SINGLE-PEAKED PREFERENCES WITH SEVERAL COMMODITIES*

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Abstract

We consider the problem of allocating m $(m \geq 2)$ infinitely divisible commodities among agents with single-peaked preferences. In the two-agents case any strategy-proof and efficient solution is dictatorial. First, we propose a solution that, in the two-agents case, is the only one that satisfies strategy-proofness, no-envy and a weak requirement related to efficiency. Moreover, it is implementable in dominant strategies and satisfies consistency properties. Second, we propose an extension of the Mas-Colell's Walrasian equilibrium with slack to characterize the efficient allocations. This new solution allow us to associate with each efficient allocation an income redistribution necessary to obtain it. We prove that the original solution proposed by Mas-Colell is the efficient selection which requires an income redistribution with smallest range, and that it satisfies consistency properties.

Keywords: Consistency; Single-peaked preferences; Strategy-proofness; Walrasian equilibrium with slack.

JEL Classification Numbers: D50, D63, D71.

1. INTRODUCTION

The problem of allocating an infinitely divisible commodity among a group of agents who have single-peaked preferences has been extensively studied in the literature. In this model it is assumed that a fixed amount of a unique commodity has to be distributed, and that each agent has a critical consumption level (called his peak): above (below) that level, increases (decreases) in the consumption make him worse off. The total amount of the good must be allocated. Examples of these situations are exchange/production at fixed prices/wages (see Sprumont [15]), or the redistribution of toxic residuals in different villages in such a way that each village will be paid a fixed price per ton stored.

In this paper we study the possibility of extending the single-peaked preferences model to the case of more than one commodity. The notion of single-peaked preferences in several dimensions was first introduced in a discrete model by Barberà, Gul, and Stacchetti [1]. In a continuous setting, the notion was introduced by Barberà, Massó, and Serizawa [4]. See also Barberà, Massó, and Neme [3]. Following these papers, we say that a preference relation defined over the m commodities is single-peaked if it is single-peaked commodity by commodity, according to the definition for just one good: there is a critical consumption level for each good; above (below) that level, increases (decreases) in the consumption of this good, the consumption of the rest of commodities fixed, makes the agent worse off.

There are many economic situations in which this model makes sense. Think for example on the monetary contributions of the countries of the European Union (EU) to different research projects, such as the European Space Agency, the European Fighter Plane, etc. In order to carry out each project, a fixed amount of money is necessary. The EU has to decide the percentage of the total cost of each project that will be paid by each country. Depending on these contributions, each country will be benefited from the resulting scientific discoveries. The preference relation of each country over these economic contributions will be single-peaked. Another type of examples are the cases in which various perishable goods have to be distributed among a group of agents. The agents have single-peaked preferences defined over these goods. Here, the critical consumption level of each good is the amount that he can consume before it goes bad.

The question is to find solutions (i.e. allocation rules that associate, with each profile of single-peaked preferences, some distribution of the good) satisfying *desirable* properties. The first axiomatic analysis in the one-good case (Sprumont [15]) was concerned with strategy-proofness (every agent's best interest is to announce his true preferences) and Pareto-efficiency, in addition to some other properties related to fairness (anonymity or no-envyness). Sprumont proved that, in economies with one good and single-peaked preferences, strategy-proofness, efficiency and anonymity (alternatively, no-envyness) characterize a unique solution: the Uniform Rule (see also Ching [6] and [7]). It gives everyone his preferred consumption within an upper or lower bound (the same for all agents), which are given by the feasibility condition that the total amount of good must be distributed.

These successful results encouraged researchers to attack the single-peaked preferences model from other angles. Thomson focused on equity properties. One of his results (Thomson [16]) characterizes the Uniform Rule making use of consistency and other related properties (such as bilateral consistency or replication invariance). There are other characterizations of the Uniform Rule based on properties such as resource-monotonicity (Thomson [17]) or population-monotonicity (Thomson [18]).

Following the steps given in the case of one good, we first focus on strategy-proof solutions. We prove that, in the two agents and m-commodities ($m \geq 2$) single-peaked preferences model, any strategy-proof and efficient solution is such that one agent always gets his most preferred allocation (i.e. it is dictatorial).

How can we relax the requirement of efficiency in order to obtain strategy-proof solutions? We would like that our solutions verify at least a weak efficiency requirement that we call Condition E. We say that a solution satisfies Condition E if it is not possible to redistribute the amount of only one good in such a manner that no agent is worse off and at least one agent is better off.

We propose a strategy-proof solution verifying Condition E: the Generalized Uniform Rule. It involves applying the original definition of the Uniform Rule in each separate commodity. This solution satisfies some interesting properties related to fairness, such as Pareto-domination of equal division, no-envyness and most consistency related properties. Obviously, it fails to be efficient. However, we provide a characterization of this solution for economies with only two agents based on strategy-proofness, no-envyness and Condition E. Moreover, we prove that it is implementable in dominant strategies by means of its associated manipulation game.

Next, we focus on efficient solutions. Unlike the one-good case, there is no simple way to characterize the Pareto-efficient allocations when there are several commodities and preferences are single-peaked. However, we can solve this difficulty by making use of a concept introduced by Mas-Colell [9]: the Walrasian equilibrium with equal slacks. This is an extension of the Walrasian equilibrium for economies with possibly satiated preferences, by giving all consumers an identical extra amount of income (called slack).

We use a generalization of this concept, by allowing the slack to be different among agents and adding up to zero. We call it Walrasian equilibrium with balanced slacks. One can interpret these slacks as income redistribution among consumers: agents with negative slacks subsidize agents with positive ones. This solution allow us to extend the first and second welfare theorems to the single-peaked preferences model, and then to characterize the efficient allocations.

If we look for solutions satisfying additional properties, we just have to impose additional restrictions over the slacks. An example of that is the original Walrasian equilibrium with equal slacks (any Walrasian allocation with equal slacks can be obtained as a Walrasian allocation with balanced slacks). We prove that, given a Walrasian allocation with equal slacks, there is not any other Walrasian allocation with balanced slacks in which the income redistribution that allows us to achieve it is smaller (i.e. there is not any other efficient allocation in which, either the agent who makes the larger income contribution pays less, or the agent who receives more income obtains less). In this sense, we can think of Walrasian allocation with equal slacks as the efficient allocation which requires smaller income redistribution. We have also checked that when applying this solution from equal division, it satisfies most of the consistency related properties. It is also easy to see that this solution coincides with the Uniform Rule when $m = 1$.

The paper is organized as follows. The model and some basic definitions are presented in Section 2. In Section 3 we prove that strategy-proofness and efficiency are not compatible. In Section 4 we focus on strategy-proofness and present the Generalized Uniform Rule. Section 5 deals with efficiency and Walrasian solutions with slacks. In Section 6 we study consistency related properties. Concluding comments are gathered in Section 7.

2. THE MODEL

Let $M = \{1, ..., m\}$ be the set of m infinitely divisible commodities. For each commodity $r \in M$ there is an amount $\Omega_r \in \mathbb{R}_+$ that has to be distributed among the agents in a set $N = \{1, ..., n\}$. Let $\Omega = (\Omega_1, ..., \Omega_m) \in \mathbb{R}^m_+$. Each agent $i \in N$ has a preference relation R_i defined over $\mathbb{R}^m_+$ which is single-peaked over $[0, \Omega] \subset \mathbb{R}^m_+$. Single-peakness means that there is a vector $p(R_i) = (p_1(R_i), ..., p_m(R_i)) \in [0, \Omega]$ called agent i's peak such that for all $x_i = (x_{i1}, ..., x_{im})$, $x'_i = (x'_{i1}, ..., x'_{im})$ with $x_i, x'_i \in [0, \Omega]$ and $x_i \neq x'_i$, if for each $r \in M$ either $p_r(R_i) \geq x_{ir} \geq x'_{ir}$ or $p_r(R_i) \leq x_{ir} \leq x'_{ir}$, then $x_i P_i x'_i$ (where P_i denotes the strict preference relation associated with R_i; in the same way, I_i denote the indifference relation). Let $\Re$ denote the class of all these preference relations. Figures 2.1 and 2.2 show some examples.

An *economy* is a profile $R \in \Re^n$. A *feasible allocation* for $R \in \Re^n$ is a list $x = (x_i)_{i \in N} \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$ such that $\sum x_i = \Omega$ (notice that free disposal of the commodities is not assumed). Let X denote the set of feasible allocations. A *solution* is a mapping, φ, which associates with every preference profile $R \in \Re^n$ a non-empty subset $\varphi(R)$ of the set of feasible allocations. Some examples are the following:

Pareto-efficient solution, PE. Given $R \in \Re^n$, $x \in PE(R)$ if $x \in X$ and there is no other feasible allocation $x' \in X$ such that for all $i \in N$, $x'_i R_i x_i$, and for some $j \in N$, $x'_j P_j x_j$.

No-envy solution, F. Given $R \in \Re^n$, $x \in F(R)$ if $x \in X$ and for all $i, j \in N$, $x_i R_i x_j$.

Pareto-dominant of equal division solution, D. Given $R \in \Re^n$, $x \in D(R)$ if $x \in X$ and for all $i \in N$, $x_i R_i (\Omega/n)$.

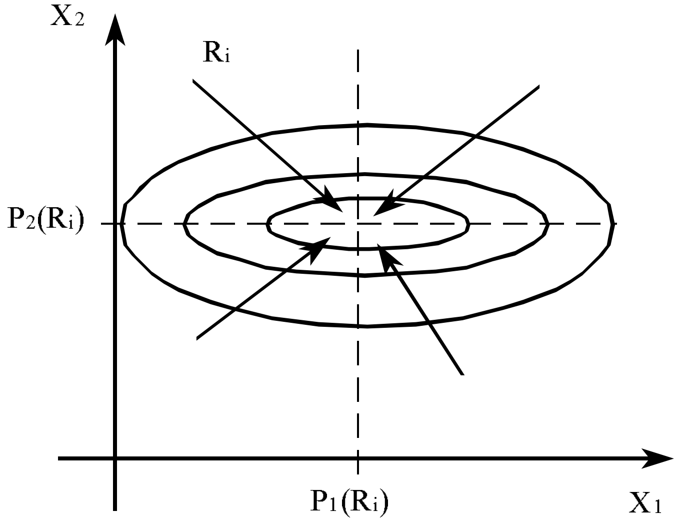


Figure 2.1:

Occasionally, we will find it convenient to make additional assumptions on economies. Given $R \in \Re^n$, $i \in N$ an $x_i \in \mathbb{R}_+^m$, let $U_i(x_i, R_i)=\{x_i' \in \mathbb{R}_+^m : x_i' R_i x_i\}$ and $L_i(x_i, R_i)=\{x_i' \in \mathbb{R}_+^m : x_i R_i x_i'\}$ be the upper and lower contour sets of R_i at x_i respectively (when it is clear we will write $U_i(x_i)$ and $L_i(x_i)$). Replacing preference by strict preference we have the strict upper and lower contour sets: $SU_i(x_i, R_i)$ and $SL_i(x_i, R_i)$ respectively. Agent i's preference relation $R_i \in \Re$ is *continuous* if, for all $x_i \in \mathbb{R}_+^m$, $U_i(x_i, R_i)$ and $L_i(x_i, R_i)$ are closed sets (equivalently, $SU_i(x_i, R_i)$ and $SL_i(x_i, R_i)$ are open sets). We say that R_i is *weakly convex* if, for all $x_i, x_i' \in \mathbb{R}_+^m$ with $x_i R_i x_i'$ and for all $\mu \in (0, 1)$, $(\mu x_i + (1-\mu)x_i')R_i x_i'$. R_i is *strictly convex* if, for all $x_i, x_i' \in \mathbb{R}_+^m$ with $x_i R_i x_i'$ and for all $\mu \in (0, 1)$, $(\mu x_i + (1-\mu)x_i')P_i x_i'$. We will say that $R_i \in \Re$ is *monotone* in $[0, \Omega]$ when $p(R_i) = \Omega$. Finally, R_i is *smooth* in the interior of $[0, \Omega]$ ($Int[0, \Omega]$) if for all $x_i \in Int[0, \Omega]$ with $x_i p(R_i)$, the indifference curve passing through x_i can be supported by only one hyperplane.

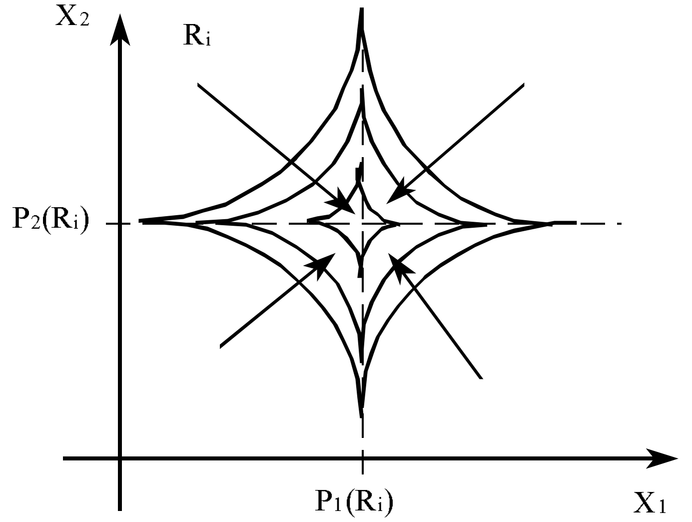


Figure 2.2:

3. INCOMPATIBILITY BETWEEN STRATEGY-PROOFNESS AND EFFICIENCY

For all preference profile $R \in \Re^n$ and all agent $i \in N$, we denote by R_{-i} the list of preferences for all agents except agent i, that is, $R_{-i} \equiv (R_j)_{j \in N \setminus \{i\}}$.

