MODELS À LA LANCASTER AND À LA HOTELLING: WHEN THEY ARE THE SAME*

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A B S T R A C T

Consumer behavior in differentiated product markets can be specified following the Lancastrian characteristics approach or following Hotelling's approach of spatial competition. In the case of unit demand I present a class of models of (heterogeneous) consumer behavior which can be written as models à la Lancaster as well as models à la Hotelling.

Keywords: Product Differentiation, Characteristics Approach, Spatial Competition

JEL-Classification: D11.

1 Introduction

Recently, the theory of product differentiation has raised much interest as a fruitful branch of economics to study situations of imperfect competition. In this note, I relate two different approaches of the theory of product differentiation to each other.[1]

Following Gorman (1980), which was written in 1956, and Lancaster (1966, 1979) consumers only derive utility from the (desirable) characteristics which are embodied in the consumption goods and a consumer is characterized by his preferences defined on the characteristics space. Lancaster (1979) provides a long discussion of noncombinability of goods in this framework.

An alternative view is offered by the work on spatial competition which was initiated by Hotelling (1929). An overview is provided e.g. by Gabszewicz and Thisse (1992). In models of spatial competition à la Hotelling consumers are characterized by their locations or ideal points on the line. Also goods are characterized by their location on the line. Consumers buy one unit of that good for which the price of the good plus transportation costs, which depends on the distance between good and consumer, is the lowest.

The first attempt to relate the two approaches has been made by Lancaster (1979). He compares the properties of a particular Lancaster model with the original Hotelling model, i.e. with linear transportation costs and unit demand. He observes some "striking" differences between the two approaches which, due to his specifications, is not surprising.

The second attempt has been made by Archibald and Eaton (1989). They define pseudo-metrics in the Lancastrian characteristics space which allows one to speak in a meaningful way of distances between goods. They point out several similarities between the two approaches. In their summary they write: "The problem of characterizing diverse preferences in an interesting and tractable manner in the two models remains, but is beyond the scope of this paper." In particular, they did not try to present the equivalent of a model à la Hotelling with convex transportation costs in the Lancastrian characteristics approach.

The objective of this note is to merge Hotelling and Lancaster models such that the heterogeneity of consumer tastes is well-understood in both approaches. In the literature (e.g. Friedman, 1983, pp. 81, Ireland, 1987, p. 18) it has been suggested that locations on the Hotelling line can be interpreted as ratios of two technologically related characteristics. In this note I develop a model for which this suggestion is correct. The model is similar to the Defender model which is used in the marketing literature (see Hauser, 1988).

I restrict the analysis to a two-dimensional characteristics space in which characteristics are seen as perfect substitutes. Consumers have unit demand. Section 2 contains properties of consumer behavior which I will impose, Section 3 the "equivalence" result. I impose a technological constraint on the two-dimensional characteristics space: goods have to be chosen

[1]Note that there are other approaches around, namely the representative consumer approach and the random utility approach. On this see Anderson, de Palma and Thisse (1992) and the references therein.

from an arc of a circle in the positive orthant. This restriction is essential because, as I show in Section 3, the loss in utility for each consumer due to the deviation of the chosen good from his most preferred position is independent of the orientation and one can directly rewrite the utility functions for the location model.
In Section 4 I establish the existence of equilibrium when firms compete in prices.

2 Lancaster and Hotelling: a Characterization

2.1 Models à la Lancaster

According to the characteristics approach consumers derive utility from a vector of characteristics which are objectively measurable and represented by some real numbers. I will consider two-dimensional models à la Lancaster with unit demand and characteristics as perfect substitutes. They fulfill the following requirements:

(L1) Goods are bundles of two characteristics; the set of goods is given by $\{(\gamma_n, \delta_n)_n\}$ where $\gamma_n, \delta_n \in \Re$.
(L2) Consumers buy one unit of the differentiated goods which gives the highest utility, i.e. $\max_n u^L(n, x_0)$ where the Hicksian composite commodity 0 is bought in budget exhausting quantity. The conditional utility function of one unit of good n is given by $u^L(n, x_0) = h(\gamma_n, \delta_n) + x_0$.
(L3) The function $h : C \subseteq \Re^2 \longrightarrow \Re_+$ is nondecreasing in γ_n and δ_n and $\forall(\gamma, \delta) \in C$ $\exists(\gamma', \delta') \in N_\epsilon(\gamma, \delta)$, $(\gamma', \delta') > (\gamma, \delta)$: $h(\gamma', \delta') > h(\gamma, \delta)$.
(L4) A consumer is characterized by his substitution rate θ between γ and δ. If $\theta \in (0, \infty)$ take any $\alpha > 0$, $\beta \equiv \frac{\alpha}{\theta}$, one has $h(\gamma, \delta) = h(\gamma + \beta, \delta - \alpha)$. For a consumer of type 0 one has $h(\gamma, \delta) = h(\gamma', \delta)$, $\gamma' \geq 0$, and for a consumer of type ∞ one has $h(\gamma, \delta) = h(\gamma, \delta')$, $\delta' \geq 0$.

With (L2) I impose unit demand: consumers evaluate the two characteristics embodied in one unit of a differentiated good and the amount of the composite commodity. Furthermore, preferences on $\Re^2 \times \Re_+$ are quasilinear. It is implicitly assumed that all goods entering the composite commodity do not share any of the characteristics of the goods in the differentiated market. (L3) says that characteristics are desirable and, for given x_0, preferences are locally nonsatiated. (L4) is a very special assumption saying that goods are perfect substitutes. The consumer's substitution rate is the absolute value of the slope of an indifference curve. A consumer with $\theta = 0$ does not derive any utility from characteristic γ whereas a consumer with $\theta = \infty$ does not derive any utility from characteristic δ. (L4) is very helpful because the distribution over substitution rates fully describes the heterogeneity of the consumers when all consumers buy in the differentiated market.

2.2 Models à la Hotelling

Hotelling (1929) observes that consumers cannot move costlessly from one location to another. Hotelling assumes that all consumers are identical apart from being located at different points in space. In particular, they have the same transportation cost function.

(H1) The conditional indirect utility function v^H (for type $\omega \in \Re$) of consuming one unit of good n at location $l_n \in \Re$ is $v^H(n, p_n) = r - t(|\omega - l_n|) - p_n$, where r is a positive constant and p_n the price of good n.
(H2) The transportation cost function $t : D \subseteq \Re_+ \to \Re_+$ is twice continuously differentiable and $t(0) = 0$,
(H3) $t'(0) \geq 0$, $t'(d) > 0$, $d \in D \setminus \{0\}$,
(H4) $t''(d) \geq 0$, $d \in D$.

(H1) to (H4) are standard. The transportation cost function is increasing in distance and convex.

3 The Equivalence Result

This section contains the main results of the paper (Propositions 1 and 2). I begin within the Lancastrian approach such that (L1) is satisfied. In the Lancastrian model with perfect substitutes consumers have been characterized by their substitution rate between the two characteristics. Alternatively, they are described by the angle ω in radians of the ray from the origin which is perpendicular to their indifference curves. One has $\theta = \cot \omega$ and $\omega \in [0, \frac{\pi}{2}]$. I will now specify a utility function which is shown to satisfy (L2) to (L4) and, under a restriction on the possible set of goods, (H1) to (H4). The description of a good (γ_n, δ_n) is rewritten in polar coordinates $(l_n; ||(\gamma_n, \delta_n)||)$ where l_n is the angle in radians. Consumers have unit demand and buy one unit of the good for which the conditional utility is maximal. For convenience, I exclude the possibility that consumers do not buy in the differentiated market (the analysis can be extended to cover this case). The conditional utility functions of a consumer of type ω, $\omega \in [\underline{\omega}, \bar{\omega}] \subset [0, \frac{\pi}{2}]$, is defined as

$$u^S(n, x_0) = ||(\gamma_n, \delta_n)|| \cos(|\omega - l_n|) + x_0.$$

Note that $||(\gamma_n, \delta_n)|| \cos(|\omega - l_n|)$is the scalar product of the vectors $(\omega; 1)$and $(l_n; ||\gamma_n, \delta_n)||)$, both in polar coordinates.

Lemma 1.
u^Ssatisfies (L2) to (L4).

