RATIONALITY OF BARGAINING SOLUTIONS[*]

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A B S T R A C T

We analyze the rationality of two person bargaining solutions by considering conditions which are weaker than those used by Peters and Wakker (1991) or Bossert (1994). As a particular consequence of their results, the rationality of the Nash solution is obtained, although they can not be applied to other well known bargaining solutions. The aim of this paper is, on the one hand, to prove that a choice function defined on the usual bargaining domain which satisfies Independence of Irrelevant Alternatives, Weak Pareto Optimality and Pareto Continuity is also rationalized by a preorder (reflexive, complete and transitive binary relation). Moreover, the representability of this relation is analyzed. These results can be applied, in particular, to the *Nash solution* and moreover to the *egalitarian* (Kalai, 1977), *monotone path solutions* and their *lexicographic extensions*. On the other hand, and by substituting Pareto Continuity for Monotonicity assumptions, rationality is also analyzed. As a consequence, a result along the same lines as Bossert's (1994) is obtained.

KEYWORDS: Rational Choice; Bargaining Solution.

0. INTRODUCTION

A *bargaining solution* is a special case of *choice function* which selects a unique outcome for each *choice situation (bargaining problem)* and which is achieved by means of cooperation by the agents involved. One of the interpretations of bargaining solutions is to consider that the outcomes of the bargaining problems are given by the recommendations of an *"impartial arbitrator"* whose preferences represent, in some sense, the preferences of the agents as a group. So, the agreement reached in each bargaining situation may be thought of as the most preferred alternative within the set of feasible outcomes according to the arbitrator's preferences.

Thus, *rationality of bargaining solutions* can be analyzed by making use of some results obtained for general rational choice functions. Along these lines we have to mention the works of Lensberg (1987), Peters and Wakker (1991) and Bossert (1994) who provide a set of sufficient conditions in order to ensure the rationality of bargaining solutions by making use of standard axioms used in cooperative bargaining models (*Pareto Optimality, Independence of Irrelevant Alternatives* and *Continuity*). In order to do that, bargaining solutions are considered to be single-valued choice functions whose domain is given by the family of convex, comprehensive and compact subsets of $\mathbb{R}^2_+$ and, in this context, rationality by means of a transitive binary relation is analyzed.

Peters and Wakker (1991) assume that the domain of the choice function is given by the family of convex and compact subsets of $\mathbb{R}^2_+$ and prove that a single-valued choice function which satisfies *Pareto Optimality, Pareto Continuity* and *Independence of Irrelevant Alternatives* is rationalized by means of a transitive binary relation. Recently, Bossert (1994) has provided an alternative proof to this result by considering convex, compact and comprehensive subsets of $\mathbb{R}^2_+$ (which is the usual domain in bargaining theory) and by imposing *Continuity* instead of *Pareto Continuity*. As a particular consequence of this result, it is obtained that the Nash solution (1950) is rationalized by a *preorder* on the set of alternatives which are chosen in some bargaining problem.

As Bossert (1994) mentions in his work, there are other important bargaining solutions which are also rational(in the sense of choice functions) but whose rationality is not obtained by applying that result. This is the case, for instance, of the *egalitarian solution* (Kalai, 1977), *monotone path solutions* and *the lexicographic extensions of these solutions* which satisfy axioms which are different to those used in Peters and Wakker's or Bossert's works (axiomatic characterizations of these solutions can be found in Thomson, forthcoming).

In this paper it is proved that a single-valued choice function defined in the usual bargaining domain which satisfies *Pareto Continuity, Weak Pareto Optimality* and *Independence of Irrelevant Alternatives* is rational, generalizing Bossert's result (1994). The existence of a "numerical representation" of the choice function is also analyzed.

Moreover, it is also proved that, in the same context, a choice function which satisfies *Independence of Irrelevant Alternatives* and *Strong Monotonicity* is rational on the set of alternatives chosen in some choice situation. Finally we present a different rationality result by weakening the monotonicity assumption. In particular it is proved that a choice function which satisfies *Interior Monotonicity, Independence of Irrelevant Alternatives and Pareto Optimality* is also rational. As a consequence of all of these results the rationality of the *egalitarian, monotone path solutions* as well as their *lexicographic extensions* is obtained.

The paper is organized as follows: in Section 1 notation and assumptions which will be used through the paper are introduced. In Section 2 rationality results are presented and applied in order to obtain the rationality and representability of well known bargaining solutions and finally, in Section 3, rationality results without continuity are obtained.

1. PRELIMINARIES.

A *bargaining problem* is usually described by a pair (S,d) where $S \subset \mathbb{R}^2_+$ is the family of feasible utility vectors which individuals can afford and $d \in \mathbb{R}^2_+$ represents the disagreement point, that is the vector of utilities which individuals would obtain if they do not reach an agreement. As is usual in bargaining theory we assume that d=(0,0) and we restrict our attention only to subsets of $\mathbb{R}^2_+$ (gains of the agents over the disagreement point).

Moreover we consider the usual domain in bargaining theory which is given by

$$D = \{ A \subseteq \mathbb{R}^2_+ | \text{ A is convex, compact, comprehensive}$$
$$\text{and there exists } x \in A \text{ such that } x > 0 \}$$

where by $x > 0$ we mean that $x_i > 0$ $\forall i=1,2$. From now on, by x>y we mean $x_i > y_i$ $\forall i=1,2$ and by $x \geq y$, $x_i \geq y_i$ $\forall i=1,2$.

In order to simplify notation, for every subset A of $\mathbb{R}^2_+$ we will denote by <A> the *comprehensive and convex hull* of A, which is defined in the usual way, that is

$$<A> = \cap \{B \in \mathbb{R}^2_+ | A \subseteq B, \text{ B convex and comprehensive} \}$$

Formally a *bargaining solution* is a function which selects a unique outcome for each and every subset of outcomes which is considered (*bargaining problem*) within the domain D defined above. That is

$$F:D \longrightarrow \mathbb{R}^2_+$$
$$\forall S \in D \qquad F(S) \in S$$

We will denote by F(*D*) the set of alternatives which are chosen in some bargaining problem by the solution F, that is

$$F(D) = \{ a \in \mathbb{R}^2_+ | \exists S \in D \text{ such that } F(S) = a \}$$

Therefore a *bargaining solution* is a particular case of single-valued choice function in which the universal set of alternatives is considered to be $\mathbb{R}^2_+$ and whose domain is the usual one in bargaining theory as defined previously. From now on, this is the context in which we are going to work.

The notion of *rationality* of a (single-valued) choice function states the existence of some kind of binary relation whose maximization determines the choice set for each choice situation. Formally, F is *rational* if there exists a *reflexive, complete* and *transitive* binary relation R^* such that $\forall S \in D$,

$$\{F(S)\} = \{ a \in S | a R^* z \quad \forall z \in S \}$$

This notion of rational choice function corresponds to the notion of *regular-rational* choice function used by Richter (1971), who proves that this kind of rationality is equivalent to the transitive one in the single-valued case.

Next we present the different assumptions (all of them standard ones in bargaining theory) which will be used to obtain the rationality results.

(PO). Pareto Optimality:

$$\forall S \in D, \qquad F(S) \in PO(S) = \{\ x \in S \mid y \geq x, y \neq x \ \Rightarrow \ y \notin S\ \}$$

(WPO). Weak Pareto Optimality:

$$\forall S \in D, \qquad F(S) \in WPO(S) = \{\ x \in S \mid y > x \ \Rightarrow \ y \notin S\ \}$$

(IIA). Independence of Irrelevant Alternatives:

$$\forall S_1, S_2 \in D, \ S_1 \subset S_2 \ \text{and} \ F(S_2) \in S_1 \ \Rightarrow \ F(S_1) = F(S_2)$$

It is important to point out that (IIA) is necessary for the rationality of a single-valued choice function (Peters and Wakker, 1991), so any bargaining solution which does not satisfy this assumption can not be rational in the sense we have just defined above. For instance, this is the case of the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975), the Equal-Loss solution (Chun, 1988) or the Rational Equal-Loss solution (Herrero and Marco, 1993).

With respect to continuity assumptions, we will make use of *Pareto Continuity* (Peters, 1986) which is the condition also used by Peters and Wakker (1991) to present their results. This condition is weaker than usual *Continuity* (slight changes in the subset of alternatives presented for choice imply that the solution outcome does not change radically) and it takes into account not only changes in the set of alternatives presented for choice but also in its Pareto Optimal boundary.

