ROAD REVISITED:
UNIQUENESS AND SUPERMODULARITY *

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THE CIRCULAR ROAD REVISITED:

UNIQUENESS AND SUPERMODULARITY

Martin Peitz

A B S T R A C T

The model of the circular road has proved to be a popular model of oligopolistic interaction, yet its theoretical properties are not fully explored. In this paper I extend the uniqueness result of price equilibrium in the circular road with equidistant locations from quadratic transportation costs to a class of convex power transportation costs, i.e. I show uniqueness in oligopoly for a class of utility functions of the consumers. I show that the associated game is supermodular and dominance solvable. The paper also discusses possible extensions and limitations of the model.

KEYWORDS: Circular Road, Localized Competition, Uniqueness of Price Equilibrium, Supermodular Games

1 Introduction

In this paper I analyze a particular model of localized competition. Under localized in contrast to global competition a price change of a firm only affects the demand of some of the firms in the market. In the model of the circular road, which has first been studied by Vickrey (1964) and later by Salop (1979), it is possible to study localized competition. In comparison to models of competition on an interval the advantage of the circle is that there are no end points so that the model with equidistant locations is (locally) symmetric. Each firm has two neighbors for strictly convex transportation costs when every consumer buys in the market. Competition on the circle can be interpreted as competition around a lake or mountain range or as competition by companies offering daily services at a particular time of the day. The model is the reference model of localized competition with more than two firms and is widely used in industrial organization and regional science.

The model is presented in section 2. For the specification with equidistant locations I give a new interpretation as a model of limited consumer information. Economides (1989) provides existence and uniqueness results in the model with quadratic transportation costs. The framework of Caplin and Nalebuff (1991) only applies to this specification of the transportation cost function. It seems to be important to generalize the results to show that results do not critically depend on one particular form of the transportation cost function, i.e. one particular form of the utility function. Existence of equilibrium in the model with power transportation costs has been established by Anderson, de Palma and Thisse (1992) for the power between 1 and 6.2.[1] Furthermore, they show that there exists only one symmetric equilibrium but they do not rule out asymmetric equilibria.

In section 3 I show that for sufficiently small price caps uniqueness holds in the model with power transportation costs. Uniqueness is shown by proving that the dominant diagonal property holds. This enables me to establish the uniqueness of equilibrium for power transportation cost functions with a power between 2 and 3 in the model with equidistant location given an upper bound on the maximal individual expenditure in the differentiated market. As far as I am aware of, this is the first uniqueness result for a class of address models of product differentiation with more than two firms, which is not restricted to linear or quadratic disutilities in the product-consumer distance.

[1] In the Hotelling duopoly power transportation costs have been introduced by Economides (1986). He characterizes price equilibria and relocation tendency in an earlier location stage.

In section 4 I show that the associated game is supermodular and dominance solvable. Uniqueness here follows from the fact that there exists a unique symmetric equilibrium candidate and from supermodularity. The properties of the profit functions needed for cardinal supermodularity coincides in this model with the required properties for the dominant diagonal. This implies dominance solvability of the game, provided the power of the transportation costs function is in $[2, 3)$. I shortly discuss the difficulties which one encounters when trying to extend the results to more general models of localized competition.

2 The Model

Firms and consumers are described by addresses in the product space, which is a circle of circumference L. Distances between goods and consumers are measured by the arc distance on the circle. Formally the arc distance between two points on the circle with circumference L labeled $a, b \in [0, L)$ is denoted by $|a - b| \equiv \min\{\text{abs}(a - b), \text{abs}(a - b + L), \text{abs}(a - b - L)\}$ where abs is the absolute value function. The set of firms $N = \{1, \ldots, \#N\}$ is fixed. Firms set prices p_n, $n \in N$, and they are assumed to have identical constant marginal cost of production c. Firm n has its good at z_n somewhere on the circle and $z_{n-1} < z_n$.

A consumer at $\hat{z}$ either buys one unit of the good at one location and zero everywhere else or not in the market at all. When a consumer buys good n his utility depends negatively on the distance $|\hat{z} - z_n|$. Interpreting the model as a location model this disutility is called the transportation cost and may be thought of as travel time. Hence the transportation cost does not affect the budget of a consumer.