Proof. Clearly, (L2) is satisfied. Since $\omega \in [0, \frac{\pi}{2}]$, $u^S(n', x_0) \geq u^S(n, x_0)$when $(\gamma_{n'}, \delta_{n'}) \geq (\gamma_n, \delta_n)$. Furthermore, $\exists(\gamma_{n'}, \delta_{n'}) \in N_\epsilon(\gamma_n, \delta_n) : u^S(n', x_0) > u^S(n, x_0)$. Hence, (L3) is satisfied. Since the scalar product is determined by the length of the orthogonal projection of (γ_n, δ_n)onto the ray with angle ω, indifference curves are straight lines and (L4) is satisfied. □

Not all goods are technologically feasible. Assume that characteristics can be combined in different quantities but that there is a trade-off between the two desirable characteristics. I impose the particular technological constraint that $||(\gamma, \delta)|| = 1$, i.e. the characteristics of each possible good are located on the unit circle with positive coordinates. A good $n$now is completely represented by a number $l_n \in [\underline{l}, \bar{l}] \subseteq [0, \pi/2]$. Other technological constraints than the unit circle will be discussed below.
Can one speak in a meaningful way of a distance between goods? Since each possible good has a unique characteristics ratio, the distance between good $j$and good $n$can be defined as $|l_n - l_j|$which measures how similar goods are in their characteristics ratios.

Lemma 2.
For $||\gamma_n, \delta_n|| = 1$, $n = 1, \ldots, N$, u^Ssatisfies (H1) to (H4).

Proof. The utility function from above can be incorporated into a spatial model where goods are located on the interval $[\underline{l}, \bar{l}] \subset [0, \pi/2]$. The conditional utility function is defined as $u^H(n, x_0) = 1 - t(|\omega - l_n|) + x_0$and the transportation cost function $t : \Re_+ \to \Re_+$takes values $t(|\omega - l_n|) = -\cos(|\omega - l_n|) + 1$. For $||(\gamma, \delta)|| = 1$, $u^S(n, x_0) = u^H(n, x_0)$. The indirect utility function corresponding to $u^H(n, x_0)$satisfies (H1). The transportation cost function from above has the following properties: $t(0) = 0$, $t'(0) = 0$, $t'(d) > 0$, $d \in (0, \pi/2]$, $t''(d) > 0$, $d \in [0, \frac{\pi}{2})$and $t''(\frac{\pi}{2}) = 0$. □

Consequently, one can represent the model as a two-dimensional Lancastrian model with a technological constraint and as a one-dimensional location model. A Hotelling model is called an *equivalent representation* of a Lancastrian model if they both describe the same consumer behavior and locations on the Hotelling line are characteristics ratios in the characteristics space (measured as angles in radians). This is what Friedman (1983) probably had in mind. Putting Lemmas 1 and 2 together gives the first result.

Proposition 1.
The consumer demand model à la Lancaster which satisfies (L1) to (L4) has an equivalent representation as a model à la Hotelling which satisfies (H1) to (H4) if the technological

constraint is $T \subseteq \{(\gamma, \delta) \geq 0 : ||(\gamma, \delta)|| = 1\}$.

To fully specify the population of consumers one has to make a distributional assumption on the location of the consumers on the interval (see the following section). I require that the set of consumers contains a non-empty, open interval. Consider a technological constraint which is any arc of the unit circle in the positive orthant. When consumers are distributed on $[\underline{\omega}, \bar{\omega}] \supseteq [\underline{l}, \bar{l}]$, the model is one of horizontal product differentiation. If one assumes that $[\underline{\omega}, \bar{\omega}] \cap [\underline{l}, \bar{l}] = \emptyset$one has a model of vertical product differentiation. Remark that the literal interpretation of the latter specification in the approach of spatial competition may seem unsatisfactory (because it means that shops cannot be located between consumers) whereas thinking in terms of the Lancastrian characteristics approach leads to a more satisfactory interpretation.[2]

If a characteristic is a “bad” ((L3) violated) for some or all consumers the analysis is easily modified. Again the heterogeneity of consumers$'tastesandthepotentialofproductdifferentiationarenotallc$
$thereexists$e $\in \Re$with $[\underline{l}, \bar{l}] \cup [\underline{\omega}, \bar{\omega}] \subset [e, e + \pi/2]$.

