condition for a stationary competitive allocation to be CPO in the class of *all* allocations is $\lambda(Q) \leq 1$. The argument of the proof of his result is rather different from ours and is based on the Martingale Convergence Theorem.

5. SEQUENTIALLY INCOMPLETE MARKETS: A SPECIAL CASE

We consider in this section the case in which agents are no longer able to fully insure against every possible realization of the uncertainty when old; markets are then said to "sequentially incomplete". A general analysis of the efficiency properties of competitive equilibria with sequentially incomplete markets is beyond the scope of this paper;[24] here we only examine the special case which arises when there is only one asset, a riskless bond, available at each date-event.

CPO appears to be too demanding as an efficiency criterion for competitive equilibrium allocations when markets are sequentially incomplete. Consequently, we consider a weaker criterion, ex post Pareto optimality (denoted EPPO).[25] According to this notion the agents' welfare is evaluated by conditioning on the realization of the uncertainty not only in the first period of their lives but also the second period. Efficiency then only concerns the allocation of commodities at each date-event pair, and between the two periods of the agents' lifetime, not the allocation of risk. Such a notion appears to be particularly well suited for the analysis of efficiency in situations where there is no asset whose payoff is contingent on the state.

In order to be able to apply this criterion, we need to be able to evaluate agents' preferences over consumption conditional upon both the event when young and the event when old. To this end, we will impose here the additional condition that preferences are additively separable over time and across the various possible events when old:

ASSUMPTION 2:

[24] See Cass, Green and Spear [7] and Gottardi [14] for some preliminary steps on this issue.

[25] Lucas [16] was the first to use such a criterion to evaluate the welfare properties of competitive equilibria in OLG models under uncertainty.

For all $(\sigma, h) \in \Sigma \times H$, $u_{\sigma,h}(x(\sigma,h)) = u_{\sigma,\sigma,h}(x(\sigma;\sigma,h)) + \sum_{\sigma' \in \sigma^+} u_{\sigma',\sigma,h}(x(\sigma';\sigma,h))$.

Under this assumption the preferences of agent (σ, h) naturally induce a specification of preferences over consumption conditional upon both the events, that when young and that when old.

The previous definitions of feasible allocations remains unchanged. We are now ready to define the new welfare criterion:

DEFINITION 6 (EPPO): Let x be a feasible allocation. x is *ex post Pareto optimal (EPPO)* if there does not exist another feasible allocation $\widehat{x}$ such that

(i) for all $(\sigma, h) \in \Sigma_1 \times H$, $u_{\sigma,h,o}(\widehat{x}(\sigma;h,o)) \geq u_{\sigma,h,o}(x(\sigma;h,o))$,
for all $(\sigma, h) \in \Sigma \times H$, $\sigma' \in \sigma^+$,
$u_{\sigma,\sigma,h}(\widehat{x}(\sigma;\sigma,h)) + u_{\sigma',\sigma,h}(\widehat{x}(\sigma';\sigma,h)) \geq u_{\sigma,\sigma,h}(x(\sigma;\sigma,h)) + u_{\sigma',\sigma,h}(x(\sigma';\sigma,h))$,

(ii) either for some $(\tilde{\sigma}, \tilde{h}) \in \Sigma_1 \times H$, $u_{\tilde{\sigma},\tilde{h},o}(\widehat{x}(\tilde{\sigma};\tilde{h},o)) > u_{\tilde{\sigma},\tilde{h},o}(x(\tilde{\sigma};\tilde{h},o))$,

or for some $(\tilde{\sigma}, \tilde{h}) \in \Sigma \times H$, $\tilde{\sigma}' \in \tilde{\sigma}^+$,
$u_{\tilde{\sigma},\tilde{\sigma},\tilde{h}}(\widehat{x}(\tilde{\sigma};\tilde{\sigma},\tilde{h})) + u_{\tilde{\sigma}',\tilde{\sigma},\tilde{h}}(\widehat{x}(\tilde{\sigma}';\tilde{\sigma},\tilde{h})) > u_{\tilde{\sigma},\tilde{\sigma},\tilde{h}}(x(\tilde{\sigma};\tilde{\sigma},\tilde{h})) + u_{\tilde{\sigma}',\tilde{\sigma},\tilde{h}}(x(\tilde{\sigma}';\tilde{\sigma},\tilde{h}))$.

Thus EPPO applies the usual Pareto criterion to the larger set of agents (relative to the CPO criterion) where every agent is distinguished by the event faced at each of the two dates at which he is alive. When carrying out the EPPO optimality analysis, we will use the notation (σ', σ, h) to identify an agent.

To obtain an improvement according to the EPPO criterion, an agent who receives a negative transfer when young must be compensated with a positive transfer in every possible realization of the uncertainty when old. By iterating the argument we see that we must have a nonzero transfer at all nodes which are successors to a node at which a transfer is made. Consequently, we define a *full sub-tree* with root $\tilde{\sigma}$, denoted by $\Gamma(\tilde{\sigma})$, as a sub-tree with the property that every node $\sigma \in \Gamma$ which is a successor of $\tilde{\sigma}$ is an element of $\Gamma(\tilde{\sigma})$.

We can relate the transfers that generate EPPO and CPO improvements by comparing Definitions 2 and 6. Consider a transfer defined on some full sub-tree; call it the "initial" transfer. For each path in the sub-tree, induce a "new" transfer which takes the value zero at all nodes off the path that has been fixed and takes the same value as the "initial" transfer at all nodes on the path. This procedure generates a collection of "new" transfers, one for each path in the sub-tree. Suppose that each of these "new" transfers is CPO improving; then the "initial" transfer must be EPPO improving.[26]

REMARK 5: To properly evaluate the consequences of the sequential incompleteness of markets, we provide first a necessary and sufficient condition for competitive equilibria with sequentially complete markets to be EPPO. So, let (x^*, p^*) be an S-CE. By an immediate extension of the argument of Lemma 1 we obtain that, under Assumptions 1 and 2, for $x^*(\sigma, h) \in R_{++}^{L(1+S(\sigma))}$, the following version of the non-vanishing Gaussian curvature condition holds for the agent (σ', σ, h): for $k^* = 4KHL$, there exists $\underline{\rho}_{\sigma',\sigma,h}(k^*) > 0$ such that for $\| \hat{x}(\sigma';\sigma,h) - x^*(\sigma';\sigma,h) \| \leq k^*$,

$$u_{\sigma,\sigma,h}(\widehat{x}(\sigma;\sigma,h)) + u_{\sigma',\sigma,h}(\widehat{x}(\sigma';\sigma,h)) \geq u_{\sigma,\sigma,h}(x^*(\sigma;\sigma,h)) + u_{\sigma',\sigma,h}(x^*(\sigma';\sigma,h))$$

[26] Clearly, this is *not* the same as requiring a CPO improvement on only one path.

$$\Rightarrow \qquad \delta_2(\sigma', \sigma, h) \geq -\delta_1(\sigma, h) + \rho \frac{(\delta_1(\sigma, h))^2}{\| \, p^*(\sigma) \, \|}$$

for all $0 \leq \rho \leq \underline{\rho}_{\sigma', \sigma, h}(k^*)$ (where we recall that $\delta_1(\sigma, h) := p^*(\sigma) \cdot [\widehat{x}(\sigma; \sigma, h) - x^*(\sigma; \sigma, h)]$, $\delta_2(\sigma', \sigma, h) := p^*(\sigma') \cdot [\widehat{x}(\sigma'; \sigma, h) - x^*(\sigma'; \sigma, h)]$, for $\sigma' \in \sigma^+$). A similar expression can be obtained for the bounded Gaussian curvature condition.

One can show that, under conditions analogous to Theorems 1 and 2, an S-CE allocation is ex post Pareto optimal if and only if there exists a full sub-tree $\Gamma(\tilde{\sigma})$ and an $A < \infty$ such that, for every path $\sigma^\infty(\Gamma(\tilde{\sigma}))$ in the full sub-tree, $\sum_{t=t(\tilde{\sigma})}^{\infty} \frac{1}{\|p^*(\sigma_t^\infty)\|} \leq A$.

The proof of this result follows closely the proofs of Theorems 1 and 2 since, as was noted earlier, the conditions obtained for EPPO improvements at an S-CE are simply the conditions for having a CPO improvement on every path contained in a full sub-tree.

It is easy to see that the necessary and sufficient conditions for an S-CE allocation to be EPPO are less demanding than the corresponding conditions for CPO. As we noticed, EPPO is in fact a weaker notion of optimality than CPO.

We turn to the particular sequentially incomplete market structure in which there is only one asset, a riskless bond, promising the delivery of a fixed amount (set equal to one without loss of generality) of the numeraire commodity in the subsequent period, independent of the realization of the uncertainty.

In order to be able to directly compare the conditions obtained here with the results obtained in the earlier sections of this paper, we use the no arbitrage condition to write the price of the asset in terms of a vector of state prices and then substitute away the portfolio holdings in the budget constraints. It follows that the variables appearing in an agent's budget constraints are, as before, the contingent commodity prices $(p(\sigma), (p(\sigma'))_{\sigma' \in \sigma^+})$ and the consumption bundle $x(\sigma, h)$, but, given market incompleteness, the agent now faces a multiplicity of budget constraints.

