VALUATION EQUILIBRIUM REVISITED*

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ABSTRACT

This paper extends the notion of *valuation equilibrium* which applies to market economies involving the choice of a public environment. Unlike some other recent work, it is assumed here that consumers and firms evaluate alternative environments taking market prices as given (hence this notion is closer to that of competitive equilibria). It is shown that valuation equilibria with balanced tax schemes yield efficient allocations and that efficient allocations can be decentralized as valuation equilibria, with tax schemes that may be unbalanced.

Key words: Valuation Equilibrium; Non-Convexities; Public Goods.

1 INTRODUCTION

Market economies can be regarded as collections of economic agents that take individual decisions over their private feasible sets. Like budget sets in the Walrasian model, these sets depend on prices and income from profits. But there are also influenced by conditioning variables that may be determined outside the market mechanism. Examples of these conditioning variables are the prevailing social rules (e.g. laws, and in particular the assignment of property rights), the provision of public goods by the public sector (roads, health care, unemployment benefits), the regulation of economic activities (the presence of quotas, the regulation of quality standards or labour conditions), the working of a tax system, etc. We can refer to the collection of these conditioning variables as the *public environment.* It is worth stressing that the concept of public environment allows many different problems to be treated within one common setting. The cost of this generality is that little structure can be imposed on the set of public environments. The chief question is the analysis of efficient allocations in such a framework.

Mas-Colell's (1980) contribution to the pure theory of public goods proposes the notion of *valuation equilibria* to address this problem. He considers an economy involving a single private good (to be interpreted as a Hicksian composite commodity), and the choice of a single public project from a space where no linear structure is imposed. This equilibrium notion may be regarded as an application to this abstract framework of the Lindahlian approach to the provision of public goods. A valuation equilibrium corresponds to a unanimously agreed choice of public project, the role of the public sector being to design a mechanism that induces such an efficient agreement. In a valuation equilibrium every consumer prefers her consumption plan, consisting of some amount of the private good and a public project, to any other which is affordable given the valuation function (to be interpreted as tax system). Taking the private good as the *numéraire*, Mas-Colell shows that valuation equilibria satisfy the standard properties of competitive equilibria.

Later, Mas-Colell & Silvestre (1989) introduced the concept of *cost-share equilibrium* which extends the work of Kaneko (1977) and Mas-Colell (1980), to allow many public goods. Even though in some cases this model allows for many private goods, the authors point out that "the extension that seems to us very difficult is to allow for produced intermediate goods." (cf. p. 255) [see also Diamantaras & Wilkie (1994)].

Allowing several private goods offers a choice of how to extend the notion of valuation equilibrium. One possibility is to assume that agents evaluate alternative actions *taking the equilibrium prices as given.* The alternative is to assume that they *compute the new prices* that would emerge in different environments. In the first case, agents compare alternatives disregarding the effect on prices that results from a change in the environment (as in the standard competitive or Lindahl equilibrium model). In the second case, they are highly sophisticated and able to calculate the set of conditional equilibria associated with an alternative public environment (and to coordinate on which one if there are several).

Diamantaras & Gilles (1996) and Hammond & Villar (1998) extend Mas-Colell's work by taking the second avenue. They present slightly different models with an abstract set of public projects and several private commodities that may be both inputs and outputs. Their notion of valuation equilibria requires agents to maximize their payoff functions taking into account the changes in the prices of private commodities resulting from changes in the public environment. They show that these equilibria satisfy the two fundamental welfare theorems.

The purpose of this paper is to discuss the efficiency and decentralizability of valuation equilibria in the case when prices of private commodities are treated as fixed. Besides completing the analysis, the interest of this second approach is twofold. On the one hand, it allows us to analyse the efficiency of equilibrium allocations with fewer informational requirements. On the other hand, it remains closer to the notion of competitive and Lindahl equilibria and also permits several models to be encompassed within one common framework, as will be illustrated later on. This does not mean that the case of sophisticated consumers is uninteresting. On the contrary, if there are significant nonconvexities in the provision of public goods, changes in the public environment may induce substantial changes in the prices of private goods that should not be neglected. What we claim here is that both possibilities are worth investigating.

We consider an economy with an abstract set $\mathbf{Z}$ of public environments lacking any formal structure. Agents' choice sets and payoff functions are defined conditional on the public environment in such a way that, for each given $\mathbf{z}$ in $\mathbf{Z}$, there is a standard convex economy in the subspace of private commodities. Agents' choices are also affected by a *tax system* that modifies net profits and budget sets depending on the public environment. A valuation equilibrium is defined as a public environment, a feasible allocation, a price vector and a tax system such that: (a) Every consumer prefers her consumption bundle and the public environment to any other which is affordable given the equilibrium prices and the tax system; (b) Firms' profits are maximized over their production sets; and (c) The aggregate net taxes (i.e. taxes less subsidies or environmental charges) add up to zero.

Our main findings are as follows:

(i) A valuation equilibrium is Pareto optimal (provided the tax system is balanced).

(ii) Every Pareto optimal allocation can be decentralized as a (compensated) valuation equilibrium.

(iii) Every valuation equilibrium yields a core allocation, provided the tax system does not allow for intercoalitional transfers.