Consumers are identical apart from their address or location in space. They are uniformly distributed over the circle and the population has mass M. With y I denote the income of the consumer or, more generally, the maximal budget reserved for a good in the differentiated market. Prices p_n, $n \in N$, and y are measured in units of the Hicksian composite commodity 0. Conditional utility functions of a consumer at $\hat{z}$ are written as

$$
\begin{aligned}
u_0 &= x_0, \\
u_n &= x_0 + R - \tau(z_n - \hat{z})^{\beta}, \; n \in N,
\end{aligned}
$$

where x_0 is the quantity of the composite commodity, R is the reservation price, τ is a scaling parameter, and β the power of the transportation cost function. The budget constraint reads $x_0 + p_n \leq y$ when a consumer buys good n and $x_0 \leq y$ when he does not buy in the market.

Consumers buy at the shop where they obtain maximal utility. I am interested in situations where firms in the market are in direct competition to each other. In the terminology of Economides (1989) I only study 'competitive' equilibria and exclude 'kink' equilibria and 'local monopolistic' equilibria from the analysis. Otherwise, demand functions have to be modified to allow for local monopoly, i.e. there is no competition between some neighboring firms in the differentiated market. For $p_n \leq y$ and $R \geq y + \tau \max_{n \in N}(\frac{1}{2}(z_n - z_{n-1}))^\beta$, $n \in N$, each consumer chooses according to arg $\max_{n \in N} v_n$, where $v_n = R - p_n - \tau|z_n - \hat{z}|^\beta$. The model with this choice rule for all p_n I call the *competitive specification.*[2]

The transportation cost function $\tau|z_n - \hat{z}|^\beta$ is quadratic when $\beta = 2$. It is strictly convex for $\beta > 1$ and linear for $\beta = 1$. If all firms are active, prices will be $c \leq p_n \leq \min\{y, R\}$ for all firms $n \in N$. For $\beta < 1$ the market area served by a firm is not necessarily convex; this case is therefore excluded from the analysis.

For $\beta > 1$ there is exactly one *marginal consumer* who is indifferent between buying from two particular neighboring firms with positive market share. Suppose firm $n-1$ is the direct competitor to the left of firm n, firm $n+1$ the direct competitor to the right, and $z_{n+1} > z_n > z_{n-1}$. The marginal consumer $\hat{z}_L$ to the left of firm n is located according to

$$p_n + \tau|z_n - \hat{z}_L|^\beta = p_{n-1} + \tau|\hat{z}_L - z_{n-1}|^\beta$$

and analogously for the marginal consumer $\hat{z}_R$ to the right of firm n. Firm $\#N$ is the neighbor to the left of firm $n = 1$. Demand for good n equals the market share of firm n, which is the length of the market segment between $\hat{z}_L$ and $\hat{z}_R$ relative to the circumference of the circle, multiplied by the mass of the consumers, i.e. $X_n(p_{n-1}, p_n, p_{n+1}) = \frac{M}{L}(\hat{z}_R - \hat{z}_L)$ and profits of firm n are $\pi_n(p_{n-1}, p_n, p_{n+1}) = \frac{M}{L}(p_n - c)(\hat{z}_R - \hat{z}_L)$. Firm n maximizes its profits with respect to p_n. A vector of prices (p_n^*) is a Nash equilibrium if for all $n \in N$, $\pi_n(p_{n-1}^*, p_n^*, p_{n+1}^*) \geq \pi_n(p_{n-1}^*, p_n, p_{n+1}^*)$, for all $p_n > 0$.

Before turning to the results in this model I provide a variant of the circular road with equidistant locations. This variant I call the circular road with limited consumer information. Consider a market of differentiated goods which are ordered as $1, 2, \ldots, \#N$. Each pair of goods $(n-1, n)$ with $n < \#N$ and $(\#N, 1)$ is observed by an equal share of consumers $1/\#N$ i.e. consumers draw a good n from the set N with

[2] I will look at this specification only at some intermediate steps. In the main results customers are described by maximizing u_n with respect to $n \in N \cup \{0\}$, x_0 subject to the budget constraint.

probability $1/\#N$ (independently over consumers) and then with probability $1/2$ one of the neighboring goods. Hence, localized competition arises because of correlated information. This consumer information can be best understood as arising when consumers who enter a shop select only among a pair of neighboring goods they encounter in the shelf. Consumers make comparisons between the two goods they "analyze" and only purchase one of them.