(CONT). Continuity: If $\{A_n\} \subset D$, $\{A_n\} \longrightarrow \{A\}$ in the Hausdorff topology, then $\{F(A_n)\} \longrightarrow \{F(A)\}$.

(PC). Pareto Continuity: If $\{A_n\} \subset D$, $\{A_n\} \longrightarrow \{A\}$ and $\{PO(A_n)\} \longrightarrow \{PO(A)\}$ in the Hausdorff topology, then $\{F(A_n)\} \longrightarrow \{F(A)\}$.

The binary relation which will be used to obtain the rationality of bargaining solutions is the *revealed preference relation* which is formally defined as follows,

$$\forall x,y \in \mathbb{R}^2_+, \ x \neq y \qquad x \ P \ y \ \Leftrightarrow \ \exists A \in D \mid x,y \in A \ \text{and} \ F(A) = x$$

So this is an irreflexive binary relation although it is not generally complete. From this we define the *transitive closure of the revealed preference* (P*) as follows,

$$x \ P^* \ y \ \Leftrightarrow \ \exists x_1, x_2, ..., x_n \in X \text{ such that } x = x_1 \ P \ x_2 \ P \ ... \ P \ x_n = y$$

The non-existence of cycles of length 2 for the revealed preference is known as the *Weak Axiom of the Revealed Preference (WARP)*, which is

equivalent in our context (domain closed under intersection) to (IIA) (Hansson, 1968). Moreover whenever (IIA) is imposed we can use the following equivalent formulation of the revealed preference relation which will be used in the proofs:

$$\forall x,y \in \mathbb{R}^2_+, \quad x \neq y \qquad x \, P \, y \quad \Leftrightarrow \quad F(<x,y>) = x$$

In order to obtain rationality results for bargaining solutions we will make use of the well-known *Strong Axiom of Revealed Preference (SARP)* (acyclicity of the revealed preference). Formally,

(SARP). Strong Axiom of Revealed Preference: The revealed preference relation does not have cycles, that is

$$x_1 \, P \, x_2 \, P \, ... \, P \, x_n \qquad \text{implies} \qquad no[x_n \, P \, x_1]$$

Ville (1946), and independently Houthakker (1950), proved that, in general, this axiom is equivalent to rationality by means of a transitive binary relation. However, in the particular case of considering single-valued choice functions, the Strong Axiom of Revealed Preference also characterizes the rationality by means of a reflexive, transitive and complete binary relation [see Richter (1971); Corollary 1, Chapter 2]. Therefore, in order to obtain some of the rationality results, it will in fact be proved that the Strong Axiom of Revealed Preference is satisfied.

2. RATIONALITY RESULTS.

Peters and Wakker's (1991) or Bossert's (1994) results can be applied in order to obtain (in different contexts) that the Nash solution is rational. If we wish to analyze the rationality of other well-known solutions, then either Pareto Optimality or Continuity conditions must be removed. The first result of the paper makes use of Weak Pareto Optimality instead of Pareto Optimality and proves that this assumption together with Independence of Irrelevant Alternatives and Pareto Continuity also implies rationality. In particular this result generalizes the one obtained by Bossert (1994) and allows us to prove the rationality of any monotone path solutions as well as of their lexicographic extensions.

THEOREM 1. *If* $F:D \longrightarrow \mathbb{R}^2_+$ *is a choice function which satisfies (IIA), (WPO) and (PC), then it satisfies (SARP).*

Proof. (See Appendix A).

Remark 1: It is important to note that, although the usual bargaining solutions satisfy at least Weak Pareto Optimality, the previous result can also be proved by dropping this condition and by imposing only *Independence of Irrelevant Alternatives* and *Pareto Continuity*. Of course, some of the arguments used in the way of reasoning should change. So, although (IIA) is not enough to ensure the rationality of bargaining solutions (see Peters and Wakker, 1991), if we add (PC), this rationality can be obtained.

As we mentioned above and as a consequence of this result, not only the Nash solution but any bargaining solution satisfying (PC), (WPO) and (IIA) are rational, which we formulate as a corollary.

COROLLARY 1. *Nash, Egalitarian, monotonce path solutions and their lexicographic extensions are rational.*

Now we want to know some properties of the relation which rationalizes these bargaining solutions. Along these lines, Peters and Wakker analyze the "representability" of the binary relation: a function $f:\mathbb{R}^2_+ \longrightarrow \mathbb{R}$ *represents* a binary relation P, if $\forall x,y \in \mathbb{R}^2_+$ x P y implies $f(x) > f(y)$. In particular they prove the existence of a function which represents the transitive closure of the revealed preference associated to a choice function F (which satisfies (CONT), (PO) and (IIA)) and which is *maximized by F*, that is $f(F(A)) > f(x) \ \forall A \in D, x \in A, x \neq F(A)$.

The following result shows that this representability result can be also obtained in the context we have considered if Pareto Optimality conditions are removed. Therefore, not only the Nash solution works as if maximizing a group utility function, but also the egalitarian and monotone path solutions.

THEOREM 2. *If F is a choice function which satisfies (CONT) and (IIA), then there exists a real function $f:\mathbb{R}^2_+ \longrightarrow \mathbb{R}$ such that F maximizes f over D.*

Proof. Consider the following subset,

$$A = \{ a \in \mathbb{R}^2_+ \mid a = F(<z>) \text{ for some } z \in \mathbb{Q}^2_+ \}$$

Since it is a countable set, we rename its elements as follows,

$$A = \{a_i\}_{i \in I} \qquad I \neq \varnothing, I \subseteq \mathbb{N}$$

Now by considering the transitive closure of the revealed preference (which is asymmetric by applying Remark 1) we define, for each alternative x in $\mathbb{R}^2_+$, the set $I_x = \{i \in I \mid x \ P^* \ a_i\}$ and the following function:

$$f(x) = \begin{cases} \sum_{i \in I_x} \dfrac{1}{2^i} & \text{if } I_x \neq \varnothing \\ 0 & \text{if } I_x = \varnothing \end{cases}$$

First we are going to prove that x P y implies $f(x) > f(y)$. Consider $x,y \in \mathbb{R}^2_+$ such that x P y. By (IIA) we know that $F(<x,y>) = x$ and $F(<x>) = x$. Moreover it is clear that if x P^* y then $I_y \subseteq I_x$ and therefore $f(x) \geq f(y)$. We analyze two different cases.

Case 1. $y \notin F(D)$.

In this case and by definition of P, we know that there does not exist $z \in \mathbb{R}^2_+$ such that y P z. In particular, there does not exist $z \in A$ such that y P^* z, therefore $I_y = \varnothing$ and $f(y) = 0$.

With respect to alternative x, since $F(<x>) = x$, we know that x P a $\forall a \in <x>$. So if we consider $b \in <x> \cap \mathbb{Q}^2_+$ and we denote $F(<b>) = w$, then we obtain that $w \in A$ and x P w, so we can ensure that $f(x) > 0$.

Case 2. $y \in F(D)$.

From (IIA) this implies that $F(<y>) = y$. We are going to analyze the different configurations of alternatives $\{x,y\}$.

If $x \leq y$, then $x \in <y>$ and we obtain y P x, contradicting the asymmetry of P.

If $y \leq x$ and $y \in \text{int}(<x>)$, then it is enough to consider an alternative $z \in \text{int}(<x>) \cap \mathbb{Q}^2_+$ such that $y \in \text{int}(<z>)$. Since $F(<z>) \in A$, we denote $F(<z>) = a_j$, thus we obtain that $j \in I_x$ due to $a_j \in <z> \subseteq <x>$, but $j \notin I_y$ since $y \in <z>$ therefore if $a_j \neq y$ it implies that a_j P y and if not $(a_j = y)$ then no(y P a_j) from the definition of P. So we can conclude that $f(x) > f(y)$.