DEFINITION 7 (B-CE): $(\bar{x}, \bar{p})$ is a *competitive equilibrium with complete spot markets and a riskless bond (B-CE)* if $\bar{x}$ is a feasible allocation, and,

(i) for all $(\sigma, h) \in \Sigma_1 \times H$,

$\bar{p}(\sigma) \cdot \bar{x}(\sigma; h, o) \leq \bar{p}(\sigma) \cdot \omega(\sigma; h, o)$

$u_{\sigma,h,o}(x(\sigma; h, o)) > u_{\sigma,h,o}(\bar{x}(\sigma; h, o)) \quad \Rightarrow \quad \bar{p}(\sigma) \cdot x(\sigma; h, o) > \bar{p}(\sigma) \cdot \omega(\sigma; h, o)$

(ii) for all $(\sigma, h) \in \Sigma \times H$,

$(\bar{p}(\sigma), \bar{p}(\sigma')) \cdot (\bar{x}(\sigma; \sigma, h), \bar{x}(\sigma'; \sigma, h)) \leq (\bar{p}(\sigma), \bar{p}(\sigma')) \cdot (\omega(\sigma; \sigma, h), \omega(\sigma'; \sigma, h))$ for all $\sigma' \in \sigma^+$,

$u_{\sigma,h}(x(\sigma, h)) > u_{\sigma,h}(\bar{x}(\sigma, h)) \quad \Rightarrow$

$(\bar{p}(\sigma), \bar{p}(\sigma')) \cdot (x(\sigma; \sigma, h), x(\sigma'; \sigma, h)) > (\bar{p}(\sigma), \bar{p}(\sigma')) \cdot (\omega(\sigma; \sigma, h), \omega(\sigma'; \sigma, h))$ for some $\sigma' \in \sigma^+$.

In what follows we provide a characterization of the efficiency properties, according to the EPPO criterion, of equilibria with complete spot markets and a riskless bond, in terms of the prices $\bar{p}$.

A crucial step in the argument is the derivation of appropriate curvature conditions, a more subtle problem since agents now face a multiplicity of budget constraints. We begin by noting that under Assumptions 1 and 2, for all $(\sigma, h) \in \Sigma \times H$, the optimal choice $\bar{x}(\sigma, h)$, if it is interior, i.e. $\bar{x}(\sigma, h) \in R_{++}^{L(1+S(\sigma))}$, satisfies the following first order conditions:

$Du_{\sigma,\sigma,h}(\bar{x}(\sigma;\sigma,h)) = (\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))\cdot\bar{p}(\sigma)$,
$Du_{\sigma',\sigma,h}(\bar{x}(\sigma';\sigma,h)) = \bar{\mu}(\sigma',\sigma,h)\dot{\bar{p}}(\sigma')$ for all $\sigma'\in\sigma^+$, and
$(\bar{p}(\sigma),\bar{p}(\sigma'))\dot{[}(\bar{x}(\sigma;\sigma,h),\bar{x}(\sigma';\sigma,h)) - (\omega(\sigma;\sigma,h),\omega(\sigma';\sigma,h))] = 0$,
where $Du_{\sigma,\sigma,h}$ (respectively, $Du_{\sigma',\sigma,h}(\cdot)$) is the vector of derivatives of $u_{\sigma,\sigma,h}(\cdot)$ (respectively, $u_{\sigma',\sigma,h}(\cdot)$) with respect to its L arguments, and where $((\bar{\mu}(\sigma',\sigma,h))_{\sigma'\in\sigma^+})\in R^{S(\sigma)}_{++}$ is the vector of Lagrange multipliers associated with the $S(\sigma)$ constraints faced by (σ,h) (positivity follows from the strict monotonicity of preferences).

¿From the above expressions we see that the "personalized" price vector $((\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))\cdot\bar{p}(\sigma),(\bar{\mu}(\sigma',\sigma,h)\dot{\bar{p}}(\sigma'))_{\sigma'\in\sigma^+})$ supports the same choice $\bar{x}(\sigma,h)$ of agent (σ,h) were he to maximize utility subject to a single intertemporal budget constraint (as he does when markets are sequentially complete). This allows us to obtain the non-vanishing Gaussian curvature condition in terms of these "personalized" prices by essentially the same argument as in the proof of Lemma 1: under Assumptions 1 and 2, for $\bar{x}(\sigma,h)\in R^{L(1+S(\sigma))}_{++}$, and $k^*=4KHL$, there exists $\underline{\rho}_{\sigma',\sigma,h}(k^*)>0$ such that for $\parallel\hat{x}(\sigma';\sigma,h)-\bar{x}(\sigma';\sigma,h)\parallel\leq k^*$,

$$u_{\sigma,\sigma,h}(\widehat{x}(\sigma;\sigma,h))+u_{\sigma',\sigma,h}(\widehat{x}(\sigma';\sigma,h))\geq u_{\sigma,\sigma,h}(\bar{x}(\sigma;\sigma,h))+u_{\sigma',\sigma,h}(\bar{x}(\sigma';\sigma,h))$$

$$\Rightarrow\qquad \bar{\mu}(\sigma',\sigma,h)\delta_2(\sigma',\sigma,h)\geq-(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))\delta_1(\sigma,h)+\rho\frac{((\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))\delta_1(\sigma,h))\hat{2}}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))\parallel\bar{p}(\sigma)\parallel}$$

for all $0\leq\rho\leq\underline{\rho}_{\sigma',\sigma,h}(k^*)$ (where $\delta_1(\sigma,h):=\bar{p}(\sigma)\cdot[\widehat{x}(\sigma;\sigma,h)-\bar{x}(\sigma;\sigma,h)]$, $\delta_2(\sigma',\sigma,h):=\bar{p}(\sigma')\cdot[\widehat{x}(\sigma';\sigma,h)-\bar{x}(\sigma';\sigma,h)]$, for $\sigma'\in\sigma^+$). Since $\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h)>0$, we can rewrite the second inequality in the above expression as follows:

$$\frac{\bar{\mu}(\sigma',\sigma,h)}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))}\delta_2(\sigma',\sigma,h)\geq-\delta_1(\sigma,h)+\rho\frac{(\delta_1(\sigma,h))\hat{2}}{\parallel\bar{p}(\sigma)\parallel}$$

for all $0\leq\rho\leq\underline{\rho}_{\sigma',\sigma,h}(k^*)$.

In a similar manner, we can obtain the bounded Gaussian curvature condition.

Let $\bar{\mu}(\sigma')$ denote the average of the agents' Lagrange multipliers at a given node σ':

$$\bar{\mu}(\sigma'):=\frac{1}{H}\sum_{h\in H}\frac{\bar{\mu}(\sigma',\sigma,h)}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))}\quad\text{for }\sigma'\in\cup_{t\geq1}\Sigma_t.$$

In the following result, we obtain separate necessary and sufficient conditions for competitive equilibria with complete spot markets and a riskless bond to be EPPO.

THEOREM 5: *Let $(\bar{x},\bar{p})$ be a B-CE and let $\{\bar{\mu}(\sigma',\sigma,h)\}_{\sigma\in\Gamma,\sigma'\in\sigma^+,h\in H}$ be the system of associated Lagrange multipliers. Suppose Assumptions 1 and 2 hold and assume that there are numbers $K>0$, $\underline{\rho}>0$, $\varepsilon>0$, $\bar{\rho}>0$, and $\bar{k}>0$, such that*
(i) $\omega_l(\sigma)\leq 2KH$ for all $l=1,\cdots,L$ for all nodes $\sigma\in\Sigma$,
(ii) for every $(\sigma,h)\in\Sigma\times H$, and every $\sigma'\in\sigma^+$,
$\bar{x}(\sigma,h)\geq\varepsilon\dot{1}_{L(1+S(\sigma))\times1}$, $\underline{\rho}\leq\underline{\rho}_{\sigma',\sigma,h}(4KHL)$ and $\bar{\rho}_{\sigma',\sigma,h}(\bar{k})\leq\bar{\rho}$.
A) If the equilibrium allocation is not ex post Pareto optimal then there exists a full sub-tree $\Gamma(\tilde{\sigma})$ and an $A<\infty$ such that, for every path $\sigma^\infty(\Gamma(\tilde{\sigma}))$ in the full sub-tree,

$$A(\ \sigma^\infty(\Gamma(\tilde{\sigma}))):=\sum_{t=t(\tilde{\sigma})}^{\hat{\infty}}\frac{1}{H^{t-t(\tilde{\sigma})}(\Pi_{\tau=t(\tilde{\sigma})+1}^{t}\bar{\mu}(\sigma_\tau^\infty))\parallel\bar{p}(\ \sigma_t^\infty)\parallel}\leq A.$$

B) If there exists a full sub-tree $\Gamma(\tilde{\sigma})$, with an agent $h_\sigma \in H$ at every node $\sigma \in \Gamma(\tilde{\sigma})$, and there exists an $A < \infty$ such that, for every path $\sigma^\infty(\Gamma(\tilde{\sigma}))$ in the full sub-tree,

$$A(\ \sigma^\infty(\Gamma(\tilde{\sigma}))) := \sum\nolimits_{t=t(\tilde{\sigma})} \hat{\infty}\Big(\Pi_{\tau=t(\tilde{\sigma})+1}^{t} \frac{\sum_{\sigma' \in (\sigma_{\tau-1}^\infty)^+} \bar{\mu}(\sigma', \sigma_{\tau-1}^\infty, h_{\sigma_{\tau-1}^\infty})}{\bar{\mu}(\sigma_\tau^\infty, \sigma_{\tau-1}^\infty, h_{\sigma_{\tau-1}^\infty})}\Big) \frac{1}{\parallel \bar{p}(\ \sigma_t^\infty)\ \parallel} \leq A,$$

then the equilibrium allocation is not ex post Pareto optimal.

In the present set-up, the prices appearing in the Gaussian curvature conditions are agent-specific and, since markets are incomplete, will typically vary in a non-trivial way with h. Hence, the sufficient condition with many agents per generation is stated in terms of the average of these agent-specific prices.[27] Evidently, this issue does not arise when there is a single agent per generation as in this case we can simply use the prices supporting the choice of that agent. So with $H = 1$, the two conditions in Theorem 5 reduce to a single necessary and sufficient condition for a B-CE to not be EPPO:

$$A(\ \sigma^\infty(\Gamma(\tilde{\sigma}))) := \sum\nolimits_{t=t(\tilde{\sigma})} \hat{\infty}\Big(\Pi_{\tau=t(\tilde{\sigma})+1}^{t} \frac{\sum_{\sigma' \in (\sigma_{\tau-1}^\infty)^+} \bar{\mu}(\sigma', \sigma_{\tau-1}^\infty)}{\bar{\mu}(\sigma_\tau^\infty, \sigma_{\tau-1}^\infty)}\Big) \frac{1}{\parallel \bar{p}(\ \sigma_t^\infty)\ \parallel} \leq A,$$

where $\{\bar{\mu}(\sigma', \sigma)\}_{\sigma \in \Gamma, \sigma' \in \sigma^+}$ are the agent's Lagrange multipliers at the B-CE.