Goods are heterogeneous due to horizontal differentiation which is modeled for each pair $(n-1, n)$ with $n < \#N$ and $(\#N, 1)$: consumers' ideal points are distributed uniformly on the interval $[0, L/\#N]$ where goods are located at the endpoints of this interval. Then with the transportation cost function from above one has reconstructed the circle with the notable difference that in such a model there do not exist prices such that a consumer who is informed about a particular good and whose ideal point coincides with the location of this good is in the interior of the market area of a competing firm. In other words, firms can only sell to a consumer whose location on the circle is within distance $L/\#N$. This property facilitates the analysis because a firm can at most serve the whole market to be shared between itself and each of the two firms with neighboring goods on the shelf. For prices such that every marginal consumer between two neighboring firms has $\hat{z} \in (0, L/\#N)$ profit functions are identical to those in the model above. Existence and uniqueness results of the following sections hold. Hence, the circular road with equidistant locations can be interpreted as a model of product differentiation with limited consumer information.

3 Uniqueness: Dominant Diagonal

Economides (1989) has proved that in the model of the circular road with quadratic transportation costs there exists a unique price equilibrium for given locations. Economides (1989) also has shown that in the location-then-price game of the circular road with quadratic transportation costs there exists a subgame perfect equilibrium with equidistant locations. As shown by Salop (1979), in the model of the circular road with linear transportation costs there exists a unique price equilibrium for equidistant locations.

In the case of power transportation costs the marginal consumer $\hat{z}_L$ who is indifferent whether to buy from firm $n-1$ or firm n is determined by the price p_n for the good of firm n given the other prices.

$$\frac{d\hat{z}_L}{dp_n} = \frac{1}{\beta\tau}\left(|z_n - \hat{z}_L|^{\beta-1} + |\hat{z}_L - z_{n-1}|^{\beta-1}\right)^{-1}$$

Anderson, de Palma, and Thisse (1992) have shown that in the competitive specification of the model of the circular road with power transportation costs and equidistant location there exists a only one symmetric price equilibrium if $\beta \in [1, 6.2]$. Furthermore, the unique candidate for a symmetric equilibrium in the model with equidistant location has the following equilibrium prices:

$$p_n^* = c + \tau\beta 2^{1-\beta}\left(\frac{L}{\#N}\right)^{\beta}$$

Prices p_n^* are increasing in the parameters τ and L. Since

$$\frac{\partial p_n^*}{\partial \beta} = \tau\left(1 + \beta \log\frac{L}{2\#N}\right) 2^{1-\beta}\left(\frac{L}{\#N}\right)^{\beta},$$

prices are increasing in β if $L \geq 2\#N$, i.e. there is sufficient space between the firms. Prices are decreasing in the number of firms $\#N$. The profit function of firm n is quasi-concave in p_n for given $p_{n-1} = p_{n+1} = p_n^*$ when β is smaller than 5.8. For larger values of β there exists a second local maximum at which firm n sets a higher price $p_n > p_n^*$. This is due to the fact that at high β market share is not very sensitive in prices. For $\beta > 6.2$, the local maximum at $p_n = p_n^*$ fails to be the global maximum implying that there does not exist a symmetric equilibrium.

The findings are represented by the contour curves in Figure 1. The ordinate represents the market share λ of firm n on the interval between its two neighbors. The contour curves are the iso-profit curves in the (β, λ)-plain. For any $\beta \geq 1$ there is a local maximum in profits at $p_n = p_n^*$ $(\lambda = 0.5)$ which is the unique candidate for a symmetric equilibrium. Hence if one restricts the strategy sets of the firms such that higher profits than at $p_n = p_n^*$ cannot be made, one obtains the existence of only one symmetric equilibrium with equilibrium prices in the interior of the strategy sets for all $\beta \geq 1$. Anderson, de Palma, and Thisse (1992) only consider symmetric equilibria. This leaves the question open whether asymmetric equilibria exist.

To derive uniqueness results I will for the moment impose condition (C) which is a joint restriction on the parameters of the model. I introduce a price cap $\bar{p}_n > c$ for each firm, which depends upon β. Hence, I restrict the strategy set of firm n to $[0, \bar{p}_n]$. At a later stage I replace these price caps by a restriction on the parameters of the model.