In other cases, that is if $\{y \leq x$ with $y \in <x>-\text{int}(<x>)\}$ or $\{no[x \leq y]$ and $no[y \leq x]\}$, we can consider a sequence of alternatives as follows,

$$\{z_j\} \in \mathbb{Q}^2_+ \cap [\text{int}(<x,y>)-<y>] \mid \{z_j\} \longrightarrow x$$

By applying the continuity of F we can ensure on the one hand that

$$F(<y,z_j>) \longrightarrow F(<x,y>) = x$$

and, on the other hand, that there exists $k \in \mathbb{N}$ such that

$$F(<y,z_k>) \notin \{x,y\}$$

since $x \notin <y,z_j>$ $\forall j \in \mathbb{N}$ and if $F(<y,z_j>) = y$ $\forall j$ we obtain a contradiction with the continuity of F. Therefore if we denote $b_k = F(<y,z_k>)$ we have that b_k P y. If we now consider an alternative $s \in \mathbb{Q}^2_+ \cap <x>$ such that $s > b_k$ and denote $F(<s>) = a_k$, we obtain that $a_k \in A$ and x P a_k P b_k P y, so x P* a_k P* y and therefore $f(x) > f(y)$.

Finally it is clear that F maximizes f: if $S \in D$ and $F(S) = w$, then w P s $\forall s \in S-\{w\}$ and therefore $f(w) > f(s)$ $\forall s \in S-\{w\}$. ∎

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3. RATIONALITY RESULTS WITHOUT CONTINUITY.

In this Section we present two different rationality results by dropping continuity conditions and by imposing monotonicity assumptions instead. The first one is along the same lines as Bossert's result (1994) in which it is proved that any choice function which satisfies (CONT), (IIA) and (PO) is not only rational on $F(D)$, but the rationalization satisfies some regularity conditions (monotonicity, strict quasi-concavity,...). Along these lines we will prove that a choice function which satisfies Independence of Irrelevant Alternatives and *Strong Monotonicity* (if we expand a set, then the outcome selected by the bargaining solution in the new set is greater than or equal to the initial one) is also rational on $F(D)$ and that, under these assumptions, the rationalization satisfies monotonicity and strict quasiconcavity. So we drop Pareto Optimality and Continuity, although we need to require Strong Monotonicity in their place. Formally the monotonicity assumption is as follows:

(SM). Strong Monotonicity:

$$\forall A,B \in D \text{ such that } A \subseteq B, \quad F(A) \leq F(B))$$

THEOREM 3. *If F is a choice function satisfying (IIA) and (SM), then it is rational on F(D).*

Proof. First we are going to prove that the revealed preference P is transitive and then it will in fact be proved that it is complete when we restrict our attention to $F(D)$. As in the previous cases, by (IIA) we obtain the asymmetry of the revealed preference.

Assume that x P y P z for some $x,y,z \in \mathbb{R}^2_+$. Then by (IIA) we know that $F(<x,y>) = x$, $F(<y,z>) = y$, $F(<x>) = x$ and $F(<y>) = y$. We consider now the subset $<x,y,z>$ and analyze $F(<x,y,z>) = w$. If $w = x$, then x P z and we obtain the transitivity of P.

By contradiction, assume that $w \neq x$. If $w = y$ then y P x, contradicting the asymmetry of P. If $w = z$ then z P y contradicting the asymmetry of P once again. If $w \in <x,y>$ then by (IIA) $w = x$, a contradiction. If $w \in <y,z>$ then by (IIA) $w = y$, a contradiction. Therefore the only possibility is that $w \in <x,z>$, so by (IIA) $F(<x,z>) = w$. We now analyze the configuration of $\{x,z\}$. If $x \leq z$ then $<x,y,z> = <y,z>$, a contradiction since $w \neq y$; if $z \leq x$ then $<x,z> = <x>$, a contradiction since $w \neq x$. Therefore no$[x \leq z]$ and no$[z \leq x]$, but by applying Strong Monotonicity we know that $w \geq x$ ($<x,y> \subseteq <x,y,z>$), therefore the only possibility is that $w = x$, a contradiction.

To show that P is complete on $F(D)$, consider $x,y \in F(D)$, $x \neq y$. It is clear that it implies that $x \leq y$ or $y \leq x$: if no$[x \leq y]$ and no$[y \leq x]$ and we denote by $w = F(<x,y>)$, we know (by applying Strong Monotonicity) that since $<x> \subseteq <x,y>$, $<y> \subseteq <x,y>$ and by (IIA) $F(<x>) = x$ and $F(<y>) = y$, then $x \leq w$ and $y \leq w$, which implies that $w \notin <x,y>$, a contradiction. But if $x \leq y$ then $x \in <y>$, so y P x and if $y \leq x$ then $y \in <x>$ and therefore x P y.

Finally and in order to ensure the reflexivity we modify this relation by considering

$$x \: R \: y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x \: P \: y$$

which is obviously complete, reflexive, transitive and also a rationalization of F on F(*D*). ∎

With respect to the additional properties which this rationalization on F(*D*) satisfies, we can ensure that under the same assumptions [(IIA) and (SM)] it satisfies *Monotonicity* and *Strict Quasi-concavity*. First we formally present these properties.

Monotonicity: $\forall x,y \in F(D)$, if $x > y$ then $x \: P \: y$.

Strict Quasi-concavity: $\forall x,y \in F(D)$ if $x \: P \: y$ and for some $\alpha \in (0,1)$ $[\alpha x + (1-\alpha)y] \in F(D)$, then $[\alpha x + (1-\alpha)y] \: P \: y$.

THEOREM 4. *If F is a choice function satisfying (IIA) and (SM), then there exists a rationalization on F(D) which satisfies Monotonicity and Strict Quasi-concavity.*

Proof. By applying Theorem 3 we know that there exists a rationalization on F(*D*) given by R ($x \: R \: y \Leftrightarrow x = y$ or $x \: P \: y$). We only need to show that this relation satisfies the conditions mentioned above.

Monotonicity: Consider $a,b \in F(D)$ such that $a > b$. Since $a,b \in F(D)$ we know by (IIA) that $F(<a>) = a$ and $F(<b>) = b$; so $b \in <a>$ and therefore $a \: P \: b$.

Strict Quasi-concavity: Consider $a,b \in F(D)$ such that $a \: P \: b$ and $\alpha \in (0,1)$ such that $[\alpha a + (1-\alpha)b] \in F(D)$. By (IIA) we know that $F(<a>) = a$, $F(<b>) = b$ and $F(<\alpha a+(1-\alpha)b>) = \alpha a+(1-\alpha)b$. Moreover and by reasoning as in Theorem 3 we know that $a \leq b$ or $b \leq a$; but $a \leq b$ implies that $a \in <b>$ and therefore $b \: P \: a$, in contradiction with the asymmetry of P. So the only possibility is $b \leq a$, which implies $\alpha a+(1-\alpha)b \geq b$, so $b \in <\alpha a+(1-\alpha)b>$ and therefore $\alpha a+(1-\alpha)b \: P \: b$. ∎

Remark 2: It is easy to prove that if we impose continuity to the choice function then the rationalization is also *Upper Semicontinuous* [that is $\forall x \in F(D)$, $\{y \in F(D) \mid x \: P \: y\}$ is open in F(*D*)].

As an immediate consequence of this result it is also obtained that egalitarian and monotone path solutions are rationalized by an order on F(*D*) satisfying those conditions.

COROLLARY 2. *Egalitarian and Path monotone solutions are rational on F(D) and the rationalization satisfies Monotonicity, Strict Quasi-concavity and Upper Semicontinuity.*

In order to obtain an alternative characterization result without considering continuity conditions and by weakening the monotonicity

assumption, we present the following result. First we need the notion of *Interior Pareto Optimal point* and *Interior Monotonicity*.

If $S \in D$, an alternative $x \in S \subset \mathbb{R}^2$ is an *interior Pareto optimal point* if it is satisfied that $x \in \text{int}_r[PO(S)]$, where by $\text{int}_r[PO(S)]$ we mean the relative interior of subset $PO(S)$.

So the notion of *Interior Monotonicity* in the two-dimensional case states that if given a subset of alternatives, the outcome solution is an interior Pareto optimal point, then if we expand the feasible set, the new solution has to be greater than or equal to the initial one. Formally,

(IM). Interior Monotonicity: $\forall A,B \in D$, if

$$\left. \begin{array}{l} A \subset B \quad \text{and} \\ F(A) \text{ is an interior Pareto optimal point in } A \end{array} \right\} \Rightarrow F(A) \leq F(B)$$

The next result shows that if we require Interior Monotonicity together with Independence of Irrelevant Alternatives and Pareto Optimality of the choice function, then it is obtained that the revealed preference is acyclic.

