

**TWO-STAGE MODELS OF PRODUCT DIFFERENTIATION  
WITH UNIT-ELASTIC DEMAND\***

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# TWO-STAGE MODELS OF PRODUCT DIFFERENTIATION WITH UNIT-ELASTIC DEMAND

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## ABSTRACT

Models of product differentiation try to provide answers to the question which good will be provided in an imperfectly competitive market and how it will be priced. In such models consumers have been modeled as buying one unit of one good in the market. I construct counterparts to frequently used models of product differentiation in which consumers have unit-elastic demand, provide conditions which guarantee the existence of a unique price equilibrium, and characterize the unique subgame perfect equilibrium of the two-stage game in which firms first specify products and then compete in prices. In these models the discrete choice is affected by relative as opposed to absolute price changes. Models analyzed in this paper are counterparts to the Hotelling models, the circle model, and a model of vertical product differentiation.

**Keywords:** Hotelling model, vertical product differentiation, product choice, variable individual demand.



# 1 Introduction

Address models of product differentiation have become successful modeling tools in the literature of industrial organization. They provide an understanding of the forces which operate in markets in which firms enjoy market power. Basic models are variants of Hotelling's (1929) model of horizontal product differentiation (D'Aspremont, Gabszewicz, and Thisse, 1979, Neven, 1985, Salop, 1979, Economides, 1989) and various specifications of models of vertical product differentiation (Mussa and Rosen, 1978, Gabszewicz and Thisse, 1979, 1980, Shaked and Sutton 1982, 1983, and Tirole, 1988). For surveys see Eaton and Lipsey (1989) and Gabszewicz and Thisse (1992). In all these models

- consumers make a discrete choice between the differentiated goods (*discrete choice*) and
- consumers buy 0 or 1 unit of the differentiated goods (*unit demand*).

Firms play a two-stage game: at stage 1 they simultaneously choose variety, at stage 2 price.

In this paper I develop corresponding models in which consumers do not satisfy (unit demand). Instead of presenting models in which consumers have unit demand I present models in which consumers have unit-elastic demand. This work is motivated by a theoretical and an empirical observation:

*Empirical observation:* in several markets consumer choice is characterized by discrete choice and variable individual demand. Such markets include potato chips, particular categories of milk (e.g. full fat UHT milk), cigarettes and many other consumer non-durables.

*Theoretical observation:* unit demand models run into difficulties when income heterogeneity among consumers is taken into account. As shown in Peitz (1998a) there are serious problems of existence of price equilibrium (stage 2) due to an inside kink of the aggregate demand functions which firms face. In addition, a model with multiple imperfectly competitive markets, in each of which consumers buy one unit, has the property that the ability to pay in one market depends upon the prices in the other markets. This makes the model intractable and it is not clear how the unit demand

specification can be extended to model demand for all commodities in the market.<sup>1</sup>

These two observations suggest that a framework in which consumers have variable individual demand may be useful for the understanding of differentiated product markets. The present paper is not the first in the analysis of address models of product differentiation with variable individual demand. Several papers (e.g. Smithies, 1941, Greenhut, 1974, Capazzo and Van Order, 1978, Novshek, 1980, Greenhut, Norman and Hung, 1987, and Norman and Thisse, 1996) presented a Hotelling model of product differentiation with linear individual demand which is discussed at the end of section 2. Also in these models income heterogeneity and multiple markets lead to the theoretical difficulties pointed out above.

According to my specification of utility functions, discrete choice is derived and individual expenditure is constant in the differentiated market. This means that households make their budgeting decisions independent of the prevailing prices. It seems to be an interesting benchmark case to consider competition between firms in a market of fixed dollar-size. When in such a market only the (weighted) number of consumers in the turf of one firm determines its market share, individual demand should be modeled as unit-elastic. With this specification I provide counterparts to the Hotelling model with quadratic transportation costs, the circle model with quadratic transportation costs and a model of vertical product differentiation. I characterize the unique sub-game perfect equilibrium of the location-then-price game. Calculations turn out to be more complicated than in the corresponding unit demand models but formulas result to be simple along the equilibrium path in models of horizontal product differentiation. With my specifications extensions to multiple imperfectly competitive markets and/or heterogeneous income are straightforward. The models can also be extended to address other questions in industrial organization (such as entry, advertising, non-linear pricing etc.) and international trade.

What can we learn from the analysis? Concerning the price setting one might expect firms to compete less strongly than under unit demand because demand is downward

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<sup>1</sup>For this reason models with Cobb Douglas or CES consumers have been used in the international trade literature (see e.g. Helpman and Krugman, 1985, chapter 6).

sloping. However, under unit-elastic demand a monopolist would set an infinite price. Hence firms, when freed from the competitive pressure raise prices without bounds. The same happens in a unit demand model except that firms are constrained by the reservation price of the consumers. The mechanism which determines whether firms agglomerate are similar to the ones in the corresponding unit demand models. Firms prefer to locate apart in order to reduce the competitive pressure. In the specification of horizontal differentiation this pressure is not strong enough to make firms maximally differentiate their products. With vertical product differentiation the low quality firm has an incentive to produce positive quality because otherwise consumers would not derive any utility from the purchase of the good. I extend this example to allow for a correlation between buying decisions in the differentiated market and the individual expenditure in the market. This adds to the applicability of the model because in many real markets there seems to be a relationship between expenditure and preference for quality of the consumers.

The paper is organized as follows. In Section 2 I present a collection of models to be analyzed and establish the existence of a unique equilibrium of the second stage. Section 3 analyzes the location-then-price game for models of horizontal product differentiation. As examples I chose the counterpart of the Hotelling model with quadratic transportation costs and the circle model with 2 and 3 firms. Vertical product differentiation is analyzed in section 4. Section 5 concludes.