(C) Prices are restricted by price caps $\bar{p}_n > c$, $n \in N$, which satisfy

$$\begin{aligned} \bar{p}_n - c &< \tau \frac{\beta}{\beta - 1} 2^{2-\beta} (z_n - z_{n-1})^{\beta}, \\ \bar{p}_n - c &< \tau \frac{\beta}{\beta - 1} 2^{2-\beta} (z_{n+1} - z_n)^{\beta}. \end{aligned}$$

Uniqueness of equilibrium is shown by applying the contraction mapping theorem. The corresponding contraction mapping property is implied by the following dominant diagonal property

$$\left| \frac{\partial^2 \pi_n}{\partial p_n^2} \right| > \sum_{j \neq n} \left| \frac{\partial^2 \pi_n}{\partial p_n \partial p_j} \right| .$$

When all firms are active, the expression on the right-hand side is

$$\sum_{j \neq n} \left| \frac{\partial^2 \pi_n}{\partial p_n \partial p_j} \right| = \left| \frac{\partial^2 \pi_n}{\partial p_n \partial p_{n-1}} \right| + \left| \frac{\partial^2 \pi_n}{\partial p_n \partial p_{n+1}} \right| .$$

Lemma 1. *Suppose condition (C) is satisfied. In the competitive specification of the model of the circular road with power transportation costs and $\beta \geq 2$, profit functions satisfy the dominant diagonal property on the set of prices above marginal costs where all firms are active.*

The proof of the lemma is delegated to the appendix. Note that since firms have identical marginal costs there cannot exist an equilibrium at which one or several firms are inactive. From Lemma 1 follows the existence of a unique equilibrium.

Lemma 2. *In the competitive specification of the model of the circular road with power transportation cost, $\beta \geq 2$, and price caps satisfying condition (C), a Nash equilibrium in prices exists and is unique.*

Proof. It is a strictly dominated strategy for each firm to be inactive for any strategy profile of its competitors which is in $[c, \infty)^{\# N - 1}$. Prices $p_n < c$ for at least one $n \in N$ are serially dominated. Hence the result follows from Lemma 1. ■

This result will be used to show that the symmetric equilibrium in the competitive specification is the unique equilibrium under an assumption on income y. Given equidistant locations, condition (C) reduces to

$$\bar{p}_n - c < \tau \frac{\beta}{\beta - 1} 2^{2-\beta} \left(\frac{L}{\#N} \right)^{\beta}$$

Since prices cannot be larger than the price caps this also has to hold for the unique candidate of a symmetric equilibrium. This implies that $\beta < 3$. Whenever $y < \bar{p}_n$, prices which violate (C) are dominated by some prices in $[c, y]$.

Assumptions under which the proposition is applied are (A.1) that all consumers want to buy in the market when they can afford it and (A.2) that the maximal budget in units of the composite commodity that consumers spend in the market is inside a particular interval.

(A.1). $R > y + \left(\frac{L}{2\#N} \right)^{\beta}$.

(A.2). $c + \tau \beta 2^{1-\beta} \left(\frac{L}{\#N} \right)^{\beta} < y < c + \tau \frac{\beta}{\beta-1} 2^{2-\beta} \left(\frac{L}{\#N} \right)^{\beta}$.

The following result extends the uniqueness result for quadratic transportation costs to β's in the interval $[2, 3)$.

Proposition 1. *Assume (A.1) and (A.2). In the model of the circular road with power transportation costs and equidistant locations, for $\beta \in [2, 3)$, there exists a symmetric price equilibrium, which is determined by equation (1). The equilibrium is unique.*

Proof. The existence of a symmetric equilibrium follows from Anderson, de Palma, and Thisse (1992). Uniqueness of the equilibrium follows from Lemma 2 and the argument above. ■

For lower income y, $p_n^* = y$, $n \in N$. This is made precise in the following theorem.

(A.3). $c \leq y \leq c + \tau \beta 2^{1-\beta} \left(\frac{L}{\#N} \right)^{\beta}$.

Proposition 2. *Assume (A.1) and (A.3). In the model of the circular road with power transportation costs and equidistant locations, for $\beta \in [2, 3)$, there exists a symmetric price equilibrium $p_n^* = y$, $n \in N$. The equilibrium is unique.*

Proof. Prices according to (1) are greater or equal to y. Note that strategy profiles which are not in $[c, y]^{\#N}$ are serially dominated. It follows from the proof of Lemma 1 that on this set profit functions π_n are strictly concave in p_n. Hence if $\left.\frac{\partial \pi_n}{\partial p_n}\right|_{p_j=y, j\in N} \geq 0$, $p_n = y$, $n \in N$ is an equilibrium. This is shown in Lemma 3 in the appendix. Concavity and Lemma 3 also imply that there does not exist an equilibrium with $p_n \in [c, y)$ for at least one $n \in N$ because each firm wants to set a higher price than the neighbor with the lowest price whenever $\min\{p_{n-1}, p_{n+1}\} \in [c, y)$. ■

I end this section with a remark on endogenous locations. At equidistant locations $\bar{z}_n$, $n \in N$, firms have no tendency to relocate in the location-then-price game when (A.3) holds with strict inequality because continuation profits are constant in z_n in a neighborhood around $\bar{z}_n$ for each firm n.

