PREEMINENCE AND SUSTAINABILITY IN BANKRUPTCY PROBLEMS*

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ABSTRACT

This paper focuses on two new properties for bankruptcy rules: *preeminence* and *sustainability.* They pertain to situations when the claims of some agents are "much larger" than the claims of other agents. They differ in the way they recommend agents with "small" claims to be treated. Preeminence requires that the agents with "very small" claims should not be alloted anything. Sustainability takes the opposite side: claims "small enough" should be fully honored. Our main results are that the *constrained equal-loss rule* is the only rule that satisfies *equal treatment of equals, composition* and *preeminence*, and that the *constrained equal-award rule* is the only rule that satisfies *equal treatment of equals, path independence* and *sustainability.*

Key words: Bankruptcy problems, constrained equal loss, preeminence and sustainability.

1 INTRODUCTION

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? In this paper we deal with such *bankruptcy problems,* and search for well-behaved methods, or *rules*, of associating with each bankruptcy problem a division of the liquidation value of the firm.

This is a major practical issue and, as such, it has a long history. The best-known rule is the *proportional rule,* which recomends awards to be proportional to the claims. Some other rules were generated trying to acommodate examples appearing in the literature. This is the case of the *talmudic rule* that generates the numbers proposed in the Talmud as solutions to some examples of bankruptcy problems [see Rabinovitch (1973), O'Neill (1982), Aumann and Maschler (1985)].

Modern economic analysis has addressed this problem from two different perspectives. One is the game theoretical, in which bankruptcy problems are formulated either as TU coalitional games, or as bargaining problems, and rules are derived from solutions to coalitional games and from bargaining solutions, respectively [see O'Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1988), Dagan and Volij (1993)]. Most of the recent literature follows the axiomatic framework, in which appealing properties of rules are formulated, rules are compared on the basis of these properties, and the existence of rules satisfying various combinations of these properties together is investigated [see O'Neill (1982), Young (1987), Chun (1988a), Dagan (1996), Herrero, Maschler and Villar (1998)]. The reader is referred to Thomson (1995) for a survey of this literature. Here we follow this approach.

The *proportional rule* satisfies a number of appealing properties, and when compared with other rules, it has much to recommend itself. The idea of *equality* underlies another well-known rule: the *constrained equal-award rule*. It makes awards as equal as possible to all creditors, subject to the condition that no creditor receives more than her claim. A dual formulation of equality, focusing on the losses creditors incur, as opposed to what they receive, underlies the *constrained equal-loss rule*. It proposes losses as equal as possible for all creditors, subject to the condition that no creditor ends up with a negative award. Although this is a well known rule,[1] it has not yet

[1] Aumann & Maschler (1985) trace back this notion to Maimonides. In the context of

been characterized in this context.

The proportional, constrained equal-award and constrained equal-loss rules satisfy three basic properties. The first one, *equal treatment of equals,* requires identical agents to be treated identically.

To motivate the next two properties, consider a situation when the actual worth of a bankrupt firm is being evaluated, and the arbitrator proposes a tentative division based on a forecast of the firm's net worth. After this is done, the firm's assets are reevaluated and found to be worth more than originally thought. Then, two options are open: either the first division is cancelled altogether and the rule is applied to the actual problem, or the rule is applied to the problem of dividing the incremental value of the firm after adjusting the claims down by the amounts asigned in the first division. *Composition* requires the recommendation given by the rule to be independent of the chosen option. Assume now, on the contrary, that after solving a problem by using some forecast on the firm's net worth, it turns out that the actual worth of the firm falls short of what was expected. Then, two options are open: either the first division is cancelled and the actual problem is solved, or the initial claims are adjusted down to the awards initially assigned, and this revised problem is solved. *Path independence* requires both options to give the same answer.