## 2 The Model

In a differentiated market with  $n$  goods consumers can buy quantities  $x_i$ ,  $i \in N = \{1, \dots, n\}$ , and they may spend some part of their income (or budget) on the Hicksian composite commodity with index 0. The price of good 0 is normalized to 1. Hence, prices  $p_i$ ,  $i = 1, \dots, n$ , and income  $y$  are measured in units of the composite commodity. The budget constraint reads  $x_0 + \sum_{i \in N} p_i x_i \leq y$ .

The following choice behavior will be rationalized:

- consumers buy one good in the differentiated market;

- their budget for the differentiated market is fixed;
- differences between goods are perceived as one-dimensional.

The product space  $L$  will be defined below.  $l_i$  denotes the location of the good in the product space. With  $\Omega$  I define the space of taste parameters of the consumers which determines the relative evaluation of the differentiated goods. Taste parameter  $\alpha \in A \subseteq [0, 1]$  determines the relative evaluation of any differentiated good relative to the composite commodity. The utility function is defined as

$$u(x_0, x, l) = \left( \sum_i x_i z(\omega, l_i) \right)^\alpha x_0^{1-\alpha}$$

where  $z(\omega, l_i)$  is a factor which specifies the attractiveness of good  $l_i$  for consumer  $\omega$ . Note that  $x_i z(\omega, l_i)$  is linear and hence convex in  $x_i$ . This implies that consumers maximize utility with discrete choice. Note also that each consumer spends  $\alpha y$  in the differentiated market. When a consumer buys variant  $i$  a monotone transformation of utility is

$$\alpha \log x_i + (1 - \alpha) \log x_0 + \alpha z(\omega, l_i)$$

Maximizing utility gives conditional indirect utility

$$v_i(p_i, y) = a - \log p_i + \log z(\omega, l_i)$$

where  $a$  is a constant which depends on  $\alpha$  and  $y$ . Note that not buying in the market gives indirect utility  $v_0 = -\infty$  and hence variant  $i$  is chosen if  $i = \arg \max_{j \in N} \{(a - \log p_j + \log z(\omega, l_j))_j\}$ . When comparing two goods, the consumer takes logarithmic price differences into account. Hence, the discrete choice responds to relative prices as opposed to price differences as in standard models of product differentiation. Also note that the logarithm of the function  $z$  matters when comparing the indirect utility of two goods. Two different specifications of the function  $z(\omega, l_i)$  will be considered.

*Specification 1: Horizontal Product Differentiation.*  $z(\omega, l_i) = e^{-(\omega - l_i)^2}$ . The product space is  $L \in \{\Re, [\underline{l}, \infty), (-\infty, \bar{l}], [\underline{l}, \bar{l}], C_k\}$  where  $\underline{l}, \bar{l} \in \Re$  and  $C_k$  is the circle with circumference  $k$ . The space of taste parameters of the consumers is assumed to satisfy

$\Omega = L$  if  $L = C_k$  and  $\Omega \subseteq \mathfrak{R}$  else.

*Specification 2: Vertical Product Differentiation.*  $z(\omega, l_i) = l_i^\omega$ . The product space is  $[L, \bar{l}] \subset \mathfrak{R}_+$  and the space of taste parameters is  $[\underline{\omega}, \bar{\omega}] \subset \mathfrak{R}_+$  with  $0 = \underline{\omega}$ .

In both specifications the disutility which is due to a difference between actual and ideal good takes an exponential form (in specification 2 one can write  $z(\omega, l_i) = \exp\{\omega \log l_i\}$ ). In the first specification the function is similar to Samuelson's iceberg transportation cost which has recently been applied in economic geography (see e.g. Krugman, 1991, and for an exponential specification Krugman, 1994, and Fujita and Krugman, 1995). In a spatial economy the interpretation of this function is that only a share of the sent good finally arrives at the consumer location (and takes the form of the function  $z$  in my specification). The function  $\log z(\omega, l_i) = -(\omega - l_i)^2$  depresses the utility if the location of good  $i$  does not coincide with the ideal variant of consumer  $\omega$  and will play the same role as the transportation cost function in the Hotelling model with  $d(0) = 0$  and  $(\omega - l_i)^2 > 0$ ,  $\omega \neq l_i$ .

In specification 2,  $l_i$  is the quality of good  $i$ . If the good is of 0 quality, consumers do not derive positive utility from it. The formulation corresponds to the vertical differentiation model of Mussa and Rosen (1978) because there is a multiplicative interaction between consumer taste and a function of quality.

For simplicity, consumers are assumed to have identical expenditure  $\alpha y$  in the differentiated market. This assumption is easily relaxed (see Peitz, 1998b). For positive demand one can write the demand of firm  $i$  as

$$X_i = \frac{1}{p_i} \alpha y \int_{m_{iL}}^{m_{iR}} g(\omega) d\omega$$

where  $m_{iL}$  and  $m_{iR}$  are the marginal consumers of firm  $i$  to the left and to the right and  $g(\omega)$  is a logconcave density function which is continuously differentiable on its support. The firm sells only to the consumers between  $m_{iL}$  and  $m_{iR}$ . If a firm has only one neighboring firm one has to integrate from or until the end point of the space of taste parameters. The formulas for  $m_{iL}$  and  $m_{iR}$  are derived in the sections below. The goal is to study the two-stage game in which at

*stage 1:* firm  $i = 1, \dots, n$  chooses location  $l_i \in L$  and at

*stage 2:* firm  $i = 1, \dots, n$  sets price  $p_i > 0$ .