4 Uniqueness: Supermodularity

In this section I provide an alternative proof of uniqueness in the circular road which is based on results in supermodular games (see e.g. Vives, 1990, and Milgrom and Roberts, 1990). Since firms choose prices from a closed interval of the real line, supermodularity (in its ordinal form) of the profit functions is implied by the single crossing property (see Milgrom and Shannon, 1994). A profit function satisfies the *single crossing property* if for $p'_n \geq p''_n$ and $p'_{-n} \geq p''_{-n}$

$$\pi_n(p'_n, p''_{-n}) \geq \pi_n(p''_n, p''_{-n}) \text{ implies } \pi_n(p'_n, p'_{-n}) \geq \pi_n(p''_n, p'_{-n}).$$

Supermodularity implies that best responses are nondecreasing. First, I consider a cardinal and differentiable version of this property. By Topkis characterization theorem π_n is supermodular if

$$\frac{\partial \pi_n}{\partial p_n \partial p_j} \geq 0, \; j \neq n.$$

This property says that the incentive to increase price increases in the competitors' prices. It means that goods are strategic complements. In the circle model one only has to check whether this condition holds for $j = n-1, n+1$. These conditions are

$$\begin{aligned} \frac{d\hat{z}_L}{dp_n} - (p_n - c)\frac{d^2\hat{z}_L}{dp_n dp_{n-1}} &\geq 0, \\ -\frac{d\hat{z}_R}{dp_n} + (p_n - c)\frac{d^2\hat{z}_R}{dp_n dp_{n+1}} &\geq 0. \end{aligned}$$

An important observation in the circle model or indeed any product differentiation model on the line with identical disutitiliy functions is that changes of the marginal consumer satisfy

$$\frac{d^2 \hat{z}_L}{dp_n^2} = -\frac{d^2 \hat{z}_L}{dp_n\, dp_{n-1}} \text{ and } \frac{d^2 \hat{z}_R}{dp_n^2} = -\frac{d^2 \hat{z}_R}{dp_n\, dp_{n+1}}$$

which are typically different from zero. The only parameter choice for which these second derivatives are always zero is $\beta = 1$. In this case profit functions are supermodular and uniqueness of equilibrium follows from Lemma 5 below (this is the version of Salop, 1979). Since these second derivatives can change sign on the domain of prices for $\beta > 1$, *demand* functions cannot be shown to be supermodular. Then summing the two conditions and comparing them with the second order conditions of profit maximization shows that the property of supermodularity is *stronger* than concavity of profit functions in the present example. Hence, supermodularity can only be a property of the profit function for the power of the transportation cost function sufficiently small. Furthermore, as follows from the proof of Lemma 1, the same property on admissible prices is needed as in the case of the dominant diagonal property.

Lemma 4. *Suppose condition (C) is satisfied. In the competitive specification of the model of the circular road with power transportation costs and $\beta \geq 2$, the associated game is supermodular on the set of prices above marginal costs where all firms are active.*

For the proof see appendix. The circular road with equidistant locations is an example of a locally symmetric game. For this class of games it is shown in Peitz (1998) that supermodularity together with the existence of a single symmetric equilibrium candidate is sufficient for uniqueness of equilibrium and dominance solvability of the game.

Lemma 5. *For any parameter $\beta \leq 6.2$ for which profit functions are supermodular on the set of possible equilibrium price vectors, the equilibrium of the circular road with equidistant locations is unique and the game is dominance solvable.*

This implies that uniqueness of equilibrium can be shown without applying the contraction mapping theorem. Hence the results of Propositions 1 and 2 can be shown with the proof relying on supermodularity. The result is strengthened to dominance solvability, i.e. the unique equilibrium is the only price vector which survives the serial elimination of strictly dominated strategies.

Proposition 3. *Assume (A.1) and (A.2) or (A.1) and (A.3). In the model of the circular road with power transportation costs and equidistant locations, for $\beta \in [2, 3)$, there exists a unique equilibrium and the game is dominance solvable.*

With Propositions 3 it has essentially been shown that the uniqueness result for the quadratic specification is robust to a more convex specification of the transportation costs. Although this result is not surprising, it is far from obvious because properties such as the supermodularity of demand (which are satisfied in several model of price competition) is not satisfied in the circle model.

First, with the arguments in Peitz (1998) one can extend the previous two results to locally symmetric markets in which some of the consumers only know of the existence of one good. It is then possible to construct monotone comparative statics in the share of consumers who only know of the existence of a single good.