The rest of the paper proceeds as follows. Section 2 presents the model (including the equilibrium notion and the assumptions). Section 3 contains the results. Section 4 provides some examples of standard economic problems that can be considered as particular instances of valuation equilibria.

2 THE MODEL

Consider an economy with ℓ private commodities. The vector $\omega \in \mathbb{R}^\ell$ represents the aggregate initial endowments of private goods. There is an abstract set $\mathbf{Z}$ whose members are vectors of those variables defining the *public environment*. Each agent's feasible set and payoff function may be affected by the values taken by these vectors $\mathbf{z} \in \mathbf{Z}$. No particular structure will be postulated on the set $\mathbf{Z}$. We assume that there is some kind of public agency or *public sector* that can determine these variables and/or affect consumers' budget sets and firms' profit functions via taxes and subsidies. Even though the tax system itself can be thought of as part of the public environment, we find it more convenient to treat these variables separately. For the sake of generality, agents' feasible sets will be defined as subsets of $\mathbb{R}^\ell \times \mathbf{Z}$ (where $\mathbb{R}^\ell$ stands for the commodity space and $\mathbf{Z}$ for the public environment space), even though they may actually be independent of many of the variables in $\mathbf{Z}$. A point $\mathbf{p} \in \mathbb{R}^\ell_+$ is a *price vector* relative to private commodities.

There are n firms in the economy. Let $Y_j \subset \mathbb{R}^\ell \times \mathbf{Z}$ be the jth firm's production set, and denote by $Y_j(\mathbf{z})$ the jth firm's *conditional production set* when the environment variables take the value $\mathbf{z} \in \mathbf{Z}$. That is,

$$Y_j(\mathbf{z}) := \{\mathbf{y}_j \in \mathbb{R}^\ell \ / \ (\mathbf{y}_j, \mathbf{z}) \in Y_j\}$$

For each firm $j = 1, 2, ..., n$, the mapping $\sigma_j : \mathbb{R}^\ell_+ \times \mathbf{Z} \to \mathbb{R}$ is assumed to represent the net subsidy this firm receives (or, if negative, pays as taxes), as a function of prevailing prices and the value of the environment variables. Thus for each given $(\mathbf{p}, \mathbf{z})$ in $\mathbb{R}^\ell_+ \times \mathbf{Z}$, the jth firm's conditional profits are given by $\pi_j(\mathbf{p}, \mathbf{z}) = \sup\{\mathbf{p}\mathbf{y}_j + \sigma_j(\mathbf{p}, \mathbf{z}) \ / \ \mathbf{y}_j \in Y_j(\mathbf{z})\}$. In equilibrium this supremum should be attained. This implies that, for each given $(\mathbf{p}, \mathbf{z})$ in $\mathbb{R}^\ell_+ \times \mathbf{Z}$, the jth firm selects a production plan $\mathbf{y}_j(\mathbf{p}, \mathbf{z})$ that maximizes (conditional) profits.

There are m consumers. The ith consumer is characterized by the collection $[X_i, u_i, \omega_i, (\theta_{ij})]$, where $X_i \subset \mathbb{R}^\ell \times \mathbf{Z}$, $u_i : X_i \to \mathbb{R}$, $\omega_i \in \mathbb{R}^\ell$ denote the consumption set, utility function and initial endowments, respectively, and θ_{ij} stands for the ith consumer's share in the jth firm's profits. By definition, $0 \leq \theta_{ij} \leq 1$, and $\sum_{j=1}^{n} \theta_{ij} = 1$, for all i, j. For a given point $\mathbf{z} \in \mathbf{Z}$ the ith consumer's *conditional consumption set* is given by:

$$X_i(\mathbf{z}) = \{\mathbf{x}_i \in \mathbb{R}^\ell \ / \ (\mathbf{x}_i, \mathbf{z}) \in X_i\}.$$

Evidently, a preference ordering is induced on each of these conditional sets, which can be described by a conditional utility function $u_i^z : X_i(\mathbf{z}) \to \mathbb{R}$ given by $u_i^z(\mathbf{x}_i) = u_i(\mathbf{x}_i, \mathbf{z})$.

A mapping $\tau_i : \mathbb{R}_+^\ell \times \mathbf{Z} \to \mathbb{R}$, for $i = 1, 2, ..., m$, describes the ith consumer's (net) tax mapping. Given a pair $(\mathbf{p}, \mathbf{z}) \in \mathbb{R}_+^\ell \times \mathbf{Z}$, the ith consumer's behaviour $\mathbf{x}_i(\mathbf{p}, \mathbf{z})$ is obtained by solving the following program:

$$\begin{array}{rrcl} & & & Max \ \ u_i^z \\ s.t. & \mathbf{x}_i & \in & X_i(\mathbf{z}) \\ & \mathbf{p}\mathbf{x}_i & \leq & \mathbf{p}\omega_i + \sum_{j=1}^{n} \theta_{ij}\pi_j(\mathbf{p}, \mathbf{z}) + \tau_i(\mathbf{p}, \mathbf{z}) \end{array}$$

Consider now the following definitions. The first makes precise the notion of a tax system, whereas the second specifies a relevant restriction:

Definition 1 *A* ***tax system*** *is a collection of mappings* $T = [(\sigma_j)_{j=1}^n, (\tau_i)_{i=1}^m]$, *where each* σ_j *and* τ_i *is a function from* $\mathbb{R}_+^\ell \times \mathbf{Z}$ *into* $\mathbb{R}$, *and such that for every pair* $(\mathbf{p}, \mathbf{z}) \in \mathbb{R}_+^\ell \times \mathbf{Z}$ *one has* $\sum_{j=1}^n \sigma_j(\mathbf{p}, \mathbf{z}) + \sum_{i=1}^m \tau_i(\mathbf{p}, \mathbf{z}) \leq 0$.