Hence, a strategy of a firm is a location  $l_i$  and a price function  $P_i : L^n \rightarrow \mathfrak{R}_+$  which gives a price for any locational choice of the firms. At stage 2 firm  $i$  maximizes profits  $\pi_i(p_1, \dots, p_n; l_1, \dots, l_n) = (p_i - c)X_i(p_1, \dots, p_n; l_1, \dots, l_n)$  with respect to its price  $p_i$  where  $c$  denotes the constant marginal cost of production which is identical between firms and  $X_i$  is mean demand for good  $i$ . At stage 1 firm  $i$  maximizes continuation profits  $\tilde{\pi}_i(l_1, \dots, l_n)$  with respect to  $l_i$  in which prices are replaced by their (unique) equilibrium values at stage 2. With continuation profits as the payoff functions at the first stage, subgame perfect Nash equilibria are determined.

In this section I provide a result on the existence and uniqueness of equilibrium for given locations. This result is a special case of results in Peitz (1998b). An equilibrium of the second stage given locations are prices  $p_1^*, \dots, p_n^*$  such that  $\pi_i(p_1^*, \dots, p_n^*; l_1, \dots, l_n) \geq \pi_i(p_i, p_{-i}^*; l_1, \dots, l_n)$  for all  $p_i$  and  $i = 1, \dots, n$ , where  $p_{-i}^*$  is the vector of the competitors' prices.

**Proposition 1.** *(Peitz, 1998b) In the presented model of product differentiation there exists a unique price equilibrium for any finite number of firms with  $l_i \neq l_j$ ,  $j \neq i$  and  $L \subseteq \Omega$  in specification 1. The associated game is dominance solvable, i.e. only the vector  $(p_1^*, \dots, p_n^*)$  survives the serial elimination of strictly dominated strategies.*

The reader is referred to Peitz (1998b) for the proof of a more general result (see also Caplin and Nalebuff, 1991, Dierker, 1991, and Peitz, 1997a). In most of the paper I will assume that consumers are uniformly distributed on  $\Omega$ . The results can possibly be extended to non-uniform densities (see Anderson, Goeree, and Ramer, 1997). Before turning to the location-then-price game I discuss an alternative address model of product differentiation with variable individual demand.

*Remark on other models with variable individual demand.* As mentioned in the introduction other authors have proposed models of product differentiation with vari-

able individual demand. The model studied is the Hotelling model with linear transportation costs and linear demand (e.g. Smithies, 1941, Greenhut, 1974, Capazzo and Van Order, 1978, Novshek, 1980, Greenhut, Norman and Hung, 1987, and Norman and Thisse, 1996). Consider the conditional utility function of the form  $u_i = ax_i - x_i^2/2 - t(|\omega - l_i|)x_i + x_0$ , where  $t(|\omega - l_i|)$  is the transportation cost function and enters as a disutility,  $a$  a positive constant. Consumers maximize this utility function with respect to  $x_0$  and  $x_i$  subject to the budget constraint. This implies that demand functions are linear:  $x_i = a - t(|\omega - l_i|) - p_i$  for income sufficiently high. If  $t(|\omega - l_i|) = b|\omega - l_i|$ ,  $b > 0$ , this is the linear demand function used by the authors above.

When consumers have discrete choice, they choose good  $j$  as

$$j = \arg \max_i \{ (\max_{(x_i, x_0)} u_i(x_0, x_i) \text{ s.t. } p_i x_i + x_0 \leq y) \}.$$

It is easily calculated that this implies the same choice as minimizing the “delivered price”  $p_i + t(|\omega - l_i|)$ . Now allow consumers to choose any number of goods they want. For this, consider the utility function  $u = [a \sum x_i - (\sum_i x_i)^2/2] - \sum_i t(|\omega - l_i|)x_i + x_0$ . The term in square bracket refers to the utility derived from consuming some quantity of the sum of the differentiated products. When a differentiated product is not the ideal good, consumers suffer a utility loss which is linear in quantity and additive over the goods consumed. Utility maximization leads to linear demand and discrete choice.

For reasons of tractability, the location-then-price game has been studied with linear transportation costs. However, with this specification one encounters similar problems as in the Hotelling model with linear transportation costs (for the latter see d’Aspremont, Gabszewicz and Thisse, 1979). Independent of the functional form of the transportation cost function the critique of Peitz (1998a) applies and price equilibria may fail to exist when income is heterogeneous. Also, in a set-up with multiple markets the demand function in one market depends on the prices which prevail in other markets.

Alternatively, Böckem (1994) has introduced heterogeneous reservation prices into the Hotelling model with quadratic transportation costs. Hence, individual demand is unit demand but demand of a locational type depends on price. Again the critique

concerning heterogeneous income and multiple markets applies.

### 3 Horizontal Product Differentiation

Before analyzing two particular specifications I characterize the consumers which are indifferent between goods. For the one-dimensional markets under consideration there exists for each good a marginal consumer  $m_{iL} \in \mathfrak{R}$  who is indifferent between buying good  $i$  or the neighboring good to the left and analogously the consumer at  $m_{iR}$  separates the market areas for good  $i$  and the neighboring good to the right. Suppose that goods are labeled according to  $l_i < l_{i+1}$ ,  $i = 1, \dots, n-1$ . Then the marginal consumers for good  $i$  in specification 1 are

$$\begin{aligned} m_{iL}^{(1)} &= \frac{\log p_i - \log p_{i-1}}{2(l_i - l_{i-1})} + \frac{l_i + l_{i-1}}{2} \text{ and} \\ m_{iR}^{(1)} &= \frac{\log p_{i+1} - \log p_i}{2(l_{i+1} - l_i)} + \frac{l_{i+1} + l_i}{2} \end{aligned}$$

with the exceptions  $m_{1L}^{(1)} = m_{nR}^{(1)}$  if  $\Omega = C_k$  and  $m_{1L}^{(1)} = \underline{\omega}$  else; analogously for  $m_{nR}^{(1)}$ . Clearly,  $m_{iR}^{(1)} = m_{i+1,L}^{(1)}$ ,  $i = 1, \dots, n-1$ .