Second, one may be tempted to conclude that a monotone transformation of the profit function leads to a more general result. Consider the logarithmic transformation of profit functions. I show that this monotone transformation does not help in extending the previous result to a general uniqueness result. Clearly, if the logarithmic transformation is cardinal supermodular then profit functions are ordinal supermodular. The condition of log-supermodularity can be written as

$$\frac{d\widehat{z}_L}{dp_{n-1}} \left(\frac{d\widehat{z}_R}{dp_n} - \frac{d\widehat{z}_L}{dp_n} \right) \geq \frac{d^2 \widehat{z}_L}{dp_n dp_{n-1}} (\widehat{z}_R - \widehat{z}_L).$$

The inequalities for log-supermodularity (and also for supermodularity) are strictly satisfied for all β in any point $p_n = p_{n-1}$. This implies that the symmetric equilibrium is locally unique (follows from Lemma 5).

I restrict the set of admissible prices to those prices at which all consumers buy only from a neighboring firm (as in the model with limited consumer information). Since $d^2 \widehat{z}_L / dp_n dp_{n-1}$ is unbounded on $[z_{n-1}, z_n]$ for $\beta \in (1, 2)$ this inequality is not satisfied for these parameters. Also, one can show that for $\beta \geq 2$ the restriction (by finding upper and lower bounds for numerator and denominator) reduces to $2^{4-2\beta} \geq \beta - 1$. This inequality is only satisfied for $\beta = 2$. Hence, profit functions are log-supermodular only in the linear and quadratic specification of the profit function.

The property of supermodularity can also be shown for asymmetric locations. Since the game is asymmetric, uniqueness of equilibrium no longer automatically follows (for this it is sufficient to show that the dominant diagonal property of profit function also holds). Similarly, in the case of product differentiation on an interval of the real line, the game is never locally symmetric and the property of supermodularity does not imply uniqueness of equilibrium.

In summary, the analysis in this paper reinforces the circular road with equidistant locations to remain the most popular oligopoly model of horizontal product differentiation. However, properties of the specification with quadratic transportation costs are not general properties. In particular, strategic complementarity could only be shown for the power of the transportation cost function in $[2, 3)$.

Since supermodularity holds in the neighborhood of the symmetric equilibrium independent of the parameter β one can derive monotone comparative statics of this selection for any $\beta \in [1, 6.2]$.

Appendix

Proof of Lemma 1. The dominant diagonal property is:

$$\left|\frac{\partial^2\pi_n}{\partial p_n^2}\right| > \left|\frac{\partial^2\pi_n}{\partial p_n \partial p_{n-1}}\right| + \left|\frac{\partial^2\pi_n}{\partial p_n \partial p_{n+1}}\right|$$

where

$$\begin{aligned}
\frac{\partial^2\pi_n}{\partial p_n^2} &= -2\frac{M}{L}\frac{d\hat{z}_L}{dp_n} - \frac{M}{L}(p_n - c)\frac{d^2\hat{z}_L}{dp_n^2} \\
&\quad +2\frac{M}{L}\frac{d\hat{z}_R}{dp_n} + \frac{M}{L}(p_n - c)\frac{d^2\hat{z}_R}{dp_n^2} \\
\frac{\partial^2\pi_n}{\partial p_n \partial p_{n-1}} &= -\frac{M}{L}\frac{d\hat{z}_L}{dp_{n-1}} - \frac{M}{L}(p_n - c)\frac{d^2\hat{z}_L}{dp_n\, dp_{n-1}} \\
\frac{\partial^2\pi_n}{\partial p_n \partial p_{n+1}} &= \frac{M}{L}\frac{d\hat{z}_R}{dp_{n+1}} + \frac{M}{L}(p_n - c)\frac{d^2\hat{z}_R}{dp_n\, dp_{n+1}}
\end{aligned}$$

Note that

$$\frac{d\hat{z}_L}{dp_{n-1}} = -\frac{d\hat{z}_L}{dp_n},\ \frac{d\hat{z}_R}{dp_{n+1}} = -\frac{d\hat{z}_R}{dp_n},$$

$$\frac{d^2\hat{z}_L}{dp_n^2} = -\frac{d^2\hat{z}_L}{dp_n\, dp_{n-1}},\ \text{and}\ \frac{d^2\hat{z}_R}{dp_n^2} = -\frac{d^2\hat{z}_R}{dp_n\, dp_{n+1}}$$

The case $\beta = 2$ has been analyzed before (see Lemma 1 in Economides, 1989). Hence only $\beta > 2$ has to be analyzed. Note that $\frac{\partial^2\pi_n}{\partial p_n \partial p_j}$, $j = n-1, n, n+1$, are continuous because the derivatives above are continuous at all locations $\hat{z}_L$ and $\hat{z}_R$. The marginal consumer $\hat{z}_L$ can be located to the left of firm $n-1$, between firms n and $n-1$ and to the right of firm n. I treat these cases separately.