Definition 2 *A tax system* T *is* ***balanced*** *if for any pair* $(\mathbf{p}, \mathbf{z}) \in \mathbb{R}_+^\ell \times \mathbf{Z}$ *one has* $\sum_{j=1}^n \sigma_j(\mathbf{p}, \mathbf{z}) + \sum_{i=1}^m \tau_i(\mathbf{p}, \mathbf{z}) = 0$.

A tax system is a collection of functions from $\mathbb{R}_+^\ell \times \mathbf{Z}$ into $\mathbb{R}$ such that aggregate subsidies do not exceed aggregate taxes. This amounts to saying that the economy is *financially viable* in the sense that the tax system cannot rely on resources from outside. Note that this definition is very general as it can depend on many variables apart from private income and gross profits. A tax system is balanced whenever aggregate taxes equal aggregate subsidies (e.g. a cost-share system). In particular, the case of $\sigma_j(\mathbf{p}, \mathbf{z}) = \tau_i(\mathbf{p}, \mathbf{z}) = 0$ for all i, j, each $(\mathbf{p}, \mathbf{z})$ in $\mathbb{R}_+^\ell \times \mathbf{Z}$ constitutes a (degenerate) balanced tax system. Note that balancedness ensures that Walras Law $\mathbf{p}[\sum_{i=1}^m \mathbf{x}_i(\mathbf{p}, \mathbf{z}) - \sum_{i=1}^m \omega_i - \sum_{j=1}^n \mathbf{y}_j(\mathbf{p}, \mathbf{z})] = 0$ holds, provided consumers are not satiated.

The following defines the equilibrium notion (which is reminiscent of the modern version of Lindahl's equilibrium):

Definition 3 *A **valuation equilibrium** is a price vector* $\mathbf{p}^*$ *in* $\mathbb{R}_+^\ell$*, an environment* $\mathbf{z}^* \in \mathbf{Z}$*, an allocation* $[(\mathbf{x}_i^*), (\mathbf{y}_j^*)]$ *in* $\prod_{i=1}^m X_i(\mathbf{z}^*) \times \prod_{j=1}^n Y_j(\mathbf{z}^*)$ *and a tax system* T^* *such that:*

(i) For all $i = 1, 2, ..., m$, $(\mathbf{x}_i^*, \mathbf{z}^*)$ *maximizes* u_i *over the set of points* $(\mathbf{x}_i, \mathbf{z})$ *in* X_i *that satisfy* $\mathbf{p}^*\mathbf{x}_i \leq \mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\pi_j(\mathbf{p}^*, \mathbf{z}^*) + \tau_i(\mathbf{p}^*, \mathbf{z})$.

(ii) For all $j = 1, 2, ..., n$, $\mathbf{p}^*\mathbf{y}_j^* + \sigma_j(\mathbf{p}^*, \mathbf{z}^*) \geq \mathbf{p}^*\mathbf{y}_j + \sigma_j(\mathbf{p}^*, \mathbf{z})$ *for all* $(\mathbf{y}_j, \mathbf{z}) \in Y_j$.

(iii) $\sum_{i=1}^m \mathbf{x}_i^* - \sum_{i=1}^m \omega_i = \sum_{j=1}^n \mathbf{y}_j^*$.

(iv) $\sum_{j=1}^n \sigma_j(\mathbf{p}^*, \mathbf{z}^*) + \sum_{i=1}^m \tau_i(\mathbf{p}^*, \mathbf{z}^*) = 0$.

That is, a valuation equilibrium is a price vector, a tax system and a feasible allocation such that no agent finds it individually beneficial to choose an alternative allocation that is affordable *at given prices*. Note that we assume that every consumer computes her budget by taking prices of private commodities and firms' profits as given (she only allows for changes in the public environment through the effect on her own taxes).[1] Also observe that, by definition, aggregate taxes equal aggregate subsidies in equilibrium. Indeed, in some cases the tax system may be thought of as an incentive scheme that sustains the equilibrium environment, so that we can picture the situation as agents choosing the environment and the public sector choosing the incentive scheme.

When utility maximization is relaxed to expenditure minimization one gets the standard notion of *compensated* equilibrium, which will be used in order to discuss the second welfare theorem.

Definition 4 *A **compensated valuation equilibrium** is defined like a valuation equilibrium except that condition (i) is replaced by:*

(i') For every $i = 1, 2, ..., m$, $\mathbf{p}^*\mathbf{x}_i^* \leq \mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\pi_j(\mathbf{p}^*, \mathbf{z}^*) + \tau_i(\mathbf{p}^*, \mathbf{z}^*)$ *and* $\mathbf{p}^*\mathbf{x}_i \geq \mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\pi_j(\mathbf{p}^*, \mathbf{z}^*) + \tau_i(\mathbf{p}^*, \mathbf{z})$ *whenever* $(\mathbf{x}_i, \mathbf{z}) \in X_i$ *with* $u_i(\mathbf{x}_i, \mathbf{z}) \geq u_i(\mathbf{x}_i^*, \mathbf{z}^*)$.