Notice that in the present set-up, with one agent per generation, markets are effectively sequentially complete; hence, the equilibrium allocation at a B-CE is also an S-CE equilibrium, supported by the "personalized" prices $((\sum_{\sigma' \in \sigma^+} \bar{\mu}(\sigma', \sigma, h)) \cdot \bar{p}(\sigma), (\bar{\mu}(\sigma', \sigma, h) \dot{\bar{p}}(\sigma'))_{\sigma' \in \sigma^+})$. Clearly, once prices are replaced by these "personalized" prices, the condition above (for B-CE to be EPPO when $H = 1$) reduces to the necessary and sufficient condition given in Remark 5 for S-CE to be EPPO.

[27] Notice that the number of agents did not play any role in our earlier results.

6. PROOFS

PROOF OF LEMMA 1

We follow Benveniste [3] and Balasko and Shell [2].

We will prove a more general result from which Lemma 1 follows.

For A and B positive integers, let $R\hat{A}_+ \times R\hat{B}_+$ be an agent's consumption set, $U : R\hat{A}_+ \times R\hat{B}_+ \to R$ be his utility function, and $(p_A, p_B) \cdot (x_A, x_B) \leq m$ be his budget restriction, where $p = (p_A, p_B)$ is a price vector, $x = (x_A, x_B)$ is a consumption vector, and $m > 0$.

Also, let $V_x := \{y \in R\hat{A}_+ \times R\hat{B}_+ | U(y) \geq U(x)\}$, the weak upper contour set at x, and, for $k \in R_+$, $\rho \in R$, let $K(x,k) := \{y \in R^A_+ \times R\hat{B}_+ | \parallel y - x \parallel \leq k\}$, $K(x,k,p) := \{y \in K(x,k) | p_A \cdot (y_A - x_A) < 0\}$, and $R(\rho, x, p) := \{y \in R^A \times R\hat{B} | p_B \dot{(} y_B - x_B) \geq -p_A \dot{(} y_A - x_A) + \rho \frac{(p_A \dot{(} y_A - x_A))\hat{2}}{\|p_A\|}\}$, a paraboloid of curvature ρ which has x on its boundary and p as the support vector at x.

LEMMA 1A: *Let $x = (x_A, x_B) \in R\hat{A}_{++} \times R\hat{B}_{++}$ be the solution to the agent's utility maximization problem at prices $p = (p_A, p_B) \in R\hat{A}_{++} \times R\hat{B}_{++}$. Under the assumption that $U(\cdot)$ is C^2, strictly monotone, and differentiably strictly quasi-concave, so that $\forall z \in R\hat{A} \times R\hat{B}, z \neq 0,\ z \cdot \nabla_x U = 0 \ \Rightarrow\ z\hat{\imath} D\hat{2}_x U z < 0$, we have*
(i) for any $k > 0$, $\underline{\rho} > 0$ where $\underline{\rho} := \sup\{\rho | V_x \cap K(x,k) \subset R(\rho, x, p) \cap K(x,k)\}$;
(ii) for any $k > 0$, $\infty > \bar{\rho} \geq 0$ where $\bar{\rho} := \inf\{\rho | R(\rho, x, p) \cap K(x,k,p) \subset V_x \cap K(x,k,p)\}$.

PROOF: (i) Under the assumptions of the lemma, for any $k > 0$, there is a closed ball of radius $\underline{r}$, for $1/\underline{r} > 0$, with center at $c = x + \frac{\underline{r}}{\|p\|} p$, denote it by $B_{(c,\underline{r})}$, which is (a) tangent to the budget plane at x, (b) tangent to the upper contour set, V_x, at the point x, and (c) contains the set $V_x \cap K(x,k)$ (see, e.g. Mas-Colell [18, page 40]).

Now take any point $y = (y_A, y_B)$ in the set $V_x \cap K(x,k)$; so $y \in B_{(c,\underline{r})}$ and $\parallel y - c \parallel \leq \underline{r}$. Let $\lambda \neq 0$ be such that $\parallel \lambda(y - x) - \frac{\underline{r}}{\|p\|} p \parallel = \underline{r}$; since $c = x + \frac{\underline{r}}{\|p\|} p$, we have $\parallel \lambda(y - x) + x - c \parallel = \underline{r}$, and hence $\lambda \geq 1$ as $\parallel y - c \parallel \leq \underline{r}$. Now

$$\lambda(y - x) \cdot (\lambda(y - x) - \tfrac{2\underline{r}}{\|p\|} p) = (\lambda(y - x) - \tfrac{\underline{r}}{\|p\|} p + \tfrac{\underline{r}}{\|p\|} p) \cdot (\lambda(y - x) - \tfrac{\underline{r}}{\|p\|} p - \tfrac{\underline{r}}{\|p\|} p)$$
$$= \parallel \lambda(y - x) - \tfrac{\underline{r}}{\|p\|} p \parallel^2 - \parallel \tfrac{\underline{r}}{\|p\|} p \parallel^2 = \underline{r}^2 - \underline{r}^2 = 0.$$

Thus $\lambda \parallel y - x \parallel^2 = (y - x) \cdot \frac{2\underline{r}}{\|p\|} p$, which implies $p \cdot (y - x) \geq \frac{1}{2\underline{r}} \parallel y - x \parallel \hat{2} \parallel p \parallel$ since $\lambda \geq 1$. Since $\parallel (p_A, p_B) \parallel \geq \parallel p_A \parallel$, and $\parallel y - x \parallel \geq \parallel y_A - x_A \parallel$, we have $p \cdot (y - x) \geq \frac{1}{2\underline{r}} \parallel y_A - x_A \parallel \hat{2} \parallel p_A \parallel$. But $\parallel y_A - x_A \parallel \parallel p_A \parallel \geq |p_A \cdot (y_A - x_A)|$, so we have $\parallel y_A - x_A \parallel \hat{2} \parallel p_A \parallel \geq \frac{|p_A \cdot (y_A - x_A)|\hat{2}}{\|p_A\|}$, and $p \cdot (y - x) \geq \frac{1}{2\underline{r}} \frac{|p_A (y_A - x_A)|\hat{2}}{\|p_A\|}$. Letting $\rho = \frac{1}{2\underline{r}}$ we see that for $y \in K(x,k)$,

$$U(y) \geq U(x) \quad \Rightarrow \quad p_B \dot{(} y_B - x_B) \geq -p_A \dot{(} y_A - x_A) + \rho \frac{(p_A \dot{(} y_A - x_A))\hat{2}}{\|p_A\|}.$$

So, for $\rho = \frac{1}{2\underline{r}} > 0$, $V_x \cap K(x,k) \subset R(\rho, x, p) \cap K(x,k)$, implying that $\sup\{\rho | V_x \cap K(x,k) \subset R(\rho, x, p) \cap K(x,k)\} > 0$. Thus, $\underline{\rho} > 0$.

(ii) For any $k > 0$, by smoothness and strict monotonicity of preferences there exists a paraboloid with curvature ρ which is locally contained in the set V_x and is tangent to V_x at x (see, e.g. Benveniste [3]). So, for any $k > 0$, there is a ρ for which $R(\rho, x, p) \cap K(x,k,p) \subset V_x \cap K(x,k,p)$; furthermore, by local non-satiation, $\rho \geq 0$ (in fact, by (i) above, $\rho > 0$). Hence, $\inf\{\rho | R(\rho, x, p) \cap K(x,k,p) \subset V_x \cap K(x,k,p)\}$ is bounded. Thus, $\infty > \bar{\rho}$. ■

PROOF OF THEOREM 1

As indicated in the text, the proof of Theorem 1 starts by showing that if there exists a CPO improvement then necessarily there exists a sub-tree with the property that the per capita transfer to the old agents at each node σ is strictly positive. We then construct a weight function, by disaggregating the per capita transfer across nodes when old, and then we show that the non-vanishing Gaussian curvature condition implies that the per capita transfer, transformed by the weights, increases according to a quadratic function along every path. Finally, we use the uniform bound on endowments to show that this quadratic increase can occur only if the weighted reciprocal of the norm of the price increases sufficiently fast along every path.

Since the competitive equilibrium allocation x^* is, by assumption, not CPO, there must exist an improving allocation; let $\widehat{x}$ denote such an improving allocation. For $(\sigma, h) \in \Sigma \times H$, construct the sequences $\delta_1(\sigma, h) = p^*(\sigma) \cdot [\widehat{x}(\sigma; \sigma, h) - x^*(\sigma; \sigma, h)]$, $\delta_2(\sigma', \sigma, h) = p^*(\sigma') \cdot [\widehat{x}(\sigma'; \sigma, h) - x^*(\sigma'; \sigma, h)]$, for $\sigma' \in \sigma^+$; similarly $\delta_2(\sigma, h, o) := p^*(\sigma) \cdot [\widehat{x}(\sigma; h, o) - x^*(\sigma; h, o)]$. Define $\delta_2(\sigma, o) := \frac{1}{H} \sum_{h \in H} \delta_2(\sigma, h, o)$, $\delta_1(\sigma) := \frac{1}{H} \sum_{h \in H} \delta_1(\sigma, h)$, $\delta_2(\sigma', \sigma) := \frac{1}{H} \sum_{h \in H} \delta_2(\sigma', \sigma, h)$.