THEOREM 5. *If $F{:}D \longrightarrow \mathbb{R}^2_+$ is a choice function which satisfies (PO), (IIA) and (IM), then it satisfies (SARP).*

Proof. (See Appendix B).

Finally we can also apply this result to obtain the rationality of the lexicographic extension of the bargaining solutions mentioned above.

In order to simplify the notation, from now on whenever we write $i + j$ we mean $i + j$ if $i + j \leq n$ and $i + j - n$ if $i + j > n$ for every $i,j \in \{1,2,...,n\}$. Moreover, $\forall A \in D$, Fr(A) denotes the topological boundary of A and for any pair of alternatives $x,y \in \mathbb{R}^2_+$ seg(x,y) denotes the segment which joins x and y up without considering x and y (in another case we use seg[x,y]), that is, seg(x,y) = $\{\lambda x + (1-\lambda)y \quad \lambda \in (0,1)\}$.

To prove the acyclicity of the revealed preference we are going to apply induction, therefore first we present the following result obtained by Bossert (1994) in which the non existence of cycles of length 3 is proved. In particular it shows that (IIA) is enough to guarantee the non existence of cycles of length 3.

THEOREM 6. [Bossert, 1994]. *If $F:D \longrightarrow \mathbb{R}^2_+$ is a univalued choice function which satisfies (IIA), then there do not exist cycles of length 3 for the revealed preference.*

An analogous result was obtained by Peters and Wakker (1991) in the context of convex and compact (but not necessarily comprehensive) subsets of $\mathbb{R}^2_+$, but by imposing (PO) apart from (IIA) [see Peters and Wakker (1991); Lemma 3.5.]. Moreover they prove that (IIA) and (PO) are not enough to guarantee the non-existence of cycles of length greater than 3.

Proof of Theorem 1. We prove the acyclicity of the revealed preference relation by induction. Because (IIA) is equivalent to (WARP) in this context, there do not exist cycles of length 2 and, by Theorem 6, there do not exist cycles of length 3. Let $n \geq 4$ and suppose that there exists no cycle of length k for all k<n; we are going to prove that there are no cycles of length n either. By way of contradiction suppose that there exists a cycle of length n, that is $x_1, x_2,...,x_n \in \mathbb{R}^2_+$ such that $x_1 \, P \, x_2 \, P...P \, x_n \, P \, x_1$; by (IIA):

$$F(<x_i, x_{i+1}>) = x_i \qquad \forall i=1,2,...,n$$
$$F(<x_i>) = x_i \qquad \forall i=1,2,...,n$$

Moreover it is clear that x_i can not be in $<x_j>$ for all $j \neq i$ (if not, there exists a cycle of a length smaller than n) and, therefore

$$\{ \, x_i^1 > x_j^1 \text{ and } x_i^2 < x_j^2 \, \} \quad \text{or} \quad \{ \, x_i^1 < x_j^1 \text{ and } x_i^2 > x_j^2 \, \}$$

for all $i,j \in \{1,2,...,n\}$. Furthermore and in general, we can also ensure that

$$x_j \notin <x_p, x_{p+1}> \quad \forall p = 1,2,...,n, \; j \neq p, p+1 \qquad (1)$$

since if not we obtain a cycle of length smaller than n. For the same reason if we denote $A = \{x_1, x_2,...,x_n\}$ we can ensure that whenever we consider the comprehensive and convex hull of a subset of alternatives of A with more than two alternatives or with two non consecutive ones according

to the cycle, then the choice set will never be one of the alternatives of the subset, that is

$\forall B \subseteq A$ such that B has more than 2 alternatives or two

non consecutive ones, then $F(<B>) \notin B$ (2)

If we now analyze the choice over the convex and comprehensive hull of A, $w = F(<A>)$, it must be $w \neq x_i$ $\forall i=1,...,n$, since if not, we obtain a cycle of length 2, a contradiction. So there exist alternatives $x_i, x_{i+k} \in A$, $x_i \neq x_{i+k}$ such that $w \in <x_i, x_{i+k}>$. Moreover and by (IIA) these alternatives are non consecutive, since if $w \in <x_i, x_{i+1}>$ for some i then it is obtained that $F(<x_i, x_{i+1}>) = w$, a contradiction. Finally note that $w \in seg(x_i, x_{i+k})$ since if not by (WPO) either $w \in <x_i>$ or $w \in <x_{i+k}>$ and by (IIA) it would imply that $F(<x_i>) = w$ or $F(<x_{i+k}>) = w$ respectively, a contradiction.

Since we start from a cycle, we can assume without loss of generality that $x_i^1 < x_{i+k}^1$, so the configuration of these two alternatives would be as Fig.1 shows.
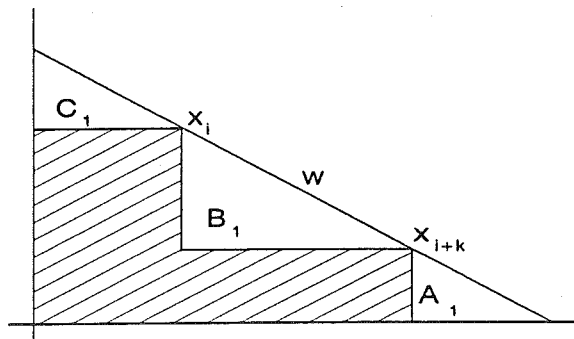
Fig. 1

Next we are going to analyze the different cases which exhaust all possible configurations of alternatives $\{x_i, x_{i+1},..., x_{i+k}\}$. Since by (WPO) $seg(x_i, x_{i+k}) \in Fr(<A>)$ we know that any other alternative of A ($x_j \in A$, $j \neq i, i+k$) can not be placed over the line which passes through x_i and x_{i+k} (see Fig.1). So, and from the considerations we have made above, we can ensure that there are only three different possibilities to place alternative x_{i+1} which have been denoted as A_1, B_1 and C_1 (see Fig.1).

CASE 1: $x_{i+1} \in A_1$.

In this case by applying (2), (IIA) and (WPO) we know that $F(<x_{i+k}, x_{i+1}>) = w_1 \in seg(x_{i+k}, x_{i+1}]$, while applying (IIA) it is satisfied that $F(<x_i, x_{i+k}, x_{i+1}>) = w$ (since $<x_i, x_{i+k}, x_{i+1}> \subset <A>$ and $F(<A>) \in <x_i, x_{i+k}, x_{i+1}>$). We are going to prove that this fact contradicts continuity.

Consider the configuration of alternatives w, x_{i+k} and x_{i+1} (see Fig.2).
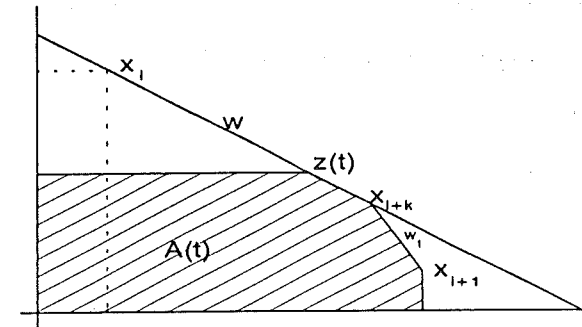
Fig. 2

For any $t \in [0,1]$ we define $z(t) = tw + (1-t)x_{i+k}$, $A(t) = <x_{i+1}, x_{i+k}, z(t)>$ and the following function:

$$g:[0,1] \longrightarrow \mathbb{R}^2_+ \qquad \text{such that} \qquad g(t) = F(A(t))$$

This is a continuous function since it is a composition of continuous ones, and moreover by (IIA) and (WPO) it is clear that $\forall t \in [0,1]$, $g(t) = F(A(t)) \notin A(0)$ or $g(t) = w_1$. We denote $B = g^{-1}(w_1)$, which is closed (due to the continuity of g), nonempty ($0 \in B$) and $B \neq [0,1]$ ($1 \notin B$). Therefore we can ensure the existence of maximum of B, namely t_0. Since $t_0 \in B$ we know that $g(t_0) = w_1$ and by definition of maximum $\forall t \in [0,1]$ such that $t > t_0$ it is satisfied that $g(t) \neq w_1$. But this is a contradiction with the continuity of g by considering $\varepsilon < \| w_1 - x_{i+k} \|$, since $g(t) \notin A(0)$ or $g(t) = w_1$ and therefore we can consider t as close as we want to t_0 but $\| g(t) - g(t_0) \| > \varepsilon \quad \forall t > t_0$.