A single-valued solution φ is *strategy-proof* if for all $R \in \Re^n$, all $i \in N$, and all $R'_i \in \Re$, $\varphi_i(R_i, R_{-i}) R_i \varphi_i(R'_i, R_{-i})$.

A single-valued solution φ is *dictatorial* when there is some $i \in N$ such that, for all $R \in \Re^n$, $\varphi_i(R) = p(R_i)$. In the case of only one good, $m = 1$, we define the Uniform Rule as follows:

Uniform Rule, U. Given $R \in \Re^n$, $x = U(R)$ if $x \in X$ and, (i) when $\sum p(R_i) \geq \Omega$, $x_i = \min\{p(R_i), \lambda\}$ for all $i \in N$, where λ solves $\sum \min\{p(R_i), \lambda\} = \Omega$, and (ii) when $\sum p(R_i) \leq \Omega$, $x_i = \max\{p(R_i), \lambda\}$ for all $i \in N$, where λ solves $\sum \max\{p(R_i), \lambda\} = \Omega$.

The Uniform Rule has been characterized making use of variety of properties (see Thomson [19], Chapter 11, for a good survey). Sprumont [15] proved that, when $m = 1$, the Uniform Rule is the only strategy-proof selection from the Pareto and no-envy solution (see also Ching [6]). Is it possible to extend a similar result to the case of more than one commodity? What we know of classical economies is not very encouraging. Zhou [20] proved that, in two-agents, m-goods ($m \geq 2$) pure exchange economies with monotone, continuous, and strictly convex preferences, any strategy-proof and efficient solution is such that one agent always receives nothing. Similar results hold in economies with public goods (Border and Jordan [5] and Zhou [21]) and economies with public and private goods (Moreno and Walker [11] and Moreno [10]).

Unfortunately, at least for the two-agents case, negative results reappear when preferences are single-peaked over more than one commodity. As a matter of fact, this can be directly deduced from the previous results for economies with public goods. Notice that, when there are only two agents, the distribution problem of the single-peaked preferences model can be reinterpreted as one of public goods (see Barberà, Jackson, and Neme [2]). Suppose that $N = \{i, j\}$. Since the total amount of all commodities has to be allocated, we have that for all $x \in X$, $x_j = \Omega - x_i$, and therefore the redistribution problem is just to select some $z \equiv x_i \in [\in, \Omega]$. Then, for all $R_i, R_j \in \Re$, we can redefine the preference relations of the agents in such a way that both of them depend on the same variable z, and they remain single-peaked.

Some of the impossibility results on strategy-proof and efficient solutions for economies with pure public goods can be extended to our model. We first introduce some notation. We say that a preference relation is separable quadratic if it can be represented by a utility function of the form $u(x_i) = -\sum_r a_r(x_{ir} - p_{ir})^2$, where $a_r > 0$ for all $r \in M$. Let $\Re^c$ be the class of separable quadratic preferences. Notice that $\Re^c \subset \Re$.

Border and Jordan [5] proved that, in economies with pure public goods where the space of admissible economies is $\Re^c$ and there are two or more commodities, any strategy-proof and Pareto-efficient solution is dictatorial[1]. Given the previous interpretation of our model in terms of public goods when $n = 2$, we can state the following theorem:

Theorem 3.1. *(Border and Jordan [5]). Let $n = 2$ and $m \geq 2$. Let $\varphi : (\Re^c)^2 \to X$ be a strategy-proof and Pareto-efficient solution. Then φ is dictatorial.*

Applying this result to our model we have the following corollary:

[1] See Border and Jordan [5], Corollary 4.

Corollary 3.2. *Let $n = 2$ and $m \geq 2$. Let $\varphi : \Re^n \to X$ be a strategy-proof and Pareto-efficient solution. Then φ is dictatorial.*

Proof. Suppose that $N = \{i, j\}$. Let $\Re^*$ be a class of preference relations defined over $\mathbb{R}^m_+$ such that $\Re^c \subset \Re^* \subset \Re$. Let $\varphi^* : (\Re^*)^2 \to X$ be a strategy-proof and Pareto-efficient solution. Let $\varphi : (\Re^c)^2 \to X$ be such that for all $R \in (\Re^c)^2$, $\varphi(R) = \varphi^*(R)$. Then, by Theorem 3.1, φ is dictatorial. Suppose w.l.o.g. that agent i is the dictator. Let $R_i \in \Re^*$ and $R_j \in \Re^c$. Then $\varphi^*(R_i, R_j) = p(R_i)$. Suppose not. Let $R'_i \in \Re^c$ be such that $p(R'_i) = p(R_i)$. Then, since $\varphi^*(R'_i, R_j) = \varphi(R'_i, R_j) = p(R'_i) = p(R_i) \neq \varphi^*(R_i, R_j)$, we have that $\varphi^*(R'_i, R_j) P_i \varphi^*(R_i, R_j)$, which contradicts strategy-proofness. Let $R_i \in \Re^c$ and $R_j \in \Re^*$. Then $\varphi^*(R_i, R_j) = p(R_i)$. Suppose not. Let $R'_j \in \Re^c$ be such that $p(R'_j) = \varphi^*(R_i, R_j)$. Then, since $\varphi^*(R_i, R'_j) = \varphi(R_i, R'_j) = p(R_i) \neq \varphi^*(R_i, R_j) = p(R'_j)$, we have that $\varphi^*(R_i, R_j) P'_j \varphi^*(R_i, R'_j)$, which contradicts strategy-proofness. Let $R_i \in \Re^*$ and $R_j \in \Re^*$. Then $\varphi^*(R_i, R_j) = p(R_i)$. Suppose not. Let $R'_i \in \Re^c$ be such that $p(R'_i) = p(R_i)$. Then, since $\varphi^*(R'_i, R_j) = p(R'_i) = p(R_i) \neq \varphi^*(R_i, R_j)$, we have that $\varphi^*(R'_i, R_j) P_i \varphi^*(R_i, R_j)$, which contradicts strategy-proofness. ■

Corollary 3.2 states that Sprumont's positive results on strategy-proofness and efficiency depend essentially on the number of commodities: if it is larger than one, the results do not hold, at least for the two-agent case[2].

The proof of this corollary is based on an argument similar to the one used in Schummer[3] [13], and it follows from the result by Border and Jordan and the fact that the class of separable quadratic preferences are included in the class of single-peaked preferences. It is easy to see that the negative result remains for all subdomain of $\Re$ including $\Re^c$.

[2]For economies with more than two agents, there are examples of strategy-proof and efficient solutions where there is no agent who always obtains his peak (see Satterthwaite and Sonnenschein [12]). However, any such solution will not be individually rational from equal division.

[3]Schummer [13] prove that, in pure exchange economies, even if we restrict the domain of admissible economies to linear and monotone preferences, any strategy-proof and Pareto-efficient solution is such that one agent always receives nothing. Then, he uses essentially the same argument of the proof of Corollary 3.2 to show that the result by Zhou [20] can be obtained just as a consequence of his own result (the domain considered by Schummer is included in the one considered by Zhou). One could think that, since linear and monotone preferences are also included in the single-peaked preferences domain, the same argument is valid to extending Schummer=s result to our model. However, in the proof of Schummer, the fact that all preference relations in the largest domain are monotone plays a crucial role.

One could think that some other stronger impossibility results on strategy-proof solutions for economies with public goods apply here (for instance, the result by Zhou [21] that states that any strategy-proof solution is dictatorial). However, this is not true, because the preference domain considered in these results contains the class of all quadratic preferences, which itself contains preferences which do not verify our definition of single-peakness.

Corollary 3.2 is related with the result by Zhou [20] (or the more general result by Schummer [13]) for pure exchange economies. Monotone preferences is the special case where agents' peaks are equal to Ω, and then Corollary 3.2 predicts that, the agent who is not the dictator obtains nothing. However, as we comment in Footnote 3, Corollary 3.2 can not be deduced from these results.

In view of this incompatibility between strategy-proofness and efficiency, we analyze both problems separately. First, we focus on strategy-proof solutions.

4. STRATEGY-PROOF SOLUTIONS: THE GENERALIZED UNIFORM RULE

A good way to look for strategy-proof solutions with a good behavior in our model is to try to extend the Uniform Rule to the multi-commodity case. There is, a priori, a natural way to do that: the application, in each separate good, of the original definition of the Uniform Rule. This is what the next solution does:

Generalized Uniform Rule, V. Given $R \in \Re^n$, $x = V(R)$ if $x \in X$ and, for all $r \in M$, (i) when $\sum_i p_r(R_i) \geq \Omega_r$, then $x_{ir} = \min\{p_r(R_i), \lambda_r\}$ for all $i \in N$, where λ_r solves $\sum_i \min\{p_r(R_i), \lambda_r\} = \Omega_r$, and (ii) when $\sum_i p_r(R_i) \leq \Omega_r$, then $x_{ir} = \max\{p_r(R_i), \lambda_r\}$ for all $i \in N$, where λ_r solves $\sum_i \max\{p_r(R_i), \lambda_r\} = \Omega_r$.

The Generalized Uniform Rule maintains some of the nice properties of the Uniform Rule. First, as the following proposition states, V is strategy-proof.

Proposition 4.1. *The Generalized Uniform Rule is strategy-proof.*

Proof. Given $R \in \Re^n$, $i \in N$, $R'_i \in \Re$ and $r \in M$, suppose that $\sum_j p_r(R_j) \geq \Omega_r$. Then $V_{ir}(R) \leq p_r(R_i)$. In case that $V_{ir}(R) < p_r(R_i)$, if $p_r(R'_i) \geq V_{ir}(R)$, $V_{ir}(R'_i, R_{-i}) = V_{ir}(R)$, and if $p_r(R'_i) < V_{ir}(R)$ then, either (i) $p_r(R'_i)+\sum_{j\neq i} p_r(R_j) \geq \Omega_r$ and $V_{ir}(R'_i, R_{-i}) = p_r(R'_i) < V_{ir}(R)$, or (ii) $p_r(R'_i) + \sum_{j\neq i} p_r(R_j) < \Omega_r$ and $V_{ir}(R'_i, R_{-i}) \leq V_{ir}(R)$. Hence, $V_{ir}(R'_i, R_{-i}) \leq V_{ir}(R) \leq p_r(R_i)$. In the same way it can be proved that, if $\sum_j p_r(R_j) \leq \Omega_r$, then $V_{ir}(R'_i, R_{-i}) \geq V_{ir}(R) \geq p_r(R_i)$.

Notice that then, for all $r \in M$, either $V_{ir}(R'_i, R_{-i}) \leq V_{ir}(R) \leq p_r(R_i)$, or $V_{ir}(R'_i, R_{-i}) \geq V_{ir}(R) \geq p_r(R_i)$. Therefore $V_i(R) R_i V_i(R'_i, R_{-i})$. ■

Furthermore, V also satisfies some interesting properties related to fairness, such as no-envy or Pareto domination of equal division:

Proposition 4.2. *The generalized uniform allocation is no-envy.*

Proof. Given $R \in \Re^n$, $x = V(R)$, $i, j \in N$ and $r \in M$, suppose that $\sum_k p_r(R_k) \geq \Omega_r$. Then, there exists some $\lambda_r \geq 0$ such that, for all $k \in N$, $x_{kr} = \min\{p_r(R_k), \lambda_r\}$. Therefore, either $x_{ir} = p_r(R_i)$, or $x_{jr} \leq x_{ir} = \lambda_r < p_r(R_i)$. Suppose now that $\sum_i p_r(R_i) \leq \Omega_r$. Then, there is $\lambda_r \geq 0$ such that, for all $k \in N$, $x_{kr} = \max\{p_r(R_k), \lambda_r\}$. Therefore, either $x_{ir} = p_r(R_i)$, or $p_r(R_i) < x_{ir} = \lambda_r \leq x_{jr}$. In any case, either $p_r(R_i) \geq x_{ir} \geq x_{jr}$ or $p_r(R_i) \leq x_{ir} \leq x_{jr}$. Therefore, if $x_i \neq x_j$, $x_i P_i x_j$. ■

Proposition 4.3. *The generalized uniform allocation Pareto-dominates equal division.*

Proof. Given $R \in \Re^n$, $x = V(R)$, $i \in N$ and $r \in M$, suppose that $p_r(R_i) \leq \Omega_r/n$. If $\sum_j p_r(R_j) \geq \Omega_r$, $x_{ir} = p_r(R_i) \leq \Omega_r/n$, and if $\sum_j p_r(R_j) \leq \Omega_r$, $p_r(R_i) \leq x_{ir} \leq \Omega_r/n$. Suppose now that $p_r(R_i) \geq \Omega_r/n$. If $\sum_j p_r(R_j) \geq \Omega_r$, $\Omega_r/n \leq x_{ir} \leq p_r(R_i)$, and if $\sum_j p_r(R_j) \leq \Omega_r$, $\Omega_r/n \leq p_r(R_i) = x_{ir}$. Then, for all $r \in M$, either $p_r(R_i) \leq x_{ir} \leq \Omega_r/n$, or $p_r(R_i) \geq x_{ir} \geq \Omega_r/n$. Therefore $x_i R_i \Omega/n$. ■

From Corollary 3.2 of the previous section and Proposition 4.1 it follows that V is not a Pareto selection. However it is easy to see that it satisfies the following weaker requirement related to efficiency:

Condition E. A solution φ satisfies Condition E when, for all $R \in \Re^n$ and $r \in M$, (i) if $\sum_i p_r(R_i) \geq \Omega_r$, then $\varphi_{ir}(R_i) \leq p_r(R_i)$ for all $i \in N$, and (ii) if $\sum_i p_r(R_i) \leq \Omega_r$, then $\varphi_{ir}(R_i) \geq p_r(R_i)$ for all $i \in N$.