$z_{n-1} \le \hat{z}_L \le z_n$.

$$\begin{aligned}
\frac{d^2\hat{z}_L}{dp_n\, dp_{n-1}} &= -\frac{1}{\beta\tau}\left((z_n - \hat{z}_L)^{\beta-1} + (\hat{z}_L - z_{n-1})^{\beta-1}\right)^{-2} \\
&\quad \left(-(\beta-1)(z_n - \hat{z}_L)^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}} + (\beta-1)(\hat{z}_L - z_{n-1})^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}}\right)
\end{aligned}$$

It is to be shown that this term is bounded from above. Note that this expression becomes unbounded for $\beta < 2$ when $\hat{z}_L \to z_n$ or $\hat{z}_L \to z_{n-1}$. Therefore, the analysis had to be restricted to $\beta \ge 2$.

Condition (C) can be rewritten:

$$\bar{p}_n - c < \frac{\beta\tau}{\beta-1}2^{2-\beta}(z_n - z_{n-1})^{\beta} = \frac{\beta\tau}{\beta-1}2^{2-\beta}\frac{\left((z_n - z_{n-1})^{\beta-1}\right)^2}{(z_n - z_{n-1})^{\beta-2}}$$

Note that the function $f\colon\ [z_{n-1}, z_n] \to \Re_+$ with $f(\hat{z}_L) \equiv (z_n - \hat{z}_L)^{\beta-1} + (\hat{z}_L - z_{n-1})^{\beta-1}$ reaches its minimum at $\hat{z}_L = (z_n + z_{n-1})/2$ for $\beta > 2$. Hence the following implication holds.

$$\begin{aligned}
\Longrightarrow \quad p_n - c \quad &< \quad \frac{\beta\tau}{\beta-1}\frac{\left((z_n-\hat{z}_L)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^2}{(z_n-z_{n-1})^{\beta-2}} \\
&\le \quad \frac{\beta\tau}{\beta-1}\frac{\left((z_n-\hat{z}_L)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^2}{-(z_n-\hat{z}_L)^{\beta-2}+(\hat{z}_L-z_{n-1})^{\beta-2}} \qquad (1)
\end{aligned}$$

This inequality holds for $\beta > 2$ when the denominator is strictly positive and will be needed below. Two cases remain to be considered.

$\hat{z}_L \le z_{n-1} < z_n$:

$$\begin{aligned}
\frac{d\hat{z}_L}{dp_n} \quad &= \quad \frac{1}{\beta\tau}\left((z_n-\hat{z}_L)^{\beta-1}+(z_{n-1}-\hat{z}_L)^{\beta-1}\right)^{-1} \\
\frac{d^2\hat{z}_L}{dp_n dp_{n-1}} \quad &= \quad -\frac{1}{\beta\tau}\left((z_n-\hat{z}_L)^{\beta-1}+(z_{n-1}-\hat{z}_L)^{\beta-1}\right)^{-2} \\
&\quad \left((-\beta+1)(z_n-\hat{z}_L)^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}}+(-\beta+1)(z_{n-1}-\hat{z}_L)^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}}\right) \\
&= \quad -\frac{\beta-1}{\beta\tau}\left((z_n-\hat{z}_L)^{\beta-1}+(z_{n-1}-\hat{z}_L)^{\beta-1}\right)^{-2} \\
&\quad \left((z_n-\hat{z}_L)^{\beta-2}+(z_{n-1}-\hat{z}_L)^{\beta-2}\right)\frac{d\hat{z}_L}{dp_n} \\
&< \quad 0 \qquad (2)
\end{aligned}$$