So far units for $\gamma$and δhave been taken as given. The description of goods, the shape of the technological constraint, and the shape of the indifference curves depend on the scale in which the characteristics are measured. Does rescaling allow for an equivalent representation?[3] An *equivalent representation under rescaling* now interprets locations as characteristics ratios under a particular rescaling after which (L1)-(L4) hold. In a generalized version of Proposition 1 the restriction on $T$is imposed under the rescaling.

According to this more general version (L4) is not necessarily satisfied on the original scale. This allows one to construct examples in which $T$is not the arc of a unit circle and (L4) does not hold on the original scale. Take for example indifference curves of a Cobb-Douglas consumer (generated e.g. by $h(\gamma, \delta) = \gamma^\alpha \delta^\beta$, $\gamma, \delta \geq 0$). Under an exponential rescaling preferences represent perfect substitutes with substitution rate $\theta = \frac{\alpha}{\beta}$.

Now I only allow for a rescaling under which one does not leave the class of models à la Lancaster which satisfy (L1)-(L4). An (L4)-preserving rescaling is linear affine. As the result shows the technological constraint Thas to be a linear affine transformation of an arc of the unit circle.

Proposition 2.

The consumer demand model à la Lancaster which satisfies (L1) to (L4) has an equivalent representation under (L4)-preserving rescaling as a model à la Hotelling which satisfies (H1)

[2] In a location context vertical differentiation may arise from government regulation, see Gabszewicz and Thisse (1986).

[3] A *rescaling* is described by an increasing function $f : C \subseteq \Re^2 \rightarrow C' \subseteq \Re^2$ with $f(\gamma, \delta) = (f_1(\gamma), f_2(\delta))$ where C is the characteristics space under the original scale and C' after rescaling.

to (H4) only if the technological constraint $T$is an arc of the unit circle under some (L4)-preserving rescaling.

Sketch of Proof. The proposition also reads: if there does not exist an (L4)-preserving rescaling such that $T$is an arc of the unit circle there is no equivalent representation under (L4)-preserving rescaling. Assume for any scale such that (L1)-(L4) are satisfied, $T$is not an arc of the unit circle. Since (L1)-(L4) hold under linear rescaling, $T$is not the arc of any circle. Then there exist consumers $\omega_1, \omega_2$and technologically feasible goods $(l_1, ||(\gamma_1, \delta_1)||), (l_2, ||(\gamma_2, \delta_2)||) \in$ Tsuch that $h^{\omega_1}(\gamma_1, \delta_1) = h^{\omega_2}(\gamma_2, \delta_2)$and $|\omega_1 - l_1| \neq |\omega_2 - l_2|$where h^{ω}denotes the 'subutility$'from(L3)ofacons$ $\cot \omega$. In particular, if there are consumers $\omega \in [\underline{l}, \bar{l}]$then: $\exists \omega, l_1, l_2$: $h^{\omega}(\gamma_1, \delta_1) = h^{\omega}(\gamma_2, \delta_2)$and $|\omega - l_1| \neq |\omega - l_2|$. Consequently, there cannot exist a Hotelling model as an equivalent representation. □

For different technological constraints one can extend the class of spatial models by making the transportation cost dependent upon distance and *direction.* In such a framework consumers have different transportation cost functions in the spatial representation of the model whereas they may have the same functional form of the utility function in the Lancastrian representation. When preferences do not satisfy (L4) there may be other ways to link the two approaches which allow one to speak about heterogeneity in a meaningful way (for example the Cobb-Douglas case presented above).

4 Existence of Equilibrium

The problem of showing the existence of equilibrium arises from the fact that only quadratic transportation costs fit within the framework of Caplin and Nalebuff (1991).
By applying a result by Champsaur and Rochet (1988) I show the existence of an equilibrium where prices are chosen as pure strategies. I confine myself to the uniform distribution of consumers$'idealpoints.Theexistenceresultisrobusttosmalldeviationsfromtheuniformdistribution.$
$Thereisanexogenousnumberoffirms$n = 1, . . . , N.$Eachfirm$n$hasconstantmarginalcostsofproduction$c.$I$
$l_n \in [\underline{l}, \bar{l}]$and $l_n < l_{n+1}$, $n = 1, \ldots, N-1$.[4]
Let $G: [\underline{\omega}, \bar{\omega}] \to [0,1]$denote the cumulative distribution function of consumers$'locationswithdensity$g$which$

$$\alpha \equiv \max_{\omega \in [\underline{\omega}, \bar{\omega}]} \left[\frac{g'(\omega)}{g(\omega)} - 2\frac{g(\omega)}{G(\omega)} \right],$$

$$\beta \equiv \min_{\omega \in [\underline{\omega}, \bar{\omega}]} \left[\frac{g'(\omega)}{g(\omega)} + 2\frac{g(\omega)}{1 - G(\omega)} \right],$$