We focus in this paper on two new properties, *preeminence* and *sustainability.* They pertain to situations when the claims of some agents are, informally speaking, "much larger" than the claims of other agents. They differ in the way they recommend agents with "small" claims to be treated. Preeminence requires that the agents with "small" claims should not be alloted anything. Sustainability takes the opposite side: claims "small enough" should be fully honored. Our main results are that the *constrained equal-loss rule* is the only rule that satisfies *equal treatment of equals, composition* and *preeminence*, and that the *constrained equal-award rule* is the only rule that satisfies *equal treatment of equals, path independence* and *sustainability.*

Suppose that, when solving a problem, we start by temporarily awarding every agent his claim. Since it is not feasible, now we apply a particular rule to allocate losses. By this procedure we obtain a new rule, *the dual rule* of the initially used. When a rule coincides with its dual, it is called *self-dual.* The proportional rule is self-dual, and the constrained equal-loss and the constrained equal awards rules are dual from each other. This duality under-

taxation, this rule corresponds to the *leveling tax.*

lies the properties they fulfil, and help to better understand their different behavior.

The paper is organized as follows. In Section 2, we introduce the model, the proportional, constrained equal-award and constrained equal-loss rules, as well as several properties of rules. In Section 3 we present the characterizations of the constrained equal-loss and the constrained equal-award rules, in terms of the properties presented in Section 2. In Section 4 we conclude by introducing the duality relations and comparing previous rules and properties from this perspective.

2 THE MODEL

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. A **bankruptcy problem** is a pair $b = (E, c)$, where $E \in \mathbb{R}_+$ represents the **net worth** of a firm, and $c \in \mathbb{R}^n_+$ is a **vector of claims:** c_i represents the **claim** of creditor $i \in N$. Moreover, $\sum c_i > E$. We denote by $\mathbb{B}$ the family of all such problems.

The model describes the situation faced by a bankruptcy court. An alternative interpretation of the model is the division of an estate E among a group of heirs when the estate is insufficient to cover all the bequeathed amounts, c_i, $i \in N$. The same model can also be interpreted as a formalization of a class of tax assessment problems: there, the cost E of a project has to be divided among a group of taxpayers, where c_i stands for agent $i's$ income.

A **rule** is a mapping F that associates with every $b = (E, c) \in \mathbb{B}$ a unique point $F(b) \in \mathbb{R}^n$ such that: (i) For all $i \in N$, $0 \leq F_i(b) \leq c_i$, and (ii) $\sum F_i(b) = E$. The point $F(b)$ is to be interpreted as a desirable way of dividing E. Requirement (i) is that each creditor receive an award that is non-negative and bounded above by his claim. Requirement (ii) is that the entire net worth of the firm be allocated.

Next we introduce three well-known rules. The **proportional rule** is the most widely used rule. It makes awards proportional to claims.[2] Formally:

Proportional rule, P: For all $b = (E, \mathbf{c}) \in \mathbb{B}$ and all $i \in N$, $P_i(b) = \lambda c_i$, where $\lambda \geq 0$ solves $\sum \lambda \;\, c_i = E$.

[2]Hence, provided that $\sum_i \; c_i \neq 0$, it equalizes the ratios between claims and awards.

The **constrained equal-award rule** makes awards as equal as possible, subject to the condition that no agent receives more than his claim. This rule has been advocated by many authors and it was accepted as law by rabbinical authorities. Formally:

Constrained equal-award rule, CEA: For all $b = (E, \mathbf{c}) \in \mathbb{B}$ and all $i \in N$, $CEA_i(b) = \min\{c_i, \lambda\}$, where λ solves $\sum \min\{c_i, \lambda\} = E$.

The **constrained equal-loss rule** makes awards such that losses are as close to equal as possible, subject to the condition that no creditor ends up with a negative award.[3] Formally:

Constrained equal-loss rule, CEL: For all $b = (E, c) \in \mathbb{B}$ and all $i \in N$, $CEL_i(b) = \max\{0, c_i - \lambda\}$, where λ solves $\sum \max\{0, c_i - \lambda\} = E$.

Next, we formulate several properties for rules. The first one is a basic equity requirement: agents with identical claims should be treated identically. Hence, we exclude differentiating between agents on the basis of their names, gender, religion, political ideas, etc. Formally:

Equal treatment of equals: For all $b = (E, c) \in \mathbb{B}$ and all $i, j \in N$, if $c_i = c_j$, then $F_i(b) = F_j(b)$.

