

**DIFFERENTIATED BERTRAND DUOPOLY  
WITH VARIABLE DEMAND<sup>\*</sup>**

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# **DIFFERENTIATED BERTRAND DUOPOLY WITH VARIABLE DEMAND**

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## **A B S T R A C T**

Two one-product firms compete in prices on a market with differentiated products. Goods are differentiated because customers switch from one good to the other at different relative prices. With the specification that mean demand in the market is unit-elastic I provide conditions on the shape of the customer density which guarantee the existence of a unique Bertrand equilibrium.

**KEYWORDS:** Duopoly Equilibrium; Price Competition;  
Product Differentiation.

# 1 Introduction

In an oligopolistic market with differentiated goods firms compete in prices. Individual demand functions and the taste heterogeneity between the customers determine market demand. A large number of studies have looked at sufficient properties of the taste heterogeneity of customers, described by a customer density on a parameter space, in order to show the existence of equilibrium for particular specifications of individual demand.

The literature on product differentiation started with models in which customers have unit demand and are uniformly distributed in a one-dimensional space (e.g. Hotelling, 1929, Gabszewicz and Thisse, 1979, D'Aspremont, Gabszewicz and Thisse, 1979, and Salop, 1979). Another popular functional form on the taste heterogeneity is presented in the logit model (for an overview see Anderson, de Palma, and Thisse, 1992). To my knowledge the first articles dealing with shape assumptions on the customer density guaranteeing the existence of Bertrand equilibrium are Neven (1986) and Caplin and Nalebuff (1986). Both assume that the customer density is concave, Neven (1986) in a Hotelling model of unit demand and Caplin and Nalebuff (1986) in a model with unit elastic demand. Further work has focused on shape assumptions in models of unit demand which give rise to the existence of equilibrium (Champsaur and Rochet, 1988, Caplin and Nalebuff, 1991b, Dierker and Podczeck, 1992, Allen and Thisse, 1992, Bester, 1992). Ansari, Economides, and Ghosh (1994) and Goeree and Ramer (1994) then studied location-then-price games for non-uniform customer densities. Unit demand seems to be the adequate specification for many durables such as cars and microwaves. However, other goods such as soft drinks, beer, cigarettes, and potato chips are bought in variable quantities.

Models in which customers buy a variable quantity have received less attention (Caplin and Nalebuff, 1991b, Dierker, 1991, Anderson, de Palma, and Thisse, 1992, Peitz, 1995). In this literature it is assumed that taste heterogeneity can be described by some log-concave customer density.

I restrict attention to a duopoly with unit-elastic demand. The behavior of customers and firms is presented in section 2. Section 3 contains the existence and uniqueness results. I show for a class of "flat" density functions and for a class of log-concave density functions the existence of a unique equilibrium. All proofs are collected in the appendix.

## 2 Behavior of Customers and Firms

Let me call the market under consideration the fruit market. I assume that there are only two kinds of fruit, apples and oranges. For each of the fruits there exists a positive price:  $p_1$  is the price of an apple,  $p_2$  is the price of an orange. There are other goods in the economy but their prices are fixed. These goods are summarized by a Hicksian

composite commodity  $x_0$ ; its price  $p_0$  is normalized to 1. Hence income and prices  $p_1, p_2$  are measured in units of the composite commodity.

## 2.1 Customers

A customer has to make two decisions: in the first step, how much to spend on fruit and, in the second step, how to divide the fruit expenditure between apples and oranges. A customer with fixed income  $y > 0$  has a demand function for apples  $\xi_1(p_1, p_2)$  and oranges  $\xi_2(p_1, p_2)$  with the property that  $0 \leq p_1 \xi_1(p_1, p_2) + p_2 \xi_2(p_1, p_2) \leq y$ . I assume that a customer either buys apples or oranges. Expenditure on apples and oranges is fixed at  $b$  and it is assumed that the point  $\hat{p}_{12}$  at which the customer switches from one to the other is independent of expenditure in the market. Expenditure on apples is

$$p_1 \xi_1(p_1, p_2) = \begin{cases} b & \text{if } p_1/p_2 \leq \hat{p}_{12} \\ 0 & \text{else} \end{cases}$$

A customer only buys apples if  $p_1/p_2 \leq \hat{p}_{12}$ .<sup>1</sup>  $\hat{p}_{12}$  is one of the characteristics of a customer. It will be convenient to use logs:  $\theta \equiv -\log \hat{p}_{12}$ . A customer switches from oranges to apples when  $\log p_2 - \log p_1 \geq \theta$ . Customers evaluate apples and oranges differently, i.e. they have different  $\theta$ .