CASE 2: $x_{i+1} \in B_1$.

Given the configuration of alternatives x_{i+1}, x_i and x_{i+k} (see Fig.3), we analyze the possible location of alternative x_{i+2}. The different possibilities for locating this alternative have been denoted as B_2, C_2, D_2, E_2 and F_2. We are going to prove that the only one which does not give us a contradiction is D_2.
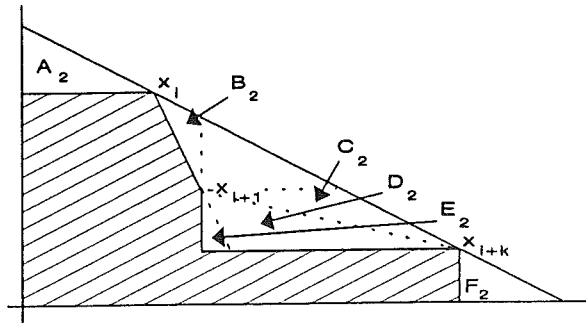
Fig. 3

2.1. If $x_{i+2} \in A_2$, B_2, C_2 or F_2, then by reasoning in an analogous way to the previous case, by considering the following subsets in each case, a contradiction with continuity is obtained (see Fig.4 and Fig.5):

In case of $x_{i+2} \in A_2$, consider $<x_{i+2}, x_i, w>$ and $<x_{i+2}, x_i>$; if $x_{i+2} \in B_2$, consider $<x_{i+2}, x_{i+1}, F(<x_i, x_{i+2}, x_{i+1}>)>$ and $<x_{i+2}, x_{i+1}>$; if $x_{i+2} \in F_2$ consider $<x_{i+2}, x_{i+k}, w>$ and $<x_{i+2}, x_{i+k}>$; and, finally, if $x_{i+2} \in C_2$, then consider $F(<x_{i+2}, x_{i+1}, F(<x_{i+2}, x_{i+1}, x_{i+k}>)$ and $<x_{i+2}, x_{i+1}>$.

Fig. 4

$w^* = F(<x_{i+2}, x_{i+1}, x_{i+k}>)$
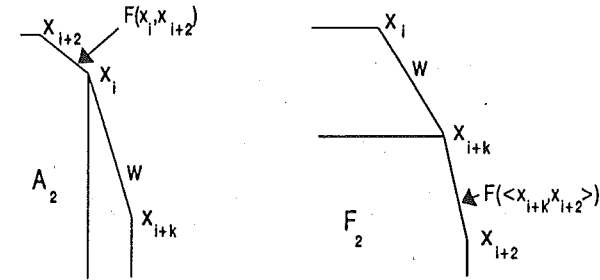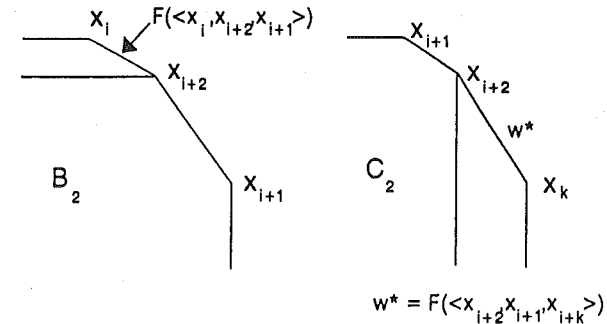
Fig. 5

2.2. Finally if $x_{i+2} \in E_2$, then $F(<x_i,x_{i+1},x_{i+2}>)$ belongs to $<x_i,x_{i+1}> \cup <x_{i+1},x_{i+2}>$, which implies by (IIA) a contradiction with (2).

Therefore, as we mentioned above, the only possibility for alternative x_{i+2} is D_2. The final configuration of alternatives x_i, x_{i+1}, x_{i+2} and x_{i+k} is shown in Fig.6.
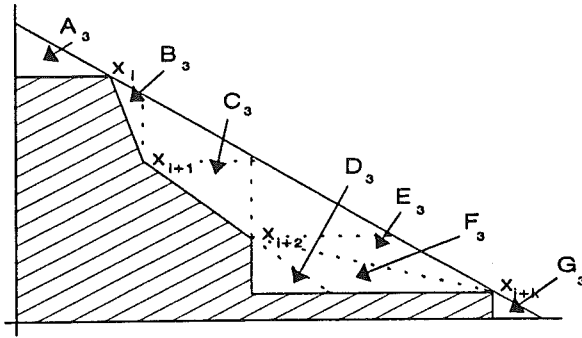

Fig. 6

By following an analogous way of reasoning, we are going to show that there is also a unique possibility for locating alternative x_{i+3}, which has been denoted as F_3 (see Fig.6). Notice that this zone is defined by the line which passes through alternatives x_{i+2} and x_{i+1} and the line which goes through alternatives x_{i+2} and x_{i+k}. Therefore the last two previous alternatives which have been located together with x_{i+k} are the references to determine which this area is.

2.3. If x_{i+3} belongs to A_3, C_3, E_3 and G_3 a contradiction with continuity is obtained [by considering the following subsets in each case: if $x_{i+3} \in A_3$, consider $<x_{i+3},x_i,x_{i+k}>$ and $<x_i,x_{i+3}>$; if $x_{i+3} \in C_3$, consider

$<x_{i+1},x_{i+3},x_{i+2}>$ and $<x_{i+2},x_{i+3}>$; if $x_{i+3} \in E_3$, consider $<x_{i+2},x_{i+3},x_{i+k}>$ and $<x_{i+2},x_{i+3}>$ and, finally, if $x_{i+3} \in G_3$ then consider $<x_i,x_{i+k},x_{i+3}>$ and $<x_{i+k},x_{i+3}>$].

2.4. If $x_{i+3} \in B_3$ then $x_{i+1} \in <x_{i+2},x_{i+3}>$, a contradiction.

2.5. Finally if $x_{i+3} \in D_3$ then $F(<x_{i+1},x_{i+2},x_{i+3}>)$ belongs to $<x_{i+1},x_{i+2}> \cup <x_{i+2},x_{i+3}>$, in contradiction with (2) by (IIA).

Therefore, and by following an analogous argument for the rest of alternatives $(x_{i+4},x_{i+5},...,x_{i+k-1})$ we can ensure that the final configuration of those alternatives in this case would be as shown in Fig.7.
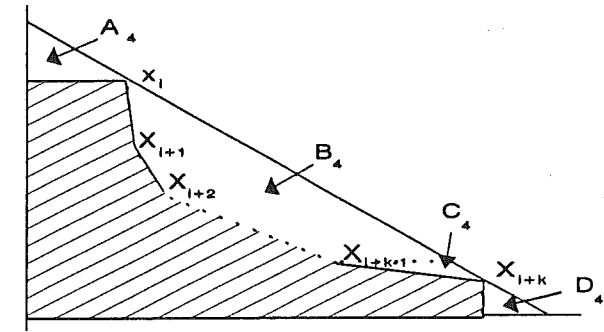

Fig. 7

The final contradiction which is obtained is that there does not exist any possibility of locating alternative x_{i+k+1}. On the one hand we know that the only possibilities for locating this alternative are the areas denoted as A_4, B_4, C_4 y D_4 (see Fig.7). But in all of these cases a contradiction is obtained. In particular if $x_{i+k+1} \in A_4$, then $F(<x_{i+k+1},x_i,x_{i+k}>) = w$ and $F(<x_{i+k+1},x_i>) \in seg(x_{i+k+1},x_i)$, in contradiction with continuity; if $x_{i+k+1} \in D_4$ then $F(<x_{i+k-1},x_{i+k},x_{i+k+1}>)$

belongs to $\langle x_{i+k-1}, x_{i+k}\rangle) \cup \langle x_{i+k}, x_{i+k+1}\rangle$ in contradiction with (2) by (IIA); if $x_{i+k+1} \in B_4$ then some of the alternatives x_{i+j} for $j=1,2,...,k-1$ belongs to $\langle x_{i+k}, x_{i+k+1}\rangle$ a contradiction; and finally, if $x_{i+k+1} \in C_4$ then by considering $\langle x_{i+k-1}, x_{i+k}, x_{i+k+1}\rangle$ a contradiction with continuity is obtained once again $(F(\langle x_{i+k-1}, x_{i+k}, x_{i+k+1}\rangle) \in seg(x_{i+k-1}, x_{i+k+1})$ and $F(\langle x_{i+k}, x_{i+k+1}\rangle) = x_{i+k})$.