In order to motivate the next property, consider a bankrupt firm with two types of assets (buildings and machinery, say). Suppose that these assets are sold to different buyers on different dates. Two options are open concerning the way awards are calculated. Either we wait until both assets are sold, and then distribute the total net worth of the firm, or we distribute it in two steps, as follows: First, we distribute the amount obtained after selling the first asset. In a second step, we distribute the amount obtained from the second asset, after adjusting down the creditors' claims by the amount they just obtained. **Composition** states that the rule be invariant with respect to the chosen option. This precludes any disagreement on the procedure by which the net worth of the firm should be distributed. Formally:

[3] The principle underlying this rule, the *equal-loss* principle, has been applied to other distribution problems, such as cost-sharing, taxation or axiomatic bargaining [see for instance Young (1987), (1988), Chun (1988b), Herrero and Marco (1993)].

Composition (Young, 1988): For all $b = (E, c) \in \mathbb{B}$ and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$, if $b_1 = (E_1, c)$ and $b_2 = [E_2, c - F(b_1)]$, then $F(b) = F(b_1) + F(b_2)$.

Consider now the case in which, after solving a problem, it turns out that the actual worth of the firm falls short of what was expected. Consider a group of creditors with claims c, and a worth firm's forecast E. If we solve the problem (E,c), the vector of awards is z. Assume now that the net worth turns out to be smaller than expected, $\widetilde{E} < E$. **Path independence** requires that the solution of the problem $(\widetilde{E}, c)$ be the same as that of the problem (E, z), namely, if we adjust claims down to z, the final awards do not change. Formally:

Path Independence (Moulin, 1987): For any problem $b = (E, c) \in \mathbb{B}$, and any $E' > E$, we have $F(b) = F(E, F(E', c))$

Remark. It is easy to see that if a rule satisfies either *composition* or *path independence* it is *monotonic with respect to the net worth*, that is, for all $b = (E, c)$ and $b' = (E', c) \in \mathbb{B}$, and all $i \in N$, if $E \leq E'$ then $F_i(b) \leq F_i(b')$. Furthermore, it is *continuous with respect to the net worth* (see Herrero and Villar; 1997, Lemma 2).

In some cases, when the claims of some agents are, informally speaking, "much larger" than the claims of other agents, it may be sensible to give full priority to agents with those "very large" claims, or, in other words, not to allocate anything to agents with "very small" claims. As an example of the previous situation, suppose that, in the taxation interpretation of the model, some agent $i's$ income is greater than the sum of the cost of the project plus some other agent $j's$ income. Then, it may be fair to require that agent j should not contribute to the cost of the project.

As another example, consider a group of agents suffering from the same disease, and suppose that a fixed amount of money per year is needed in order to keep each of them healthy. Suppose, furthermore, that agents' claims are made in terms of the amount of money they need in order to be healthy for their expected remaining lifetime. Thus, the young agents' claims are greater than the old agents' claims. Consider two agents such that, even if we devoted the total amount available to the youngest one, his expected lifetime would still be below the actual age achieved by the oldest agent. In this case, it may be reasonable to require that the youngest agent should be given absolute priority, namely, the oldest one should not be alloted any money.

We next propose a specific formulation of the previous idea.

Let $b = (E, c) \in \mathbb{B}$ and let $j \in N$. **Creditor i's claim is dominated in b** if there is some $i \in N$ such that $c_i \geq c_j + E$. The next property, **preeminence,** states that agents whose claims are dominated should not be allocated anything. Formally:

Preeminence: For all $b = (E, \mathbf{c}) \in \mathbb{B}$ and all $j \in N$, if $j's$ claim is dominated in b, then $F_j(b) = 0$.

There are, nonetheless, situations where, having some agents with claims "much larger" than the claims of other agents, it would be more reasonable to take the opposite side, namely to fully honor "small" claims. As an example, imagine that a savings bank goes bankrupt, and there are two types of creditors: households and firms. The claims of households are usually "much smaller" than the claims of the firms, but represent a higher share of their wealth. In this situation, it may be fair to give priority to households, fulfilling their claims in full.

As another example, imagine the case of a man who dies, and his estate is insufficient to honor the bequeathed amounts in his will. Then , if one of his heirs has a "very small" claim, he may argue the following way: I claim c_i. If we all are alloted c_i, there is still money left. So, my claim can be considered as "enforceable".

Next, we propose a specific formulation of previous idea.

Let $b = (E, c) \in \mathbb{B}$ and $i \in N$. Now, consider the problem $b^i = (E, c^i(b))$, where, for all $j \in N$, $c^i_j(b) = \min\{c_i, c_j\}$. That is, we truncate all claims by agent $i's$ claim. **Agent i's claim is sustainable in b** if $\sum_j c^i_j \leq E$.