#### 3.1 Horizontal Differentiation on an Interval

This subsection provides a counterpart to the Hotelling model with quadratic transportation costs (d'Aspremont, Gabszewicz, and Thisse, 1979) in the framework of variable demand. The product space is assumed to be an interval, without loss of generality it is assumed to be  $[0, 1]$ . The taste parameters are assumed to be distributed uniformly over the same space. Competition takes place between two firms. Since the consumer density is symmetric, intuition suggest that solutions to the restricted game  $l_1 \leq l_2$  are the same as in the unrestricted game. This intuition turns out to be correct.

Individual demand for the composite commodity is  $\xi_0(p_1, p_2) = (1 - \alpha)y$ . Individual demand for good 1 is  $\xi_1(p_1, p_2) = \frac{\alpha y}{p_1}$  if  $(\omega - l_1)^2 + \log p_1 \leq (\omega - l_2)^2 + \log p_2$  and 0 otherwise. The marginal consumer  $m$  who is indifferent between good 1 and good 2 is

determined by

$$m = \frac{\log p_2 - \log p_1}{2(l_2 - l_1)} + \frac{l_1 + l_2}{2}.$$

This is the same formula as in d'Aspremont, Gabszewicz and Thisse (1979) except that in my model prices are replaced by logarithmic prices. Hence, an assumption on the distribution of types  $\omega$  tells how market shares change due to *relative* price changes.

When both firms are active, the market share of firm 1 simply is equal to  $m$  and the market share of firm 2 is equal to  $1 - m$ . Profit functions are

$$\begin{aligned} \pi_1(p_1, p_2, l_1, l_2) &= \frac{p_1 - c}{p_1} \alpha y \left( \frac{\log p_2 - \log p_1}{2(l_2 - l_1)} + \frac{l_1 + l_2}{2} \right) \\ \pi_2(p_1, p_2, l_1, l_2) &= \frac{p_2 - c}{p_2} \alpha y \left( 1 - \frac{\log p_2 - \log p_1}{2(l_2 - l_1)} - \frac{l_1 + l_2}{2} \right) \end{aligned} \quad (1)$$

**Proposition 2.** *For symmetric locations  $(l_1, 1 - l_1)$ ,  $l_1 \leq \frac{1}{2}$ , equilibrium prices are  $p_1^* = p_2^* = c(2 - 2l_1)$ .*

From the explicit formula of equilibrium prices it follows that the markups  $\frac{p_i - c}{c}$  are constant in marginal costs. This is different from the Hotelling model with quadratic transportation costs where the price-cost margin  $p_i - c$  is constant.

For symmetric locations, continuation profits of firm 1 are

$$\tilde{\pi}_1(l_1, l_2) = \frac{l_2^2 - l_1^2}{1 + l_2^2 - l_1^2} \alpha y \frac{l_1 + l_2}{2} = \frac{1 - 2l_1}{1 - l_1} \frac{\alpha y}{4}$$

which are maximal at  $l_1 = 0$  on  $[0, 1]$  if firm 2 reacts symmetrically to firm 1. Given the competitor's location  $l_2 = 1$ , firm 1 does *not* choose maximal differentiation (the result is qualitatively the same as Böckem, 1994). It prefers to be located closer to the market center, because increased market share compensates for tougher competition. The next proposition characterizes locations in the location-then-price game.

**Proposition 3.** *Firms choose approximately  $(0.12, 0.88)$  as locations in the unique subgame perfect equilibrium of the location-then-price game.*

It is worth noting that it is sufficiently attractive to locate “close” to the market center so that maximal differentiation does not result although both firms would have preferred maximal differentiation. It implies that joint product specification by the firms increases profits of the firms and consequently collusion at stage 1 leads to higher prices. This is not the case in the corresponding unit demand model.

Equilibrium profits are approximately  $\tilde{\pi}_1^* = \tilde{\pi}_2^* = 0.213\alpha y$ . Unlike in the Hotelling model with quadratic transportation costs there is no explicit equilibrium price function for asymmetric locations. However, the model provides explicit functions for symmetric locations and locations are symmetric in subgame perfect equilibrium.

### 3.2 Horizontal Differentiation on the Circle

The product space is  $C_k$  and it is assumed that consumers are uniformly distributed on  $C_k$ . In the first part I will analyze the duopoly on the circle. Without loss of generality, firm 1 is located at 0 and firm 2 at  $l_2$ . When both firms are active there are two marginal consumers, one to the right and one to the left of firm 1, who are indifferent between good 1 and good 2.

$$\begin{aligned} m_R &= \frac{l_2}{2} + \frac{\log p_2 - \log p_1}{2l_2} \\ m_L &= \frac{l_2 + k}{2} + \frac{\log p_2 - \log p_1}{2(l_2 - k)} \end{aligned}$$

Demand for firm 1 is  $X_1(p_1, p_2, l_1, l_2) = m_R + L - m_L$  and, for firm 2,  $L - X_1$ . As in the model with unit demand firms choose locations at distance  $\frac{k}{2}$  which maximizes the minimal distance and minimizes the maximal distance between the firms on the circle. With the convention that 0 on the circle is where firm 1 is located one obtains that there exists a unique subgame perfect equilibrium.<sup>2</sup> This is shown by the following proposition.

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<sup>2</sup>Clearly, without giving firm 1 location 0 there exists a continuum of subgame perfect equilibria with  $l_1, l_1 + \frac{k}{2}$  for  $l_1 \leq \frac{k}{2}$  and  $l_1, l_1 + \frac{k}{2} - L$  for  $l_1 > \frac{k}{2}$ . Under sequential product choice there does not exist a coordination problem: firm 1 as the first mover locates somewhere on the circle and firm 2 locates  $\frac{k}{2}$  apart.