[4]Two firms located at the same location are excluded from the analysis. Equilibrium existence can also be shown for this case.

where $g'(\underline{\omega}) \equiv \lim_{\omega \searrow \underline{\omega}} g'(\omega)$and $g'(\bar{\omega}) \equiv \lim_{\omega \nearrow \bar{\omega}} g'(\omega)$. Denote Das the interval $[0, \bar{d}]$where $\bar{d}$is the maximal distance between a consumer and a firm, i.e. $\bar{d} = \max_{\omega \in [\underline{\omega}, \bar{\omega}]} \max_{n \in N} |\omega - l_n|$. When prices are such that each firm has a positive market share and all consumers buy in the market, profits of firm $n$are

$$\Pi_n(p_1, p_2) = (p_n - c) \int_{m_{n-1}(p_{n-1}, p_n)}^{m_n(p_n, p_{n+1})} g(\omega)\, d\omega,$$

with $m_0 \equiv \underline{\omega}$and $m_N \equiv \bar{\omega}$. m_n, $n = 1, \ldots, N-1$, denotes the marginal consumer between firm $n$and $n+1$which is determined by

$$p_n + t(|m_n - l_n|) = p_{n+1} + t(|m_n - l_{n+1}|).$$

Lemma 3. (Champsaur and Rochet, 1988)
If the transportation cost function $t$is strictly convex and thrice continuously differentiable in an open interval including $D$and if

$$\alpha < \frac{t'''(d)}{t''(d)} < \beta \text{ for all } d \in D$$

then there exists a price equilibrium in pure strategies.

In order to interpret their result it is helpful to notice that this implies the existence of equilibrium in the Hotelling model with quadratic transportation costs, a uniform distribution of consumers, and an arbitrary and exogenous number of firms.
With the following result I establish the existence of equilibrium when the maximal distance $\bar{d}$is $\frac{\pi}{3}$. It contains a joint restriction on the support of consumers$'locationsandthepotentialofproductdiffere$

Proposition 3.
Consider the Hotelling-Lancaster model with consumers$'utilities$u^S. Assume that there exists an $e \in \Re_+$such that $[\underline{l}, \bar{l}] \cup [\underline{\omega}, \bar{\omega}] \subseteq [e, e + \frac{\pi}{3}]$and that $g$is uniform on $[\underline{\omega}, \bar{\omega}]$then there exists a price equilibrium in pure strategies.

Proof. Consider the case of a uniform distribution on $[e, e + \frac{\pi}{3}]$. Hence the density is $g = 3/\pi$on its support. When all firms are located in this interval the transportation cost function only has to be defined on $[0, \frac{\pi}{3}]$. Computing the relevant variables gives

$$\alpha = -\frac{6}{\pi}, \;\; \beta = \frac{6}{\pi}, \text{ and } -\sqrt{3} \;\leq\; \frac{t'''(d)}{t''(d)} \;\leq\; 0, \; d \in [0, \frac{\pi}{3}].$$

Since $-\frac{6}{\pi} < -\sqrt{3}$and $0 < \frac{6}{\pi}$, the condition of Lemma 3 is satisfied and there exists a price equilibrium. Now assume that $[\underline{\omega}, \bar{\omega}] \subset [e, e + \frac{\pi}{3}]$and there may be firms located outside this interval. One has $\alpha < -\frac{6}{\pi}$and $\beta > \frac{6}{\pi}$. $\Box$

The following remarks show how this result can be generalized. There are other densities $g$which satisfy the conditions. For a smaller support of $g$the existence result holds for a larger interval than $[e, e + \frac{\pi}{3}]$. The smaller $\bar{d}$the larger the admissible deviation from the uniform distribution, i.e. $\max_{[\underline{\omega},\bar{\omega}]} |g'(\omega)|$can become larger.

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