3 MAIN RESULTS

The first main result says that a valuation equilibrium is Pareto optimal, provided the tax system is balanced. The second establishes that any Pareto efficient allocation can be decentralized as a valuation equilibrium. Finally, it will be shown that a valuation equilibrium is in the core, provided that inter-coalitional transfers are excluded.

[1] It can be easily shown that the results in section 3 remain valid if we consider that consumers are more sophisticated, and can also compute the effect on their budget constraint of changes in firm's profits.

Theorem 1 *Let* $[\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ *be a valuation equilibrium, and suppose that consumers are locally non-satiated. If* T^* *is a balanced tax system the resulting allocation is Pareto optimal.*

Proof.

Let $[\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ be a valuation equilibrium, and suppose that $[(\mathbf{x}_i), (\mathbf{y}_j), \mathbf{z}]$ together satisfy $\mathbf{x}_i \in X_i(\mathbf{z})$ and $u_i(\mathbf{x}_i, \mathbf{z}) \geq u_i(\mathbf{x}_i^*, \mathbf{z}^*)$ for all i, with strict inequality for at least one agent, and $\mathbf{y}_j \in Y_j(\mathbf{z})$. Then local non-satiation and the definition of valuation equilibrium imply that:

$$\sum_{i=1}^{m} \mathbf{p}^*\mathbf{x}_i > \mathbf{p}^*\omega + \sum_{j=1}^{n} \left[\mathbf{p}^*\mathbf{y}_j^* + \sigma_j(\mathbf{p}^*, \mathbf{z}^*)\right] + \sum_{i=1}^{m} \tau_i(\mathbf{p}^*, \mathbf{z})$$

and also that $\sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j^* + \sum_{j=1}^{n} \sigma_j(\mathbf{p}^*, \mathbf{z}^*) \geq \sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j + \sum_{j=1}^{n} \sigma_j(\mathbf{p}^*, \mathbf{z})$. These inequalities imply that

$$\sum_{i=1}^{m} \mathbf{p}^*\mathbf{x}_i > \mathbf{p}^*\omega + \sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j + \sum_{j=1}^{n} \sigma_j(\mathbf{p}^*, \mathbf{z}) + \sum_{i=1}^{m} \tau_i(\mathbf{p}^*, \mathbf{z})$$

But T^* is balanced by assumption, so $[\sum_{j=1}^{n} \sigma_j(\mathbf{p}^*, \mathbf{z}) + \sum_{i=1}^{m} \tau_i(\mathbf{p}^*, \mathbf{z})] = 0$. Hence $\sum_{i=1}^{m} \mathbf{p}^*\mathbf{x}_i > \mathbf{p}^*\omega + \sum_{j=1}^{n} \mathbf{p}^*\mathbf{y}_j$, which implies that $[(\mathbf{x}_i), (\mathbf{y}_j), \mathbf{z}]$ cannot be a feasible allocation satisfying $\sum_{i=1}^{m} \mathbf{x}_i \leq \omega + \sum_{j=1}^{n} \mathbf{y}_j$. It follows that no feasible allocation can be Pareto superior. ■

Observe that an allocation may fail to be optimal if we drop balancedness. Indeed it is easy to produce a tax system that induces a waste of resources *in* equilibrium and an even greater waste *out* of equilibrium (e.g., the tax system rewards a firm which destroys part of the initial endowments).

Remark 1 *It follows from this theorem that, in a valuation equilibrium, the tax system cannot be arbitrary once we put some structure on the set* $\mathbf{Z}$*. This is because, besides balancedness, the first-order necessary conditions for efficiency must be satisfied in equilibrium. Informally, one can say that consumers' taxes should be equivalent to a non-linear system of Lindahl prices.*

In order to obtain the second welfare theorem for this model we need three assumptions:

Assumption 1 For every $i = 1, 2, ..., m$ and every $\mathbf{z} \in \mathbf{Z}$:

(i) $X_i(\mathbf{z})$ is a non-empty, closed and convex subset of $\mathbb{R}^\ell$;

(ii) $u_i^z : X_i(\mathbf{z}) \to \mathbb{R}$ is continuous, quasi-concave, and satisfies local non-satiation.

Assumption 2 For every $j = 1, 2, ..., m$ and each $\mathbf{z} \in \mathbf{Z}$, the set $Y_j(\mathbf{z})$ is closed and convex in $\mathbb{R}^\ell$, with $Y_j(\mathbf{z}) \cap \mathbb{R}^\ell_+ = \{\mathbf{0}\}$, and $Y_j(\mathbf{z}) - \mathbb{R}^\ell_+ \subset Y_j(\mathbf{z})$.

Assumptions 1 and 2 essentially say that, for any given value of the public environment, the resulting conditional economy is standard (except in that we do not impose bounded consumption sets here). Note that this is compatible with the presence of non-convexities (an example will be discussed later). They also allow public goods to affect both production and consumption possibilities.

Observe that these assumptions involve no restriction on the set $\mathbf{Z}$ whose members may therefore contain all kind of variables.