(A.1). There exists a continuous distribution function  $G$  over  $\theta \in \Re$  with  $G(0) > 0$ .  $G$  has a density  $g$  which is continuous on  $[\underline{\theta}, \bar{\theta}]$  and positive and continuously differentiable on  $(\underline{\theta}, \bar{\theta})$ .  $g$  has bounded support, i.e.  $\theta < \underline{\theta}$  implies  $G(\theta) = 0$  and  $\theta > \bar{\theta}$  implies  $G(\theta) = 1$ .

Without loss of generality  $G(0) > 0$ .  $g'(\underline{\theta})$  is defined as  $\lim_{\theta \searrow \underline{\theta}} g'(\theta)$  and  $g'(\bar{\theta}) \equiv \lim_{\theta \nearrow \bar{\theta}} g'(\theta)$ . The assumption of a bounded support is taken because I want to include the uniform distribution. Also it facilitates the proof that firms choose out of a compact strategy set.<sup>2</sup> The assumption of a bounded support implies that for a given orange price one can always find apple prices sufficiently large such that mean apple expenditure is equal to zero. All customers buy only apples if  $\log p_2 - \log p_1 \geq \bar{\theta}$  and only oranges if  $\log p_2 - \log p_1 < \underline{\theta}$ . If the interval degenerated to a single point all customers would be identical with respect to the switching point. In such a case also mean expenditure is discontinuous; goods are homogeneous. On the other hand, if  $g$

<sup>1</sup>To avoid correspondences I assume that the customer only buys apples at a relative price equal to the switching point. As customers will be assumed to be different and mass points for a distribution over  $\hat{p}_{12}$  will be excluded demand can be arbitrary at the switching point without changing aggregate demand.

<sup>2</sup>See Caplin and Nalebuff (1991b) for a result with unbounded support.

had unbounded support there would be a positive demand left for any price combination  $(p_1, p_2)$  and each type of fruit.

Demand relevant customers characteristics are  $b$  and  $\theta$ .  $B$  denotes conditional mean expenditure of all customers of type  $\theta$  which is assumed to be constant in  $\theta$ .<sup>3</sup>

According to the description of the population of customers mean demand functions for good 1 and 2 are of the form

$$\begin{aligned} X_1(p_1, p_2) &= \frac{B}{p_1} G(\log p_2 - \log p_1), \\ X_2(p_1, p_2) &= \frac{B}{p_2} (1 - G(\log p_2 - \log p_1)), \end{aligned}$$

where  $B > 0$  is a constant. Next I present three examples which generate the mean demand functions from above. Customers maximize their utility function  $u$  subject to the budget constraint  $x_0 + p_1 x_1 + p_2 x_2 \leq y$ .

*Example 1 in the goods-are-goods framework.* Sattinger (1984) proposed a utility functions which generates non-combinability and fixed individual expenditure  $b$ .

$u(x_0, x_1, x_2) = \left(\sum_{i=1,2} \frac{x_i}{q_i}\right)^\alpha x_0^{1-\alpha}$ . At his critical price ratio  $\frac{q_1}{q_2}$  a customer is indifferent between apples and oranges. A distribution over  $(q_1, q_2)$  generates a distribution over  $\theta$ . Each customer spends  $\alpha y$  on apples and oranges.

*Example 2 following the Lancasterian characteristics approach.* Example 1 is slightly modified in the spirit of Lancaster (1979). Assume that each good can be described by a vector  $\gamma = (\gamma^1, \dots, \gamma^n) \in \mathfrak{R}_+^n$  of characteristics and that customers only derive utility from the total amount in each characteristic,  $u(x_0, x_1, x_2) = \tilde{u}(x_0, \sum_i x_i \gamma_i^1, \dots, \sum_i x_i \gamma_i^n)$ . One specification is that the characteristic are perfect substitutes and  $u(x_0, x_1, x_2) = \left(\sum_k \lambda^k \sum_i \gamma_i^k x_i\right)^\alpha x_0^{1-\alpha}$  where  $\lambda$  defines the type of the customer and is an element of the  $n$ -dimensional unit simplex. The distribution on  $\theta$  depends on the distribution over weights  $\lambda$  and on the products' characteristics  $\gamma$ ,

$$\theta = \log \frac{\sum_k \lambda^k \gamma_1^k}{\sum_k \lambda^k \gamma_2^k}.$$

A different specification is provided by Caplin and Nalebuff (1986).

*Example 3 following the Hotelling approach.* The utility function is defined as  $u = (\sum_i \tilde{u}_i)^\alpha x_0^{1-\alpha}$  with  $\tilde{u}_i = x_i(1 + d(|\omega - l_i|))^{-1}$  and  $d(|\omega - l_i|) = e^{((\omega - l_i)^2)} - 1$ .  $\omega \in \mathfrak{R}$  denotes the type of the customer and  $l_i \in \mathfrak{R}$  the location of the good in the product space. The devaluation factor  $d(|\omega - l_i|)$  depresses the utility if the location of good

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<sup>3</sup> $B$  being constant in  $\theta$  is implied by stochastic independence of  $b$  and  $\theta$ . In the subsection on log-concave densities the results still hold if  $B(\theta)$  is log-concave in  $\theta$ .