As Sprumont [15] shows, Condition E is a characterization of the Pareto-efficient solution when $m = 1$. However, when $m \geq 2$ Condition E is only a necessary condition for Pareto-efficiency. If a solution does not satisfy Condition E, there are economies for which the allocation selected by the solution can be improved in a simple way: it is possible to redistribute the amount of only one good in such a manner that no agent is worse off and at least one agent is better off.

In this sense Condition E can be interpreted as a minimum efficiency requirement. Given the results of the previous section, if we insist on strategy-proofness, we have to study how efficiency can be relaxed in order to obtain solutions other than the dictatorial. Condition E is an obvious way to do that.

Although Condition E is a weak requirement, when combined with strategy-proofness and other properties related with fairness like no-envyness we obtain a characterization of the Generalized Uniform Rule. Theorem 4.4 states that in two-agents and m-commodities economies, there is no strategy-proof and no-envy solution satisfying Condition E other than the Generalized Uniform Rule.

Theorem 4.4. *The Generalized Uniform Rule is the only no-envy and strategy-proof solution satisfying Condition E when* $n = 2$.

Notice that, although our notions of single-peakedness, Condition E and Generalized Uniform Rule are just multidimensional extensions of the corresponding one-dimensional concepts of single-peakedness, Pareto-efficiency and Uniform Rule respectively, Theorem 4.4 can not be deduced from the characterization of the Uniform rule given by Sprumont [15], since the multidimensional concepts of non-envyness and strategy-proofness are not one-dimensionally based properties. This is also the reason that our result is restricted to the $n = 2$ case, whereas the result of Sprumont holds for all $n \in \mathbb{N}$.

In order to prove the theorem we need five lemmas. The first lemma give us conditions under which, given three bundles, there exists a single-peaked preference relation with its peak equal to the first bundle, and such that the second bundle is strictly preferred to the third one.

Lemma 4.5. *For all agent* $i \in N$ *and all three bundles,* $x_i^*, x_i', x_i'' \in \mathbb{R}_+^m$ *with* $x_i' \neq x_i''$, *and such that it does not happen that, for all* $r \in M$, *either* $x_{ir}^* \leq x_{ir}'' \leq x_{ir}'$ *or* $x_{ir}^* \geq x_{ir}'' \geq x_{ir}'$, *then there exists some* $R_i \in \Re$ *with* $p(R_i) = x_i^*$ *and* $x_i' P_i x_i''$.

Proof. (Figures 4.1 and 4.2). Let $x_i^*, x_i', x_i'' \in \mathbb{R}_+^m$ be as defined. Let $\overline{a} = (a_r^1, a_r^2)_{r \in M} \in \mathbb{R}_+^{2m} \setminus \{0\}$. Let $u_{\overline{a}} : \mathbb{R}_+^m \to \mathbb{R}$ be an utility function such that, for all $x_i \in \mathbb{R}_+^m$,

$$u_{\overline{a}}(x_i) = -\sum_r a_r^{x_i}(x_{ir} - x_{ir}^*)^2$$

where, for all $r \in M$,

$$a_r^{x_i} = \begin{cases} a_r^1, \text{ if } x_{ir} \geq x_{ir}^* \\ a_r^2, \text{ if } x_{ir} < x_{ir}^* \end{cases}$$

Let $R_i(\overline{a})$ be the preference relation represented by the former utility function. Now we will prove that, for all $\overline{a} \in \mathbb{R}_+^{2m} \backslash \{0\}$, $R_i(\overline{a})$ is single-peaked. Let $z_i, z_i' \in \mathbb{R}_+^m$ be such that $z_i \neq z_i'$ and, for all $r \in M$, either $x_{ir}^* \leq z_{ir} \leq z'_{ir}$ or $x_{ir}^* \geq z_{ir} \geq z'_{ir}$. Notice that for all $r \in M$, $(z_{ir} - x_{ir}^*)^2 \leq (z'_{ir} - x_{ir}^*)^2$, and that at least one of these inequalities is strict. Moreover, for all $\overline{a} \in \mathbb{R}_+^{2m} \setminus \{0\}$ and all $r \in M$, either $(z_{ir} - x_{ir}^*)^2 = 0$ or $a_r^{z_i} = a_r^{z_i'}$. Therefore $u_{\overline{a}}(z_i) > u_{\overline{a}}(z_i')$, and $R_i(\overline{a}) \in \Re$ with peak $p(R_i) = x_i^*$. Now, we will prove that there exists some $\overline{a} \in \mathbb{R}_+^{2m} \setminus \{0\}$ such that $u_{\overline{a}}(x_i') > u_{\overline{a}}(x_i'')$. Let $M' = \{r \in M$: neither $x'_{ir} \geq x''_{ir} \geq x_{ir}^*$ nor $x'_{ir} \leq x''_{ir} \leq x_{ir}^*\}$ First notice that, for all $\overline{a} \in \mathbb{R}_+^{2m} \setminus \{0\}$ and all $r \in M$ such that either $x''_{ir} > x'_{ir} \geq x_{ir}^*$ or $x''_{ir} < x'_{ir} \leq x_{ir}^*$, we have that $-a_r^{x_i'}(x'_{ir} - x_{ir}^*)^2 > -a_r^{x_i''}(x''_{ir} - x_{ir}^*)^2$. Let now $\overline{a} = (a_r^1, a_r^2)_{r \in M} \in \mathbb{R}_+^{2m} \setminus \{0\}$ be such that for all $r \in M$ with $x'_{ir} \neq x''_{ir}$ (i) if $x'_{ir} \geq x_{ir}^*$ and $x''_{ir} \leq x_{ir}^*$, then $a_r^1 < a_r^2$ with $(a_r^2 - a_r^1)$ large enough for $-a_r^1(x'_{ir} - x_{ir}^*)^2 > -a_r^2(x''_{ir} - x_{ir}^*)^2$ and, (ii) if $x'_{ir} \leq x_{ir}^*$ and $x''_{ir} \geq x_{ir}^*$, then $a_r^1 > a_r^2$ with $(a_r^1 - a_r^2)$ large enough for $-a_r^2(x'_{ir} - x_{ir}^*)^2 > -a_r^1(x''_{ir} - x_{ir}^*)^2$. Then, for all $r \in M'$, a_r^1 and a_r^2 are such that $-a_r^{x_i'}(x'_{ir} - x_{ir}^*)^2 > -a_r^{x_i''}(x''_{ir} - x_{ir}^*)^2$. Moreover, it is clear that for all $r \in M \backslash \{M'\}$ and for some $s \in M'$, we can choose $\overline{a} \in \mathbb{R}_+^{2m} \backslash \{0\}$ in such a manner that $-\sum_{r \in M \backslash \{M'\}} a_r^{x_i'}(x'_{ir} - x_{ir}^*)^2 + a_s^{x_i'}(x'_{is} - x_{is}^*)^2 > -\sum_{r \in M \backslash \{M'\}} a_r^{x_i''}(x''_{ir} - x_{ir}^*)^2 + a_s^{x_i''}(x''_{is} - x_{is}^*)^2$. Therefore, we can construct a utility function $u_{\overline{a}}(.)$ as defined for which $u_{\overline{a}}(x_i') > u_{\overline{a}}(x_i'')$. ■

Now we introduce some additional notation. For all $i \in N$, all $R_i, R_i' \in \Re$ and all $x_i \in \mathbb{R}_+^m$, let $M^*(R_i, R_i', x_i)$=$\{r \in M$: either (i) $p_r(R_i) < x_{ir}$ and $p_r(R_i') \leq x_{ir}$, or (ii) $p_r(R_i) > x_{ir}$ and $p_r(R_i') \geq x_{ir}$, or (iii) $p_r(R_i) = x_{ir}$ and $p_r(R_i') = x_{ir}\}$, and $M(R_i, R_i', x_i)$=$\{r \in M$: either (i) $p_r(R_i) < x_{ir}$ and $p_r(R_i') \leq x_{ir}$, or (ii) $p_r(R_i) > x_{ir}$ and $p_r(R_i') \geq x_{ir}$, or (iii) $p_r(R_i) = x_{ir}\}$. In words, commodity r is in set $M^*(R_i, R_i', x_i)$ when, (1) if the peak for this commodity in R_i is strictly smaller (greater) than the amount of that commodity in x_i, the peak in R_i' remains smaller (greater), or, (2) if the peak for this commodity in R_i is equal to the amount of that

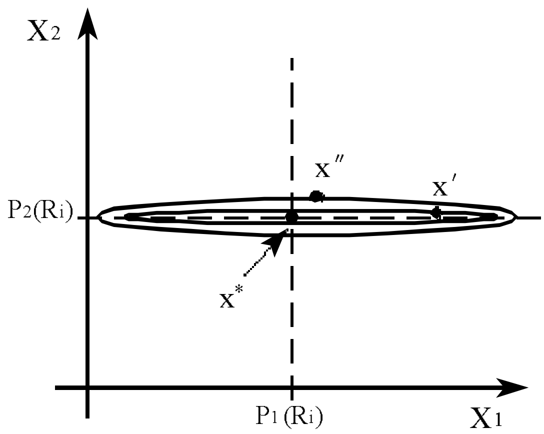


Figure 4.1:

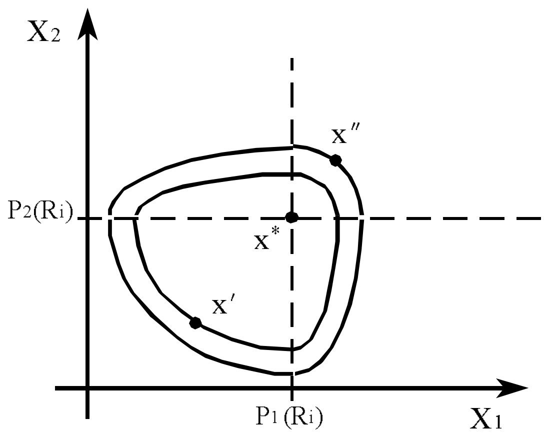


Figure 4.2:

commodity in x_i, the peak in R_i' is the same. Commodity r is in set $M(R_i, R_i', x_i)$ when it satisfies (1).

The next four lemmas apply only when there are two agents. Lemma 4.6 establishes a limited invariance property of strategy-proof solutions satisfying Condition E. It states that, given any initial announcements, if agents change them in such a manner that one agent announces his new peak just in the bundle selected by the solution for him with the initial announcements, his consumption of any commodity does not change if the other agent announces a new peak which, for this good, remains smaller, greater or equal (depending on the initial announcements) than his own initial consumption.

Lemma 4.6. *Let $N = \{i, j\}$. Let φ be a strategy-proof solution satisfying Condition E. Then, for all $R \in \Re^n$, all $R_i^* \in \Re$ with $p(R_i^*) = \varphi_i(R)$, all $R_j' \in \Re$ and all $r \in M^*(R_j, R_j', \varphi_j(R))$, we have $\varphi_{ir}(R_i^*, R_j') = \varphi_{ir}(R)$.*

Proof. Let $N = \{i, j\}$. Given $R \in \Re^n$, let $R_i^* \in \Re$ be such that $p(R_i^*) = \varphi_i(R)$.
Step 1. Since φ is strategy-proof, $\varphi_i(R_i^*, R_j) = \varphi_i(R) = p(R_i^*)$, and then $\varphi(R_i^*, R_j) = \varphi(R)$.
Step 2 (Figure 4.3). For all $R_j' \in \Re$ with $p(R_j') = p(R_j)$, $\varphi(R_i^*, R_j') = \varphi(R)$. Notice that, for all $r \in M$, $p_r(R_i^*)+p_r(R_j') = p_r(R_i^*)+p_r(R_j)$. Since φ satisfies Condition E, for all $r \in M$, (i) if $p_r(R_j) = p_r(R_j') < \varphi_{jr}(R_i^*, R_j')$ then $p_r(R_i^*) \leq \varphi_{ir}(R_i^*, R_j')$, and (ii) if $p_r(R_j) = p_r(R_j') > \varphi_{jr}(R_i^*, R_j)$, then $p_r(R_i^*) \geq \varphi_{ir}(R_i^*, R_j')$. Therefore, if $p_r(R_j) < \varphi_{jr}(R_i^*, R_j')$, $\varphi_{jr}(R_i^*, R_j') \leq \varphi_{jr}(R_i^*, R_j)$ (otherwise, by feasibility, $\varphi_{ir}(R_i^*, R_j') < \varphi_{ir}(R_i^*, R_j) = p_r(R_i^*)$, which contradicts (i)). In the same way, if $p_r(R_j) > \varphi_{jr}(R_i^*, R_j')$, $\varphi_{jr}(R_i^*, R_j') \geq \varphi_{jr}(R_i^*, R_j)$ (otherwise, by feasibility, $\varphi_{ir}(R_i^*, R_j') > \varphi_{ir}(R_i^*, R_j) = p_r(R_i^*)$, which contradicts (ii)). Therefore, for all $r \in M$, either $p_r(R_j) \leq \varphi_{jr}(R_i^*, R_j') \leq \varphi_{jr}(R_i^*, R_j)$ or $p_r(R_j) \geq \varphi_{jr}(R_i^*, R_j') \geq \varphi_{jr}(R_i^*, R_j)$. Hence $\varphi_j(R_i^*, R_j') = \varphi_j(R_i^*, R_j)$ (otherwise $\varphi_j(R_i^*, R_j') Pj \varphi_j(R_i^*, R_j)$, which contradicts strategy-proofness), and so $\varphi(R_i^*, R_j') = \varphi(R_i^*, R_j)$.
Step 3 (Figure 4.4). For all $R_j' \in \Re$ and $r \in M^*(R_j, R_j', \varphi_j(R))$, $\varphi_{ir}(R_i^*, R_j') = \varphi_{ir}(R)$. If $p_r(R_i^*)+p_r(R_j) \leq \Omega_r$, by Condition E and Step 1, $p_r(R_j) \leq \varphi_{jr}(R_i^*, R_j) = \varphi_{jr}(R)$, and then, since $p_r(R_i^*) = \varphi_{ir}(R)$ and $r \in M^*(R_j, R_j', \varphi_j(R))$, we have $p_r(R_i^*) + p_r(R_j') \leq \varphi_{ir}(R) + \varphi_{jr}(R) = \Omega_r$. Similarly, if $p_r(R_i^*) + p_r(R_j) \geq \Omega_r$, then $p_r(R_i^*) + p_r(R_j') \geq \Omega_r$. Given this, by Condition E, (i) if $p_r(R_j) < \varphi_{jr}(R_i^*, R_j)$, $p_r(R_i^*) \leq \varphi_{ir}(R_i^*, R_j')$, and (ii) if $p_r(R_j) > \varphi_{jr}(R_i^*, R_j)$, $p_r(R_i^*) \geq \varphi_{ir}(R_i^*, R_j')$. Suppose by contradiction that $\varphi_{jr}(R_i^*, R_j') \neq \varphi_{jr}(R)$. Then $p_r(R_j) \neq \varphi_{jr}(R)$ (otherwise $p_r(R_j) = \varphi_{jr}(R) = p_r(R_j')$, and since $p_r(R_i^*) = \varphi_{ir}(R)$, then $\varphi_{jr}(R_i^*, R_j') = p_r(R_j') = \varphi_{jr}(R)$). Moreover, we can not have $p_r(R_j) < \varphi_{jr}(R) < \varphi_{jr}(R_i^*, R_j')$