By strict monotonicity (Assumption 1 (ii)), and the fact that $\widehat{x}$ improves over x^*, we have that for every agent (σ, h, o) of the initial generation $\delta_2(\sigma_1, h, o) \geq 0$, while for every agent (σ, h) of the successive generations $\delta_1(\sigma, h) + \sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma, h) \geq 0$; a strict inequality holds if $\widehat{x}(\sigma, h) \neq x^*(\sigma, h)$ (by strict quasi-concavity). So, averaging across $h \in H$,

$\delta_2(\sigma, o) \geq 0$ for all $\sigma \in \Sigma_1$, $\quad \delta_1(\sigma) + \sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma) \geq 0$ for all $\sigma \in \Sigma$,

and the inequality is strict if at least one of the agents born at node σ is being strictly improved.

Feasibility of $\widehat{x}$ and x^* implies that

$\delta_1(\sigma) + \delta_2(\sigma, o) \leq 0$, for $\sigma \in \Sigma_1$, $\delta_1(\sigma) + \delta_2(\sigma, \sigma_{-1}) \leq 0$, for $\sigma \in \cup_{t \geq 2} \Sigma_t$.

Furthermore, since $\widehat{x}$ and x^* differ, there is a finite $\underline{t}$, and a set $\widehat{\Sigma_t \times H} \subseteq \Sigma_t \times H$, for $t \geq \underline{t}$, such that for every agent $(\sigma, h) \in \cup_{t < \underline{t}} (\Sigma_t \times H)$, $\widehat{x}(\sigma, h) = x^*(\sigma, h)$, while for $(\sigma_t, h) \in \widehat{\Sigma_t \times H}$, $\widehat{x}(\sigma_t, h) \neq x^*(\sigma_t, h)$. Hence for all $(\sigma, h) \in \cup_{t < \underline{t}} (\Sigma_t \times H)$, and $\sigma' \in \sigma^+$, we have $\delta_1(\sigma, h) = \delta_2(\sigma', \sigma, h) = 0$. On the other hand, for $\sigma_{\underline{t}}$ such that $(\sigma_{\underline{t}}, h) \in \widehat{\Sigma_{\underline{t}} \times H}$, by feasibility, $\delta_1(\sigma_{\underline{t}}) + \delta_2(\sigma_{\underline{t}}, \sigma_{\underline{t}-1}) \leq 0$; since $\delta_2(\sigma_{\underline{t}}, \sigma_{\underline{t}-1}) = 0$, we have then $\delta_1(\sigma_{\underline{t}}) \leq 0$. Since $\widehat{x}$ was assumed to be an improving allocation, from the strict quasi-concavity and monotonicity of preferences it follows $\sum_{\sigma' \in \sigma_{\underline{t}}^+} \delta_2(\sigma', \sigma_{\underline{t}}) > 0$; hence for *some* $\tilde{\sigma} \in \sigma_{\underline{t}}^+$, $\delta_2(\tilde{\sigma}, \sigma_{\underline{t}}) > 0$. A Pareto improving sequence of reallocations of resources, starting from generation $\sigma_{\underline{t}}$ must therefore be characterized by a positive per capita transfer to the members of this generation in at least one state when old, $\tilde{\sigma} \in \sigma_{\underline{t}}^+$.

Note that $\tilde{\sigma} \in \Sigma_{\underline{t}+1}$. Using again feasibility, we find $\delta_1(\tilde{\sigma}) < 0$, so that by monotonicity, $\sum_{\sigma' \in \tilde{\sigma}^+} \delta_2(\sigma', \tilde{\sigma}) > 0$ and hence $\delta_2(\sigma'', \tilde{\sigma}) > 0$ for *some* $\sigma'' \in \tilde{\sigma}^+$. Identify *all* the nodes that are immediate successors of the node $\tilde{\sigma}$ and at which the old receive a positive per capita transfer. Iterating the argument, we can find a collection of nodes which are successors of $\tilde{\sigma} \in \Sigma_{\underline{t}+1}$ and are characterized by (i) a positive level of per capita transfers to the old agents and (ii) the property that the set of successors to any given node in the set includes all nodes at which the per capita transfer is positive. It is easy to verify that this sequence defines a sub-tree; let it be denoted by $\Gamma_{\tilde{\sigma}}$. By construction, for all $\sigma \in \Gamma_{\tilde{\sigma}}$,

$\delta_2(\sigma, \sigma_{-1}) > 0$.

By strict monotonicity of preferences, $p^*(\sigma) >> 0$ at all nodes; so $\| \hat{p*}(\sigma) \| > 0$ for all $\sigma \in \Sigma$. Furthermore, by condition (i) of Theorem 1 and Assumption 1 (i), $\| \hat{x}(\sigma, h) - x^*(\sigma, h) \| \leq 2KHL(1 + \bar{S})$ for all $(\sigma, h) \in \Sigma \times H$. Hence, by interiority of the allocation and by Lemma 1, for all agents $(\sigma, h) \in \Gamma_{\tilde{\sigma}} \times H$ we have

$$\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma, h) \geq -\delta_1(\sigma, h) + \underline{\rho}_{\sigma,h}(2KHL(1 + \bar{S})) \frac{(\delta_1(\sigma, h))^2}{\| \hat{p*}(\sigma) \|}. \tag{2}$$

Using condition (ii) of Theorem 1, the inequality in (2) continues to hold if we replace $\underline{\rho}_{\sigma,h}(2KHL(1+\bar{S}))$ by $\underline{\rho}$. In addition, notice that the set of pairs $(\delta_1(\sigma, h), \sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma, h)) \in R^2$ which satisfy (2) is a convex set; hence, if we consider the per capita transfer to the H members of generation $\sigma \in \Sigma$, the inequality is still valid:

$$\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma) \geq -\delta_2(\sigma) + \underline{\rho} \frac{(\delta_1(\sigma))^2}{\| \hat{p*}(\sigma) \|}. \tag{3}$$

Now, define the function $\lambda_{\Gamma_{\tilde{\sigma}}} : \Gamma_{\tilde{\sigma}} \to [0, 1]$ by the rule
$\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma') := \frac{\delta_2(\sigma', \sigma)}{\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma)}$.
Given the way in which the sub-tree $\Gamma_{\tilde{\sigma}}$ was constructed, the function $\lambda_{\Gamma_{\tilde{\sigma}}}$ is well defined on its domain, is positive, and satisfies the restriction $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \lambda_{\Gamma_{\tilde{\sigma}}}(\sigma') = 1$ for all $\sigma \in \Gamma_{\tilde{\sigma}}$.

So we have a pair $(\Gamma_{\tilde{\sigma}}, \lambda_{\Gamma_{\tilde{\sigma}}})$ consisting of a sub-tree and a weight function, the latter determined by the relative distribution of the per capita transfers across successor nodes.

For the rest of the proof, we concentrate on $\sigma^\infty(\Gamma_{\tilde{\sigma}})$, an arbitrary path in the sub-tree.

Using the definition of the function $\lambda_{\Gamma_{\tilde{\sigma}}}$, the inequality in the curvature condition, (3), can be rewritten as

$$\frac{1}{\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma')} \delta_2(\sigma', \sigma) \geq -\delta_1(\sigma) + \underline{\rho} \frac{(\delta_1(\sigma))^2}{\| \hat{p*}(\sigma) \|}, \tag{4}$$

i.e. as a condition at a given pair of nodes, and (4) holds for all $\sigma', \sigma \in \Gamma_{\tilde{\sigma}}$. Using the fact that, by feasibility, $\delta_2(\sigma, \sigma_{-1}) + \delta_1(\sigma) = 0$, and that $\delta_2(\sigma, \sigma_{-1}) > 0$ for all $\sigma \in \Gamma_{\tilde{\sigma}}$, we can invert (4) to obtain

$$\frac{\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma')}{\delta_2(\sigma', \sigma)} \leq \frac{1}{\delta_2(\sigma, \sigma_{-1})} - \frac{1}{\| \hat{p*}(\sigma) \|} \frac{\underline{\rho}}{1 + \underline{\rho} \frac{\delta_2(\sigma, \sigma_{-1})}{\| \hat{p*}(\sigma) \|}} \tag{5}$$

for all $\sigma \in \Gamma_{\tilde{\sigma}}$.

By condition (i) of Theorem 1, the per capita endowment of every commodity is bounded at each node. Hence, $0 < \delta_2(\sigma, \sigma_{-1}) \leq 2KL \| \hat{p*}(\sigma) \|$ so that

$$\frac{\underline{\rho}}{1 + 2KL\underline{\rho}} \leq \frac{\underline{\rho}}{1 + \underline{\rho} \frac{\delta_2(\sigma, \sigma_{-1})}{\| \hat{p*}(\sigma) \|}}. \tag{6}$$

Substituting (6) in (5) we have

$$\frac{\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma')}{\delta_2(\sigma', \sigma)} + \frac{\underline{\rho}}{1 + 2KL\underline{\rho}} \frac{1}{\| \hat{p*}(\sigma) \|} \leq \frac{1}{\delta_2(\sigma, \sigma_{-1})}. \tag{7}$$

But by iterating on the inequality (7) along $\sigma^{\infty}(\Gamma_{\tilde{\sigma}})$, the path in the sub-tree that we fixed, we have

$$\frac{\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma_T^{\infty})\cdots\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma_{t(\tilde{\sigma})}^{\infty})}{\delta_2(\sigma_T^{\infty},\sigma_{T-1}^{\infty})}+\frac{\underline{\rho}}{1+2KL\underline{\rho}}\sum_{t=t(\tilde{\sigma})}^{T-1}\frac{\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma_t^{\infty})\cdots\lambda_{\Gamma_{\tilde{\sigma}}}(\sigma_{t(\tilde{\sigma})}^{\infty})}{\parallel p\hat{*}(\sigma_t^{\infty})\parallel}\leq\frac{1}{\delta_2(\tilde{\sigma},\tilde{\sigma}_{-1})} \tag{8}$$

so that the series on the left-hand side of (8), being positive and monotonically increasing, must converge (the other two terms being positive since $\underline{\rho}>0$). Recalling the definition of the function $\widehat{\lambda}_{\Gamma_{\tilde{\sigma}}}$, and defining $A=:\frac{1}{\delta_2(\tilde{\sigma},\tilde{\sigma}_{-1})}$, we have

$$A(\ \sigma^{\infty}(\Gamma_{\tilde{\sigma}});(\Gamma_{\tilde{\sigma}},\lambda_{\Gamma_{\tilde{\sigma}}})):=\sum_{t=t(\tilde{\sigma})}^{\infty}\frac{\widehat{\lambda}_{\Gamma_{\tilde{\sigma}}}(\sigma_t^{\infty})}{\parallel p^*(\ \sigma_t^{\infty})\parallel}\leq A.$$

This completes the proof of the theorem since the argument above was made for an arbitrary path in the sub-tree. ■

PROOF OF THEOREM 2

The proof of Theorem 2 consists in (i) proposing a candidate collection of transfers (of income) which will support an improvement, (ii) verifying that there exists a corresponding collection of commodity transfers which is feasible, i.e. generates commodity bundles in the consumption set of every agent and is compatible with aggregate endowments, and (iii) verifying that the transfers satisfy the bounded Gaussian curvature condition so that the associated commodity transfers generate a Pareto improvement.