Therefore in this case a contradiction is also obtained and so the only possibility for the final configuration of alternative x_{i+1} is C_1 (see Fig.1).

CASE 3: $x_{i+1} \in C_1$.

The different possibilities for alternative x_{i+2} have been denoted as A_5, B_5, C_5, D_5 and E_5 (see Fig.8).
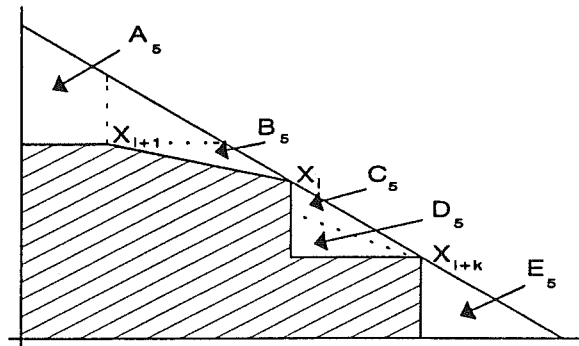
Fig. 8

But in A_5, B_5, C_5 and E_5 a contradiction with continuity is obtained by reasoning in a similar way to the previous cases [in particular by considering the following subsets in each case: if $x_{i+2} \in A_5$, consider $\langle x_{i+2}, x_i, x_{i+k}\rangle$ and $\langle x_i, x_{i+2}\rangle$; if $x_{i+2} \in B_5$, consider $\langle x_{i+1}, x_{i+2}, x_i\rangle$ and

$\langle x_{i+1}, x_{i+2}\rangle$; if $x_{i+2} \in C_5$ consider $\langle x_{i+1}, x_{i+2}, x_{i+k}\rangle$ and $\langle x_{i+1}, x_{i+2}\rangle$; and finally, if $x_{i+2} \in E_5$ then consider $\langle x_i, x_{i+k}, x_{i+1}\rangle$ and $\langle x_{i+1}, x_{i+k}\rangle$].

So the only possibility for alternative x_{i+2} is D_5 (see Fig.9), but by reasoning as in case 2 the only possibility for x_{i+3} is G_6 and the final configuration of alternatives $x_i, x_{i+1},...,x_{i+k}$ is shown in Fig.10. A contradiction is now obtained when we try to locate alternative x_{i+k+1} (it can not be located in any place).
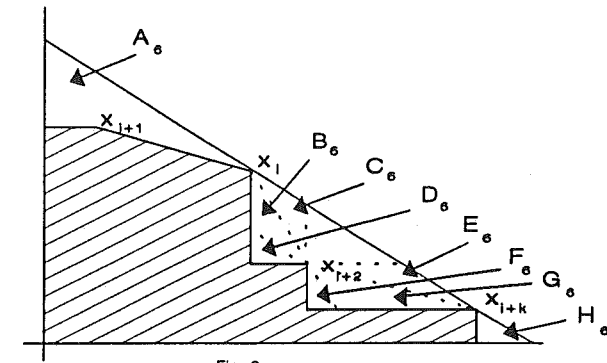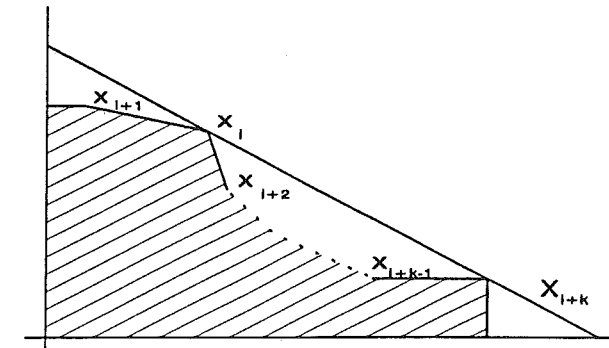

Fig. 9

Fig. 10

Therefore we can conclude that there do not exist cycles of length n and the Strong Axiom of the Revealed Preference is satisfied. ∎

APPENDIX B

In order to simplify the proof of Theorem 5, first we present the following lemma in which the non existence of cycles of length 4 is proved by requiring (PO), (IIA) and (IM).

LEMMA 1. *If $F:D \longrightarrow \mathbb{R}^2_+$ is a univalued choice function which satisfies (PO), (IIA) and (IM), then there do not exist cycles of length 4 for the revealed preference relation.*

Proof: By contradiction, assume that there exist alternatives x_1, x_2, x_3, $x_4 \in X$ such that $x_1 \, P \, x_2 \, P \, x_3 \, P \, x_4 \, P \, x_1$; so by (IIA) we know that for every i=1,2,3,4:

$$F(<x_j, x_{i+1}>) = x_i$$
$$F(<x_i>) = x_i$$

By applying the same way of reasoning as in Theorem 1 we can ensure that:

(a). $x_j \notin <x_i> \quad \forall i \neq j$;

(b). $x_i \notin <x_j, x_{j+1}> \qquad \forall j=1,2,3,4, \quad i \neq j, j+1$;

(c). $\forall B \subseteq A$ such that B has more than 2 alternatives or two non consecutive ones, $F(<B>) \notin B$.

Moreover in this case we also know that

(d). $F(<x_1, x_3>) \leq w$ and $F(<x_2, x_4>) \leq w$ by applying (IM) since condition (c) together with (PO) implies that $F(<x_1, x_3>)$ and $F(<x_2, x_4>)$ are Interior Pareto Optimal points in $<x_1, x_3>$ and $<x_2, x_4>$ respectively.

By reasoning as in Theorem 1, if we consider the subset $A = \{x_1, x_2, x_3, x_4\}$ and denote $F(<A>) = w$, we can also ensure that $w \notin A$ and that either $w \in <x_2, x_4>$ or $w \in <x_1, x_3>$. Without loss of generality we assume that $w \in <x_1, x_3>$, $x_1^1 < x_1^3$ and we are going to analyze the possible location of alternatives of A. The configuration of alternative x_1, x_3 and w as well as the different possibilities to locate alternative x_2 (which have been denoted as A_1, B_1, C_1, D_1 and E_1) are shown in Fig. 11.
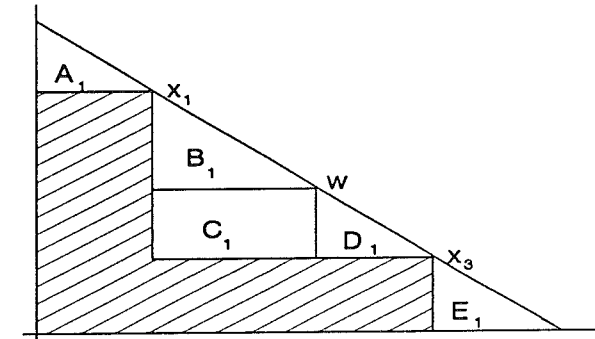

Fig. 11

CASE 1: $x_2 \in A_1$.

By applying (d), the possibilities to locate x_4 are C_1, D_1 or E_1. However in all of these situations by considering $<x_1, x_2, x_4>$ and by applying (PO) together with (c), it is obtained that $F(<x_1, x_2, x_4>)$ belongs to $seg(x_2, x_1)$ or $seg(x_1, x_4)$, both cases in contradiction with (IIA)).

CASE 2: $x_2 \in B_1$.

By reasoning as above, possibilities for x_4 are given by C_2, D_2 or E_2 (see Fig.12). However if x_4 belongs to C_2 or D_2, it implies that $x_2 \in <x_1, x_4>$, in contradiction with (b), and if $x_4 \in E_2$, then by considering $<x_2,x_3,x_4>$ and reasoning as in Case 1, a contradiction is also obtained.
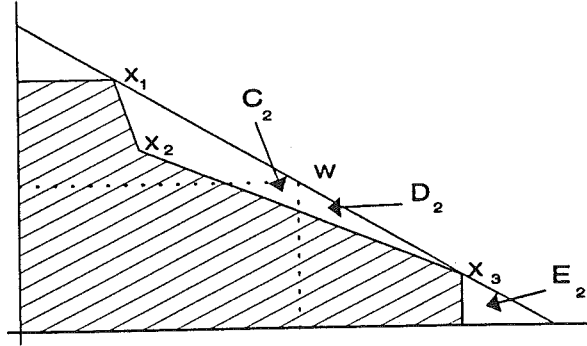

Fig. 12

CASE 3: $x_2 \in C_1$.