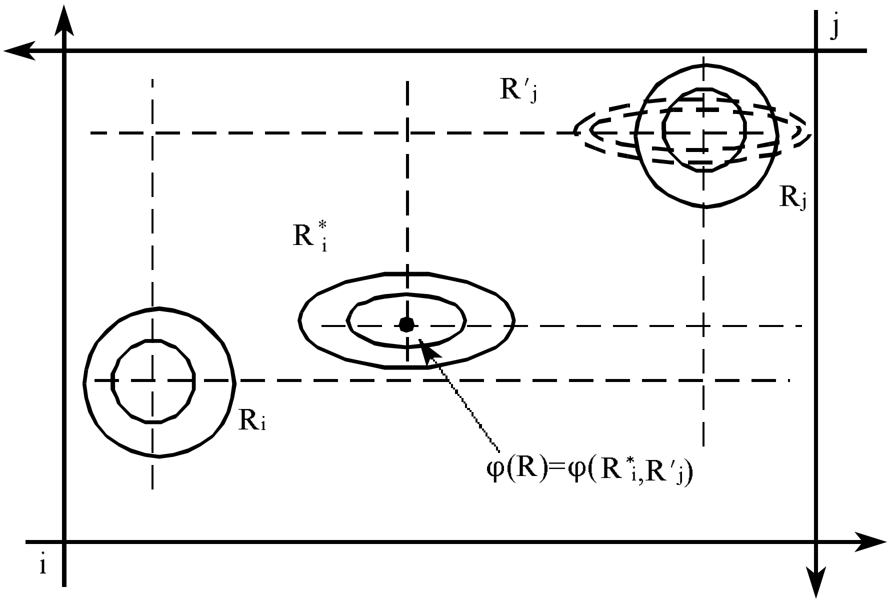


Figure 4.3:

(otherwise, by Step 1, $p_r(R_j) < \varphi_{jr}(R_i^*, R_j)$, and by definition of R_i^* and feasibility, $p_r(R_i^*) = \varphi_{ir}(R) > \varphi_{ir}(R_i^*, R_j')$, which contradicts (i)). In the same way, we can not have $p_r(R_j) > \varphi_{jr}(R) > \varphi_{jr}(R_i^*, R_j')$ (otherwise we have a contradiction with (ii)). Therefore, by Lemma 4.5 and Step 2, there is some $R_j'' \in \Re$ with $p(R_j'') = p(R_j)$ such that $\varphi_j(R_i^*, R_j')P_j''\varphi_j(R) = \varphi_j(R_i^*, R_j'')$, a contradiction with strategy-proofness. Then $\varphi_{jr}(R_i^*, R_j') = \varphi_{jr}(R)$, and by feasibility, $\varphi_{ir}(R_i^*, R_j') = \varphi_{ir}(R)$. ■

Now we show that the consumption of any commodity for any agent is monotonic with respect to his peak for this good. We say that a single-valued solution φ is *own-peak monotonic* when for all $R \in \Re^n$, $i \in N$, $R_i' \in \Re$ and $r \in M$, if $p_r(R_i') \leq p_r(R_i)$ then $\varphi_{ir}(R_i', R_{-i}) \leq \varphi_{ir}(R)$.

Lemma 4.7. *Let $n = 2$. If a solution satisfies strategy-proofness and Condition E, then it is own-peak monotonic.*

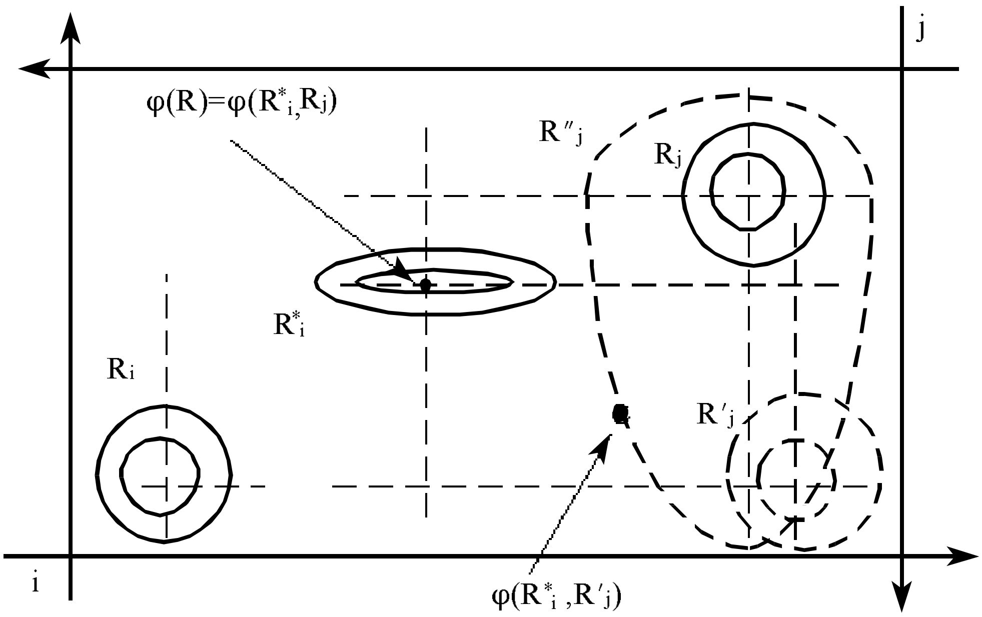


Figure 4.4:

Proof. (Figure 4.5). Let $N = \{i, j\}$, and suppose by contradiction that for some $R \in \Re^n$, $R'_i \in \Re$ and $r \in M$, $p_r(R'_i) \leq p_r(R_i)$ but $\varphi_{ir}(R'_i, R_j) > \varphi_{ir}(R)$. Suppose first that $p_r(R_i) + p_r(R_j) \leq \Omega_r$. Then $p_r(R'_i) + p_r(R_j) \leq \Omega_r$. By Condition E, $p_r(R_j) \leq \varphi_{jr}(R)$. Moreover, $p_r(R_j) \neq \varphi_{jr}(R)$ (otherwise, since $\varphi_{ir}(R'_i, R_j) > \varphi_{ir}(R)$, by feasibility $\varphi_{jr}(R'_i, R_j) < \varphi_{jr}(R) = p_r(R_j)$, which contradicts Condition E). Therefore $p_r(R_j) < \varphi_{jr}(R)$. Let $R^*_i, R^*_j \in \Re$ be such that $p(R^*_i) = \varphi_i(R)$ and $p(R^*_j) = \varphi_j(R'_i, R_j)$. Then, since by feasibility $p_r(R^*_j) = \varphi_{jr}(R'_i, R_j) < \varphi_{jr}(R)$, we have $r \in M^*(R_j, R^*_j, \varphi_j(R))$. Hence, by Lemma 4.6, $\varphi_{ir}(R^*_i, R^*_j) = \varphi_{ir}(R)$. On the other hand, since, by Condition E, $p_r(R'_i) \leq p_r(R_i) \leq \varphi_{ir}(R) < \varphi_{ir}(R'_i, R_j)$ and $p_r(R^*_i) = \varphi_{ir}(R) < \varphi_{ir}(R'_i, R_j)$, we have that $r \in M^*(R'_i, R^*_i, \varphi_i(R'_i, R_j))$. Then, by Lemma 4.6, $\varphi_{jr}(R^*_i, R^*_j) = \varphi_{jr}(R'_i, R_j)$. But then, $\varphi_{ir}(R^*_i, R^*_j) + \varphi_{jr}(R^*_i, R^*_j) = \varphi_{ir}(R) + \varphi_{jr}(R'_i, R_j) < \varphi_{ir}(R'_i, R_j) + \varphi_{jr}(R'_i, R_j) = \Omega_r$, which contradicts feasibility. Suppose now that $p_r(R_i) + p_r(R_j) > \Omega_r$. Then, by Condition E, $p_r(R_i) \geq \varphi_{ir}(R)$ and $p_r(R_j) \geq \varphi_{jr}(R)$. Moreover, $p_r(R_i) \neq \varphi_{ir}(R)$ (otherwise, $p_r(R'_i) \leq p_r(R_i) = \varphi_{ir}(R)$, and then $\varphi_{ir}(R'_i, R_j) > \varphi_{ir}(R) \geq p_r(R'_i)$, and by feasibility $\varphi_{jr}(R'_i, R_j) < \varphi_{jr}(R) \leq p_r(R_j)$ which contradicts Condition E). Since $p_r(R_i) > \varphi_{ir}(R)$ and $p(R^*_i) = \varphi_i(R'_i, R_j) > \varphi_{ir}(R)$ we have that $r \in M^*(R_i, R^*_i, \varphi_i(R))$, and therefore, by Lemma 4.6, $\varphi_{jr}(R^*_i, R^*_j) = \varphi_{jr}(R)$. On the other hand, by feasibility, $p_r(R^*_j) = \varphi_{jr}(R) > \varphi_{jr}(R'_i, R_j)$. Moreover, $p_r(R_j) \geq \varphi_{jr}(R) > \varphi_{jr}(R'_i, R_j)$. Then $r \in M^*(R_j, R^*_j, \varphi_j(R'_i, R_j))$, and by Lemma 4.6 $\varphi_{ir}(R^*_i, R^*_j) = \varphi_{ir}(R'_i, R_j)$. Hence, $\varphi_{ir}(R^*_i, R^*_j) + \varphi_{jr}(R^*_i, R^*_j) = \varphi_{ir}(R'_i, R_j) + \varphi_{jr}(R) > \varphi_{ir}(R) + \varphi_{jr}(R) = \Omega_r$, which contradicts feasibility. ■

We say that a single-valued solution is *peak-only* when, for all $R \in \Re^n$, $i \in N$, $R'_i \in \Re$ and $r \in M$, if $p_r(R'_i) = p_r(R_i)$ then, for all $j \in N$, $\varphi_{jr}(R'_i, R_{-i}) = \varphi_{jr}(R)$. If a solution is peak-only, the only relevant information in order to calculate the allocation are the peaks of the agents. An obvious consequence of Lemma 4.7 is the following:

Lemma 4.8. *Let $n = 2$. Let φ be a strategy-proof solution satisfying Condition E. Then φ is peak-only.*

Finally, we say that a single-valued solution is *uncompromising* if for all $R \in \Re^n$, $R'_i \in \Re$ and $r \in M$, (i) if $p_r(R_i) < \varphi_{ir}(R)$ and $p_r(R'_i) \leq \varphi_{ir}(R)$ then $\varphi_{ir}(R'_i, R_j) = \varphi_{ir}(R)$, and (ii) if $p_r(R_i) > \varphi_{ir}(R)$ and $p_r(R'_i) \geq \varphi_{ir}(R)$ then

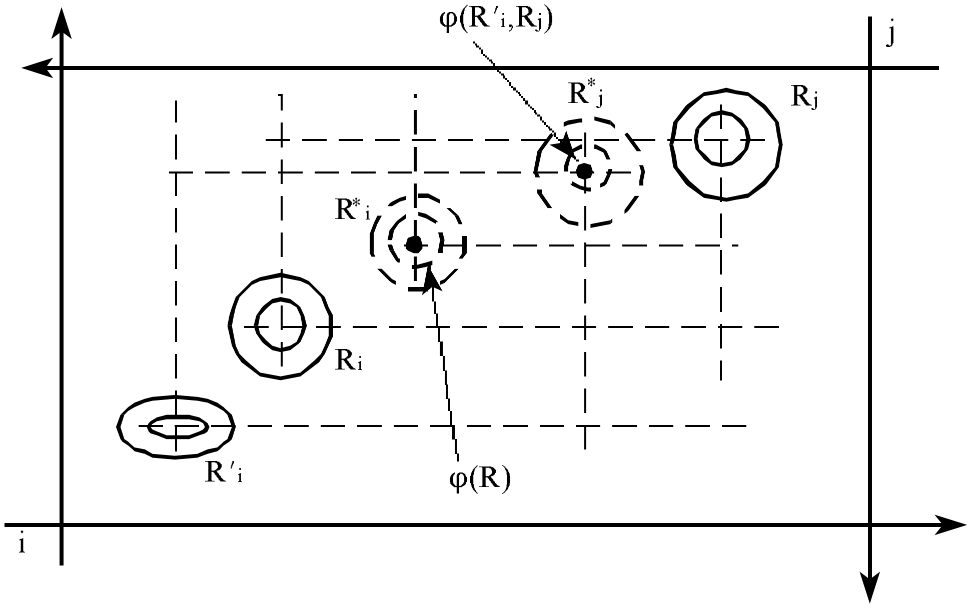


Figure 4.5:

$\varphi_{ir}(R_i', R_j) = \varphi_{ir}(R)$. That is, suppose that given any initial announcements one agent changes his announcement in such a manner that, for some commodity, if the initial peak was strictly smaller (greater) than his consumption, the new peak remains smaller (greater). Then, if the solution is uncompromising, his consumption of that commodity remains the same. Given the previous results we can prove the next lemma[4]:

Lemma 4.9. *Let $N = \{i, j\}$. Let φ a strategy-proof solution satisfying Condition E. Then φ is uncompromising.*

Proof. (Figure 4.6). Let $N = \{i, j\}$, and suppose by contradiction and w.l.o.g. that for some $R \in \Re^n$, $R_i' \in \Re$ and $r \in M$, $p_r(R_i) < \varphi_{ir}(R)$ and $p_r(R_i') \leq \varphi_{ir}(R)$ but $\varphi_{ir}(R_i', R_j) \neq \varphi_{ir}(R)$. Then by Lemma 4.8 $p_r(R_i') \neq p_r(R_i)$. Suppose first that $p_r(R_i') < p_r(R_i)$. By Lemma 4.7 $\varphi_{ir}(R_i', R_j) < \varphi_{ir}(R)$. Then, neither $p_r(R_i) \leq \varphi_{ir}(R) \leq \varphi_{ir}(R_i', R_j)$ nor $p_r(R_i) \geq \varphi_{ir}(R) \geq \varphi_{ir}(R_i', R_j)$ happens. So, by Lemma 4.5, there is some $R_i'' \in \Re$ with $p(R_i'') = p(R_i)$ such that $\varphi_i(R_i', R_j) P_i'' \varphi_i(R)$. Since,

[4]The concept of uncompromising, as one of the consequences of strategy-proofness, was suggested and studied in the public goods context by Border and Jordan [5]

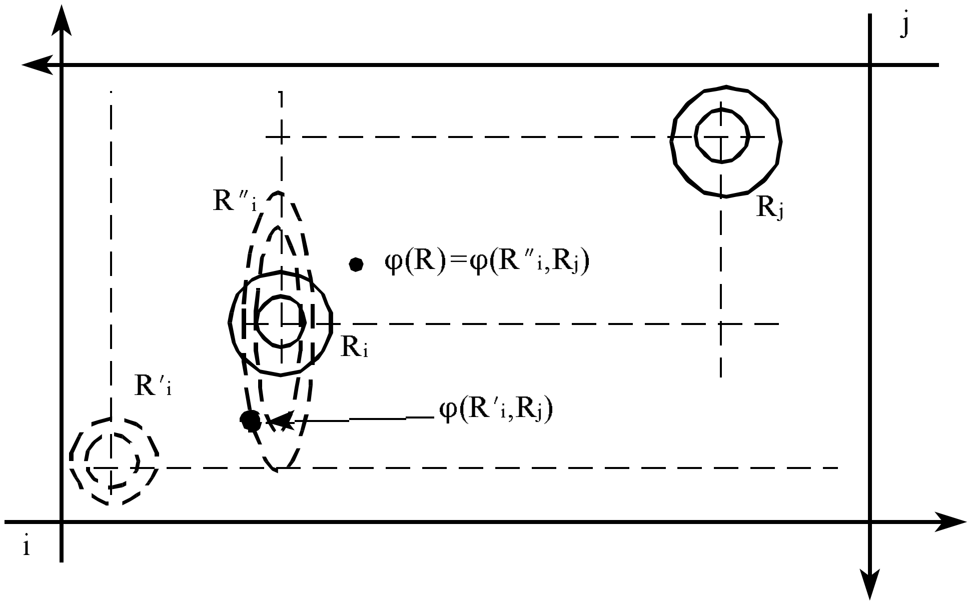


Figure 4.6:

by Lemma 4.8 $\varphi_i(R''_i, R_j) = \varphi_i(R)$, we have $\varphi_i(R'_i, R_j)P''_i\varphi_i(R''_i, R_j)$, a contradiction with strategy-proofness. Suppose now that $p_r(R'_i) > p_r(R_i)$. By Lemma 4.7 $\varphi_{ir}(R) < \varphi_{ir}(R'_i, R_j)$, and then $p_r(R_i) < p_r(R'_i) \leq \varphi_{ir}(R) < \varphi_{ir}(R'_i, R_j)$. Then, neither $p_r(R'_i) \leq \varphi_{ir}(R'_i, R_j) \leq \varphi_{ir}(R)$ nor $p_r(R'_i) \geq \varphi_{ir}(R'_i, R_j) \geq \varphi_{ir}(R)$ happens. Therefore, by Lemma 4.5, there is some $R''_i \in \Re$ with $p(R''_i) = p(R'_i)$ such that $\varphi_i(R)P''_i\varphi_i(R'_i, R_j)$. Since, by Lemma 4.8 $\varphi_i(R''_i, R_j) = \varphi_i(R'_i, R_j))$, we have $\varphi_i(R)P''_i\varphi_i(R''_i, R_j)$, which contradicts strategy-proofness. ■

Now we can prove the theorem[5]:

[5]For economies with pure public goods, Border and Jordan [5] proved that, in a reduced domain included in the single-peaked one, a solution is strategy-proof and unanimous (i.e. for all admissible economy with $\sum p(R_i) = \Omega$, then $\varphi_i(R) = p(R_i)$ for all $i \in N$) if and only if it can be decomposed into a product of one-dimensional mechanisms (one for each commodity) which are strategy-proof, unanimous, peak-only and uncompromising. Since, when $n = 2$ the

Proof of Theorem 4.4 (Figure 4.7). We have already proved that V is no-envy, strategy-proof and satisfies Condition E. Let $N = \{i, j\}$ and let φ be a solution satisfying all these properties. Let $R \in \Re^N$ and $r \in M$ and suppose w.l.o.g. that $p_r(R_i) \leq p_r(R_j)$. If $p_r(R_i) + p_r(R_j) = \Omega_r$ then by Condition E, for all $i \in N$, $\varphi_{ir}(R) = p_r(R_i) = V_{ir}(R)$. Suppose w.l.o.g. that $p_r(R_i) + p_r(R_j) < \Omega_r$. Suppose first that $p_r(R_j) > \Omega_r/2$. Then $V_{jr}(R) = p_r(R_j)$. Suppose by contradiction that $\varphi_{jr}(R) \neq p_r(R_j)$. Then, by Condition E, $\varphi_{jr}(R) > p_r(R_j)$, and so, by feasibility, $\varphi_{jr}(R) > p_r(R_j) > \Omega_r/2 > \varphi_{ir}(R)$. Let $R'_j \in \Re$ be such that (i) $p_r(R'_j) \leq \varphi_{ir}(R) < p_r(R_j)$, and (ii) for all $s \in M \setminus \{r\}$ $p_s(R'_j) = p_s(R_j)$. By Lemma 4.8, for all $s \in M \setminus \{r\}$, $\varphi_{js}(R_i, R'_j) = \varphi_{js}(R)$, and by Lemma 4.9, since $p_r(R_j) < \varphi_{jr}(R)$ and $p_r(R'_j) < \varphi_{jr}(R)$, $\varphi_{jr}(R_i, R'_j) = \varphi_{jr}(R)$. Therefore $\varphi_j(R_i, R'_j) = \varphi_j(R)$. On the other hand, since $p_r(R'_j) \leq \varphi_{ir}(R) < \varphi_{jr}(R)$, neither $p_r(R'_j) \leq \varphi_{jr}(R) \leq \varphi_{ir}(R)$ nor $p_r(R'_j) \geq \varphi_{jr}(R) \geq \varphi_{ir}(R)$ happens, and so, by Lemma 4.5, there is some $R''_j \in \Re$ with $p(R''_j) = p(R'_j)$ such that $\varphi_i(R) P''_j \varphi_j(R)$. However, by Lemma 4.8, $\varphi_j(R_i, R''_j) = \varphi_j(R_i, R'_j) = \varphi_j(R)$, and then $\varphi_i(R_i, R''_j) = \varphi_i(R)$. Hence $\varphi_i(R_i, R''_j) P''_j \varphi_j(R_i, R''_j)$, a contradiction with no-envy. Suppose now that $p_r(R_j) \leq \Omega_r/2$. Since $p_r(R_i) \leq p_r(R_j)$, $V_{jr}(R) = \Omega_r/2$. Suppose by contradiction and w.l.o.g. that $\varphi_{ir}(R) > \Omega_r/2 > \varphi_{jr}(R)$. Since $p_r(R_i) \leq p_r(R_j) \leq \Omega_r/2 < \varphi_{ir}(R)$, neither $p_r(R_i) \leq \varphi_{ir}(R) \leq \varphi_{jr}(R)$ nor $p_r(R_i) \geq \varphi_{ir}(R) \geq \varphi_{jr}(R)$ happens, and then, by Lemma 4.5, there is some $R'_i \in \Re$ with $p_r(R'_i) = p_r(R_i)$ such that $\varphi_j(R) P'_i \varphi_i(R)$. Moreover, by Lemma 4.8 $\varphi_i(R'_i, R_j) = \varphi_i(R)$, and then $\varphi_j(R'_i, R_j) = \varphi_j(R)$. Therefore $\varphi_j(R'_i, R_j) P'_i \varphi_i(R'_i, R_j)$, which contradicts no-envy. ■

The good behavior of V with respect to manipulation properties goes beyond strategy-proofness. It is also implementable in dominant strategies by means of the manipulation game associated to it. This game is the result of allowing each agent to announce a preference relation for himself, and selecting for each

single-peaked preferences model can be reinterpreted as one of public goods, an alternative way of showing Lemmas 4.8 and 4.9 is by proving that the previous result can be extended to the single-peaked preferences domain. One could think that our Theorem 4.4 can be deduced from this extension together with the characterization of the Uniform Rule given by Sprumont [15]. However, the fact that a solution is no-envy does not imply that it is no-envy commodity by commodity.

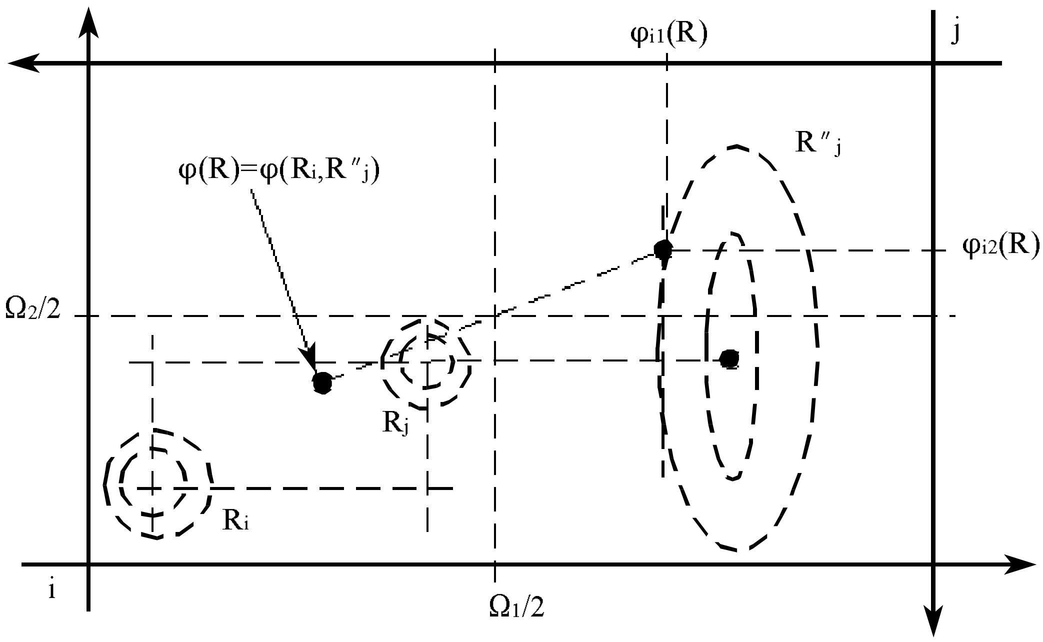


Figure 4.7:

of these possible strategies profiles the allocation to be chosen by V. It can be proved that in this game, not only is truth telling a dominant strategy (by strategy-proofness), but also there is no other dominant strategy whose associated allocation is different to the one associated to the truth[6]. All these results point at the Generalized Uniform Rule as a well-behaved solution as far as strategic-manipulation properties.

5. PARETO-EFFICIENT SELECTIONS: WALRASIAN SOLUTIONS WITH SLACKS

In this section we renounce strategy-proofness and focus on efficient solutions. In the case of several commodities ($m \geq 2$) there is no simple necessary and sufficient condition for Pareto-efficiency. All we have is a necessary condition (Condition E). A good way to solve this difficulty is by extending the classical Walrasian solution concept to our domain.

Mas-Colell [9] proposed the Walrasian equilibrium with slack to extend the Walrasian equilibrium concept to economies with possibly satiated preferences. The idea is to give each agent the same additional amount of income to spend (the slack). We begin with the more general definition of this equilibrium concept, by allowing the extra amounts of income to be different for different agents.

Given an economy $R \in \Re^n$, two feasible allocations $x, \omega \in X$, a vector of prices $p \in \mathbb{R}^m$ (possibly negative) and a vector $\beta = (\beta_i)_{i \in N} \in A^n = \{(\beta'_i)_{i \in N} \in \mathbb{R}^n : \sum \beta'_i = 0\}$, we say that the profile (x, p, β) is a *Walrasian equilibrium with balanced slacks* from endowments ω for the economy R if, for all $i \in N$, x_i maximizes the preference relation R_i over the budget set $B_i(p, \beta_i, \omega_i) = \{x'_i \in \mathbb{R}^m_+ : px'_i \leq p\omega_i + \beta_i\}$. The β_i's are called slacks. Notice that the sum of agent's slacks is equal to zero. Then we can think of the slacks as income redistribution among consumers. Agents with positive slacks receive income from agents with negative ones, and the sum of the amounts of income paid are equal to the sum of the amounts of income received[7].