$i$  does not coincide with the location of the most preferred good of customer  $\omega$ . It plays the same role as the transportation cost function in the Hotelling model. The marginal customer  $\tilde{\omega}$  who is indifferent between good 1 and good 2 is determined by  $\tilde{\omega} = \frac{\log p_2 - \log p_1}{2(l_2 - l_1)} + \frac{l_1 + l_2}{2}$  and there is a linear mapping from  $[\underline{\omega}, \bar{\omega}]$  into  $[\underline{\theta}, \bar{\theta}]$ . Further examples in a one-dimensional product space are provided by Peitz (1996).

## 2.2 Firms

Firm behavior is standard: Good  $i$ ,  $i = 1, 2$ , is produced by firm  $i$  at constant marginal cost  $c_i > 0$ . Each firm faces a mean demand function  $X_i(p_1, p_2)$  depending on the prices for apples and for oranges. Firms maximize profits, i.e.  $\max_{p_i} \pi_i(p_1, p_2)$  where  $\pi_i(p_1, p_2) = (p_i - c_i)X_i(p_1, p_2)$ ,  $i = 1, 2$ . Note that prices below marginal costs are weakly dominated. They are serially (strictly) dominated if assumption (A.2) is satisfied. (A.2) says that if each firm sets price equal marginal costs both firms have positive market shares.

(A.2).  $\underline{\theta} < \log c_2 - \log c_1 < \bar{\theta}$ .

The strategy set of firm  $i$  is  $\mathfrak{R}_+$ . Throughout the paper I am primarily concerned with the existence of a unique pure strategy equilibrium in prices. A *pure strategy Bertrand-Nash equilibrium* is a pair of prices  $(p_1^*, p_2^*) \in \mathfrak{R}_+^2$  such that  $\pi_i(p_i^*, p_j^*) \geq \pi_i(p_i, p_j^*)$  for all  $p_i \in \mathfrak{R}_+$ ,  $i, j = 1, 2$ ,  $j \neq i$ . In other words,  $p_i^*$  is element of the best response correspondence for  $p_j^*$ .

## 3 Existence and Uniqueness of Equilibrium

### 3.1 Log-Concave Densities

Since log-concavity of  $G$  implies that profit functions are quasi-concave this property is important to show the existence of equilibrium. As shown in Lemma 1 in the Appendix log-concavity of  $g$  carries over to  $G$ . The result corresponds to Lemma 1 in Dierker (1991). To prove the existence of an equilibrium it remains to be shown that the strategy set is compact.

A quite broad class of distributions have log-concave densities.<sup>4</sup> It includes such distributions as normal, exponential, and Weibull (for a larger list, illustrations and the references see Caplin and Nalebuff, 1991a). Truncated forms of the above mentioned distributions have also log-concave densities for a convex support and hence are examples of log-concave densities  $g$  with bounded support.

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<sup>4</sup>In his Proposition 2 Dierker (1991) states that log-concavity of  $g$  is implied by Schur-concavity of  $g(\theta)g(\theta')$  which expresses the idea that for a random draw out of the population more equal perturbations are more likely and shows the relationship between Schur-concavity and unimodality

To obtain uniqueness, two other properties, namely dominant diagonality under logarithmic transformation and log-supermodularity, are sufficient.<sup>5</sup>

**Proposition 1.**

Assume (A.1) and let  $G$  be log-concave on  $[\underline{\theta}, \bar{\theta}]$ . Logarithmic profits satisfy the dominant diagonal property on the set of prices above marginal costs where demand is  $C^2$ , i.e.

$$\left| \frac{\partial^2 \log \pi_i}{(\partial \log p_i)^2}(p_1, p_2) \right| > \left| \frac{\partial^2 \log \pi_i}{\partial \log p_i \partial \log p_j}(p_1, p_2) \right| \quad \text{for } i, j = 1, 2, j \neq i.$$

The following proposition establishes that profits are log-supermodular.<sup>6</sup>

**Proposition 2.**

Assume (A.1) and let  $G$  be log-concave on  $[\underline{\theta}, \bar{\theta}]$ . Profits are log-supermodular on the set of prices above marginal costs where demand is  $C^2$ .

**Theorem 1.**

Assume (A.1), (A.2), and let  $g$  be log-concave. There exists a unique pure strategy Bertrand-Nash equilibrium  $(p_1^*, p_2^*)$ . The associated game is dominance solvable.