**Proposition 4** *On the circle duopolists choose locations  $0, \frac{k}{2}$  in the unique subgame perfect equilibrium. Given these locations, equilibrium prices are  $p_1^* = p_2^* = c \left(1 + \frac{k^2}{4}\right)$ .*

Now I turn to the circle with three firms. Firm 1 is located at 0, firm 2 at  $l_1$  and firm 3 at  $l_2$  with  $l_1 \leq l_2$ . The solution to the restricted game will also be the solution to the unrestricted game. There are three marginal consumers. Consumer  $m_1$  is indifferent between good 1 and good 2, consumer  $m_2$  is indifferent between good 2 and good 3, and consumer  $m_3$  is indifferent between good 1 and good 3.

$$\begin{aligned} m_1 &= \frac{l_1}{2} + \frac{\log p_2 - \log p_1}{2l_1} \\ m_2 &= \frac{l_1 + l_2}{2} + \frac{\log p_3 - \log p_2}{2(l_2 - l_1)} \\ m_3 &= \frac{k + l_2}{2} + \frac{\log p_1 - \log p_3}{2(k - l_2)} \end{aligned}$$

When each firm sells to the consumers at its location, demand functions are written as  $X_1 = m_1 + k - m_3$ ,  $X_2 = m_2 - m_1$ , and  $X_3 = m_3 - m_2$ . Consequently, in this case the profit functions of the firms are

$$\begin{aligned} \pi_1 &= \frac{p_1 - c}{p_1} \frac{\alpha y}{2} \left( \frac{k + l_1 - l_2}{2} + \frac{\log p_2 - \log p_1}{2l_1} + \frac{\log p_3 - \log p_1}{2(k - l_2)} \right), \\ \pi_2 &= \frac{p_2 - c}{p_2} \frac{\alpha y}{2} \left( \frac{l_2}{2} + \frac{\log p_3 - \log p_2}{2(l_2 - l_1)} + \frac{\log p_1 - \log p_2}{2l_1} \right), \\ \pi_3 &= \frac{p_3 - c}{p_3} \frac{\alpha y}{2} \left( \frac{k - l_1}{2} + \frac{\log p_1 - \log p_3}{2(k - l_2)} + \frac{\log p_2 - \log p_3}{2(l_2 - l_1)} \right). \end{aligned}$$

**Proposition 5.** *On the circle three firms choose  $0, \frac{1}{3}k, \frac{2}{3}k$  in the unique subgame perfect equilibrium. Given these locations, equilibrium prices are  $p_1^* = p_2^* = p_3^* = c \left(1 + \frac{k^2}{9}\right)$ .*

In the original circle model of unit demand with quadratic transportation costs and three firms the existence of a unique subgame perfect equilibrium of the two-stage game can be shown analogously. The simple argument for uniqueness in the locational choice cannot be used for more than three firms. However, one may show numerically that

the configuration with equidistant location is a subgame perfect equilibrium (for the case of unit demand compare Economides, 1989). In the case of  $n$  firms at equidistant location equilibrium prices are  $p_i^* = c \left(1 + \frac{k^2}{n^2}\right)$ ,  $i = 1, \dots, n$ . The model is suited to address questions of entry in an industry.

The result of equidistant locations turned out to be invariant to the specification of individual demand (of course under the assumption of a uniform density). This should not come as a surprise: in the circle being close to the median consumer has no meaning and at the location stage only the distance to the other firms matter. The competitive pressure makes firms to maximize the minimal distance to their competitors.

## 4 Vertical Product Differentiation

In a model of vertical product differentiation goods are ranked unanimously. The chosen specification is motivated by the unit demand model of Mussa and Rosen (1978) which is widely used in the literature (see e.g. Tirole, 1988, and Anderson, de Palma, and Thisse, 1992). In specification 2 marginal consumers are

$$\begin{aligned} m_{1L}^{(2)} &= \underline{\omega} \\ m_{iR}^{(2)} &= \frac{\log p_{i+1} - \log p_i}{\log l_{i+1} - \log l_i}, \quad i = 1, \dots, n-1 \\ m_{nR}^{(2)} &= \bar{\omega} \end{aligned}$$

and  $m_{iR}^{(2)} = m_{i+1,L}^{(2)}$ ,  $i = 1, \dots, n-1$ .

If there is only one neighbor  $m_{iL}$  or  $m_{iR}$  is an end point of  $\Omega$ . Otherwise, the location of a marginal consumer of good  $i$  depends linearly on logarithmic price.

Consider a duopoly where the index is chosen such that  $l_1 \leq l_2$ . For  $l_1 < l_2$ , the marginal consumer is determined by  $m = (\log p_2 - \log p_1) / (\log l_2 - \log l_1)$ . Consumers are assumed to be uniformly distributed on  $[\underline{\omega}, \bar{\omega}] = [0, 1]$ . Whenever  $[\underline{\omega}, \bar{\omega}] \subset \mathfrak{R}_+$ , the location in the vertical product space can be interpreted as the quality of the good. For a population of mass 1 demand for firm 1 simply is  $X_1(p_1, p_2, l_1, l_2) = (\alpha y / p_1) m$  when both firms are active. Analogously,  $X_2(p_1, p_2, l_1, l_2) = (\alpha y / p_2) (1 - m)$ . Profit

functions then are

$$\begin{aligned}\pi_1(p_1, p_2, l_1, l_2) &= \frac{p_1 - c}{p_1} \alpha y \frac{\log p_2 - \log p_1}{\log l_2 - \log l_1}, \\ \pi_2(p_1, p_2, l_1, l_2) &= \frac{p_2 - c}{p_2} \alpha y \frac{\log l_2 - \log l_1 - \log p_2 + \log p_1}{\log l_2 - \log l_1}.\end{aligned}$$

Remark that at equal prices the low-quality firm has zero demand. Clearly, equal prices cannot occur in equilibrium at stage 2 given  $l_1 \neq l_2$  because, at prices above marginal costs, at least firm 1 can increase its profits by decreasing its prices and, at prices equal to marginal costs, firm 2 can increase its profits by increasing its price.