$z_{n-1} < z_n \le \hat{z}_L$:

$$\begin{aligned}
\frac{d\hat{z}_L}{dp_n} \quad &= \quad \frac{1}{\beta\tau}\left((\hat{z}_L-z_n)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^{-1} \\
\frac{d^2\hat{z}_L}{dp_n dp_{n-1}} \quad &= \quad -\frac{1}{\beta\tau}\left((\hat{z}_L-z_n)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^{-2} \\
&\quad \left((\beta-1)(\hat{z}_L-z_n)^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}}+(\beta-1)(\hat{z}_L-z_{n-1})^{\beta-2}\frac{d\hat{z}_L}{dp_{n-1}}\right) \\
&= \quad \frac{\beta-1}{\beta\tau}\left((\hat{z}_L-z_n)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^{-2} \\
&\quad \left((\hat{z}_L-z_n)^{\beta-2}+(\hat{z}_L-z_{n-1})^{\beta-2}\right)\frac{d\hat{z}_L}{dp_n}
\end{aligned}$$

Condition (C) implies for $\beta > 2$

$$\bar{p}_n - c \quad < \quad \frac{\beta\tau}{\beta-1}(z_n - z_{n-1})^{\beta}$$

$$< \frac{\beta\tau}{\beta-1}\frac{\left((\hat{z}_L-z_n)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^2}{(\hat{z}_L-z_n)^{\beta-2}+(\hat{z}_L-z_{n-1})^{\beta-2}} \tag{3}$$

Note that the function $g:[z_{n-1},z_n]\to\Re_+$ with

$$g(\hat{z}_L)\equiv\frac{\left((\hat{z}_L-z_n)^{\beta-1}+(\hat{z}_L-z_{n-1})^{\beta-1}\right)^2}{(\hat{z}_L-z_n)^{\beta-2}+(\hat{z}_L-z_{n-1})^{\beta-2}}$$

reaches its minimum at $\hat{z}_L=z_n$ for $\hat{z}_L\geq z_n$ and $g(z_n)=(z_n-z_{n-1})^{\beta}$. Hence the inequality above holds.
From (2), (3), and (4) it follows that

$$\frac{d\hat{z}_L}{dp_n}>(p_n-c)\frac{d^2\hat{z}_L}{dp_n dp_{n-1}}$$

Also

$$-\frac{d\hat{z}_R}{dp_n}>-(p_n-c)\frac{d^2\hat{z}_R}{dp_n dp_{n+1}}$$

where one has to use the second inequality of condition (C). This proves that the dominant diagonal property is satisfied. ■

Lemma 3. *Under (A.1),(A.3),* $\left.\frac{\partial\pi_n}{\partial p_n}\right|_{p_j=y,j\in N}\geq 0$ *at equidistant locations.*

Proof. Given that the neighbors set their price $p_{n-1}=p_{n+1}$ the market share of firm n to his left equals the market share to his right, denoted by $\lambda(p_n)$.

$$p_n=-\tau\lambda(p_n)^{\beta}\left(\frac{L}{\#N}\right)^{\beta}+\tau(1-\lambda(p_n))^{\beta}\left(\frac{L}{\#N}\right)^{\beta}+p_{n-1}$$

Profits $\pi_n\propto(p_n-c)\lambda(p_n)$. Denote $\tilde{\pi}_n$ the profit function when the argument is λ.

$$\tilde{\pi}_n(\lambda)\propto\left(-\tau\lambda^{\beta}\left(\frac{L}{\#N}\right)^{\beta}+\tau(1-\lambda)^{\beta}\left(\frac{L}{\#N}\right)^{\beta}+p_{n-1}-c\right)\lambda$$

It remains to be shown that the first derivative evaluated at $\lambda=\frac{1}{2}$ and $p_{n-1}=y$ is less or equal to 0. This is rewritten and then y is replaced by its upper bound in (A.3).

$$\tau\left(\frac{L}{\#N}\right)^{\beta}\left(-(\beta+1)2^{-\beta}-(\beta-1)2^{-\beta}\right)+y-c\leq\tau\left(\frac{L}{\#N}\right)^{\beta}\left(-2\beta 2^{-\beta}+\beta 2^{1-\beta}\right)=0 \;■$$

Proof of Lemma 4. Clearly,

$$\frac{\partial^2 \pi_n}{\partial p_n \partial p_j} = 0,\ j \neq n-1, n, n+1,$$

when firm $n-1$ and firm $n+1$ are active and hence the neighbors of firm n.

$$\frac{\partial^2 \pi_n}{\partial p_n \partial p_{n-1}} > 0$$

because it follows from (C) that

$$\frac{d\hat{z}_L}{dp_n} > (p_n - c)\frac{d^2\hat{z}_L}{dp_n dp_{n-1}}$$

(see the proof of Lemma 1 above). Similarly, $\frac{\partial^2 \pi_n}{\partial p_n \partial p_{n+1}} > 0$. ∎

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