The assumption on costs guarantees that the first-order condition of profit maximization is always satisfied in equilibrium and both firms make positive profits. If it is violated a firm can get the whole market with its price above marginal costs and, if this happens in equilibrium, the inactive competitor disciplines the firm which serves the whole market.

### 3.2 Flat Densities

I will now derive the same result as in Theorem 1 under the assumption that  $G$  is not too far from the uniform distribution. A density is called *flat* if

$$|\rho(\theta)| \equiv \left| \frac{g'(\theta)}{(g(\theta))^2} \right| \leq 1$$

on the support of  $g$ . Clearly,  $\rho(\theta) = 0$  for the uniform distribution on the interval  $[\underline{\theta}, \bar{\theta}]$ . But what does it mean that  $\theta$  is uniformly distributed? In contrast to Bester (1992) who looks at unit and not at unit elastic demand the distribution is related to relative

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<sup>5</sup>In more than one dimension one can make use of Prekopa's theorem (see Prekopa, 1973) to obtain log-concavity which was explored by Caplin and Nalebuff (1991a, 1991b) and Dierker (1991). The existence result in the case of log-concave densities hold for any finite number of firms.

<sup>6</sup>Supermodularity has recently been analyzed by Vives (1990) and Milgrom and Roberts (1990). It has already been applied by Caplin and Nalebuff (1991b). The fact that one can enhance the applicability by allowing for log-transformation is emphasized by Milgrom and Roberts.

and not to absolute price differences.<sup>7</sup> If one changes  $p_2$  from  $1p_1$  to  $2p_1$  and from  $2p_1$  to  $4p_1$  and if  $\theta$  is uniformly distributed then the share of customers who switch to good 1 is the same for the price change from  $1p_1$  to  $2p_1$  and from  $2p_1$  to  $4p_1$ . Note that multiple peaks are compatible with a flat density. Note also that a convex density can be flat. Hence the customer mass can be more concentrated around the end points of  $[\underline{\theta}, \bar{\theta}]$  than in the middle. Theorem 2 is the analogue to Theorem 1. Here, flatness gives rise to the log-concavity of  $G$ .

**Theorem 2.**

Assume (A.1), (A.2), and let  $g$  be flat. There exists a unique pure strategy Bertrand-Nash equilibrium  $(p_1^*, p_2^*)$ . The associated game is dominance solvable.

One can show that as  $(\bar{\theta} - \underline{\theta}) \rightarrow 0$ ,  $p_2^* \rightarrow c_2$  and  $p_1^* \rightarrow c_1$ , i.e. when the interval shrinks to a single point the competitive outcome is reached. Without heterogeneity the standard undercutting argument is valid. The theorem, though, is only for nondegenerate intervals for  $\theta$  because otherwise continuity of mean demand is violated.

Note that  $\partial^2 \log((p_i - c_i)/p_i)/(\partial \log p_i)^2 = -p_i/(p_i - c_i)$  which turns to infinity when price turns to marginal cost. Hence for a small support of  $g$  and high marginal costs, possible equilibrium prices have to be quite close to marginal costs so that  $\log G$  does not need to be concave to show the quasi-concavity of profit functions.

Uniqueness can be shown without exploiting the log-concavity of  $G$  and as a consequence log-supermodularity. Even under a weaker condition on  $|\rho(\theta)|$  uniqueness holds. The proof of uniqueness is elementary because I can construct a decreasing function with a zero for the equilibrium. For  $1 < |\rho(\theta)| \leq 2$ , I was not able to show the existence of equilibrium.

**Proposition 3.**

Assume (A.1), (A.2), and let  $|\rho(\theta)| \leq 2$  be flat. A pure strategy Bertrand-Nash equilibrium  $(p_1^*, p_2^*)$  is unique for strategy sets  $S_i = \mathbb{R}_+$ ,  $i = 1, 2$ .

Under (A.2) one can compute equilibrium prices. For  $|\rho(\theta)| \leq 1$  for  $\theta \in [\underline{\theta}, \bar{\theta}]$ , there exists a unique solution  $\theta^*$  to  $\frac{c_2}{c_1} \frac{1+G'(\theta)-G(\theta)}{G'(\theta)+G(\theta)} = e^\theta$ . Equilibrium prices then are

$$\begin{aligned} p_1^* &= c_1 \left(1 + \frac{G(\theta^*)}{G'(\theta^*)}\right) \quad \text{and} \\ p_2^* &= c_2 \left(1 + \frac{1-G(\theta^*)}{G'(\theta^*)}\right). \end{aligned}$$

In the model the extent of taste heterogeneity is expressed by  $g$ . According to example 3 one can view an enlargement of the support of  $g$  from  $[\underline{\theta}, \bar{\theta}]$  to  $[(1+\lambda)\underline{\theta}, (1+\lambda)\bar{\theta}]$ ,  $\lambda > 0$ , as a higher degree of product differentiation. Under the assumption that  $g$  is

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<sup>7</sup>It is the advantage of the specification in logs that the distribution is scale independent.



uniform the intuition that increased product differentiation leads to higher prices is correct. Log-profits of firm 1 are

$$\log \pi_1(p, \lambda) = \log \frac{p_1 - c_1}{p_1} + \log B + \log \frac{\frac{\log p_2 - \log p_1}{1+\lambda} - \underline{\theta}}{\bar{\theta} - \underline{\theta}}.$$

**Proposition 4.**

Assume (A.1), (A.2), and let  $g$  be uniform. Equilibrium prices  $p^*$  are increasing functions of the measure of increased product differentiation  $\lambda$ .