We can prove now:

Theorem 2 *Under assumptions 1 and 2, let $[(\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*]$ be a Pareto optimal allocation. Then, there exist a price vector $\mathbf{p}^* \in \mathbb{R}^\ell_+ - \{\mathbf{0}\}$ and a tax system T^* such that $[\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ is a compensated valuation equilibrium.*

Proof.

Take $\mathbf{z}^*$ as given and apply the standard second welfare theorem to the allocation $[(\mathbf{x}_i^*), (\mathbf{y}_j^*)]$ in the resulting convex conditional economy. This theorem ensures the existence of a price vector $\mathbf{p}^* \in \mathbb{R}^\ell_+ - \{\mathbf{0}\}$ such that $[\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*)]$ is a competitive equilibrium relative to the conditional economy resulting from $\mathbf{z}^*$. It will be shown that $[\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ is a valuation equilibrium for a tax system T^*. This requires checking parts (i), (ii) and (iv) of definition 3, part (iii) being satisfied by construction.

Define

$$\sigma_j(\mathbf{p}^*, \mathbf{z}) = \inf\{0, \ \mathbf{p}^*\mathbf{y}_j^* - \pi_j(\mathbf{p}^*, \mathbf{z})\}$$

Note that $\sigma_j(\mathbf{p}^*, \mathbf{z}^*) = 0$ because $\pi_j(\mathbf{p}^*, \mathbf{z}^*) = \mathbf{p}^*\mathbf{y}_j^*$. Also, if $(\mathbf{y}'_j, \mathbf{z}') \in Y_j$ then:

$$\mathbf{p}^*\mathbf{y}'_j + \sigma_j(\mathbf{p}^*, \mathbf{z}') \leq \mathbf{p}^*\mathbf{y}'_j + \mathbf{p}^*\mathbf{y}_j^* - \widehat{\pi}_j(\mathbf{p}^*, \mathbf{z}') \leq \mathbf{p}^*\mathbf{y}_j^* = \mathbf{p}^*\mathbf{y}_j^* + \sigma_j(\mathbf{p}^*, \mathbf{z}^*)$$

so that part (ii) of the definition is satisfied.

Taking $\mathbf{p}^*, (\mathbf{x}_i^*, \mathbf{z}^*)$ as given, define the compensation function:

$$E_i(\mathbf{z}) := \inf\{\mathbf{p}^*\mathbf{x}_i \ / \ \mathbf{x}_i \in X_i(\mathbf{z}) \textit{ and } u_i(\mathbf{x}_i, \mathbf{z}) \geq u_i(\mathbf{x}_i^*, \mathbf{z}^*)\}$$

This is the income that individual i needs to spend on private goods in order to be no worse off than at $(\mathbf{x}_i^*, \mathbf{z}^*)$, when the environment changes to $\mathbf{z}$ and prices are $\mathbf{p}^*$. Now let

$$\tau_i(\mathbf{p}^*, \mathbf{z}) = \min\{0, \ E_i(\mathbf{z}) - \mathbf{p}^*\mathbf{x}_i^*\}$$

By construction, these σ_j, τ_i constitute a tax system, with $\sigma_j(\mathbf{p}^*, \mathbf{z}^*) = \tau_i(\mathbf{p}^*, \mathbf{z}^*) = 0$, for all j, i. In particular, part (iv) of the definition is satisfied.

Finally, take a consumer i and a consumption plan $(\mathbf{x}_i, \mathbf{z}) \in X_i$ such that $u_i(\mathbf{x}_i, \mathbf{z}) \geq u_i(\mathbf{x}_i^*, \mathbf{z}^*)$. Then, $\mathbf{p}^*\mathbf{x}_i - \tau_i(\mathbf{p}^*, \mathbf{z}) \geq \mathbf{p}^*\mathbf{x}_i + \mathbf{p}^*\mathbf{x}_i^* - E_i(\mathbf{z}) \geq \mathbf{p}^*\mathbf{x}_i^*$, which implies that:[2]

$\mathbf{p}^*\mathbf{x}_i \geq \mathbf{p}^*\omega_i + \sum_{j=1}^{n} \theta_{ij}^* \pi_j(\mathbf{p}^*, \mathbf{z}^*) + \tau_i(\mathbf{p}^*, \mathbf{z})$.

Thus part (i') of the definition of compensated valuation equilibrium is also satisfied and the proof is complete. ■

Theorem 2 tells us that any efficient allocation can be decentralized as a compensated valuation equilibrium. It also tells us that a compensated valuation equilibrium *with transfers* exists, provided that there is at least one Pareto efficient allocation.

Observe that the tax system has been given an explicit form, which has an easy and sensible interpretation. Each $\sigma_j(\mathbf{p}^*, \mathbf{z})$ tells us the tax paid by the jth firm if the environment changes from $\mathbf{z}^*$ to $\mathbf{z}$. It is equal to the firm's change in profits. Similarly, every $\tau_i(\mathbf{p}^*, \mathbf{z})$ specifies the net amount that the ith consumer will have to pay, if the environment is changed from $\mathbf{z}^*$ to $\mathbf{z}$. It is equal to the equivalent variation in the sense of Hicks. Moreover, it follows that $\sigma_j(\mathbf{p}^*, \mathbf{z}^*) = \tau_i(\mathbf{p}^*, \mathbf{z}^*) = 0$, for all i, j.