Thus, *an agent's claim is sustainable* in a problem if, by truncating claims by c_i, the problem becomes feasible. The following property, **sustainability,** states that *sustainable* claims should be fully honored.

Sustainability: For all $b = (E, \mathbf{c}) \in \mathbb{B}$ and all $i \in N$, if c_i is sustainable in b, then $F_i(b) = c_i$.

3 RESULTS

In this section we present characterizations of the *constrained equal-loss* and *constrained equal-award* rules in terms of the properties introduced in Section 2.

Given $b = (E, c) \in \mathbb{B}$, let $\delta^1(c) = \max_{j \in N} c_j$, $N_1(c) = \left\{i \in N \mid c_i = \delta^1(c)\right\}$, and $n_1(c) = |N_1(c)|$. Suppose that for all $j \in N \backslash N_1(c)$, $\delta^1(c) \geq E + c_j$. Then *preeminence* says that, for all $j \in N \backslash N_1(c)$, $F_j(b) = 0$.

Lemma. *Let F be a rule satisfying equal treatment of equals, composition, and preeminence, and let $b = (E, c) \in \mathbb{B}$ be such that for all $j \in N \backslash N_1(c)$ $\delta^1(c) \geq c_j + \frac{E}{n_1(c)}$. Then, for all $i \in N \backslash N_1(c)$, $F_i(b) = 0$.*

Proof: ***Step1.*** Let $\delta^2(c) = \max_{j \in N \backslash N_1(c)} c_j$. Obviously, if for all $j \in N \backslash N_1(c)$, $\delta^1(c) \geq c_j + \frac{E}{n_1(c)}$, then $\delta^1(c) \geq \delta^2(c) + \frac{E}{n_1(c)}$.

Now, let $c^1 = c$, $E_1 = E$, and $b_1 = (E_1, c^1)$. Also, let $E_2 = \frac{1}{n_1(c)} E_1$ and $b_2 = (E_2, c^1)$. Since $\delta^1(c^1) \geq \delta^2(c^1) + \frac{E_1}{n_1(c^1)}$, by *preeminence* and *equal treatment of equals*, for all $i \in N_1(c^1)$, we have $F_i(b_2) = \frac{1}{n_1(c^1)} E_2$.

Let $\widetilde{E}_2 = E_2 - E_1$, $c^2 = c^1 - F(b_2)$, and $\widetilde{b}_2 = (\widetilde{E}_2, c^2)$. By *composition,* $F(b_1) = F(b_2) + F(\widetilde{b}_2)$.

Step 2. Note that $N_1(c^2) = N_1(c^1)$ and $\delta^1(c^2) = \delta^1(c^1) - \frac{1}{n_1(c^1)} E_1 \geq \delta^2(c^1) + \frac{1}{n_1(c^1)} E_1 - \frac{1}{n_1(c^1)} E_2 = \delta^2(c^1) + \frac{1}{n_1(c^1)} \widetilde{E}_2$.

Let $E_3 = \frac{1}{n_1(c^1)} \widetilde{E}_2$ and $b_3 = (E_3, c^2)$. By *preminence* and *equal treatment of equals*, for all $i \in N_1(c^2)$, we have $F_i(b_3) = \frac{1}{n_1(c^1)} E_3$.

Let $\widetilde{E}_3 = \widetilde{E}_2 - E_3$, $c^3 = c^2 - F(b_3)$, and $\widetilde{b}_3 = (\widetilde{E}_3, c^3)$. By *composition,* $F(\widetilde{b}_2) = F(b_3) + F(\widetilde{b}_3)$.

Step k. Suppose that $\widetilde{b}_k = (\widetilde{E}_k, c_k)$ has been defined. Note that $N_1(c^k) = N_1(c^{k-1})$ and $\delta^1(c^k) = \delta^1(c^{k-1}) - \frac{1}{n_1(c^1)} E_k \geq \delta^2(c^1) + \frac{1}{n_1(c^1)} E_{k-1} - \frac{1}{n_1(c^1)} E_k = \delta^2(c^1) + \frac{1}{n_1(c^1)} \widetilde{E}_k$.