Step 1: We begin by specifying the transfers at every $\sigma\in\Gamma_{\hat{\sigma}}$. Since each node can be identified with a coordinate of some path in the sub-tree, it suffices to define the transfers for each coordinate along each path in the sub-tree. Consequently, for $t\geq t(\hat{\sigma})$ we define

$$\delta(\sigma_{t+1}^{\infty}):=\frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^{\infty})\sum_{\tau=t(\hat{\sigma})}^{t}\frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{\tau}^{\infty})}{\parallel p^*(\ \sigma_{\tau}^{\infty})\parallel}$$

where $\kappa:=\frac{1}{(1+\bar{S})^{1/2}}\min\{\varepsilon,\bar{k}\}$ and (a) $\varepsilon>0$ is as specified in hypothesis (i) of Theorem 2 (b) $\bar{k}>0$ and $\bar{\rho}>0$ are as specified in hypothesis (ii) of Theorem 2, (c) $\bar{S}\geq 1$ is the maximal number of successor nodes (see Assumption 1 (i)), and the function $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}$ is induced by the pair $(\Gamma_{\hat{\sigma}},\lambda_{\Gamma_{\hat{\sigma}}})$, the sub-tree and weight function, specified in the theorem.

Set $\delta(\sigma):=0$ for $\sigma\notin\Gamma_{\hat{\sigma}}$.

Step 2: We now specify the commodity transfers. For $\sigma \in \Gamma$, define the vectors $\Delta x_\sigma \in R\hat{L}$ as follows: $\Delta x_\sigma := \frac{p^*(\sigma)}{\|p^*(\sigma)\|}\frac{\delta(\sigma)}{\|p^*(\sigma)\|}$. Now let $\Delta x(\sigma, h) := (-\Delta x_\sigma, (\Delta x_{\sigma'})_{\sigma' \in \sigma^+})$ for agents h_σ as defined in the statement of Theorem 2, and set $\Delta x(\sigma, h) := 0 \in R^L$ for $h \neq h_\sigma$.

The consumption vectors induced by the proposed transfers are $\widehat{x}(\sigma, h) := x\hat{*}(\sigma, h) + \Delta x(\sigma, h)$.

Evidently, these transfer vectors are compatible with the aggregate endowments of the economy. In order to be able to use the inequality defined by the bounded Gaussian curvature condition (Definition 5) we need to check that these vectors lie in the consumption sets of the agents, and satisfy the inequality $\| \Delta x(\sigma, h) \| \leq \bar{k}$. We now check these two properties.

By the hypothesis of Theorem 2, $\frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^\infty)}{\|p\hat{*}(\sigma_t^\infty)\|} \leq A$ for all nodes $\sigma_t^\infty \in \Gamma_{\hat{\sigma}}$, and $\sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^\infty)}{\|p^*(\ \sigma_\tau^\infty)\|} \leq A$, so that A is a uniform bound for the set of partial sums over paths in the sub-tree. Hence

$$A^2 \geq \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^\infty)}{\| p^*(\sigma_{t+1}^\infty) \|} \sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^\infty)}{\| p^*(\ \sigma_\tau^\infty) \|}. \tag{9}$$

Furthermore, since $\kappa > 0$ and $\bar{\rho} > 0$,

$$1 > \frac{1}{1 + \kappa\bar{\rho}} \quad \Rightarrow \quad \kappa > \frac{\kappa}{1 + \kappa\bar{\rho}}. \tag{10}$$

Combining (9) and (10), and using the definition of $\delta(\sigma)$ we have

$\kappa > \frac{\kappa}{1+\kappa\bar{\rho}} \cdot \frac{1}{A^2}\frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^\infty)}{\|p^*(\sigma_{t+1}^\infty)\|} \sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^\infty)}{\|p^*(\ \sigma_\tau^\infty)\|}$
$= \frac{\delta(\sigma_{t+1}^\infty)}{\|p^*(\sigma_{t+1}^\infty)\|}$.

So, for every node σ and for $l = 1, 2, \cdots, L$,

$\Delta x_\sigma^l := \frac{p_l^*(\sigma)}{\|p^*(\sigma)\|}\frac{\delta(\sigma)}{\|p^*(\sigma)\|} < \kappa$.

But by the definition of κ, $\varepsilon > \kappa$ so that the sequence of commodity transfer vectors, Δx_σ, is bounded above in each coordinate by ε (it is obviously bounded below by zero). Invoking hypothesis (i) of Theorem 2, we see that $\widehat{x}(\sigma, h) \in X_{\sigma,h}$ for each agent. Furthermore, we easily see that, in the Euclidean norm,

$\| \Delta x_\sigma \| = \| \frac{p^*(\sigma)}{\|p^*(\sigma)\|}\frac{\delta(\sigma)}{\|p^*(\sigma)\|} \| = \frac{\delta(\sigma)}{\|p^*(\sigma)\|} < \kappa$.

Hence, in the Euclidean norm,

$\| \Delta x(\sigma, h) \| = \| (-\Delta x_\sigma, (\Delta x_{\sigma'})_{\sigma' \in \sigma^+}) \| < \kappa\ (1 + \bar{S})^{1/2} \leq \bar{k}$

where the first inequality follows from the fact each of the $(1 + \bar{S})$ L-dimensional vectors Δx_σ and $(\Delta x_{\sigma'})$, for $\sigma' \in \sigma^+$, have norms bounded by κ, so that the Euclidean norm of $\Delta x(\sigma, h)$ is bounded by $\kappa(1 + \bar{S})^{1/2}$, and the second inequality follows from the definition of κ. Thus, we have $\| \Delta x(\sigma, h) \| \leq \bar{k}$ as required.

Step 3: We complete the proof by showing that the allocation generated by the above specification of the transfers is improving. We show that the inequality in the bounded Gaussian curvature condition is satisfied for agents who are born at nodes which belong to the sub-tree $\Gamma_{\hat{\sigma}}$ and receive a transfer, i.e. the agents h_σ; since all the other conditions imposed in Definition 5 on the vectors of transfers are readily seen to be satisfied by the vectors $\Delta x(\sigma, h)$, verification of the inequality guarantees that the agents are being

weakly improved. This is sufficient since the old agent at the node $\hat{\sigma}$, receives a positive transfer of every commodity when old and is strictly improved by strict monotonicity of preferences.

By construction, the income transfers, corresponding to the commodity transfers $\Delta x(\sigma, h_{\sigma_t})$, received by the agent h_{σ_t} in the first period of his life and in the second period at the node σ_{t+1}^{∞}, are given by, respectively[28]

$\delta_1(\sigma_t^{\infty}, h_{\sigma_t}) = -\delta(\sigma_t^{\infty}) = -\frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}) \sum_{\tau=t(\hat{\sigma})}^{t-1} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|p^*(\ \sigma_\tau^{\infty})\|}$

$\delta_2(\sigma_{t+1}^{\infty}, \sigma_t^{\infty}, h_{\sigma_t}) = \delta(\sigma_{t+1}^{\infty}) = \frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^{\infty}) \sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|p^*(\ \sigma_\tau^{\infty})\|}$.

Since, by the defining property of the function $\lambda_{\Gamma_{\hat{\sigma}}}$, we have $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} \lambda_{\Gamma_{\hat{\sigma}}}(\sigma') = 1$ for all $\sigma \in \Gamma_{\hat{\sigma}}$, and recalling that $\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^{\infty}) = \lambda_{\Gamma_{\hat{\sigma}}}(\sigma_{t+1}^{\infty}) \cdot \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty})$, we have

$\sum_{\sigma' \in \sigma_t \hat{+}} \delta_2(\sigma', \sigma_t, h_{\sigma_t}) = \frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}) \sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|p^*(\ \sigma_\tau^{\infty})\|}$

where we have used the fact that transfers are made only on nodes in the sub-tree $\Gamma_{\hat{\sigma}}$, so that

$\sum_{\sigma' \in \sigma_t \hat{+}} \delta_2(\sigma', \sigma_t, h_{\sigma_t}) = \sum_{\sigma' \in \sigma_t \hat{+} \cap \Gamma_{\hat{\sigma}}} \delta_2(\sigma', \sigma_t, h_{\sigma_t})$.

Now it is immediate from the definitions of the transfers that

$\delta_1(\sigma_t, h_{\sigma_t}) + \sum_{\sigma' \in \sigma_t \hat{+}} \delta_2(\sigma', \sigma_t, h_{\sigma_t}) = \frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}) \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty})}{\|p^*(\sigma_t^{\infty})\|}$

$\Rightarrow \quad [\delta_1(\sigma_t, h_{\sigma_t}) + \sum_{\sigma' \in \sigma_t \hat{+}} \delta_2(\sigma', \sigma_t, h_{\sigma_t})]\ \|\ p\hat{*}(\sigma_t)\ \| = \frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}} (\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}))^2$.