[6] We do not include the proof of this result in the paper becouse it is relatively long and it does not add a lot.

[7] Alternatively, we may define the Walrasian equilibrium with balanced slacks just requiring that $\beta = (\beta_i)_{i \in N} \in \mathbb{R}^n$. Let $R \in \Re^n$, $x, \omega \in X$, $p \in \mathbb{R}^m$ and $\beta = (\beta_i)_{i \in N} \in \mathbb{R}^n$ be such that, for all $i \in N$, x_i maximizes the preference relation R_i over the budget set $B_i(p, \beta_i, \omega_i) = \{x'_i \in \mathbb{R}^m_+ : px'_i \leq p\omega_i + \beta_i\}$. Then it can be shown that there exists some $\beta' = (\beta'_i)_{i \in N} \in A^n$ such that (x, p, β') is a Walrasian equilibrium with balanced slacks from endowments ω for the economy

As in the standard Walrasian equilibrium, the Walrasian equilibrium with balanced slacks applies from some endowments $\omega \in X$. In our model, all agents have the same rights over the goods to be divided. Therefore, the more appealing allocation of endowments is equal division (Ω/n). It is also the simplest one. Nevertheless, most of the results in this section do not depend on the allocation chosen as endowments. Given any endowments $\omega \in X$, we define the following solution:

Walrasian solution with balanced slacks from ω, WBS$_\omega$. Given $R \in \Re^n$, $x \in WBS_\omega(R)$ if $x \in X$ and there is some $p \in \mathbb{R}^m$ and $\beta = (\beta_i)_{i \in N} \in A^n$ such that (x, p, β) is a Walrasian equilibrium with balanced slacks from ω for the economy R.

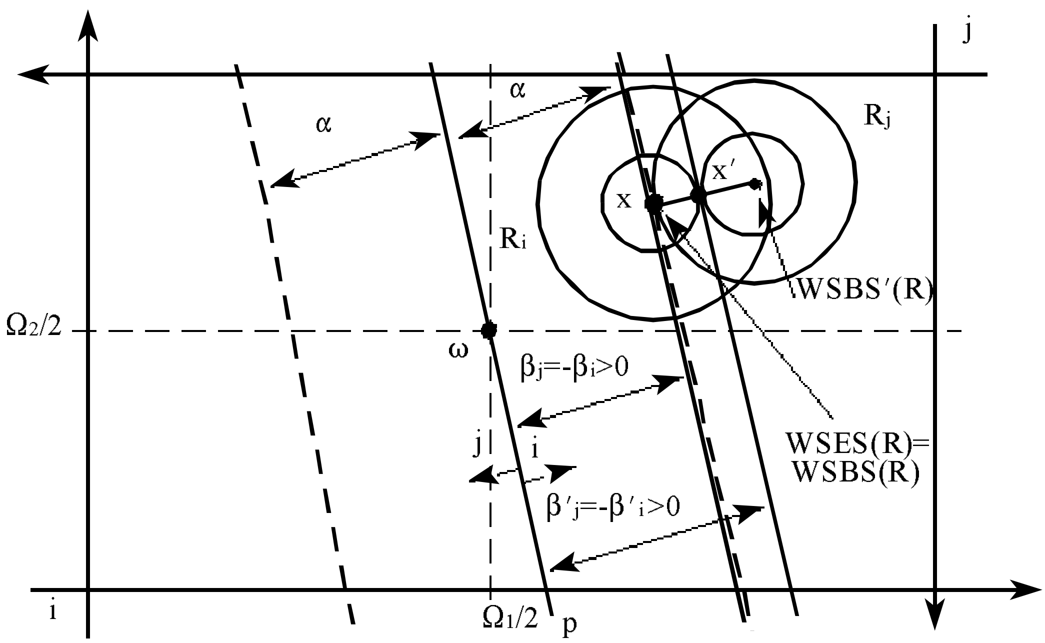
In Figure 5.1 we have an example of this solution for the two agents-two commodities case.

The following two remarks are a generalization of the well-known first and second welfare theorems. For all endowments, $\omega \in X$, Remark 1 states that all Walrasian allocation with balanced slacks from ω is efficient. Remark 2 states that, under continuity and weak-convexity, all strictly positive efficient allocation is attainable as a Walrasian allocation with balanced slacks from ω. Notice that, in contrast to the classical second welfare theorem, no mention is made of income redistribution from . This is due to the fact that, in WBS_ω, the slacks play this role. The slacks are also responsible for the fact that these results do not depend on the chosen endowments.

Remark 1. *For all $R \in \Re^n$, all $\omega \in X$ and all $x \in WBS_\omega(R)$, we have that $x \in PE(R)$.*

Remark 2. *Let $R \in \Re^n$ be such that for all $i \in N$ the preference relation R_i is continuous and weakly convex. Let $x^* \in PE(R)$ be such that, for all $i \in N$, $x_i^* > 0$. Then, for all $\omega \in X$, $x^* \in WBS_\omega(R)$.*

R. Therefore, there is no restriction on making the sum of the slacks be equal to zero. It just makes its economic interpretation easier.



j
α
α
R_j
x
x′
R_i
WSBS′(R)
$\Omega_2/2$
ω
β_j=-β_i>0
WSES(R)=
WSBS(R)
j
i
β'_j=-β'_i>0
i
$\Omega_1/2$
p

Figure 5.1:

The proof of these results are very similar to the corresponding ones in classical economies and we omit them. Remarks 1 and 2 allow us to characterize the set of Pareto-efficient allocations for all continuous and weakly-convex profile of preferences $R \in \Re^n$ by means of the Walrasian solution with balanced slacks. Any strictly positive efficient allocation can be obtained as a Walrasian allocation with balanced slacks, and any allocation like this is efficient. In Figure 5.1 we have an example of this (here, the straight line between agents' peaks is the set of Pareto-efficient allocations).

Since we are interested in solutions satisfying additional properties, for all $R \in \Re^n$ and $\omega \in X$, we will select subsets of $WBS_\omega(R)$. Proceeding in this way we can be sure that our solutions will be efficient. Moreover we know that any solution that selects efficient allocations can be obtained as a subsolution of WBS_ω.

A natural way of choosing WBS_ω's subsolutions is by demanding that the slacks satisfy additional requirements. Given the former interpretation of the slacks, this can be seen as the imposition of some properties on the procedure of income redistribution among agents necessary to achieve some efficient allocation.

As we will prove, this is precisely what makes Mas-Colell's original definition. We will call it *Walrasian equilibrium with equal slacks*. It is defined in the same manner as the Walrasian equilibrium with balanced slacks, but with the condition that all agents have identical slacks (that is, for all $i \in N$, $\beta_i = \alpha$)[8]. Then, given any endowments $\omega \in X$, we define the following solution:

Walrasian solution with equal slacks from ω, WES$_\omega$. Given $R \in \Re^n$, $x \in WES_\omega(R)$ if there is some $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}_+$ such that (x, p, α) is a Walrasian equilibrium with equal slacks from ω for the economy R.[9].

As the next proposition shows, for all fixed endowments, $\omega \in X$, the Walrasian solution with equal slacks is a subsolution of the Walrasian solution with balanced slacks[10].

[8] It is easy to see that, in all Walrasian equilibrium with equal slacks, the slack is positive ($\alpha \geq 0$).

[9] Notice that, although if agents have strict-covex preferences, for some given prices there is only one Walrasian allocation with equal slacks in each economy, the Walrasian solution with equal slacks is not generally single-valued (the same examples showing that there can be more than one Walrasian equilibrium in classical economies are valid here).

[10] Mas-Colell [9] already pointed at WS as a more general definition, but he used WES in his existence theorems.

Proposition 5.1. *Let $x, \omega \in X$, $R \in \Re^n$, $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}_+$ be such that (x, p, α) is a Walrasian equilibrium with equal slacks from ω for the economy R. Then:*

(i) there exists a unique $\beta = (\beta_i)_{i \in N} \in A^n$ such that (x, p, β) is a Walrasian equilibrium with balanced slacks from ω for the economy R, and

(ii) if preferences are smooth and $\sum p(R_i) \neq \Omega$, there is not any $p' \in \mathbb{R}^m$ with $p' \neq p$ and any $\beta' = (\beta_i')_{i \in N} \in A^n$ such that (x, p', β') is a Walrasian equilibrium with balanced slacks from ω for the economy R.

Proof. Let $x, \omega \in X$, $R \in \Re^n$, $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}_+$ be such that (x, p, α) is a Walrasian equilibrium with equal slacks from ω for the economy R. Take for each $i \in N$ some $\beta_i \in \mathbb{R}$ such that $px_i = p\omega_i + \beta_i$. Obviously x_i maximizes R_i over the budget set $B_i(p, \beta_i, \omega_i)$. Furthermore, since $p \sum x_i = p \sum \omega_i + \sum \beta_i$, $\sum \beta_i = 0$. Therefore, (x, p, β) is a Walrasian equilibrium with balanced slacks from ω for the economy R. Suppose now that there exists some $\beta' \in A^n$ with $\beta' \neq \beta$ and such that (x, p, β') is also a Walrasian equilibrium with balanced slacks from ω for the economy R. Obviously, for some agent $j \in N$, $\beta_j' < \beta_j$. Then, $p\omega_j + \beta_j' < p\omega_j + \beta_j = px_j$, and therefore $x_j' \neq x_j$, which is a contradiction. Suppose now that preferences are smooth and $\sum p(Ri) \neq \Omega$, but there is some $p' \in \mathbb{R}^m$ with $p' \neq p$ and some $\beta' \in A^n$ such that (x, p', β') is a Walrasian equilibrium with balanced slacks from ω for the economy R. Since $\sum p(Ri) \neq \Omega$, there is at least one agent $j \in N$ with $x_j \neq p(R_j)$, and then $px_j = p\omega_j + \alpha$. On the other hand, for all $i \in N$, $p'x_i = p'\omega_i + \beta_i'$ (otherwise, since $\sum \beta_i' = 0$, we have $p' \sum x_i < p' \sum \omega_i$, which is a contradiction). Then, x_j can be sustained by two different hyperplanes, which contradicts smoothness. ■

This proves that any Walrasian allocation with equal slacks from any fixed endowments $\omega \in X$ can be attained as a Walrasian allocation with balanced slacks from the same endowments and for the same prices (and, if preferences are smooth, only for these prices). Hence, $WES_\omega \subseteq WBS_\omega$[11], and then WES_ω is obtained as a subsolution of WBS_ω by additional requirements over the slacks. In Figure 5.1, the allocation x is the only Walrasian allocation with equal slacks, and it is also attainable as a Walrasian equilibrium with balanced slacks.

In a sense specified below, for all fixed endowments $\omega \in X$, WES_ω selects the efficient allocations that require smallest income redistribution between con-

[11] If agents have continuous and weak-convex preferences, existence of Walrasian equilibrium with equal slacks is guaranteed in our model. See Mas-Colell [9] for a complete proof of two existence theorems.

sumers. Actually, as Theorem 5.3 shows, given any Walrasian allocation with equal slacks, there is no other Walrasian allocation with balanced slacks for which the income redistribution that allows us to obtain it is such that, either the maximum payoff or the maximum subsidy are smaller. Since there is not any income measurement valid for any two prices vectors, if we want to compare the necessary income redistribution to achieve any two different Walrasian allocations, we have first to fix some reference prices (identical for the two allocations). That is what we do when comparing some Walrasian allocation with equal slacks with any other Walrasian allocation with balanced slacks: we take as reference prices those of the original Walrasian allocation with equal slacks.

Before proving Theorem 5.3, we need the following lemma, which establishes the sign of the prices in any Walrasian equilibrium with balanced slacks depending on the sum of the peaks. The proof of the lemma is omitted.