Let $E_{k+1} = \frac{1}{n_1(c)} \widetilde{E}_k$ and $b_{k+1} = (E_{k+1}, c^k)$. By *preeminence* and *equal treatment of equals*, for all $i \in N_1(c^k)$, we have $F_i(b_{k+1}) = \frac{1}{n_1(c^1)} E_{k+1}$.

Let $\widetilde{E}_{k+1} = \widetilde{E}_k - E_{k+1}$, $c^{k+1} = c^k - F(b_{k+1})$, and $\widetilde{b}_{k+1} = (\widetilde{E}_{k+1}, c^{k+1})$. By *composition,* $F(\widetilde{b}_k) = F(b_{k+1}) + F(\widetilde{b}_{k+1})$.

Observe that $E_{k+1} = \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^{k-1} E_2$. Consequently, $\lim_{k\to\infty}(E_2 + \cdots + E_k) = E$. Therefore, by *composition*, and since, for all $i \in N_1(c^1)$, F_i is *continuous with respect to E*,

$F_i(b) = \lim_{k\to\infty}[F_i(b_2) + \cdots + F_i(b_k)] =$

$= \lim_{k\to\infty}\left[1 + \frac{n_1(c^1)-1}{n_1(c^1)} + \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^2 + \cdots + \left(\frac{n_1(c^1)-1}{n_1(c^1)}\right)^{k-1}\right] = E_2$

$= \frac{E}{n_1(c)}$. □

The number $\delta^1(c) - \frac{E}{n_1(c)}$ is merely the average loss experienced by creditors in $N_1(c)$. Hence previous lemma can be interpreted as stating that *composition* and *equal treatment of equals* extend the power of *preeminence* to the case in which the average loss experienced by creditors in $N_1(c)$ exceeds the claims of any other creditor.

We now obtain the following result:

Theorem 1. *The constrained equal-loss rule is the only rule satisfying equal treatment of equals, composition, and preeminence.*

Proof: Trivially, the *constrained equal-loss rule* satisfies *equal treatment of equals* and *preeminence*. As for *composition*, see Herrero and Villar (1997, Proposition 1 (b)).

Let us prove the converse implication.

Let $\delta^1(c) = \max_{j\in N} c_j$, $N_1(c) = \left\{i \in N \mid c_i = \delta^1(c)\right\}$, and $n_1(c) = |N_1(c)|$. Similarly, let $\delta^2(c) = \max_{j\in N\setminus N_1} c_j$, $N_2(c) = \left\{i \in N \mid c_i = \delta^2(c)\right\}$, and $n_2(c) = |N_2(c)|$, and so forth.

(i) Let $0 \leq E \leq n_1(c)[\delta^1(c) - \delta^2(c)]$. Then, for all $j \in N\setminus N_1(c)$, we have $\delta^1(c) - \frac{E}{n_1(c)} \geq c_j$. By Lemma 1, for all $i \in N_1(c)$, we have $F_i(E, c) = \frac{E}{n_1(c)}$. Therefore, $F(b) = CEL(b)$.

(ii) Let $n_1(c)[\delta^1(c) - \delta^2(c)] < E \leq n_1(c)\delta^1(c) - n_2(c)\delta^2(c) - [n_1(c) + n_2(c)]\delta^3(c)$.

Let $E_1 = n_1(c)[\delta^1(c) - \delta^2(c)]$, $b_1 = (E_1, c)$, and $b_2 = [c - F(b_1), E - E_1]$. By *composition*, $F(b) = F(b_1) + F(b_2)$.

By (i), $F(b_1) = CEL(b_1)$, namely, for all $i \in N_1(c)$, $F_i(b_1) = \frac{E_1}{n_1(c)}$, and otherwise $F_i(b_1) = 0$. Consequently, for all $i \in N_1(c)$, $c_i - F_i(b_1) = \delta^1(c) - \frac{E_1}{n_1(c)} = \delta^2(c)$, and otherwise, $c_i - F_i(b_1) = c_i$.

Let $c' = c - F(b_1)$. That is, for all $i \in N_1(c)$, $\delta^1(c') = \delta^1(c) - F_i(b_1)$, for all $j \in N_2(c)$, $\delta^2(c') = \delta^2(c) - F_j(b_1)$, etc. By construction, $\delta^1(c') = \delta^2(c') > \delta^3(c') \geq \cdots \geq \delta^n(c')$.