**Proposition 6.** *For any parameter values  $l, \bar{l}$ , there exists a unique subgame perfect equilibrium of the location-then-price game. If  $\log \bar{l} - \log l < 4.04446$ , firms choose maximal differentiation. Otherwise,  $l_2 = \bar{l}$  and  $\log l_1 = \log \bar{l} - 4.04446$  in the unique subgame perfect equilibrium.*

If  $\log l_2 - \log l_1 > 4.04446$  equilibrium prices are approximately  $p_1 = 1.8324 c$  and  $p_2 = 4.2121 c$ . The subgame perfect equilibrium under the restriction  $l_1 < l_2$  is also a subgame perfect equilibrium of the unrestricted game. Either firm 1 or firm 2 produces the maximal quality and the quality of the other firm follows from the proposition. There exists a continuum of subgame perfect equilibria which are pay-off equivalent if  $\log \bar{l} - \log l > 4.04446$  when the quality choice is modeled as sequential.

Note that less than maximal differentiation can also be obtained in the corresponding model of unit demand if the reservation price for buying one unit is sufficiently small so that not all consumers buy in the market (see Tirole, 1988, Choi and Shin, 1992, and Wauthy, 1996). With the previous proposition it has been shown that less than maximal differentiation can prevail even if all consumers buy in the market.

Specification 2 is now modified such that consumers with a higher preference for quality spend more in the differentiated market. I assume that the preference for quality  $\omega$  and the expenditure share are positively correlated: the direct utility function is written as

$$u(x_0, x_1, x_2, l_1, l_2) = \left( \sum_i \tilde{u}_i(x_i, l_i) \right)^{\alpha f(\omega)} x_0^{1-\alpha f(\omega)}$$

where  $f : [\underline{\omega}, \bar{\omega}] \rightarrow [0, 1]$ . Alternatively, one could consider a population with heterogeneous income and introduce a positive functional relationship between mean income of types  $(\alpha, \omega)$  and  $\omega$ .

A consumer of type  $\omega$  spends  $\alpha f(\omega)y$  in the differentiated market. When both firms are active, mean demand of the firms are

$$\begin{aligned} X_1(p_1, p_2, l_1, l_2) &= \int_{\underline{\omega}}^m \frac{\alpha y}{p_1} f(\omega) g(\omega) d\omega \\ X_2(p_1, p_2, l_1, l_2) &= \int_m^{\bar{\omega}} \frac{\alpha y}{p_2} f(\omega) g(\omega) d\omega \end{aligned}$$

where  $m$  has been defined above.

Since I only want to construct examples for the two-stage model I again assume that  $\omega$  is uniformly distributed on  $[0, 1]$  and that  $f(\omega) = \omega^\beta$ ,  $\beta \geq 0$ . Since  $f(\omega)g(\omega)$  is logconcave there exist a unique equilibrium in prices for different locations. The case  $\beta = 0$  has been analyzed in the previous section. For  $\beta > 0$  consumers with a higher taste for quality spend a larger share out of income in the differentiated market. For  $\beta = 1$  the relationship is linear, for  $\beta > 1$  the expenditure share is strictly convex in the type  $\omega$ . The higher  $\beta$  the more expenditure mass is concentrated near  $\omega = 1$ .

In the duopoly with  $l_1 < l_2$  profit functions are

$$\begin{aligned} \pi_1(p_1, p_2, l_1, l_2) &= \frac{1}{1 + \beta} \frac{p_1 - c}{p_1} \alpha y \frac{(\log p_2 - \log p_1)^{1+\beta}}{(\log l_2 - \log l_1)^{1+\beta}}, \\ \pi_2(p_1, p_2, l_1, l_2) &= \frac{1}{1 + \beta} \frac{p_2 - c}{p_2} \alpha y \left( 1 - \frac{(\log p_2 - \log p_1)^{1+\beta}}{(\log l_2 - \log l_1)^{1+\beta}} \right). \end{aligned}$$

**Proposition 7.** *For any parameter values  $\beta > 0, l_1$ , and  $\bar{l}$ , there exists a unique subgame perfect equilibrium of the location-then-price game. The high-quality producer chooses maximal quality, i.e.  $l_2 = \bar{l}$ .*

This example is modified in Peitz (1997b) in order to show that equilibrium prices do not necessarily converge to marginal costs as consumer behavior becomes homogeneous.

## 5 Conclusion

With this paper I provided a specification of variable individual demand to analyze the product specification of firms in models of product differentiation with heterogeneous consumers. For applications, models of horizontal product differentiation are promising because prices in the (unique) subgame perfect equilibrium are explicit functions in the parameters of the model. Both types of models presented highlight the importance of the trade-off between market share and competitive pressure. In the model which corresponds to the Hotelling model and in the model of vertical product differentiation this implies less than maximal differentiation and shows that maximal differentiation is not a robust result even if consumers are uniformly distributed and all consumers buy in the market. Given the mechanisms at work in these models this result should not come as a surprise.

Extending the literature, I allow for the dependence of individual expenditure shares on taste parameters in a model of vertical differentiation. In the corresponding unit demand model such an assumption cannot be made at the individual level but in the aggregate (see Peitz 1997b).

As argued in the introduction the models might prove useful in the analysis of markets in which total expenditure is not very sensitive to price changes.