In this case possibilities for x_4 are A_3, B_3, C_3 or D_3 (see Fig.13). But all of them imply a contradiction:

if $x_4 \in A_3$ or D_3, the contradiction is obtained by considering $<x_1,x_2,x_4>$ or $<x_2,x_3,x_4>$, respectively, and by reasoning as in Case 1.

if $x_4 \in B_3$ or C_3, then $x_2 \in <x_3,x_4>$ or $x_2 \in <x_1,x_4>$, respectively, both cases in contradiction with (b).
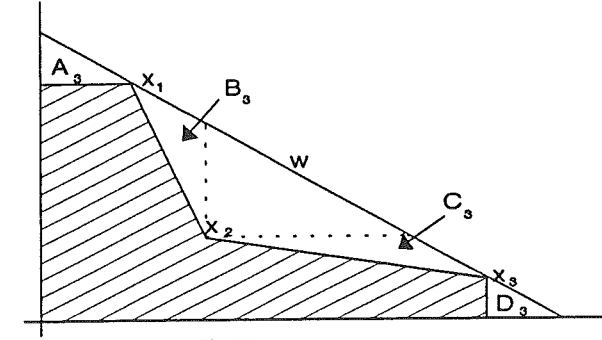
Fig. 13

CASE 4: $x_2 \in D_1$.

Now Fig.14 shows possibilities for locating x_4 (A_4 and B_4) but in both cases the same contradictions as in the previous case are obtained.
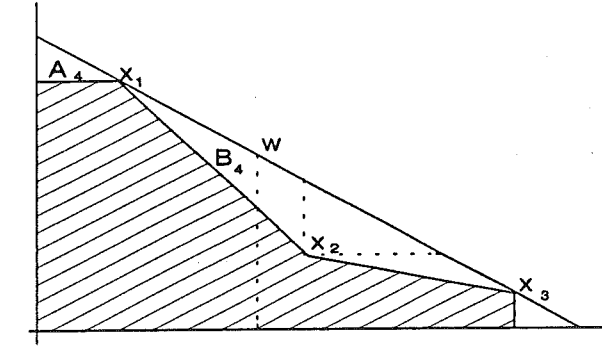
Fig. 14

CASE 5: $x_2 \in E_1$.

Finally, in this case the way of reasoning is exactly the same as in Case 1 but by considering the subset $<x_2,x_3,x_4>$.

Therefore we can conclude that there do not exist cycles of length 4. ∎

Proof of Theorem 5: As in Theorem 1 we prove the acyclicity of the revealed preference by induction. By applying Theorem 6 and Lemma 1 we know the non existence of cycles of length 3 and 4, respectively. We assume that there are no cycles of length k $\forall$k<n, n>4 and we also want to prove the non existence of cycles of length n.

By contradiction, assume that there are alternatives $x_1, x_2, ..., x_n \in \mathbb{R}_+^2$ such that $x_1 \, P \, x_2 \, P \, ... \, P \, x_n \, P \, x_1$. Therefore and by reasoning as in the previous cases it is obtained that:

(1): $x_j \notin <x_i>$ $\quad \forall i \neq j$

(2): $x_j \notin <x_i, x_{i+1}>$ $\quad \forall i=1,2,...n, \, j \neq i, i+1$

(3): $\forall B \subseteq A$ such that B has two non consecutive alternatives or $|B| \geq 3$,

$\quad F(<B>) \notin B$

(4): $F(<x_j, x_{j+s}>) \leq w$ $\quad \forall j \in \{1,...,n\}: s>1$

(5): $F(<B>) \leq w$ $\quad \forall B \subseteq A: |B| > 2$

Moreover if we consider A = $\{x_1, x_2,...,x_n\}$ and denote by w = $F(<A>) \in <A>$, we also know that w $\notin$ A and that there exist alternatives $x_j, x_{j+k} \in$ A, k>1, such that w $\in <x_j, x_{j+k}>$ (in fact, by applying (c) and (PO), w $\in seg(x_j, x_{j+k})$).

Without loss of generality we assume that $x_i^1 < x_{i+k}^1$. Now we analyze the configuration of alternatives $x_{i+1}, x_{i+2}, ..., x_{i+(k-1)}$.

CASE 1. Assume that k>2.

Possibilities for locating alternative x_{i+1} have been denoted as A_1, B_1, C_1, D_1 and E_1 (see Fig.15)
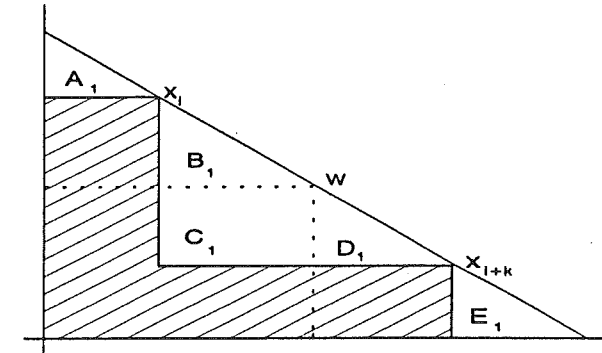

Fig. 15

CASE 1.1. $x_{i+1} \in A_1$.

Then by applying (4) we know that x_{i+n-1} can be placed in C_1, D_1 or E_1, but in all of these cases we obtain a contradiction by considering $F(<x_{i+1}, x_i, x_{i+n-1}>)$, since by (PO) and (3) we know that $F(<x_{i+1}, x_i, x_{i+n-1}>)$ belongs to $<x_{i+1}, x_i>$ or $<x_i, x_{i+n-1}>$, both cases in contradiction with (IIA).
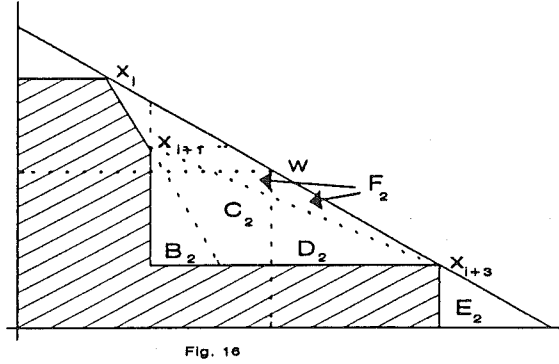
CASE 1.2. $x_{i+1} \in D_1$ or $x_{i+1} \in E_1$.

Alternative x_{i+1} can not be located in these areas since we are assuming that k>2 and by (4) we know that $F(<x_{i+1}, x_{i+k}>) \leq w$.

CASE 1.3. $x_{i+1} \in B_1$.

In this case we have to distinguish two possibilities:

1.3.a. $\boxed{k = 3}$.

In this case we analyze the location of x_{i+2}, which can be placed in B_2, C_2, D_2, E_2 or F_2 (see Fig. 16).


Fig. 16

However all of these possibilities imply a contradiction.

In concrete if $x_{i+2} \in B_2$ or $x_{i+2} \in F_2$, we obtain a contradiction by reasoning as in the Case 1.1 but by considering $<x_i,x_{i+1},x_{i+2}>$ and $<x_{i+1},x_{i+2},x_{i+3}>$ respectively.

If $x_{i+2} \in E_2$, then the contradiction appears when we locate x_{i+4} ($x_{i+4} \neq x_i$ since n>4), which has to be located in C_2 (from (4) $F(<x_{i+1},x_{i+4}>) \leq w$ and $F(<x_{i+2},x_{i+4}>) \leq w$), by considering $F(<x_{i+2},x_{i+3},x_{i+4}>)$ and by reasoning as in Case 1.1.

If $x_{i+2} \in D_2$, by reasoning as in the previous case, x_{i+4} has to be located in C_3 (see Fig. 17), but in this case $x_{i+2} \in <x_{i+3},x_{i+4}>$, in contradiction with (4).