Since $\bar{\rho}_{\sigma,h_\sigma}(\bar{k}) \leq \bar{\rho}$ and $\kappa > 0$, we have

$1 \geq \frac{\kappa\bar{\rho}_{\sigma_t,h_{\sigma_t}}(\bar{k})}{1+\kappa\bar{\rho}}$

and from the uniform bound over all paths in the sub-tree we have

$A \geq \sum_{\tau=t(\hat{\sigma})}^{t} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|p^*(\ \sigma_\tau^{\infty})\|}$.

It follows that

$$[\delta_1(\sigma_t, h_{\sigma_t}) + \sum_{\sigma' \in \sigma_t \hat{+}} \delta_2(\sigma', \sigma_t, h_{\sigma_t})]\ \|\ p\hat{*}(\sigma_t)\ \| \geq \frac{\kappa\bar{\rho}_{\sigma_t,h_{\sigma_t}}(\bar{k})}{1+\kappa\bar{\rho}} \left(\frac{\sum_{\tau=t(\hat{\sigma})}^{t-1} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|p^*(\ \sigma_\tau^{\infty})\|}}{A} \right) \hat{2} \cdot$$

$$\cdot \frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}} (\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}))^2$$

$$= \bar{\rho}_{\sigma_t,h_{\sigma_t}}(\bar{k}) \left(\frac{\kappa}{(1+\kappa\bar{\rho})A\hat{2}}\ \widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^{\infty}) \left(\sum_{\tau=t(\hat{\sigma})}^{t-1} \frac{\widehat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_\tau^{\infty})}{\|\ p^*(\ \sigma_\tau^{\infty})\ \|} \right) \right) \hat{2}$$

$$= \bar{\rho}_{\sigma_t,h_{\sigma_t}}(\bar{k})(\delta_1(\sigma_t, h_{\sigma_t}))\hat{2}.$$

So, the transfers proposed verify the inequality condition in Definition 5. ■

[28] In what follows, we will identify σ_t^{∞} and σ_t etc.

PROOF OF THEOREM 3

The proof of Theorem 3 proceeds by contradiction. As in Theorem 1, we start by showing that if there exists a CPO improvement then necessarily there exists a sub-tree with the property that the per capita transfer to the old agents at each node σ is strictly positive. We then show that given the criterion that we choose to aggregate the per capita transfers across nodes at the same point in time, the non-vanishing Gaussian curvature condition implies that the aggregated per capita transfer increases according to a quadratic function. Next we use the uniform bound on endowments to show that this quadratic increase can occur only if the sum of the norms of prices at the same point in time increases sufficiently fast. This gives a Cass-like criterion. The proof is completed by considering an arbitrary node in the sub-tree that has been identified and by applying the same argument to the sequence of aggregated per capita transfers obtained by starting from that node.

We omit some of the details of the proof of Theorem 3 referring instead to the proof of Theorem 1.

Suppose that the allocation x^* is not CPO so that there exists an improvement; let $\widehat{x}$ denote the improving allocation. As in the proof of Theorem 1, construct the sub-tree $\Gamma_{\tilde{\sigma}}$. By construction, for all $\sigma \in \Gamma_{\tilde{\sigma}}$, $\delta_2(\sigma, \sigma_{-1}) > 0$. By the same argument as in the proof of Theorem 1, we obtain, for all $\sigma \in \Sigma$

$$\frac{\sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma)}{\| \hat{p*}(\sigma) \|} \geq -\frac{\delta_1(\sigma)}{\| \hat{p*}(\sigma) \|} + \underline{\rho} \left(\frac{\delta_1(\sigma)}{\| \hat{p*}(\sigma) \|} \right) \hat{2}. \tag{11}$$

If $\sigma \in \Gamma_{\tilde{\sigma}}$, (11) holds a fortiori if on the left hand side of the expression we replace the numerator with $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma)$, i.e. if we only consider the sum of average transfers to the agents when old over the subset (defined by $\Gamma_{\tilde{\sigma}}$) of the collection of nodes at which these transfers are strictly positive.

Let σ_τ be an arbitrary node of $\Gamma_{\tilde{\sigma}}$, at period $\tau = t(\sigma_\tau)$, and consider $\Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})$. Recall that $\Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})$ is the subtree that has σ_τ as its root and includes all the nodes that are successors of σ_τ and are elements of $\Gamma_{\tilde{\sigma}}$. If we consider the average across all nodes[29] at t, $\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})$, for $t \geq \tau$, of the transfers $(\delta_1(\sigma), \sum_{\sigma' \in \sigma^+} \delta_2(\sigma', \sigma))$, the inequality in (11) is still valid so we get

$$\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \frac{\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma)}{\| \hat{p*}(\sigma) \|} \frac{\| \hat{p*}(\sigma) \|}{\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \| \hat{p*}(\sigma) \|} \geq$$

$$\geq -\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \frac{\delta_1(\sigma)}{\| \hat{p*}(\sigma) \|} \frac{\| \hat{p*}(\sigma) \|}{\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \| \hat{p*}(\sigma) \|} +$$

$$+\underline{\rho} \left(\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \frac{\delta_1(\sigma)}{\| \hat{p*}(\sigma) \|} \frac{\| \hat{p*}(\sigma) \|}{\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \| \hat{p*}(\sigma) \|} \right) \hat{2}$$

$$\Rightarrow \quad \frac{\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \sum_{\sigma' \in \sigma^+ \cap \Gamma_{\tilde{\sigma}}} \delta_2(\sigma', \sigma)}{\sum_{\sigma \in \Sigma_t \cap \Gamma(\sigma_\tau, \Gamma_{\tilde{\sigma}})} \| \hat{p*}(\sigma) \|} \geq$$

[29] By Assumption 1(i) the numbers of such nodes is always finite.

$$\geq -\frac{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\delta_1(\sigma)}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|} + \underline{\rho}\left(\frac{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\delta_1(\sigma)}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|}\right)\hat{2}$$

To simplify notation, let us write $\delta_2(t+1,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))$ for $\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\sum_{\sigma'\in\sigma^+\cap\Gamma_{\hat{\sigma}}}\delta_2(\sigma',\sigma)$, i.e. to denote the total transfer to the old at time $t+1$ on the subtree $\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})$. Notice that $\delta_2(\tau,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})) = \delta_2(\sigma_\tau,\sigma_{\tau-1}) > 0$ since $\sigma_\tau \in \Gamma_{\tilde{\sigma}}$. In addition, notice that, by feasibility $-\delta_1(\sigma) = \delta_2(\sigma,\sigma_{-1})$, so that the above inequality can be rewritten as follows:

$$\Rightarrow \quad \delta_2(t+1,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})) \geq \delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})) + \underline{\rho}\frac{\Big(\delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))\Big)\hat{2}}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|}$$

$$\Rightarrow \quad \frac{1}{\delta_2(t+1,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))} \leq \frac{1}{\delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))} -$$

$$- \frac{1}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|}\frac{\underline{\rho}}{1+\underline{\rho}\frac{\delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\|\hat{p*}(\sigma)\|}} \tag{12}$$

for all $t \geq \tau$, since, by the properties of $\Gamma_{\tilde{\sigma}}$, $\delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})) > 0$ for all $t \geq \tau$.

By condition (i) of the Theorem, the per capita endowment of every commodity is bounded at each node. Hence, for all $t \geq \tau$, $0 < \delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})) \leq 2KL\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|$. Since the inequality holds for every period, for all $t \geq 1$ we have

$$\frac{\underline{\rho}}{1+2KL\underline{\rho}} \leq \frac{\underline{\rho}}{1+\underline{\rho}\frac{\delta_2(t,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\|\hat{p*}(\sigma)\|}} \tag{13}$$

Substituting then (13) into (12) and summing the inequalities we obtain from $t = \tau$ to $t = T$ yields

$$\Rightarrow \quad \frac{1}{\delta_2(T+1,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))} + \sum_{t=\tau}^{T}\frac{1}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})}\| \hat{p*}(\sigma) \|}\frac{\underline{\rho}}{1+2KL\underline{\rho}} \leq$$

$$\leq \frac{1}{\delta_2(\tau,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))} = \frac{1}{\delta_2(\sigma_\tau,\sigma_{\tau-1})} \tag{14}$$

The terms $\delta_2(\sigma_\tau,\sigma_{\tau-1})$ and $\delta_2(T+1,\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}}))$ are, by construction, strictly positive; hence the series on the left-hand side of (14), being positive and monotonically increasing, must converge.