## 4 Conclusion

Two one-product firms compete in prices. Goods are differentiated because customers have different critical relative prices at which they revise their buying decision. I showed the existence of a unique equilibrium provided that the customer density is “flat” or log-concave.

Equilibrium profits are computable and the model may be a building block for two-stage models with endogenous product specification (see Peitz, 1996).

## Appendix

**Proof of Proposition 1.** Remark that prices are chosen from  $(\times_{i=1,2}[c_i, \infty)) \cap \{p_1, p_2 | X_i(p_1, p_2) > 0, i = 1, 2\}$ . On the interior of this set  $\log \pi_i, i = 1, 2$ , are twice continuously differentiable in  $\log p_i, i = 1, 2$ . I have to show that

$$\begin{aligned} -\frac{\partial^2 \log \pi_i}{(\partial \log p_i)^2} &> \frac{\partial^2 \log \pi_i}{\partial \log p_i \partial \log p_j} \text{ for } i, j = 1, 2, j \neq i. \\ \frac{p_1 c_1}{(p_1 - c_1)^2} - \frac{\partial^2 \log G}{(\partial \log p_1)^2} &> \frac{\partial^2 \log G}{\partial \log p_1 \partial \log p_2} = -\frac{\partial^2 \log G}{(\partial \log p_1)^2} \end{aligned}$$

Similarly for firm 2.  $\square$

**Proof of Proposition 2.** For  $C^2$  functions supermodularity is easily checked due to Topkis' Characterization Theorem.

$$\begin{aligned} \frac{\partial^2 \log \pi_i}{\partial \log p_i \partial \log p_j}(p_1, p_2) &\geq 0 \text{ for } i, j = 1, 2, j \neq i \\ \text{as } \frac{\partial^2 \log G(\log p_2 - \log p_1)}{\partial \log p_1 \partial \log p_2} &= -\frac{\partial^2 \log G(\log p_2 - \log p_1)}{(\partial \log p_1)^2} \geq 0 \end{aligned}$$

for  $G$  log-concave on  $[\underline{\theta}, \bar{\theta}]$  and in the same way for firm 2.  $\square$

### Lemma 1.

If  $g$  is log-concave in  $\theta$  for  $\theta \in [\underline{\theta}, \bar{\theta}]$  so is  $G$ .

**Proof.** (see Lemma 1 in Dierker, 1991)

$$\begin{aligned} \log G(\tilde{\theta}) &= \log \int_{\underline{\theta}}^{\tilde{\theta}} e^{h(\theta)} d\theta \text{ where } h(\theta) \equiv \log g(\theta) \\ \frac{d^2 \log G(\tilde{\theta})}{d\tilde{\theta}^2} &= -e^{h(\tilde{\theta})} \frac{e^{h(\tilde{\theta})}}{\left(\int_{\underline{\theta}}^{\tilde{\theta}} e^{h(\theta)} d\theta\right)^2} + h'(\tilde{\theta}) \frac{e^{h(\tilde{\theta})}}{\int_{\underline{\theta}}^{\tilde{\theta}} e^{h(\theta)} d\theta} \leq 0 \\ &\iff h'(\tilde{\theta}) \int_{\underline{\theta}}^{\tilde{\theta}} e^{h(\theta)} d\theta \leq e^{h(\tilde{\theta})} \end{aligned}$$

As  $h$  is concave:  $h'(\tilde{\theta}) \leq h'(\theta), \theta \in [\underline{\theta}, \tilde{\theta}]$ . Hence,

$$h'(\tilde{\theta}) \int_{\underline{\theta}}^{\tilde{\theta}} e^{h(\theta)} d\theta \leq \int_{\underline{\theta}}^{\tilde{\theta}} h'(\theta) e^{h(\theta)} d\theta < e^{h(\tilde{\theta})}. \quad \square$$

**Lemma 2.**

Assume (A.1) and let  $G$  be log-concave on  $[\underline{\theta}, \bar{\theta}]$ . Profits  $\pi_i$  are log-concave in  $\log p_i$  for all price combinations such that demand is positive and  $p_i > c_i$ .