Remark 2 *A standard argument shows that the compensated valuation equilibrium of Theorem 2 is also a valuation equilibrium provided that, whenever $(\mathbf{x}_i, \mathbf{z}) \in X_i$ satisfies $u_i(\mathbf{x}_i, \mathbf{z}) > u_i(\mathbf{x}_i^*, \mathbf{z}^*)$, there exists a cheaper point $\underline{\mathbf{x}}_i \in X_i(\mathbf{z})$ for which $\mathbf{p}^*\underline{\mathbf{x}}_i < E_i(\mathbf{z})$. This can be ensured by means of the "essentiality condition" that appears in Mas-Colell (1980, p. 626), Mas-Colell & Silvestre (1989, p. 250), or Diamantaras & Gilles (1996, p. 855).*

Our last result refers to the core, defined as follows:

Definition 5 *A feasible allocation $[(\mathbf{x}_i), (\mathbf{y}_j), \mathbf{z}]$ is in the* **core** *if there is no coalition $S \subset M = \{1, 2, ..., m\}$, with an allocation $[(\mathbf{x}_i'), (\mathbf{y}_j'), \mathbf{z}']$, such that:*

(i) $\sum_{i \in S} \left(\mathbf{x}_i' - \omega_i - \sum_{j=1}^{n} \theta_{ij} \mathbf{y}_j' \right) = 0.$

(ii) $u_i(\mathbf{x}_i', \mathbf{z}') \geq u_i(\mathbf{x}_i, \mathbf{z})$, $\forall\ i \in S$, with a strict inequality for some agent in S.

A core allocation is thus one in which no coalition can re-arrange the economy, using its own resources, so that the resulting allocation is preferred

[2]The scalars θ_{ij}^* correspond here to a distribution of profits determined by the separation argument.

by all its members. Observe that profit shares are being interpreted here as production shares (as usual when positive profits are possible).

The following definition helps characterize the tax systems which yield valuation equilibria in the core.

Definition 6 *Let* $Q^* = [\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ *be a valuation equilibrium. Say that* T^* *satisfies the* ***no-transfer condition*** *relative to* Q^* *if, for any coalition* $S \subset M$ *and each* $\mathbf{z} \in \mathbf{Z}$*, one has* $\sum_{i \in S}\left[\sum_{j=1}^n \theta_{ij}\sigma_j(\mathbf{p}^*, \mathbf{z}) + \tau_i(\mathbf{p}^*, \mathbf{z})\right] = 0$.

The no-transfer condition relative to a valuation equilibrium Q^* tells us that the restriction of T^* over any coalition corresponds to a balanced tax system. To understand better the extent of this requirement it is worth thinking of the set $\mathbf{Z}$ as containing all non-empty subsets of M, so that the tax system depends also on the coalition structure. Note that when there are neither taxes nor subsidies this condition is automatically satisfied. Another case in which this requirement holds is when the tax system is given by $\tau_i(\mathbf{p}, \mathbf{z}) = -\sum_{j=1}^n \theta_{ij}\sigma_j(\mathbf{p}, \mathbf{z})$ for all i (each individual i's net tax is equal to i's share of the subsidies paid to the producers whose shares i holds).

The following result is obtained:

Theorem 3 *Let* $Q^* = [\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{z}^*, T^*]$ *be a valuation equilibrium. Suppose that consumers are locally non-satiated and that* T^* *satisfies the no-transfer condition relative to* Q^**. Then, the resulting allocation is in the core.*

Proof.

For any $S \subset M$, consider any alternative allocation $[(\mathbf{x}_i'), (\mathbf{y}_j'), \mathbf{z}']$ satisfying (i) and (ii) of Definition 5. From the definition of valuation equilibrium and the fact that preferences are locally non-satiated, it follows that

$$\mathbf{p}^*\mathbf{x}_i' \geq \mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}[\mathbf{p}^*\mathbf{y}_j^* + \sigma_j(\mathbf{p}^*, \mathbf{z}^*)] + \tau_i(\mathbf{p}^*, \mathbf{z}') \qquad [1]$$

for all $i \in S$. Summing over S and making use of the non-transfer condition one gets:

$\sum_{i \in S} \mathbf{p}^*\mathbf{x}_i' \geq \sum_{i \in S}[\mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\mathbf{p}^*\mathbf{y}_j^* + \sum_{j=1}^n \theta_{ij}\sigma_j(\mathbf{p}^*, \mathbf{z}^*) + \tau_i(\mathbf{p}^*, \mathbf{z}')]$
$\geq \sum_{i \in S}[\mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\mathbf{p}^*\mathbf{y}_j' + \sum_{j=1}^n \theta_{ij}\sigma_j(\mathbf{p}', \mathbf{z}') + \tau_i(\mathbf{p}^*, \mathbf{z}')]$
$= \sum_{i \in S}[\mathbf{p}^*\omega_i + \sum_{j=1}^n \theta_{ij}\mathbf{p}^*\mathbf{y}_j']$

From (i) above it follows that both weak inequalities are equalities. Hence [1] also holds with equality for all $i \in S$. Because Q^* is a valuation equilibrium, $u_i(\mathbf{x}_i^*, \mathbf{z}^*) \geq u_i(\mathbf{x}_i', \mathbf{z}')$ for all $i \in S$. So S cannot be a blocking coalition. Hence, the resulting allocation is in the core. ■

4 SOME EXAMPLES

This section presents some specific models that can be interpreted as particular cases of the setting in sections 2 and 3. This illustrates how our model is flexible enough to encompass several different situations. Interestingly enough the existence of equilibrium is also guaranteed in all of these particular cases.