Fig. 17
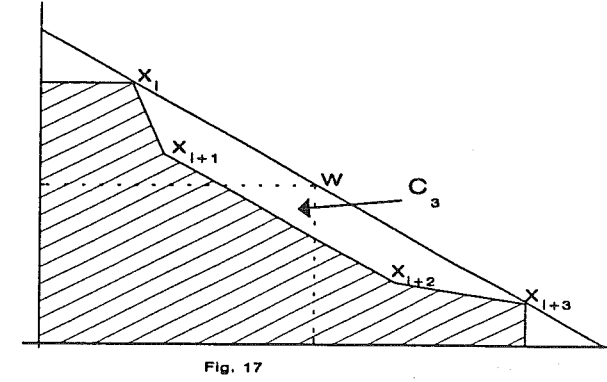
Therefore the only possibility for locating alternative x_{i+2} is C_2 and moreover $x_{i+2} \in <x_{i+1},x_{i+3}>$, if not we have the same contradiction as in Case 1.1. by considering $<x_{i+1},x_{i+2},x_{i+3}>$. So the configuration of these alternatives is as shown in Fig. 18.

Fig. 18
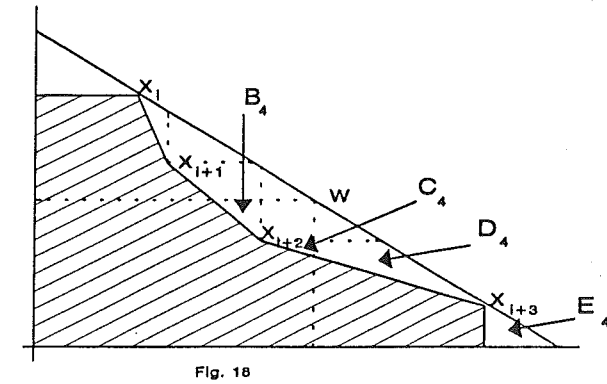
Now, by applying (4) we know that x_{i+4} can be located in B_4, C_4, D_4 or E_4.

B_4: if $x_{i+4} \in B_4$, then $x_{i+2} \in <x_{i+3}, x_{i+4}>$, in contradiction with (4).

E_4: if $x_{i+4} \in E_4$, then by considering $F(<x_{i+2}, x_{i+3}, x_{i+4}>)$ a contradiction as in Case 1.1 is obtained.

C_4, D_4: if $x_{i+4} \in C_4$ or D_4 and $x_{i+5} = x_j$, then we have a contradiction with (4) since in this case $x_{i+1} \in <x_j, x_{i+4}>$. If $x_{i+5} \neq x_j$ then the possibility for locating x_{i+5}, by reasoning in a way analogous to that of x_{i+4}, is below the horizontal line which passes through x_{i+2} (if not, a contradiction with (2) is obtained); and the same reasoning can be applied to all of the alternatives until alternative x_{i+n-1} is reached. In this moment a contradiction is obtained since this alternative has to be below the previously mentioned line, but it implies that $x_{i+1} \in <x_j, x_{i+n-1}>$, a contradiction with (2).

1.3.b. $\boxed{k \neq 3}$.

In this case and by following a similar argument to the one used in the previous case we obtain that the final configuration of alternatives x_j, x_{i+1}, ... , x_{i+k-1}, x_{i+k} is the one which is shown in Fig. 19. The contradiction which is obtained is the same as in the previous case when we try to locate alternatives x_{i+k+1}, ..., x_{i+n-1}.
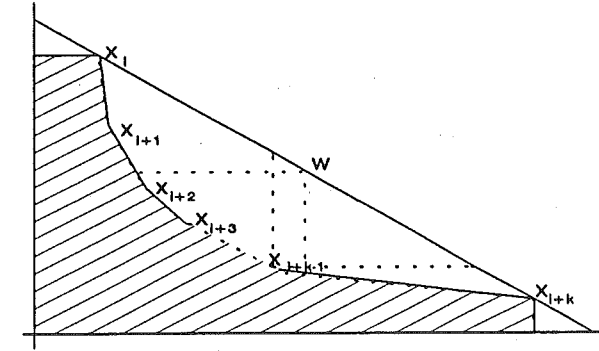

Fig. 19

CASE 1.4. $x_{i+1} \in C_1$.

The contradiction and the way of reasoning is exactly the same as in the case of B_1, although configuration of alternatives will be the one which is shown in Fig.20.
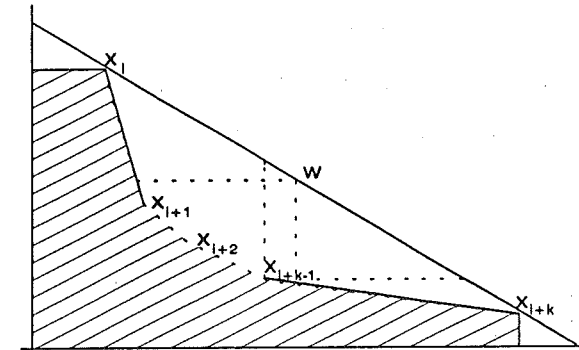

Fig. 20

CASE 2. $k = 2, n > 4$.

Possibilities for x_{i+1} have been denoted as A_5, B_5, C_5, D_5 and E_5 (see Fig.21). If $x_{i+1} \in A_5$, contradiction is obtained when we locate x_{i+n-1}, since by applying (4) it has to be placed in C_5, D_5 or E_5, but then by

considering $<x_j,x_{i+1},x_{i+n-1}>$ we obtain the same contradiction as in Case 1.1.

If $x_{i+1} \in E_5$ then x_{i+3} has to be placed in A_5, B_5, C_5 or D_5 (see Fig.21), but in each of these situations a contradiction is obtained by considering $<x_{i+1},x_{i+2},x_{i+3}>$ and by reasoning again as in Case 1.1.
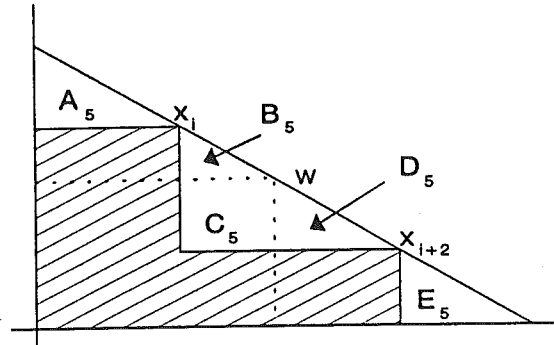

Fig. 21

With respect to B_5, C_5 and D_5, the kind of reasoning which implies the contradiction is analogous to the previous cases:

if $x_{i+1} \in B_5$ then, by applying (4), x_{i+n-1} has to be located in C_6 (see Fig.22), which is a contradiction with (2) $(x_{i+1} \in <x_{i+n-1},x_j>)$.
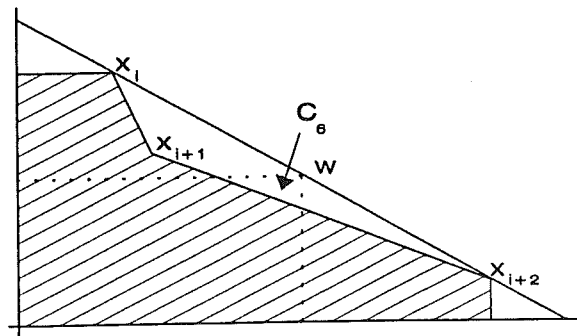

Fig. 22

if $x_{i+1} \in D_5$, then the contradiction is obtained when we locate x_{i+3}, since by (4) it has to be placed in C_7 (see Fig. 23) in contradiction with (2) $(x_{i+1} \in <x_{i+2},x_{i+3}>)$.

if $x_{i+1} \in C_5$ the way of reasoning in order to obtain the contradiction is exactly the same as in the case of C_4 or D_4.
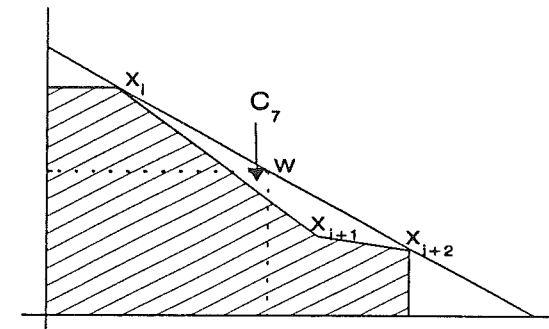

Fig. 23

So under the assumption of the existence of a cycle of length n we have proved that any of the possibilities for locating the alternatives of the cycle imply a contradiction, therefore we can conclude that there are no cycles of length n. ∎

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