Moreover, for all $j \in N \backslash [N_1(c') \cup N_2(c')]$,

$$c_j - F_j(b_1) \leq \delta^1(c') - \frac{E - E_1}{n_1(c) + n_2(c)}.$$

Again, by (i), for all $j \in N \backslash [N_1(c') \cup N_2(c')]$, we have $F_j(b_2) = 0$, and for all $i \in N_1(c') \cup N_2(c')$, we have $F_i(b_2) = \frac{E-E_1}{n_1(c)+n_2(c)}$ namely, for all $i \in N_1(c')$, $F_i(b) = \frac{E_1}{n_1(c)} + \frac{E-E_1}{n_1(c)+n_2(c)}$; for all $i \in N_2(c')$, $F_i(b) = \frac{E-E_1}{n_1(c)+n_2(c)}$, and for all other $i \in N$, $F_i(b) = 0$. Consequently, $F(b) = CEL(b)$.

We repeat the previous procedure until all possible values of the estate smaller than or equal to $\sum_i c_i$ are covered. □

To show that the three properties in Theorem 1 are independent, consider the following examples. In each case we mention the property that is not satisfied:

(i) *Equal treatment of equals.* Choose an agent $i \in N$. Let F be defined by

$$F_i(E, c) = \begin{cases} 0 & \text{if } i \notin N_1(c), \\ CEL_i(E, c) & \text{if } i \in N_1(c). \end{cases}$$

and, for all $j \in N \backslash \{i\}$, $F_j(E, c) = CEL_j(E - F_i(E, c), c)$.

(ii) *Composition*: Let F be defined by

$$F_i(E, c) = \begin{cases} CEL_i[\min\{E, \sum_{i \in N_1(c)} c_i\},\ (c_i, 0)_{i \in N_1(c)}] & \text{if } i \in N_1(c), \\ CEL_j[\max\{E - \sum_{i \in N_1(c)} c_i,\ 0\},\ (0, c_j)_{j \notin N_1(c)}] & \text{otherwise.} \end{cases}$$

(iii) *Preeminence.* The proportional rule.

As for the characterization of the *constrained equal-award rule,* we have the following:

Theorem 2. *The constrained equal-award rule is the only rule satisfying equal treatment of equals, path independence, and sustainability.*

Proof: Obviously, the *constrained equal award rule* satisfies the three properties.

Conversely, let F be a rule that satisfies the three properties. We will show that $F = CEA$.

(i) Let $b = (E, c) \in \mathbb{B}$ be such that $\sum c_i - [\delta^1(c) - \delta^2(c)] \leq E \leq \sum c_i$. By *sustainability*, for all $i \in N$ such that $i \notin N_1(c)$, $F_i(b) = c_i$. By *equal treatment of equals*, $F(b) = CEA(b)$.

(ii) Let $E_1 = \sum c_i - [\delta^1(c) - \delta^2(c)]$ and $c^1 = F(E_1, c)$. Note that for all $i \in N \backslash N_1(c)$, $c_i^1 = c_i$, and $\delta^1(c^1) = \delta^1(c) - \frac{[\delta^1(c) - \delta^2(c)]}{n_1(c)}$. Furthermore, $N_1(c) = N_1(c^1)$. Let $C^1 = \sum_N c_i^1$ and $b = (E, c) \in \mathbb{B}$ such that $C^1 - [\delta^1(c^1) - \delta^2(c)] \leq E \leq E_1$.

By *path independence*, $F(b) = F(E, c^1)$. By *sustainability*, for all $i \in N \backslash N_1(c)$, $F_i(b) = c_i$. Also, by *equal treatment of equals*, $F(b) = CEA(b)$.

Repeating this procedure, we may reduce the maximum claims, maintaining the others at their original value up to the point at which $\delta^1(c^k) = \delta^2(c)$, and obtain that $F(b) = CEA(b)$. We then repeat the procedure until all possible values of E have been covered. □

To show that the three properties in Theorem 2 are independent, consider the following examples. In each case we mention the property that is not satisfied:

(i) *Equal treatment of equals.* Let $i \in N$ and, for all $c \in \mathbb{R}^n_+$, let $c_{-i} = (c_1, c_2, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n)$. Let F be defined by:

$$F_i(E, c) = \begin{cases} c_i & \text{if } c_i \text{ is sustainable,} \\ 0 & \text{otherwise.} \end{cases}$$

and for all $j \in N \backslash \{i\}$, $F_j(E, c) = CEL_j[E - F_i(E, c), c_{-i}]$.