In order to facilitate the discussion, let us denote by $T^{\mathbf{0}}$ the (degenerate) balanced tax system given by $\sigma_j(\mathbf{p}, \mathbf{z}) = \tau_i(\mathbf{p}, \mathbf{z}) = 0$, for all i, j, every $(\mathbf{p}, \mathbf{z})$ in $\mathbb{R}^{\ell}_{+} \times \mathbf{Z}$. Consider the following assumption:

Assumption 3 For all $\mathbf{z} \in \mathbf{Z}$ one has:

(i) $X_i(\mathbf{z})$ is bounded from below;

(ii) $\omega_i \in \text{int}X_i(\mathbf{z})$ $(i = 1, 2, ..., m)$;

(iii) $[\mathbf{y} \in \mathbf{Y}(\mathbf{z}) = \sum_{j=1}^{n} Y_j(\mathbf{z})$, and $\mathbf{y} \neq \mathbf{0}] \Longrightarrow -\mathbf{y} \notin \mathbf{Y}(\mathbf{z})$.

This requires every consumer have bounded conditional consumption sets with the endowment vector in the intersection of their interiors, and the aggregate production set satisfy the irreversibility hypothesis.

4.1 Competitive equilibrium

The simplest instance of a valuation equilibrium is the standard competitive case. To see this take $\mathbf{Z}$ to be a singleton (e.g. $\mathbf{z}$ describes the assignment of property rights of a private ownership economy), let Y_j, X_i be fixed convex sets for all i, j, and let $T = T^{\mathbf{0}}$. The resulting economy is the standard private ownership model of Arrow and Debreu (1954), and every private competitive equilibrium corresponds to a valuation equilibrium. Hence, under assumptions 1, 2 and 3 these equilibria exist, are in the core, and also every Pareto optimal allocation can be decentralized as a competitive equilibrium.

A variant of this model is that in which externalities are allowed [e.g., Debreu (1952)]. It is well known that under standard conditions a competitive equilibrium exists, though it may fail to be Pareto optimal. Optimality can however be ensured if markets are combined with a suitable tax system (or a system of personalized prices). If one takes $\mathbf{Z}$ as the set of allocations and makes the tax system equivalent to a Lindahl price system, a valuation equilibrium corresponds to an efficient market equilibrium. In this case our equilibrium notion corresponds to what Bergstrom (1970) calls a *distributive Lindahl equilibrium.*

Another case worth mentioning is that of an economy in which commodities can be of different qualities [see for instance Drèze and Hagen

(1978)]. Members of the set $\mathbf{Z}$ can be thought of as different *quality standards*. Changes in these standards will typically affect consumers' feasible sets and utilities, as well as firms' production possibilities. For an arbitrary $\mathbf{z} \in \mathbf{Z}$, and $T = T^{\mathbf{0}}$, a private competitive equilibrium is well defined, and exists under assumptions 1, 2 and 3. Yet it may be inefficient (e.g., some consumers may be willing to pay for higher quality). The results in section 3 show that any efficient allocation can be decentralized as a valuation equilibrium.[3]

4.2 Economies with public goods

Consider now an economy with ℓ private commodities and k public goods. There are $n-1$ private firms and a publicly owned technology (the nth firm) that produces the public goods using private goods as inputs. Let $\mathbf{Z} = (Y_n \cap \mathbb{R}^k_+) \times \mathbb{R}^{km}_+$, where Y_n stands for the public firm, $\mathbb{R}^k_+$ for the space of public goods, and $\mathbb{R}^{km}_+$ for the space of personalized prices. Hence, a point $\mathbf{z} = (\mathbf{s}, \mathbf{q}) \in \mathbf{Z}$ describes a vector of public goods $\mathbf{s} \in \mathbb{R}^k_+$ and a vector of personalized prices $\mathbf{q} = (\mathbf{q}^1, ..., \mathbf{q}^m)$, with $\mathbf{q}^i \in \mathbb{R}^k_+$ for all i. There is a mapping $\mathbf{c} : \mathbb{R}^\ell_+ \times \mathbb{R}^k_+ \to \mathbb{R}^\ell$ such that, for each given pair $(\mathbf{p}, \mathbf{s})$ in $\mathbb{R}^\ell_+ \times \mathbb{R}^k_+$ the private goods input vector $\mathbf{c}$ minimizes the cost of producing $\mathbf{s}$. Assume also that this is a well defined single-valued vector mapping, to make things simpler.