**Proof.**

$$\log \pi_1 = \begin{cases} \log(p_1 - c_1) + \log B - \log p_1 + \log G \log \frac{p_2}{p_1} & \text{if } \log \frac{p_2}{p_1} \in [\underline{\theta}, \bar{\theta}] \\ \log(p_1 - c_1) + \log B - \log p_1 & \text{if } \log \frac{p_2}{p_1} \in (\bar{\theta}, \infty) \end{cases}$$

$\log(p_1 - c_1) + \log B - \log p_1$  is concave in  $\log p_1$ . What remains to be shown is that the last term is also concave in  $\log p_1$  for  $(\log p_2 - \log p_1) \in [\underline{\theta}, \bar{\theta}]$ . Since  $\tilde{\theta}(p_1, p_2) = \log p_2 - \log p_1$ ,

$$\frac{d^2 G(\tilde{\theta})}{d\tilde{\theta}^2} = \frac{\partial^2 G(\tilde{\theta}(p_1, p_2))}{(\partial \log p_i)^2}.$$

Similarly for firm 2.  $\square$

**Lemma 3.**

Assume that profits  $\pi_i$  are log-concave in  $\log p_i$  for all price combinations where demand is positive and  $p_i > c_i$ . Profits are quasi-concave in its own price for  $p_i \geq c_i$ .

**Proof.** Quasi-concavity is violated if there exists a  $p_{i0}$ ,  $p_{i1}$ , and  $p_{i\lambda}$  with  $c_i \leq p_{i0} < p_{i1}$  and  $p_{i\lambda} = \lambda p_{i0} + (1 - \lambda)p_{i1}$ ,  $\lambda \in (0, 1)$  such that

$$\begin{aligned} (p_{i0} - c_i)X_i(p_{i0}) &> (p_{i\lambda} - c_i)X_i(p_{i\lambda}) \\ \text{and } (p_{i1} - c_i)X_i(p_{i1}) &> (p_{i\lambda} - c_i)X_i(p_{i\lambda}) \end{aligned}$$

Case i)  $p_{i0} = c_i$ . The first inequality requires that  $p_{i\lambda} < c_i$  which is a contradiction.

Case ii)  $p_{i0} > c_i$ . Look at  $i = 1$ ,  $p_2$  fixed. For  $X_1(p_{11} = 0)$  the second inequality requires  $p_{1\lambda} < c_1$  which is a contradiction. By assumption, profit functions are log-concave in its logarithmic price where demand is positive and prices above marginal cost and Log-concavity implies quasi-concavity.  $\square$

**Notation.**

$$z \equiv \max \left\{ \bar{\theta} - \underline{\theta} - \log G(0) + \log c_2, \bar{\theta} - \underline{\theta} - \log(1 - G(0)) + \log c_1 \right\}$$

**Definition.** The best response correspondence  $R$  is defined as

$$\begin{aligned} R : [c_1, e^z] \times [c_2, e^z] &\longrightarrow [c_1, e^z] \times [c_2, e^z] \\ (p_1, p_2) &\longmapsto (r_1(p_2), r_2(p_1)) \equiv R(p_1, p_2) \end{aligned}$$

$$\text{where } r_i(p_j) = \begin{cases} \{p_i^+ | \pi(p_i^+, p_j) = \max_{p_i} \pi(p_i, p_j)\} \cap [c_i, e^z] & \text{if } \neq \emptyset \\ \{e^z\} & \text{else.} \end{cases}$$

**Lemma 4.**

Let profit functions be quasi-concave in their own price. There exists a pure strategy Bertrand-Nash equilibrium for  $\log c_i \leq \log p_i \leq z$ .

**Proof.** The best response correspondence  $R$  is a correspondence with compact convex domain into itself. Since profit functions are continuous the best response correspondence is upper-hemicontinuous. The quasi-concavity of the profit functions guarantees that  $R$  is convex-valued. Hence, one can apply Kakutani's fixed point theorem.  $\square$

**Lemma 5.**

Assume (A.1) and (A.2). There is no  $\log p_i > z$  such that  $\pi_1(p_1, p_2^*) > \pi_1(p_1^*, p_2^*)$  or  $\pi_2(p_1^*, p_2) > \pi_2(p_1^*, p_2^*)$ .

**Proof.** Suppose there is  $\log p_1 > z$  such that  $\pi_1(p_1, p_2^*) > \pi_1(p_1^*, p_2^*)$ .

Case i).  $\log p_2^* \leq z + \underline{\theta}$ . Then  $\pi_1(p_1, p_2^*) = 0$  for all  $\log p_1 > z$  contradicting  $\pi_1(p_1, p_2^*) > \pi_1(p_1^*, p_2^*)$ .