(ii) *Path independence*: Let F be defined by:

$$F_i(E, c) = \begin{cases} CEL_i\left[\min\{E, \sum_{i \notin N_1(c)} c_i\},\ (c_i, 0)_{i \notin N_1(c)}\right] & \text{if } i \notin N_1(c), \\ [E - \sum_{i \notin N_1(\mathbf{c})} F_i(E, c)]/n_1(c) & \text{otherwise.} \end{cases}$$

(iii) *Sustainability.* The proportional rule.

4 FINAL REMARKS

A requirement closely related to *sustainability* is that any claim exceeding the net worth of the firm should be ignored. This property, introduced by Dagan (1996) under the name of *independence of irrelevant claims,* together with *equal treatment of equals* and *composition* characterizes the constrained equal-award rule (cf. Dagan, 1996).

The *constrained equal-award* and the *constrained equal-loss* rules make recommendations from an opposite viewpoint. Whereas the *constrained* *equal-award rule* fully satisfies small claims, the *constrained equal-loss* rule does not allocate anything to those agents with small claims. The relationship between these two rules is formally analyzed by introducing the idea of **duality.**

Given a rule F, we define **its dual**, F^*, as follows: For all $b = (E, c) \in \mathbb{B}$, $F^*(E, c) = c - F(\sum c_i - E, c)$. Notice that F^* is well defined, since, for all $b = (E, c) \in \mathbb{B}$, the problem $(\sum c_i - E, c) \in \mathbb{B}$, that is, $\sum c_i - E \in \mathbb{R}_+$, and $\sum c_i > (\sum c_i - E)$. The dual operator is idempotent, namely, $(F^*)^* = F$. Rules F and F^* are related in a simple way: F^* splits E units of gains in the same way as F splits E units of losses.

It is immediate that $CEL = CEA^*$, whereas $P^* = P$. If a rule coincides with its dual, it is **self-dual** (cf. Aumann and Maschler, 1985).

Young (1988), characterizes the *proportional rule* as the only rule that satisfies *equal treatment of equals, composition* and *self-duality.*

Similarly, by abuse of language, a property is **self-dual** if whenever it is satisfied by a rule, it is also satisfied by its dual. *Equal treatment of equals* and *self-duality* are examples of self-dual properties.[4] Given two properties, $\mathcal{P}$ and $\mathcal{Q}$, they are **dual properties** if whenever a rule satisfies $\mathcal{P}$, its dual rule satisfies $\mathcal{Q}$. Again, the duality operator on properties is idempotent.

It is easy to see that *composition* and *path independence* are dual properties. So are *preeminence* and *sustainability.* Consequently, we may view Theorems 1 and 2 as *dual* results. An alternative characterization of the *proportional rule* by means of the duality relations can be immediately derived from Young's result, by substituting *composition* by *path independence.*

Hence, the three rules *proportional, constrained equal-award* and *constrained equal-loss* share the properties of *equal treatment of equals, composition,* and *path independence*,[5] and differ with respect to *self-duality, sustainability,* and *preeminence.* Whereas *sustainability* requires that small claims should be fully honored, *preeminence* requires that those agents whose claims exceed the worth of the firm should be given priority. *Self-duality* can be regarded as a *neutrality* requirement.

[4] Other examples of self-dual properties are *continuity, net worth monotonicity* and *consistency.* For formal definitions of all this properties, see Thomson (1995a,b).

[5] P, CEA and CEL share many additional properties. These three rules are the only rules that satisfy *equal treatment of equals, composition, path independence, consistency and scale invariance* [cf. Moulin (1997)]

These results shed some light on the type of problems for which each rule is better adapted. The constrained equal-loss rule may be particularly sensible for taxation problems, or for problems where claims represent unalienable rights. On the other hand, the constrained equal awards rule may be more appropriate for problems where claims represent maximal aspirations (as in inheritance). The proportional solution may be particularly appealing for bankruptcy problems in which the creditors are shareholders, and the claims correspond to their shares.

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