The following balanced tax system can now be defined: For $i = 1, 2, ..., m$, let $\tau_i(\mathbf{p}, \mathbf{z}) = -\sum_{r=1}^k q_r^i s_r$, where s_r is the amount of the rth public good supplied, and q_r^i the ith consumer's personalized price for the rth public good. And let $\sigma_n(\mathbf{p}, \mathbf{z}) = -\sum_{i=1}^m \tau_i(\mathbf{p}, \mathbf{z})$ and $\sigma_j(\mathbf{p}, \mathbf{z}) = 0$, for all $j \neq n$. A *Lindahl equilibrium* for this economy (which exists under the assumptions of the model) is a valuation equilibrium $Q^* = [\mathbf{p}^*, (\mathbf{x}_i^*), (\mathbf{y}_j^*), \mathbf{s}^*, \mathbf{q}^*, T^*]$ with $\sum_{i=1}^m \mathbf{x}_i^* = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \mathbf{y}_j^* - \mathbf{c}(\mathbf{p}^*, \mathbf{s}^*)$ and $-\sum_{i=1}^m \tau_i(\mathbf{p}^*, \mathbf{z}^*) = \mathbf{p}^* \mathbf{c}(\mathbf{p}^*, \mathbf{s}^*)$.

4.3 Equilibrium with increasing returns to scale

Following the work of Scarf (1986), consider now a market economy with a single firm having production set $Y \subset \mathbb{R}^\ell$. Assume that there is a group of commodities that can be identified *a priori* as pure inputs. The set of commodity indices $I = \{1, 2, \ldots, \ell\}$ is partitioned into two disjoint subsets, $I^K = \{1, 2, \ldots, k\}$, for some integer $k < \ell$, and I^S (its complement). We shall refer to goods in I^K as *capital goods*. Commodities in I^S are *standard*

[3] See Corchón (1994) for an analysis of the dual case in which prices are taken as the public environment, and qualities take the role of balancing the markets.

commodities (i.e. they can be consumption goods, inputs and outputs). It is useful to write production plans in the form: $\mathbf{y} = (\mathbf{a}, \mathbf{b})$, where $\mathbf{a} \in -\mathbb{R}^k_+$ is a vector of capital goods, and $\mathbf{b} \in \mathbb{R}^{\ell-k}$ is a vector of standard commodities. Consider now:

Assumption 4 For every $\mathbf{a}' \in -\mathbb{R}^k_+$ the set

$$B(\mathbf{a}') \equiv \{\mathbf{b} \in \mathbb{R}^{\ell-k_j} \ / \ (\mathbf{a}, \mathbf{b}) \in Y \text{ for some } \mathbf{a} \geq \mathbf{a}'\}$$

is convex.

This weakens the classical assumption of convex production sets. It says that for any given vector of capital goods $\mathbf{a}'$, the projection on $\mathbb{R}^{\ell-k}$ of those production plans not using more capital goods than those in $\mathbf{a}'$ is a convex set. This allows us to interpret these special inputs as types of fixed capital that give rise to non-convexities. Observe that this assumption is compatible with the presence of firms with convex production sets, increasing returns to scale, set-up costs or S-shaped production functions.

Let $\omega^K \in \mathbb{R}^k_+$ denote the initial endowments of capital goods, and define $\mathbf{Z}$ as follows:

$$\mathbf{Z} = \{\mathbf{a} \in -\mathbb{R}^k_+ \ / \ \mathbf{a} \geq -\omega^K\}$$

The set $\mathbf{Z}$ describes the feasible allocations of the available capital goods. Define $Y(\mathbf{a}')$ as the constrained production set, namely,

$Y(\mathbf{a}') = \{\, \mathbf{y} = (\mathbf{a}, \mathbf{b}) \in Y \text{ with } \mathbf{a} \geq \mathbf{a}'\}$.

Given a price vector $\mathbf{p} \in \mathbb{R}^\ell_+$ and a vector of capital goods $\mathbf{a}'$, the firm chooses a profit maximizing production plan $\mathbf{y}^*$ within its attainable set — that is, it satisfies $\mathbf{p}\mathbf{y}^* \geq \mathbf{p}\mathbf{y}$, for all $\mathbf{y} \in Y(\mathbf{a}')$.

It can be shown [see Villar (1997)] that under assumptions 1, 2, (i) of 4 and 5 an equilibrium exists for this economy. Yet this equilibrium may fail to be Pareto optimal. Indeed, Scarf (1986) shows that, under general assumptions, one can always find economies with empty cores unless all production sets are convex cones. A key element in that result is the possibility that all commodities are consumed. This justifies the following assumption:

Assumption 5 Capital goods are pure inputs, so they do not enter consumers' preferences or feasible sets, and also $\mathbf{a} \geq \mathbf{a}'$ implies $B(\mathbf{a}') \subset B(\mathbf{a})$..

Assumption 5 implies that the only $\mathbf{z}^*$ which is a candidate for a valuation equilibrium is that corresponding to $\mathbf{a} = -\omega^K$. Hence, making use of the existence result mentioned above we obtain:

Theorem 4 *Let E be an economy satisfying assumptions 1, 2, (i) of 3, 4 and 5. Then there exists a valuation equilibrium with $T = T^{\mathbf{0}}$.*

This equilibrium consists of a price vector and a feasible allocation in which all agents are maximizing their payoff functions within their feasible sets. These feasible sets correspond to budget sets, for the case of consumers, and the production set subject to an input constraint, for the firm.

Remark 3 *Variants of this model can accommodate public goods, as shown in Ginés (1996). A generalized Lindahl equilibrium (which exists under suitable assumptions) corresponds to a valuation equilibrium. See also Vega-Redondo (1987) for a model with both externalities and non-convexities.*

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