## Appendix

**Proof of Proposition 2.** Define  $\tilde{m} : \mathfrak{R} \rightarrow \mathfrak{R}$  with  $\tilde{m}(\phi) = \frac{\phi}{2(l_2 - l_1)} + \frac{l_1 + l_2}{2}$  where  $\phi \equiv \log p_2 - \log p_1$ . The first-order conditions of profit maximization then can be written as

$$\begin{aligned} c\tilde{m}(\phi) &= (p_1 - c)\tilde{m}'(\phi) \\ c(1 - \tilde{m}(\phi)) &= (p_2 - c)\tilde{m}'(\phi) \end{aligned}$$

Rearranging and taking ratios yields

$$\frac{1 - \tilde{m}(\phi) + \frac{1}{2(l_2 - l_1)}}{\tilde{m}(\phi) + \frac{1}{2(l_2 - l_1)}} - e^\phi = 0. \quad (2)$$

Clearly,  $\phi = 0$  is a solution for  $l_2 = 1 - l_1$ . One can show that the left-hand side is a decreasing function in  $\phi$ . Hence,  $\phi = 0$  is the unique zero and  $p_1^* = p_2^*$ . Substituting  $p_1 = p_2$  and  $l_2 = 1 - l_1$  into the first of the first-order conditions gives  $p_1^* = c(2 - 2l_1)$ .  $\square$

**Proof of Proposition 3.** By Proposition 1 there exists a unique price equilibrium for any locational pair  $l_1, l_2$ . Since (2) has a unique solution  $\phi^*$ , equilibrium prices at stage 2 are given by

$$\begin{aligned} p_1^* &= c(1 + \phi^* + l_2^2 - l_1^2) \\ p_2^* &= c(1 - \phi^* + l_1^2 - l_2^2 + 2(l_2 - l_1)) \end{aligned} \quad (3)$$

$\phi^*$  is determined numerically with Newton's method. With  $\phi^*$  one thus has continuation profits  $\tilde{\pi}_1(l_1, l_2) = ((p_1^*(l_1, l_2) - c)/p_1^*(l_1, l_2))m(\phi^*(l_1, l_2))$  and correspondingly for firm 2. Checking the profit functions one obtains that firm 1 maximizes its profits by locating at around 0.12 in the product space given the competitor's choice of 0.88. Analogously for firm 2. Consequently, firms choose  $l_1 = 0.12, l_2 = 0.88$  on  $[0, 1]^2$  in subgame perfect equilibrium. The numerical analysis also reveals that there are no other subgame perfect equilibria.  $\square$

**Proof of Proposition 4.** First-order conditions of profit maximization reduce to

$$\begin{aligned} c \left( \frac{k}{2} + \frac{\log p_2 - \log p_1 + 1}{2l_2} + \frac{\log p_2 - \log p_1 + 1}{2(k-l_2)} \right) &= p_1 \left( \frac{1}{2l_2} + \frac{1}{2(k-l_2)} \right) \\ c \left( \frac{k}{2} + \frac{\log p_1 - \log p_2 + 1}{2l_2} + \frac{\log p_1 - \log p_2 + 1}{2(k-l_2)} \right) &= p_2 \left( \frac{1}{2l_2} + \frac{1}{2(k-l_2)} \right) \end{aligned}$$

Taking ratios and replacing  $\log p_2 - \log p_1$  by  $\phi$  yields

$$\frac{\frac{k}{2} - \frac{\phi-1}{2l_2} - \frac{\phi-1}{2(k-l_2)}}{\frac{k}{2} + \frac{\phi+1}{2l_2} + \frac{\phi+1}{2(k-l_2)}} - e^\phi = 0$$

The function of the left-hand side has a unique zero in  $\phi$ . For any  $l_2$ ,  $\phi = 0$  is a solution. Consequently, equilibrium prices are

$$p_1^* = p_2^* = c \frac{k + \frac{1}{l_2} + \frac{1}{k-l_2}}{\frac{1}{l_2} + \frac{1}{k-l_2}} \quad (4)$$

Continuation profits  $\tilde{\pi}_i$  only depend on  $l_2$  because of the convention that firm 1 is always located at 0, which is somewhere on the circle. For given location of firm 1 on the circle, firm 2 can choose  $l_2$  by choosing his location.

$$\tilde{\pi}_2 = \frac{k}{k + \frac{1}{l_2} + \frac{1}{k-l_2}} \alpha y \frac{1}{2}$$

which is the same for firm 1. Since  $\frac{k}{2} = \arg \max_{l_2} \tilde{\pi}_i(l_2)$ , firm 2 will locate  $\frac{k}{2}$  away from firm 1. Equilibrium prices follows from inserting  $l_2 = \frac{k}{2}$  into equation (4).  $\square$

**Proof of Proposition 5.** Since marginal costs are identical in any equilibrium f.o.c.'s of profit maximization need to be satisfied. Since profit functions are quasi-concave any price vector which satisfies the f.o.c.'s is an equilibrium of the second stage. By Proposition 1 there exists a unique equilibrium at the second stage for any locational choice. Hence, a solution to the f.o.c.'s is the unique equilibrium price vector for given locations. The f.o.c.'s are given by

$$\begin{aligned} c \left( \frac{k+l_1-l_2}{2} + \frac{\log p_2 - \log p_1 + 1}{2l_1} + \frac{\log p_3 - \log p_1 + 1}{2(k-l_2)} \right) &= p_1 \left( \frac{1}{2l_1} + \frac{1}{2(k-l_2)} \right) \\ c \left( \frac{l_2}{2} + \frac{\log p_3 - \log p_2 + 1}{2(l_2-l_1)} + \frac{\log p_1 - \log p_2 + 1}{2l_1} \right) &= p_2 \left( \frac{1}{2(l_2-l_1)} + \frac{1}{2l_1} \right) \\ c \left( \frac{k-l_1}{2} + \frac{\log p_1 - \log p_3 + 1}{2(k-l_2)} + \frac{\log p_2 - \log p_3 + 1}{2(l_2-l_1)} \right) &= p_3 \left( \frac{1}{2(k-l_2)} + \frac{1}{2(l_2-l_1)} \right) \end{aligned}$$

Without using the theorem one can show that there exists a unique solution to the system of two equations which comes from taking ratios, rearranging, and replacing log differences in prices by two variables  $\phi_1$  and  $\phi_2$ . Plotting functions which result from solving for one variable while varying the other variable for each equation shows that there exists a unique solution to  $\phi_1$  and  $\phi_2$ . Substituting them into the f.o.c.'s gives the unique equilibrium price vector.