Since σ_τ was chosen arbitrarily, the same is true for any other possible choice of σ_τ. Consequently, the above argument shows that the existence of an improving allocation implies that there exists a sub-tree $\Gamma_{\tilde{\sigma}}$ such that for all nodes $\sigma_\tau \in \Gamma_{\tilde{\sigma}}$,

$$A(\sigma_\tau,\Gamma_{\tilde{\sigma}}) := \sum_{t=\tau}^{\infty}\frac{1}{\sum_{\sigma\in\Sigma_t\cap\Gamma(\sigma_\tau,\Gamma_{\tilde{\sigma}})} \| \hat{p*}(\sigma) \|} < \infty$$

So if for every sub-tree Γ_σ, there is a node $\bar{\sigma} \in \Gamma_\sigma$ for which the series $A(\bar{\sigma},\Gamma_\sigma)$ diverges, the allocation must be CPO. ■

PROOF OF COROLLARY 1

For any sub-tree Γ_σ, and $t \geq t(\sigma)$,
$\sum_{\hat{\sigma}\in\Sigma_t} \parallel p\hat{*}(\hat{\sigma}) \parallel \geq \sum_{\hat{\sigma}\in\Sigma_t\cap\Gamma_\sigma} \parallel p\hat{*}(\hat{\sigma}) \parallel$

$$\Rightarrow \qquad \mathrm{Lim}_{T\to\infty} \sum_{t=t(\sigma)}^{\hat{T}} \frac{1}{\sum_{\hat{\sigma}\in\Sigma_t\cap\Gamma_\sigma} \parallel p\hat{*}(\hat{\sigma}) \parallel} \geq$$

$$\geq \mathrm{Lim}_{T\to\infty} \sum_{t=1}^{\hat{T}} \frac{1}{\sum_{\hat{\sigma}\in\Sigma_t} \parallel p\hat{*}(\hat{\sigma}) \parallel} - \sum_{t=1}^{\hat{t}(\sigma)-1} \frac{1}{\sum_{\hat{\sigma}\in\Sigma_t} \parallel p\hat{*}(\hat{\sigma}) \parallel}. \tag{15}$$

Since for any sub-tree Γ_σ, $t(\sigma)$ is finite, the last term in (15) is always a finite number. Consequently, the divergence of the first term on the right hand side of (15) implies the divergence of the left hand side. This in turn implies the divergence of $A(\bar{\sigma},\Gamma_\sigma)$ for some $\bar{\sigma} \in \Gamma_\sigma$ (since $\sum_{\hat{\sigma}\in\Sigma_t\cap\Gamma_\sigma} \parallel p\hat{*}(\hat{\sigma}) \parallel \geq \sum_{\hat{\sigma}\in\Sigma_t\cap\Gamma(\bar{\sigma},\Gamma_\sigma)} \parallel p\hat{*}(\hat{\sigma}) \parallel$); we can then apply Theorem 3 to get the result. ■

PROOF OF CLAIM IN REMARK 3

Modifying Peled's notation slightly, let $(R(\sigma';\sigma))_{\sigma'\in\sigma^+}$ be the vector of random (real) rates of return on money, at node σ, and $(q(\sigma';\sigma))_{\sigma'\in\sigma^+}$ the vector of the supporting one-period contingent claim prices, the vector being uniquely determined by the marginal rates of substitution of the only agent in generation σ. At a monetary equilibrium, the present value, evaluated at σ, of the return on money at node $\sigma' \in \sigma^+$ is $\lambda(\sigma';\sigma) := R(\sigma';\sigma)q(\sigma';\sigma) > 0$; hence, the present value of the one-period return on money is $\sum_{\sigma'\in\sigma^+} R(\sigma';\sigma)q(\sigma';\sigma) = \sum_{\sigma'\in\sigma^+} \lambda(\sigma';\sigma) = 1$.

Peled [21] has shown that a sufficient condition for CPO, in addition to the requirement that all the numbers $R(\sigma';\sigma)$ lie in a compact subset of the set of positive real numbers, is that there exists $\epsilon > 0$ such that, for every path σ^∞ and every t, $\Pi_{\tau=0}^t R(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty) = \frac{\Pi_{\tau=0}^t \lambda(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty)}{\Pi_{\tau=0}^t q(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty)} \geq \epsilon$.

As we noticed, there is a one-to-one relationship (already exploited in Section 4) between the system of one-period contingent prices $\{q_{ss'}\}$ at all nodes and the system of Arrow-Debreu prices $\{p(\sigma)\}$; in particular, we have here $\Pi_{\tau=0}^t q(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty) = p(\sigma_{t+1}^\infty)$, so that the above sufficient condition can also be written $\Pi_{\tau=0}^t \lambda(\sigma_{\tau+1}^\infty,\sigma_\tau^\infty) \geq \epsilon p^*(\sigma_{t+1}^\infty)$. Summing both terms across the different first $t+1$ elements of all paths, we get $\sum_{\sigma_{t+1}^\infty\in\Sigma_{t+1}} \Pi_{\tau=0}^t \lambda(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty) \geq \epsilon(\sum_{\sigma_{t+1}^\infty\in\Sigma_{t+1}} p^*(\sigma_{t+1}^\infty))$. The term on the left hand side can also be written as $\sum_{\sigma_t^\infty\in\Sigma_t} \Pi_{\tau=0}^{t-1} \lambda(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty)[\sum_{\sigma'\in\sigma_t^{\infty+}} \lambda(\sigma';\sigma_t^\infty)]$; since $\sum_{\sigma'\in\sigma^+} \lambda(\sigma';\sigma) = 1$ for every σ, iterating the argument we get

$\sum_{\sigma_{t+1}^\infty\in\Sigma_{t+1}} \Pi_{\tau=0}^t \lambda(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty) = \sum_{\sigma_t^\infty\in\Sigma_t} \Pi_{\tau=0}^{t-1} \lambda(\sigma_{\tau+1}^\infty;\sigma_\tau^\infty) = \cdots = 1.$

This shows that the sufficient condition for CPO given by Peled indeed implies $1 \geq \epsilon\sum_{\sigma\in\Sigma_t} p^*(\sigma)$, or equivalently $\frac{1}{\sum_{\sigma\in\Sigma_t} p^*(\sigma)} \geq \epsilon > 0$.

PROOF OF LEMMA 2

To ease the burden of notation, throughout the proof we write p and Q instead of p^* and Q^*.

We first prove claim (ii). Consider the sum of prices over all nodes at a given date t such that the state at t is some prespecified $\tilde{s} \in \mathcal{S}$, $\sum_{(\sigma_0,s_1,\cdots,\tilde{s})\in\Sigma_t} p(\sigma_0, s_1, \cdots, \tilde{s})$, and denote this term by $p(\tilde{s}, t)$. Recalling that $p(\sigma_0, s_1, \cdots, s_{t-1}, \tilde{s}) = p(\sigma_0, s_1, \cdots, s_{t-1})q_{s_{t-1}\tilde{s}}$, we get $p(\tilde{s}, t) = \sum_{s\in S} p(s, t-1)q_{s\tilde{s}}$. But then $(p(1, t), \cdots, p(S, t))' = (p(1, t-1), \cdots, p(S, t-1))\cdot Q$, so that, iterating the argument,

$(p(1, t), \cdots, p(S, t))' = (p(1, 1), \cdots, p(S, 1)) \cdot Q^{t-1} = q_{\sigma_0}Q^{t-1}$,

where q_{σ_0} is a row of the matrix Q corresponding to the state at the initial date.[30] But then,

$\sum_{\sigma\in\Sigma_t} p(\sigma) = \sum_{s\in S} p(s, t) = (p(1, t), \cdots, p(S, t)) \cdot 1_{S\times 1} = q_{\sigma_0}Q^{t-1}1_{S\times 1}$

where $1_{S\times 1}$ is an S-dimensional vector whose elements are all equal to 1.

Since Q is a strictly positive matrix, by Perron's Theorem (see, e.g. Horn and Johnson [15, Theorem 8.2.8]) $[\lambda(Q)^{-1}Q]^t \to L$ as $t \to \infty$, where L is also a strictly positive matrix. So, for any $\epsilon > 0$, there exists $t(\epsilon)$ such that, for all $t > t(\epsilon)$, $\epsilon > \| [\lambda(Q)^{-1}Q]^t - L \|$ for any norm. Hence, the same is true when we consider the difference of the two matrices componentwise, so that

$L_{ij} - \epsilon < [\lambda(Q)^{-1}Q]^t_{ij} < L_{ij} + \epsilon$ for all $(i, j) \in S \times S$ and $t > t(\epsilon)$

$\Leftrightarrow \quad L - \epsilon 1_{S\times S} < [\lambda(Q)^{-1}Q]^t < \quad L + \epsilon 1_{S\times S}$ for all $t > t(\epsilon)$

where $1_{S\times S}$ is a matrix with all elements equal to 1. Pre- and post-multiplying respectively by q_{σ_0} and $1_{S\times 1}$ yields

$q_{\sigma_0}(L - \epsilon 1_{S\times S})1_{S\times 1} < [q_{\sigma_0}\lambda(Q)^{-t}Q^t 1_{S\times 1}] < q_{\sigma_0}(L + \epsilon 1_{S\times S})1_{S\times 1}$ for all $t > t(\epsilon)$,

$\Leftrightarrow \quad \frac{1}{q_{\sigma_0}(L-\epsilon 1_{S\times S})1_{S\times 1}}\frac{1}{\lambda(Q)^t} > \frac{1}{\sum_{\sigma\in\Sigma_t} p(\sigma)} > \frac{1}{q_{\sigma_0}(L+\epsilon 1_{S\times S})1_{S\times 1}}\frac{1}{\lambda(Q)^t}$ for all $t > t(\epsilon)$

so that (since q_{σ_0} is strictly positive)

$\lambda(Q) \leq 1 \quad \Rightarrow \quad \sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma\in\Sigma_t} p(\sigma)}$ diverges and

$\lambda(Q) > 1 \quad \Rightarrow \quad \sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma\in\Sigma_t} p(\sigma)}$ converges.

The validity of claim (i) is established next.

By Perron's Theorem, since Q is strictly positive, there exists a vector $x \in R^S_{++}$, $\| x \| = 1$ such that $Qx = \lambda(Q)x$ (see, e.g. Horn and Johnson [15, Theorem 8.2.2]). Define the weight function as follows:

$\lambda_{\Gamma_{\sigma_0}}(\sigma_0) = 1$ and $\lambda_{\Gamma_{\sigma_0}}(\sigma_0, s_1, \cdots, s_t) = \frac{q_{s_{t-1}s_t}x_{s_t}}{\lambda(Q)x_{s_{t-1}}}$.

Note that $\lambda_{\Gamma_{\sigma_0}}(.)$ will always take positive values and $\sum_{s_t\in S} \lambda_{\Gamma_{\sigma_0}}(\sigma_0, s_1, \cdots, s_t) = 1$ for all nodes $(\sigma_0, s_1, \cdots, s_{t-1})$ since $\sum_{s'\in\mathcal{S}} q_{ss'}x_{s'} = \lambda(Q)x_s$ for all $s \in \mathcal{S}$.