Case ii).  $z + \underline{\theta} \leq \log p_2^*$ .  $\log p_1 > z \geq \log p_2^*$  leads to a profit for firm 1 of

$$(p_1 - c_1) \frac{B}{p_1} G(\log p_2^* - \log p_1) \leq (p_1 - c_1) \frac{B}{p_1} G(0)$$

because  $G(0) \geq G(\log p_2^* - \log p_1)$ . For  $\log p_1 \geq \log p_2^* - \underline{\theta}$  one has  $G(\log p_2^* - \log p_1) = 0$ . Thus one must have  $\log p_1 < \log p_2^* - \underline{\theta}$ . Consequently, setting  $\log p_1 > z$  gives a payoff

$$\begin{aligned} \log \pi_1(p_1, p_2^*) &= \log \frac{p_1 - c_1}{p_1} + \log B + \log G(\log p_2^* - \log p_1) \\ &\leq \log \frac{p_1 - c_1}{p_1} + \log B + \log G(0) \\ &< \log \left( \frac{p_2^*}{e^{\underline{\theta}}} - c_1 \right) + \log B - \log p_2^* + \underline{\theta} + \log G(0) \end{aligned}$$

because  $(p_1 - c_1)/p_1$  is increasing in  $p_1$ . Set  $\log p'_1 = \log p_2^* - \bar{\theta}$ . It gives firm 1 the profit

$$\log \pi_1(p'_1, p_2^*) = \log \left( \frac{p_2^*}{e^{\bar{\theta}}} - c_1 \right) + \log B - \log \frac{p_2^*}{e^{\bar{\theta}}}.$$

It is not in the interest of the firm to set  $\log p_1 > z \geq \log p_2^*$  if  $\pi_1(p_1, p_2^*) < \pi_1(p'_1, p_2^*)$ .

$$\begin{aligned} \iff \log \left( \frac{p_2^*}{e^{\underline{\theta}}} - c_1 \right) - \log p_2^* + \underline{\theta} + \log G(0) &< \log \left( \frac{p_2^*}{e^{\bar{\theta}}} - c_1 \right) - \log \frac{p_2^*}{e^{\bar{\theta}}} \\ G(0) \left( \frac{p_2^*}{e^{\underline{\theta}}} - c_1 \right) &< e^{\bar{\theta} - \underline{\theta}} \left( \frac{p_2^*}{e^{\bar{\theta}}} - c_1 \right) \\ \iff e^{\bar{\theta} - \underline{\theta}} c_1 &< p_2^* (1 - G(0)) e^{-\underline{\theta}} \\ (\bar{\theta} - \underline{\theta}) - \log(1 - G(0)) + \underline{\theta} + \log c_1 &< \log p_2^* \\ \iff z + \underline{\theta} &< \log p_2^* \end{aligned}$$

which is a contradiction. In summary, one has  $\pi_1(p_1, p_2^*) < \pi_1(p_2^*/e^{\bar{\theta}}, p_2^*) \leq \pi_1(p_1^*, p_2^*)$  for all  $\log p_1 > z$ . For firm 2 one obtains  $\pi_2(p_1^*, p_2^*) > \pi_2(p_1^*, p_2)$  for all  $\log p_2 > z$ .  $\square$

**Proof of Theorem 1.** Lemmas 1,2, and 3 imply that profit functions are quasi-concave. Equilibrium existence for given compact strategy sets follows from Lemma 4. Lemma 5 shows that firms will always choose out of strategy sets  $[0, e^z]$ . Under (A.2) prices in  $\times_i[0, c_i]$  are serially dominated and  $p_i < c_i$  is dominated given  $p_j > c_j$ ,  $j \neq i$ . Hence an equilibrium in  $\times_i[c_i, e^z]$  is also an equilibrium in  $\mathfrak{R}_+^2$ .

Now I show uniqueness. Note that all strategy profiles  $(p_1, p_2)$  with  $\log p_2 - \log p_1 \notin [\underline{\theta}, \bar{\theta}]$  are serially dominated by some strategy with  $\log p_2 - \log p_1 \in [\underline{\theta}, \bar{\theta}]$ . Consequently, all prices for which Propositions 1 and 2 cannot be applied are serially dominated. By Proposition 1 log-profits satisfy the dominant diagonal property and there exists a unique pure strategy Bertrand-Nash equilibrium for the transformation (compare Milgrom and Roberts, 1990). Dominance solvability, which implies uniqueness, follows from the uniqueness under log-transformation and log-supermodularity.  $\square$

#### Lemma 6.

Assume (A.1) and let  $g$  be flat. Then  $G$  is log-concave.