Lemma 5.2. *Let $x, \omega \in X$, $R \in \Re^n$, $p \in \mathbb{R}^m$ and $\beta \in A^n$ be such that (x, p, β) is a Walrasian equilibrium with balanced slacks from ω for the economy R. Then, for all $r \in M$, (i) if $\sum_i p_r(R_i) > \Omega_r$, $p_r > 0$, and (ii) if $\sum_i p_r(R_i) < \Omega_r$, $p_r < 0$.*

In case the sum of the peaks of some good be different from the amount that has to be divided, Lemma 5.2 allows us to predict the sign of the price of this good in any Walrasian equilibrium with balanced slacks. Now we can state Theorem 5.3:

Theorem 5.3. *Let $x, \omega \in X$, $R \in \Re^n$, $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}_+$ be such that (x, p, α) is a Walrasian equilibrium with equal slacks from ω for the economy R. Let $\beta \in A^n$ be the unique vector in A^n such that (x, p, β) is a Walrasian equilibrium with balanced slacks from ω for the economy R. Let $x' \in WBS_\omega(R)$ be such that $x' \neq x$. For all $i \in N$, let $\beta'_i \in \mathbb{R}$ be such that $px'_i = p\omega_i + \beta'_i$. Then, $\max_{i \in N} \beta_i \leq \max_{i \in N} \beta'_i$ and $\min_{i \in N} \beta_i \geq \min_{i \in N} \beta'_i$.*

Proof. Let $x, x', \omega \in X$, $R \in \Re^n$, $p \in \mathbb{R}^m$, $\alpha \in \mathbb{R}_+$, $\beta \in A^n$, and $\beta' = (\beta'_i)_{i \in N} \in \mathbb{R}^n$ be as defined in the statement of the theorem. If $\alpha = 0$ then, for all $i \in N$, $\beta_i = 0$, and the theorem is verified. If $\sum p(R_i) = \Omega$ then, by efficiency of the Walrasian solution with balanced slacks, for all $i \in N$, $x_i = p(R_i) = x'_i$ and the theorem is obviously verified. From now, suppose that $\alpha > 0$ and $\sum p(R_i) \neq \Omega$.
<u>Step 1. $\max_{i \in N} \beta_i = \alpha$ and $\min_{i \in N} \beta_i = \min_{i \in N} p(p(R_i) - \omega_i) < \alpha$.</u> Given that $\alpha > 0$ and $\sum p(R_i) \neq \Omega$, then, (i) there is at least one $j \in N$ with $px_j < p\omega_j + \alpha$ (and so $x_j = p(R_j)$), and (ii) there is at least one $k \in N$ with $x_k \neq p(R_k)$ (and

so $pp(R_k) > p\omega_k + \alpha$). Moreover, from the proof of Proposition 5.1 it is clear that for all $i \in N$, β_i is such that $px_i = p\omega_i + \beta_i$. Then, for all $i \in N$, (i) if $pp(R_i) \geq p\omega_i + \alpha$, $px_i = p\omega_i + \alpha$, and then $\beta_i = \alpha$ and $p(p(R_i) - \omega_i) \geq \alpha = \beta_i$, and (ii) if $pp(R_i) < p\omega_i + \alpha$, $x_i = p(R_i)$, and then $\beta_i = p(p(R_i) - \omega_i) < \alpha$. Therefore, $\max_{i \in N} \beta_i = \alpha$ and $\min_{i \in N} \beta_i = \min_{i \in N} p(p(R_i) - \omega_i) < \alpha$.
Step 2. $\alpha \leq \max_{i \in N} \beta'_i$. For all $r \in M$ and $j \in N$, by efficiency of the Walrasian solution with balanced slacks, (i) if $\sum_i p_r(R_i) \geq \Omega_r$, $p_r(R_i) \geq x'_{jr}$, and (ii) if $\sum_i p_r(R_i) \leq \Omega_r$, $p_r(R_i) \leq x'_{jr}$. Then, by Lemma 5.2, for all $r \in M$ and $j \in N$ with $x'_{jr} \neq p_r(R_j)$, either (i) $p_r(R_j) > x'_{jr}$ and $p_r > 0$, or (ii) $p_r(R_j) < x'_{jr}$ and $p_r < 0$. Therefore, for all $j \in N$ with $pp(R_j) < p\omega_j + \alpha$, given that $x_j = p(R_j)$, $px'_j \leq px_j$. Suppose, by contradiction, that $\alpha > \max_{i \in N} \beta'_i$. Then, for all $j \in N$ with $pp(R_j) \geq p\omega_i + \alpha$ (and there is at least one agent like this), we have $px'_j = p\omega_j + \beta'_j < p\omega_j + \alpha = px_j$. Therefore, $p \sum x'_i < p \sum x_i$, which is a contradiction.
Step 3. $\min_{i \in N} \beta_i \geq \min_{i \in N} \beta'_i$. Suppose, by contradiction, that $\min_{i \in N} \beta_i < \min_{i \in N} \beta'_i$. Let $j \in N$ be an agent such that $\beta_j = \min_{i \in N} \beta_i$. Then $pp(R_j) < p\omega_j + \alpha$ (otherwise $px_j = p\omega_j + \alpha$, and then $\beta_j = \alpha = \max_{i \in N} \beta_i$, which is a contradiction since $\alpha \neq 0$). Then, using an argument identical to the one in Step 2, $px'_j \leq px_j$. On the other hand, since for all $i \in N$, $px'_i = p\omega_i + \beta'_i$, we have $p\sum_{i \neq j} x'_i = p\sum_{i \neq j} \omega_i + \sum_{i \neq j} \beta'_i$. Furthermore, since $\beta'_j > \beta_j$ and $\sum \beta'_i = \sum \beta_i = 0$, (and then $\sum_{i \neq j} \beta'_i < \sum_{i \neq j} \beta_i$), $p\sum_{i \neq j} x'_i < p\sum_{i \neq j} \omega_i + \sum_{i \neq j} \beta_i$. Moreover, since, for all $i \in N$, $px_i = p\omega_i + \beta_i$, then $p\sum_{i \neq j} \omega_i + \sum_{i \neq j} \beta_i = p\sum_{i \neq j} x_i$. Then $p \sum x'_i < p \sum x_i$, which is a contradiction. ■

In Figure 5.1, (x, p, α) is the only Walrasian equilibrium with equal slack, and $\beta \in A^n$ is the only vector in A^n such that (x, p, β) is a Walrasian equilibrium with balanced slacks. Any other Walrasian equilibrium with balanced slacks like (x', p, β') is such that the maximum payoff and the maximum subsidy are larger (i.e. $0 < \beta_j = \max_{k \in N} \beta_k \leq \max_{k \in N} \beta'_k = \beta'_j$ and $0 > \beta_i = \min_{k \in N} \beta_k \geq \min_{k \in N} \beta'_k = \beta_i$).

The interpretation of this result becomes more transparent in the special case in which, for some economy, any Walrasian equilibrium with balanced slacks is achieved for the same prices (for example, when the preference relations of all agents are quasi-linear). If this happens, Theorem 5.3 says that, for all fixed endowments $\omega \in X$ and all Walrasian allocation with equal slacks from ω, there is no other Walrasian allocation from ω with smaller range between the largest payoff and the largest subsidy necessary to obtain it.

Suppose now that we take equal division (Ω/n) as endowments. Let WES_{ed}

denote the Walrasian solution with equal slacks for that case. It is easy to see that WES_{ed} is no-envy and Pareto-dominates equal division.

Theorem 5.3 is a generalization of a previous result by Schummer and Thomson [14], who show that the Uniform Rule is the solution that selects the only efficient allocation for which the difference between the smallest and largest amounts any two agents are receiving is the smallest. Notice that when applied to the one-commodity case, WES_{ed} coincides with the Uniform Rule. Moreover, if $\sum_i p(R_i) > \Omega$ ($\sum_i p(R_i) < \Omega$) any strictly positive (strictly negative) price of the commodity, together with the suitable vector of balanced slacks, is part of some Walrasian equilibrium with equal slacks from equal division yielding the uniform allocation. The Schummer and Thomson's result can be deduced from Theorem 5.3 by fixing $p = 1$ when $\sum_i p(R_i) > \Omega$ ($p = -1$ when $\sum_i p(R_i) < \Omega$) to calculate the WES_{ed}.

6. CONSISTENCY AND RELATED PROPERTIES OF V AND WES

We have proposed two different solutions to the single-peaked preferences model with more than one commodity: the Generalized Uniform Rule and the Walrasian solution with equal slacks from equal division. In this section we compare these solutions studying other properties different from strategy-proofness and efficiency. In order to do this we will focus on some of the properties that have been used to characterize the Uniform Rule. Since one of our solutions (WES_{ed}) can be multi-valued, we have decided to base this comparison on those properties which are valid for this type of solutions: consistency and related properties[12]. For this we first introduce some additional notation.

Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. For all $N \in \mathcal{N}$, an *economy* is a profile $e = (\Omega, R) \in \mathbb{R}^m_+ \times \Re^n$. Let E^N denote the class of economies involving the group of agents N, and $E = \cup_{N \in \mathcal{N}} E^N$. Given $N \in \mathcal{N}$ and $e = (\Omega, R) \in E^N$, let $X(e)$ denote the set of feasible allocations for e: $X(e) = \{x = (x_i)_{i \in N} : \sum x_i = \Omega\}$. For all $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$, $x \in X(e)$ and $N' \subset N$, $x_{N'}$ is the restriction of x to the members of N': $x_{N'} = (x_i)_{i \in N'}$. Similarly, $R_{N'} = (R_i)_{i \in N'}$. For all $v \in \mathbb{N}$ and $x = (x_i)_{i \in N} \in X(e)$, let $v \times e = (v \times \Omega, v \times R) \in E^{v \times N}$ be the v-fold replica of e (i.e. there are v agents of each type $i \in N$, and the amount of goods to be allocated is v times Ω), and $v \times x$ the v-fold replica of the allocation x (i.e. each

[12] See Thomson [16] for these characterizations of the Uniform Rule.

agent of type $i \in N$ receives x_i). Given a solution φ on E associating with all $N \in \mathcal{N}$ and all $e \in E^N$ a non-empty subset of $X(e)$, we say that:

(i) φ satisfies *consistency* if, for all $N \in \mathcal{N}$, all $e = (\Omega, R) \in E^N$, all $x \in \varphi(e)$, and all $N' \subset N$, we have $x_{N'} \in \varphi(\sum_{N'} x_i, R_{N'})$.

(ii) φ satisfies *converse consistency* if, for all $N \in \mathcal{N}$, all $e = (\Omega, R) \in E^N$, all $x \in X(e)$, and all $N' \subset N$ with $|N'| \geq 2$, $x_{N'} \in \varphi(\sum_{N'} x_i, R_{N'})$, then $x \in \varphi(e)$.

(iii) φ satisfies *replication invariance* if, for all $N \in \mathcal{N}$, all $e = (\Omega, R) \in E^N$, all $x \in \varphi(e)$, and all $v \in \mathbb{N}$, $v \times x \in \varphi(v \times e)$.

In words, a solution is consistent if any recommendation it makes for any economy agrees with at least one recommendations it makes for any of its associated *reduced* economies. It is conversely consistent if, when an allocation is such that its restriction to each proper subgroup containing at least two agents is recommended for the associated reduced economy, then it is also recommended for the initial economy. Replication invariance is a weaker variation of this later property. The next three results show that WES_{ed} satisfies the three consistency related properties considered[13].

Proposition 6.1. *The Walrasian solution with equal slacks from equal division is consistent.*

Proof. Let $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$ and $x \in WES_{ed}(e)$ be given. Then, there is some $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that, for all $i \in N$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : px'_i \leq p\Omega/n + \alpha\}$. Let now $N' \subset N$ be given and take $\alpha' \in \mathbb{R}$ such that $p(\sum_{i \in N'} x_i)/n' + \alpha' = p\Omega/n + \alpha$. Then, for all i$\in N'$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : px'_i \leq p(\sum_{i \in N'} x_i)/n' + \alpha'\}$. That is, $x_{N'} \in WES_{ed}(\sum_{i \in N'} x_i, R_{N'})$. ■

Proposition 6.2. *If we restrict the economies domain to one with smooth preferences, the Walrasian solution with equal slacks from equal division is conversely consistent*[14].

[13] Dagan [8] proved that, in a model in which preferences are possibly satiated but may not be single-peaked, any consistent, replication invariance, weak-efficient and individually rational from equal division solution is a subsolution from the Walrasian solution with slacks from equal division defined by Mas-Colell (that is, our WES_{ed}). This generalizes a previous result by Thomson [16] which characterizes the Uniform Rule by means of the same properties.

[14] If non-smooth preferences are allowed, $WSES_{ed}$ is not conversely consistent. The reason is that, when preferences are not smooth, even if for each two agents $i, j \in N$ we can find one hyperplane supporting x_i and x_j, it may be the case that there is no hyperplane supporting x_k for all $k \in N$.

Proof. Let $N \in \mathcal{N}$ with $n \geq 3$, $e = (\Omega, R) \in E^N$ and $x \in X(e)$ be such that, for all $i \in N$, R_i is smooth, and, for all $N' \subset N$ with $n' \geq 2$, $x_{N'} \in WES_{ed}(\sum_{i \in N'} x_i, R_{N'})$. If, for all $i \in N$, $x_i = p(R_i)$, then it is obvious that $x \in WES_{ed}(e)$. Suppose that, at least for one $j \in N$, $x_j \neq p(R_j)$. For each $i \neq j$ let $N_i = \{j, i\}$. Since $x_{N_i} \in WES_{ed}(x_j + x_i, R_{N_i})$, there are some $p_{N_i} \in \mathbb{R}^m$ and $\alpha_{N_i} \in \mathbb{R}$ such that x_j maximizes R_j over the set $\{x' \in \mathbb{R}^m_+ : p_{N_i} x' \leq p_{N_i}(x_j + x_i)/2 + \alpha_{N_i}\}$, and x_i maximizes R_i over the same set. Moreover, since $x_j \neq p(R_j)$ then $p_{N_i} x_j = p_{N_i}(x_j + x_i)/2 + \alpha_{N_i}$. On the other hand, if $x_i \neq p(R_i)$, $p_{N_i} x_i = p_{N_i}(x_j + x_i)/2 + \alpha_{N_i} = p_{N_i} x_j$ and, if $x_i = p(R_i)$, $p_{N_i} x_i \leq p_{N_i}(x_j + x_i)/2 + \alpha_{N_i} = p_{N_i} x_j$. By smoothness of preferences, for all $i, k \in N \setminus \{j\}$, $p_{N_i} = p_{N_k}$ (otherwise x_j could be supported by two different hyperplanes). We call p^* to this value. Then, (i) for all $i, k \in N$ such that $x_i \neq p(R_i)$ and $x_k \neq p(R_k)$, then $p^* x_i = p^* x_k$; we call λ to this value; (ii) for all $i \in N$ such that $x_i \neq p(R_i)$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : p^* x'_i \leq \lambda\}$; (iii) for all $i \in N$ such that $x_i = p(R_i)$, then $p^* x_i \leq \lambda$. Take now $\alpha \in \mathbb{R}$ such that $p^* \Omega / n + \alpha = \lambda$. Then, for all $i \in N$ such that $x_i \neq p(R_i)$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : p^* x'_i \leq p^* \Omega / n + \alpha\}$, and for all $i \in N$ such that $x_i = p(R_i)$, we have $p^* p(R_i) \leq p^* \Omega / n + \alpha$ (and so x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : p^* x'_i \leq p * \Omega / n + \alpha\}$). Therefore $x \in WES_{ed}(e)$. ■

Proposition 6.3. *The Walrasian solution with equal slacks from equal division is replication invariant.*

Proof. Let $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$, $x \in WES_{ed}(e)$ and $v \in \mathbb{N}$. Then, there exists $p \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that, for all $i \in N$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : p x'_i \leq p \Omega / n + \alpha\}$. Let $e' = (\Omega', R') = v \times e = (v \times \Omega, v \times R) \in E^{v \times N}$. Since $\Omega' = v \times \Omega$ and $n' = v \times n$, then, for all $i \in N$, x_i maximizes R_i over the set $\{x'_i \in \mathbb{R}^m_+ : p x'_i \leq p \Omega' / n' + \alpha\}$. Therefore $v \times x \in WES_{ed}(e')$. ■

In order to complete the analysis, we next consider whether V verifies the consistency related properties analyzed before. As the following three propositions show, V satisfies them.