The f.o.c.'s are solved numerically using Newton's method for all possible locations with a grid of  $10^{-2}$  which shows that, comparing continuation profits of firm 3, whatever locations firms 1 and 2 choose firm 3 achieves maximal profits when locating at the center of the arc between firms 1 and 2 of maximal length. Since this holds analogously for firms 1 and 2, firms choose equidistant location in the unique equilibrium of the two-stage game. At equidistant locations,  $p_1^* = p_2^* = p_3^* = c(1 + \frac{k^2}{9})$  solves the f.o.c.'s.  $\square$

**Proof of Proposition 6.** Note that for any locations  $l_1, l_2$  there exists a unique equilibrium of the second stage. For  $l_1 \neq l_2$ , both firms are active and, for  $l_1 < l_2$  first-order conditions of profit maximization reduce to

$$\begin{aligned} c(1 + \log p_2 - \log p_1) &= p_1 \\ c(\log l_2 - \log l_1 + 1 - \log p_2 + \log p_1) &= p_2 \end{aligned}$$

Taking ratios and replacing  $\log p_2 - \log p_1$  by  $\phi$  gives  $\frac{\log l_2 - \log l_1 + 1 - \phi}{1 + \phi} = e^\phi$ . The quality difference is denoted by  $\varphi = \log l_2 - \log l_1$ . Note that continuation profits depend only on the logarithmic quality difference  $\log l_2 - \log l_1$ . Implicit differentiation yields

$$\frac{d\phi}{d\varphi} = \frac{1}{e^\phi(2 + \phi) + 1}.$$

Hence,  $\frac{d\phi}{d\varphi} > 0$  for  $\phi > 0$ , an inequality which holds in any equilibrium. It can be shown that continuation profits  $\frac{d\tilde{\pi}_2}{d\varphi} > 0$  which says that the high-quality firm prefers maximal differentiation. This does not hold for firm 1.

$$\frac{d\tilde{\pi}_1}{d\varphi} = \frac{d\tilde{\pi}_1}{d\phi} \frac{d\phi}{d\varphi} = 0 \iff (\phi^3 + 2\phi^2 - \phi - 2)e^\phi + 2 = 0$$

The function of the left-hand side of the equation has a unique zero in  $\phi$  for  $\phi > 0$  which is numerically determined as 0.83236. Since  $\varphi = e^\phi(1 + \phi) - (1 - \phi)$  one obtains 4.04446.

To summarize, firm 2 chooses  $l_2 = \bar{l}$  and firm 1 chooses  $l_1 = \underline{l}$  if  $\log \bar{l} - \log \underline{l} \leq 4.04446$  and  $\log l_1 = \log l_2 - 4.04446$  otherwise.  $\square$

**Proof of Proposition 7.** (using the notation introduced in the proof of Proposition 6) For  $l_1 < l_2$ , first-order conditions of profit maximization reduce to

$$\begin{aligned} c(1 + \beta + \log p_2 - \log p_1) &= (1 + \beta)p_1, \\ c \left( \frac{(\log l_2 - \log l_1)^{1+\beta}}{(\log p_2 - \log p_1)^\beta} + 1 + \beta - \log p_2 + \log p_1 \right) &= (1 + \beta)p_2. \end{aligned}$$

If both firms have a positive market share,  $\phi > 0$  because firm 1 produces the lower quality. A solution to the first-order conditions satisfies

$$\begin{aligned} \frac{(\varphi^{1+\beta}/\phi^\beta) + 1 + \beta - \phi}{1 + \beta + \phi} &= e^\phi \\ \left( e^\phi (1 + \beta + \phi) - (1 + \beta - \phi) \right) \phi^\beta &= \varphi^{1+\beta}. \end{aligned} \quad (5)$$

Consider now endogenous product choice. One obtains by differentiation of (6) that

$$\frac{d\phi}{d\varphi} = \frac{(1 + \beta) \left( \phi^\beta \left( e^\phi (1 + \beta + \phi) - (1 + \beta - \phi) \right) \right)^{\frac{\beta}{1+\beta}}}{\beta \phi^{\beta-1} \left( e^\phi (1 + \beta + \phi) - (1 + \beta - \phi) \right) + \phi^\beta \left( e^\phi (2 + \beta + \phi) + 1 \right)}.$$

This expression is positive for  $\phi > 0$ . Continuation profits can be expressed in terms of  $\varphi$ . It can be shown that continuation profits  $\frac{d\pi_2}{d\varphi} > 0$  which implies that the high-quality firm prefers maximal differentiation. In order to obtain prices and qualities in subgame perfect equilibrium of the two-stage game I look at the first-order condition of profit-maximization of firm 1 which reduces to

$$e^{\phi(\varphi)} \left( (2 - \phi(\varphi))(1 + \beta + \phi(\varphi))^2 - 2\phi(\varphi)(1 + \beta + \phi(\varphi)) \right) - 2(1 + \beta)^2 = 0$$

The function on the left-hand-side has a unique zero for  $\phi > 0$  for all  $\beta > 0$ . Any equilibrium with differentiated products  $l_2 > l_1$  has to satisfy  $\phi > 0$ . Since the low-quality firm always makes lower profits than the high-quality firm and profits only depend upon the quality ratio a solution to the restricted game where  $l_2 > l_1$  (firm 2 cannot undercut firm 1 in quality) is also a solution to the unrestricted game with strategic variables  $l_1, l_2$  at stage 1.  $\square$



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