**Proof.**  $\tilde{\theta}(p_1, p_2)$  was defined as  $\log p_2 - \log p_1$ .

$$\frac{\partial^2 \log G(\tilde{\theta})}{(\partial \log p_1)^2} = -\frac{\partial(g(\tilde{\theta})/G(\tilde{\theta}))}{\partial(\log p_1)} = \frac{1}{G^2(\tilde{\theta})}(g'(\tilde{\theta})G(\tilde{\theta}) - g^2(\tilde{\theta}))$$

Log-concavity of  $G$  is equivalent to  $g'(\tilde{\theta})G(\tilde{\theta}) \leq g^2(\tilde{\theta})$  which is implied by  $|g'(\tilde{\theta})|G(\tilde{\theta}) \leq g^2(\tilde{\theta})$ . Hence log-concavity of  $G$  follows from  $|\frac{g'(\tilde{\theta})}{g^2(\tilde{\theta})}| \leq 1$ . Similarly for firm 2.  $\square$

**Proof of Theorem 2.** Lemma 6 establishes log-concavity for the domain of prices above marginal costs and where demand is positive. Lemmas 2,3,4, and 5 then give existence as in Theorem 1. Uniqueness and Dominance Solvability then follow from Propositions 1 and 2.  $\square$

#### Lemma 7.

Assume (A.1) and  $|\rho(\theta)| \leq 2$ . An equilibrium  $(p_1^*, p_2^*)$  with prices above marginal costs is unique on  $\times_i(c_i, \infty)$ .

**Proof.** A situation in which both prices are above marginal costs and in which one firm has zero demand cannot be an equilibrium. Hence only prices such that demand is positive for both firms need to be considered. First-order conditions of profit maximization can be written as

$$c_1 G(\log p_2^* - \log p_1^*) - (p_1^* - c_1) G'(\log p_2^* - \log p_1^*) = 0$$

$$c_2(1 - G(\log p_2^* - \log p_1^*)) - (p_2^* - c_2)G'(\log p_2^* - \log p_1^*) = 0$$

Replacing  $\log p_2^* - \log p_1^*$  by  $\theta^*$ , rearranging, and taking ratios gives

$$\frac{c_2}{c_1} \frac{1 + G'(\theta^*) - G(\theta^*)}{G'(\theta^*) + G(\theta^*)} - e^{\theta^*} = 0$$

Define the function  $\Psi$  with  $\Psi(\theta) = \frac{c_2}{c_1} \frac{1 + G'(\theta) - G(\theta)}{G'(\theta) + G(\theta)} - e^\theta$ . It has to be shown that  $\Psi(\theta)$  has a unique zero  $\theta^* \in [\underline{\theta}, \bar{\theta}]$ .

$$\begin{aligned} \Psi'(\theta) &= D (2G''(\theta)G(\theta) - 2G'(\theta)^2 - G'''(\theta) - G'(\theta)) - e^\theta \\ &= D G'(\theta)^2 \left[ \frac{g'(\theta)[2G(\theta) - 1]}{g^2(\theta)} - 2 - \frac{1}{g(\theta)} \right] - e^\theta \end{aligned}$$

where  $D \equiv \frac{c_2}{c_1} (G'(\theta) + G(\theta))^{-2} > 0$ . Note that  $0 \leq G \leq 1$  implies  $(2G - 1) \in [-1, 1]$  and

$$\frac{g'(\theta)[2G(\theta) - 1]}{g^2(\theta)} \leq \left| \frac{g'(\theta)[2G(\theta) - 1]}{g^2(\theta)} \right| \leq \left| \frac{g'(\theta)}{g^2(\theta)} \right| \leq 2$$

Then  $\Psi'(\theta) \leq D G'(\theta)^2 [2 - 2 - \frac{1}{g(\theta)}] - e^\theta = -D g(\theta) - e^\theta < 0$ .  $\Psi$  is decreasing in  $\theta$ . Therefore,  $\theta^*$  is the only zero of  $\Psi$ .  $\square$

**Proof of Proposition 3.** Because of (A.2) prices in  $[0, c_i]$  are serially dominated and the result follows from Lemma 7.  $\square$

**Proof of Proposition 4.** For any  $\lambda > 0$  one can find a  $z(\lambda)$  such that there does not exist an equilibrium outside  $[c_1, e^{z(\lambda)}] \times [c_2, e^{z(\lambda)}]$ . Strategy sets are nondecreasing in  $\lambda$ . The associated game  $\Gamma(\lambda)$  with logarithmic profits as payoffs and logarithmic prices as strategic variables is dominance solvable (see proof of Theorem 1). Finally, I need to show that price elasticities of profits are increasing in the parameter  $\lambda$ .

$$\frac{\partial^2 \log \pi_1(p, \lambda)}{\partial \log p_1 \partial \lambda} = \frac{1}{(1 + \lambda)^2} \frac{1}{\frac{\log p_2 - \log p_1}{1 + \lambda} - \underline{\theta}} > 0$$

Analogously, for firm 2.  $\square$

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