The associated map $\widehat{\lambda}_{\Gamma_{\sigma_0}}(.)$ satisfies

$\frac{\widehat{\lambda}_{\Gamma_{\sigma_0}}(\sigma_t^\infty)}{p(\sigma_t^\infty)} = \Pi_{\tau=1}^{\tau=t} \frac{1}{q_{s_{\tau-1}s_\tau}}\frac{q_{s_{\tau-1}s_\tau}x_{s_\tau}}{\lambda(Q)x_{s_{\tau-1}}} = \frac{q_{\sigma_0 s_1}x_{s_1}\cdots q_{s_{t-1}s_t}x_{s_t}}{q_{\sigma_0 s_1}\cdots q_{s_{t-1}s_t}\lambda(Q)x_{\sigma_0}\cdots\lambda(Q)x_{s_{t-1}}} = \frac{x_{s_t}}{(\lambda(Q))^t x_{\sigma_0}}$.

So we have $\sum_{t=1}^{T} \frac{\widehat{\lambda}_{\Gamma_{\sigma_0}}(\sigma_t^\infty)}{p(\sigma_t^\infty)} = \sum_{t=1}^{T} \frac{x_{s_t}}{(\lambda(Q))^t\cdot x_{\sigma_0}}$. Since $\| x \| = 1$, it follows that if $\lambda(Q) > 1$

[30] The argument is very similar to the one yielding the conditional probability, for a Markov process with stationary transition probabilities, of the realization of a given state s at date t given an arbitrary initial distribution (see, e.g. Doob [10, pages 170-185]).

then there exists a weight function, $\lambda_{\Gamma_{s_0}}$, and a scalar $A < \infty$ such that, for every path $\sigma^\infty(\Gamma_{\sigma_0})$,

$$A(\ \sigma^\infty(\Gamma_{\sigma_0}); (\Gamma_{\sigma_0}, \lambda_{\Gamma_{\sigma_0}})) := \sum_{t=1}^{\infty} \frac{\widehat{\lambda}_{\Gamma_{\sigma_0}}(\sigma_t^\infty)}{p^*(\ \sigma_t^\infty)} \leq A.$$

But then the result in claim (i) follows easily by invoking the result in claim (ii). ■

PROOF OF THEOREM 4

The main part of the result follows almost immediately from Lemma 2, using Corollary 1 and Theorem 1.

If $\lambda(Q) \leq 1$, by Lemma 2(ii) we get in fact $\sum_{t=1}^{\infty} \frac{1}{\sum_{\sigma \in \Sigma_t} p(\sigma)}$ diverges; hence CPO follows by Corollary 1.

On the other hand, if $\lambda(Q) > 1$, applying results (ii) and (i) of Lemma 2 and then Theorem 2 we find that CPO necessarily fails.

What remains to be shown is that in the latter case (i.e. if $\lambda(Q) > 1$) there exists an alternative stationary allocation which is Pareto improving. A proof of this was already in Aiyagari-Peled [1] and Gottardi [14]. We provide here a partly different argument which more closely parallels the ones of the results of the earlier sections.

Since Q is strictly positive, by Perron's Theorem we know there exists a vector $x \in R^S_{++}$, $\| x \| = 1$ such that $Qx = \lambda(Q)x$. Since $\lambda(Q) > 1$, we have then $Qx > x$, i.e. $\sum_{s' \in \mathcal{S}} q_{ss'} x_{s'} - x_s > 0$ for all $s \in \mathcal{S}$.

Define $\bar{\rho} := \max_{s \in \mathcal{S}, h \in H} \bar{\rho}_{s,h}(2KH(1+S))$, where $\bar{\rho}_{s,h}(2KH(1+S))$ is well defined and positive by Assumption 1 and Lemma 1; $\alpha := \min_{s \in \mathcal{S}} \{\sum_{s' \in \mathcal{S}} q_{ss'} x_{s'} - x_s\}$, $\beta := \frac{\alpha}{\bar{\rho} \max_{s \in \mathcal{S}} x_s^2}$. Note that $\alpha > 0$ and $\alpha\beta = \bar{\rho}\beta^2 \max_{s \in \mathcal{S}} x_s^2$. Considering next the vector βx we see that

$$\begin{aligned} &\textstyle\sum_{s' \in \mathcal{S}} q_{ss'}(\beta x_{s'}) - (\beta x_s) \geq \beta \min_{s \in \mathcal{S}} \{\sum_{s'} q_{ss'} x_{s'} - x_s\} = \\ &\qquad = \alpha\beta = \bar{\rho}\beta^2 \max_{s \in \mathcal{S}} x_s^2 \\ \Rightarrow \quad &\textstyle\sum_{s' \in \mathcal{S}} q_{ss'}(\beta x_{s'}) - (\beta x_s) \geq \bar{\rho}(\beta x_s)^2 \ \text{ for all } \ s \in \mathcal{S} \end{aligned}$$

$$\Rightarrow \quad \sum_{s' \in \mathcal{S}} p^*(\sigma) q_{ss'}(\beta x_{s'}) - (p^*(\sigma)\beta x_s) \geq \bar{\rho} \frac{(p^*(\sigma)\beta x_s)^2}{p^*(\sigma)} \text{ for all } s \in \mathcal{S} \tag{16}$$

Define $\hat{x}(s', s; s, \tilde{h}) := x^*(s', s; s, \tilde{h}) + \gamma\beta x_{s'}$, $\hat{x}(s; s, \tilde{h}) := x^*(s; s, \tilde{h}) - \gamma\beta x_s$ for some $\tilde{h}$ and for $s, s' \in \mathcal{S}$, where $\gamma \in (0, 1]$ is suitably chosen[31] so that $\hat{x}(s, \tilde{h}) \in R^{1+S}_{++}$ and $\| \hat{x}(s, \tilde{h}) - x^*(s, \tilde{h}) \| \leq 2KH(1+S)$ for all $s \in \mathcal{S}$. Set $\hat{x}(s, h) = x^*(s, h)$ for all $h \neq \tilde{h}$. So, we have constructed a feasible stationary allocation for which, recalling again that $(p^*(\sigma_0, s_1, \cdots, s_{t-1}, s)_{s \in \mathcal{S}}) = (p^*(\sigma_0, s_1, \cdots, s_{t-1}) q^*_{s_{t-1}s})_{s \in \mathcal{S}})$ and using (16), we see that

$$\begin{aligned} &\textstyle\sum_{s \in \mathcal{S}} p^*(\sigma_0, s_1, \cdots, s_{t-1}, s)[\hat{x}(s, s_{t-1}; s_{t-1}, \tilde{h}) - x^*(s, s_{t-1}; s_{t-1}, \tilde{h})] + \\ &\quad p^*(\sigma_0, s_1, \cdots, s_{t-1})[\hat{x}(s_{t-1}; s_{t-1}, \tilde{h}) - x^*(s_{t-1}; s_{t-1}, \tilde{h})] \geq \\ &\qquad \geq \bar{\rho} \tfrac{(p^*(\sigma_0, s_1, \cdots, s_{t-1})[\hat{x}(s_{t-1}; s_{t-1}, \tilde{h}) - x^*(s_{t-1}; s_{t-1}, \tilde{h})])^2}{p^*(\sigma_0, s_1, \cdots, s_{t-1})} \ \text{ for all } (\sigma_0, s_1, \cdots, s_{t-1}), t \end{aligned}$$

i.e. the bounded Gaussian curvature condition is satisfied for the members of generations born at any t affected by the transfer. Noting that the initial old (of type $\tilde{h}$) receive a

[31] Since $x^*(s, \tilde{h}) \in R^{1+S}_{++}$, this is always possible.

positive transfer, we conclude that the alternative stationary allocation $\hat{x}$ is indeed CPO improving. ■

PROOF OF THEOREM 5

The result follows by a fairly immediate reformulation of the argument of the proofs of Theorems 1 and 2 where the reformulation reflects the new form of the Gaussian curvature conditions for the economy with complete spot markets and bonds. There is only one extra piece of argument in the proof of sufficiency which we need to add.

Note that if the non-vanishing Gaussian curvature condition holds for all agents at a given node, by taking averages across agents we get:

$$\sum_h \left[\frac{1}{H}\frac{\bar{\mu}(\sigma',\sigma,h)}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))}\delta_2(\sigma',\sigma,h)\right] \geq -\delta_1(\sigma) + \underline{\rho}\frac{(\delta_1(\sigma))^2}{\parallel \bar{p}(\sigma) \parallel}. \tag{17}$$

Since both $\bar{\mu}(\sigma',\sigma,h)$ and $\delta_2(\sigma',\sigma,h)$ are nonnegative, for all (σ',σ,h), we also have:

$$H\Big[\sum_h \frac{1}{H}\frac{\bar{\mu}(\sigma',\sigma,h)}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))}\Big]\Big[\sum_h \frac{1}{H}\delta_2(\sigma',\sigma,h)\Big] \geq \sum_h \left[\frac{1}{H}\frac{\bar{\mu}(\sigma',\sigma,h)}{(\sum_{\sigma'\in\sigma^+}\bar{\mu}(\sigma',\sigma,h))}\delta_2(\sigma',\sigma,h)\right].$$

Hence the inequality in (17) can also be rewritten as

$$H\bar{\mu}(\sigma')\delta_2(\sigma',\sigma) \geq -\delta_1(\sigma) + \underline{\rho}\frac{(\delta_1(\sigma))^2}{\parallel \bar{p}(\sigma) \parallel}.$$

The above expression is the same, except for the fact that $H\bar{\mu}(\sigma')$ appears instead of $\frac{1}{\lambda_{\Gamma_{\tilde{\sigma}}(\sigma')}}$, as condition (4) obtained in the proof of Theorem 1. The rest of the argument can then proceed as in the proof of Theorem 1, simply by replacing $\frac{1}{\lambda_{\Gamma_{\tilde{\sigma}}(\sigma')}}$ with $H\bar{\mu}(\sigma')$.■

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