Proposition 6.4. *The Generalized Uniform Rule is consistent.*

Proof. Let $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$ and $x = V(e)$. Then, for all $r \in M$, (i) if $\sum_{i \in N} p_r(R_i) \geq \Omega_r$, $x_{ir} = \min\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e)$ solves $\sum_{i \in N} \min\{p_r(R_i), \lambda_r(e)\} = \Omega_r$, and (ii) if $\sum_{i \in N} p_r(R_i) \leq \Omega_r$, $x_{ir} = \max\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e)$ solves $\sum_{i \in N} \max\{p_r(R_i), \lambda_r(e)\} =$

Ω_r. Then, for all $N' \subset N$, (i) if $\sum_{i \in N} p_r(R_i) \geq \Omega_r$, since for all $i \in N$ $p_r(R_i) \geq x_{ir}$, $\sum_{i \in N'} p_r(R_i) \geq \sum_{i \in N'} x_{ir}$, and (ii) if $\sum_{i \in N} p_r(R_i) \leq \Omega_r$, since for all $i \in N$ $p_r(R_i) \leq x_{ir}$, $\sum_{i \in N'} p_r(R_i) \leq \sum_{i \in N'} x_{ir}$. Let $e' = (\Omega', R_{N'})$ be the reduced economy where $\Omega' = \sum_{i \in N'} x_i$. Then, for all $r \in M$, (i) if $\sum_{i \in N} p_r(R_i) \geq \Omega_r$, $\sum_{i \in N'} p_r(R_i) \geq \Omega'_r$, and then, for all $i \in N'$, $V_{ir}(e') = \min\{p_r(R_i), \lambda_r(e')\}$, where $\lambda_r(e')$ solves $\sum_{i \in N'} \min\{p_r(R_i), \lambda_r(e')\} = \Omega'_r$, and (ii) if $\sum_{i \in N} p_r(R_i) \leq \Omega_r$, $\sum_{i \in N'} p_r(R_i) \leq \Omega'_r$, and then, for all $i \in N'$ $V_{ir}(e') = \max\{p_r(R_i), \lambda_r(e')\}$, where $\lambda_r(e')$ solves $\sum_{i \in N'} \max\{p_r(R_i), \lambda_r(e')\} = \Omega'_r$. Notice that, for all $r \in M$, $\lambda_r(e) = \lambda_r(e')$, and then $V_{ir}(e') = V_{ir}(e)$ for all $r \in M$ and $i \in N'$. ■

Proposition 6.5. *The Generalized Uniform Rule is conversely consistent.*

Proof. Let $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$ and $x \in X(e)$. Given $N' \subset N$ with $n' = 2$, let $\Omega' = \sum_{i \in N'} x_i$ and $e' = (\Omega', R_{N'})$. Suppose that for all $N' \subset N$ with $n' = 2$, $x_{N'} = V(e')$. Then, for all $N' \subset N$ with $n' = 2$, and for all $r \in M$, (i) if $\sum_{i \in N'} p_r(R_i) \geq \Omega'_r$, $x_{ir} = \min\{p_r(R_i), \lambda_r(e')\}$ for all $i \in N'$, where $\lambda_r(e')$ solves $\sum_{i \in N'} \min\{p_r(R_i), \lambda_r(e')\} = \Omega'_r$, and (ii) if $\sum_{i \in N'} p_r(R_i) \leq \Omega'_r$, $x_{ir} = \max\{p_r(R_i), \lambda_r(e')\}$ for all $i \in N'$, where $\lambda_r(e')$ solves $\sum_{i \in N'} \max\{p_r(R_i), \lambda_r(e')\} = \Omega'_r$. Notice that then, for all $r \in M$, either (i) $x_{ir} \leq p_r(R_i)$ for all $i \in N$, or (ii) $x_{ir} \geq p_r(R_i)$ for all $i \in N$. Moreover, for all $r \in M$ and all $j, k \in N$ such that $x_{jr} \neq p_r(R_j)$ and $x_{kr} \neq p_r(R_k)$, $x_{jr} = x_{kr}$. Then, it is easy to see that, for all $r \in M$, (i) $\sum_{i \in N} p_r(R_i) \geq \Omega_r$, $x_{ir} = \min\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e) = \min_{i \in N} x_{ir}$ solves $\sum_{i \in N} \min\{p_r(R_i), \lambda_r(e)\} = \Omega_r$, and (ii) if $\sum_{i \in N} p_r(R_i) \leq \Omega_r$, $x_{ir} = \max\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e) = \max_{i \in N} x_{ir}$ solves $\sum_{i \in N} \max\{p_r(R_i), \lambda_r(e)\} = \Omega_r$. Therefore, $x = V(e)$. ■

Proposition 6.6. *The Generalized Uniform Rule is replication invariant.*

Proof. Let $N \in \mathcal{N}$, $e = (\Omega, R) \in E^N$ and $x = V(e)$. Then, for all $r \in M$, (i) if $\sum_{i \in N} p_r(R_i) \geq \Omega_r$, $x_{ir} = \min\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e)$ solves $\sum_{i \in N} \min\{p_r(R_i), \lambda_r(e)\} = \Omega_r$, and (ii) if $\sum_{i \in N} p_r(R_i) \leq \Omega_r$, $x_{ir} = \max\{p_r(R_i), \lambda_r(e)\}$ for all $i \in N$, where $\lambda_r(e)$ solves $\sum_{i \in N} \max\{p_r(R_i), \lambda_r(e)\} = \Omega_r$. Given $v \in \mathbb{N}$, let $v \times e \in E^{v \times N}$ be the v-fold replica of e, and $v \times x \in X(v \times e)$ the v-fold replica of x. Notice that for all $r \in M$, $\sum_{i \in N} p_r(R_i) \geq \Omega_r$ ($\sum_{i \in N} p_r(R_i) \leq \Omega_r$) if and only if $\sum_{i \in v \times N} p_r(R_i) \geq v \times \Omega_r$ ($\sum_{i \in v \times N} p_r(R_i) \leq v \times \Omega_r$). Moreover, for all $r \in M$, if $\lambda_r(e)$ solves $\sum_{i \in N} \min\{p_r(R_i), \lambda_r(e)\} = \Omega_r$ ($\sum_{i \in N} \max\{p_r(R_i), \lambda_r(e)\} = \Omega_r$), then it also solves $\sum_{i \in v \times N} \min\{p_r(R_i), \lambda_r(e)\} = v \times \Omega_r$ ($\sum_{i \in v \times N} \max\{p_r(R_i), \lambda_r(e)\} = v \times \Omega_r$). Therefore, for all $r \in M$, (i) if

$\sum_{i \in v \times N} p_r(R_i) \geq v \times \Omega_r$, $x_{ir} = \min\{p_r(R_i), \lambda_r(e)\}$ for all $i \in v \times N$, and $\lambda_r(e)$ solves $\sum_{i \in v \times N} \min\{p_r(R_i), \lambda_r(e)\} = v \times \Omega_r$, and (ii) if $\sum_{i \in v \times N} p_r(R_i) \leq v \times \Omega_r$, $x_{ir} = \max\{p_r(R_i), \lambda_r(e)\}$ for all $i \in v \times N$, and $\lambda_r(e)$ solves $\sum_{i \in v \times N} \max\{p_r(R_i), \lambda_r(e)\} = v \times \Omega_r$. Hence, $v \times x \in V(v \times e)$. ■

From these results we conclude that it is not possible to determine which of the two solutions (V or WES_{ed}) is better if we base the comparison on consistency and related properties.

7. CONCLUSION

We consider the problem of allocating m ($m \geq 2$) infinitely divisible commodities among agents with single-peaked preferences. We first show that, in spite of what happens when $m = 1$, in the two agent case any strategy-proof and efficient solution is dictatorial. Then, we study strategy-proof and Pareto-efficient solutions separately.

When we focus on strategy-proofness, we propose the Generalized Uniform Rule. In the case of only two agents it is the only strategy-proof and no-envy solution satisfying a requirement related to efficiency that we call Condition E. Moreover, this solution is implementable in dominant strategies by means of its associated manipulation game. It also satisfies some interesting properties as no-envy and Pareto-domination of equal division. Obviously, by our first result, it fails to be efficient.

We characterize the Pareto-efficient allocations by means of the Walrasian solution with balanced slacks, and prove that a subsolution of this (the Walrasian solution with equal slacks) is the one that minimizes the range of income redistribution necessary to attain any efficient allocation. Moreover, when this solution applies from equal division, it coincides with the Uniform Rule when $m = 1$, and also verifies no-envy and Pareto-domination from equal division.

Finally, we compare both solutions focusing on some of the properties that enabled different characterizations of the Uniform Rule. Both solutions seem to have the same good behavior with respect to these properties. Comparison of V with WES_{ed} is summarized in Tables 1 and 2 below.

The Uniform Rule has been characterized in many different ways, making use of a variety of fairness properties, such as population-monotonicity, resource-monotonicity, or consistency. The question of whether some of these characterizations can be extended in the m-good case by V or WES_{ed} is still open.

	ConditionE	Efficiency	Strategy−Proofness	Implemt. Dominant Strategies	No−Envyness	Pareto Domination ed
V	yes	no	yes	yes	yes	yes
$\mathbf{WES}_{ed}$	yes	yes	no	no	yes	yes

Table 1

	Consistency	Converse Consistency	Replication Invariance
V	yes	yes	yes
$\mathbf{WES}_{ed}$	yes	yes (smoothness)	yes

Table 2

References

[1] BARBERÀ, S., GUL, F. , AND STACCHETTI, E., (1993): Generalized Median Voter Schemes and Committes, Journal of Economic Theory, 61, 262-289.

[2] BARBERÀ, S., JACKSON, M., AND NEME, A., (1997): Strategy-Proof Allotment Rules, Games and Economic Behaviour, 18, 1-21.

[3] BARBERÀ, S., MASSÓ J., AND NEME, A., (1997): Voting under Constraints, Journal of Economic Theory, 76, 298-321.

[4] BARBERÀ, S., MASSÓ J., AND SEIZAWA, : Strategy-Proof Voting on Compact Ranges, forthcoming in Games and Economic Behavior.

[5] BORDER, K. C. AND JORDAN, J.S., (1983): Straightforward Elections, Unanimity and Phantom Voters, Review of Economic Studies, 50, 153-170.

[6] CHING, S., (1992): A Simple Characterization of the Uniform Rule, Economics Letters, 40, 57-60.

[7] CHING, S., (1994): An Alternative Characterization of the Uniform Rule, Social Choice and Welfare, 11, 131-136.

[8] DAGAN, N., (1996): A Note on Thomson's Characterization of the Uniform Rule, Journal of Economic Theory, 69 (1), 255-261.

[9] MAS-COLELL, A., (1992): Equilibrium Theory with Possibly Satiated Preferences, Essays in honour of David Gale, Majumdar, M. ed., New York, 201-213.

[10] MORENO, D., (1994): Nonmanipulable Decision Mechanisms for Economic Environments, Social Choice and Welfare, 11, 225-240.

[11] MORENO, D. AND WALKER, M., (1991): Nonmanipulable Voting Schemes when Participant=s Interests Are Partially Decomposable, Social Choice and Welfare, 8, 221-233.

[12] SATTERTHWAITE, MA. AND SONNENSCHEIN, H., (1981): Strategy-Proof Allocation Mechanisms at Differentiable Points, Review of Economics Studies, 48, 587-597.

[13] SCHUMMER, J., (1997): Strategy-Proofness versus Efficiency on Restricted Domains of Exchange Economies, Social Choice and Welfare, 14, 47-56.

[14] SCHUMMER, J. AND THOMSON, W., (1997): Two Derivations of the Uniform Rule and an Application to Bankruptcy, Economics Letters, 55, 333-337.

[15] SPRUMONT, Y., (1991): The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Rule, Econometrica, 59, 509-519.

[16] THOMSON, W., (1994a): Consistent Solutions to the Problem of Fair Division when Preferences Are Single-Peaked, Journal of Economic Theory, 63, 219-245.

[17] THOMSON, W., (1994b): Resource-Monotonic Solutions to the Problem of Fair Division when Preferences Are Single-Peaked, Social Choice and Welfare, 11, 205-223.

[18] THOMSON, W., (1995): Population-Monotonic Solutions to the Problem of Fair Division when Preferences Are Single-Peaked, Economic Theory, 5, 229-246.

[19] THOMSON, W., (1996): Theory of Fair Allocation, mimeo, University of Rochester.

[20] ZHOU, L., (1991a): Inefficiency on Strategy-Proof Allocation Mechanism in Pure Exchange Economies, Social Choice and Welfare, 8, 247-254.

[21] ZHOU, L., (1991b): Impossibility of Strategy-Proof Mechanisms in Economies with Pure Public Goods, Review of Economic Studies, 58, 107-119.