A MODEL OF MULTIPRODUCT PRICE COMPETITION*

Yair Tauman, Amparo Urbano and Junichi Watanabe**

WP-AD 96-07

* A preliminary version of this paper was presented in the Second Summer Meeting on "Implementation and Issues in Game Theory", held in Valencia, June 22-24, 1996. We thank participants for helpful comments. We would also like to thank David Schmiedler and Yan Chen for very helpful remarks. This research has been partially supported by D.G.I.C.Y.T under Project PB93-0684. A previous draft of this paper appears as a Working Paper (1996) at the "Instituto Valenciano de Investigaciones Económicas" (IVIE).

** Yair Tauman and Junichi Watanabe: State University of New York at Stony Brook. Amparo Urbano: University of Valencia.

Editor: **Instituto Valenciano de Investigaciones Económicas, S.A.**
Primera Edición Julio 1996.
ISBN: 84-482-1259-2
Depósito Legal: V-2454-1996
Impreso por Copisteria Sanchis, S.L.,
Quart, 121-bajo, 46008-Valencia.
Printed in Spain.

A MODEL OF MULTIPRODUCT PRICE COMPETITION

Yair Tauman, Amparo Urbano & Junichi Watanabe

ABSTRACT

Strategic interaction in oligopolistic markets has been extensively studied in the literature. This literature deals mostly with the case of multiple firms which produce a homogeneous good or goods that are perfect substitutes. In this paper we provide a simple model of price competition in a multiproduct oligopoly market. We find that a pure strrategy equilibrium exists and that the equilibrium consumption sets are efficient since they maximize the total social surplus. If the willingness to pay function of the consumer is convex, the set of equilibrium prices coincides with the core of a related game and the firms extract total industry surplus. If it is concave, the only equilibrium price of a product is its marginal contribution to the consumer's total willingness to pay. If the products are perfect substitutes we obtain the standard Bertrand equilibrium.

Keywords: Multiproduct Oligopolistic Competition; Efficient Consumption Sets.

1. Introduction

Strategic interaction in oligopolistic markets has been extensively studied in the literature, the Cournot (1838) quantity competition and the Bertrand (1883) price competition models being the most well known. The literature on this subject deals mostly with the case of multiple firms which produce a homogeneous good or goods that are perfect substitutes (for an extensive summary see Shapiro (1989)). Relatively little work has been done on multiproduct oligopolistic competition.

Most of the literature on strategic multiproduct oligopolistic competition deals with product differentiation where each firm produces a single product with different characteristics. Hotelling (1929), d'Aspremont, Gabszewicz and Thisse (1979), Salop (1979) and others considered modes of spatial competition where firms differ just in their location. Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982, 1983) analyzed models of quality differentiation where a number of firms produce substitute goods that differ just in quality and sell it to a continuum of consumers, identical in tastes but differing in income. Each consumer has a unit demand and either makes no purchase or else buys exactly one unit from only one firm. In the spatial models as well as in the quality differentiation models firms try to relax price competition through product differentiation.

Another related literature on multiproduct competition is the theory of contentable markets as summarized in Baumol, Panzer and Willig (1982). Their formulation is essentially a model of price competition between multiproduct incumbent firms and potential entrants. They argue that the essential features of a perfectly competitive market emerge even if the number of incumbents is small. Baumol, Bailey and Willig (1977) showed that under certain conditions, a multiproduct natural monopoly can set prices for its products that are sustainable against competitive entry. The potential competition between the monopolist incumbent and the entrants forces the monopolist to charge zero profit prices. Mirman, Tauman and Zang (1985) showed that sustainable prices are the Nash equilibrium outcome of a standard multiproduct Bertrand game played by the incumbent and by infinitely many potential entrants. The main problem is that, in general, such sustainable prices do not exist.

Finally, Milgrom and Roberts (1990) and Milgrom and Shanon (1994) showed that under certain restrictions on the demand and cost functions pure strategy equilibrium in a standard multiproduct Bertrand competition does exist, if the products are substitutes or if they are complements.

In this paper we provide a simple model of price competition in a multiproduct oligopoly market. The products can be of a very general nature, and not necessarily substitutes or complements. To develop a tractable model, we make some simplifying assumptions. We assume that every firm produces one good and consumers are all identical. Unlike the literature on quality differentiation, the products of the firms in our model are exogeneously given and they are not strategic variables of the firms. Similar to Shaked and Sutton (1982, 1983), each consumer consumes either one or zero units of each of these products. A consumer is characterized by his willingness to pay for every subset of products. Furthermore, the marginal cost of production is fixed, but may differ from product to product. The firms are engaged in a price competition in the first stage and consumers make their consumption decisions in the second stage.

Our model resembles some of the features of the monopolistic competition (see Spence (1976) and Dixit and Stiglitz (1977)). In both models there is one consumer and every firm produces only one product — with constant marginal cost. However, our model differs from the above two models in two major aspects. First, a price change by one firm may have in our model a significant effect on the demand for the other products. Secondly, the set of firms in our model is fixed and thus we depart from the free entry assumption. Firms in our model can make positive profits.

We find that a pure strategy equilibrium always exists and that the equilibrium consumption sets are always efficient since they maximize the total social surplus. That is, they maximize the difference between the consumer's willingness to pay and total production cost. The equilibrium prices depend on the characteristic function of a consumer. This is the willingness of a consumer to pay as a function of the subset of products he consumes. We show that if the characteristic function is convex, the set of equilibrium prices coincides with the core of that characteristic function and hence the firms extract the total industry surplus. If the characteristic function is concave, the only equilibrium price of a product is the marginal contribution of this product to the total willingness to pay of a consumer. In this case the consumer obtains a positive share of the surplus. If the products are perfectly substitutes we obtain the standard Bertrand equilibrium, and the consumer extracts the entire surplus.

The results of this paper are mainly driven by the fact that there is only one type of consumer who can consume each product in quantities of zero or one. If one departs from these assumptions, then non-existence of a pure

strategy equilibrium can be obtained. Also, the equilibrium outcome, when it exists, may not be efficient.

We believe however that the existence and the efficiency results of this paper can be extended to the case where firms produce multiple products. Unfortunately, we did not succeed to prove this.

2. The Model

Consider n firms and one consumer (or a set of identical consumers). Each firm produces a single good, and different firms may produce different goods. Let $N = \{1, 2, \ldots, n\}$ be the set of firms; we will use the same notation for the set of goods. The consumer, denoted by 0, consumes either one or zero units.

A consumption set of the consumer is a subset S of N. The consumer consumes one unit of each of the goods in S. For each $S \subseteq N$ let $v(S)$ be the total willingness to pay of the consumer for the consumption set S. We will refer to v as the value function of the consumer. It is assumed that $v(\phi) = 0$. The value function, v, can be derived by standard primitives. Suppose that the consumer's utility is given by $u(x_1, \ldots, x_n, m) = f(x_1 \ldots x_n) + m$, where m is a monetary numerare and $(x_1 \ldots x_n)$ is a consumption bundle. Then $f(x_1 \ldots x_n)$ measures the monetary value of the bundle $(x_1 \ldots x_n)$. Let $S \subseteq N$

be a consumption set and let x^S be the corresponding quantities consumed. Namely, $x_k^S = 1$ if $k \in S$ and $x_k^S = 0$ if $k \notin S$. The value function v is defined to be $v(S) = f(x^s)$.

For every firm i let c_i be the (fixed) unit cost of production. The sequence of events is as follows. First, each firm i selects its price $p_i \in \mathbb{R}_+$ simultaneously and independently. Then, the consumer who observes $p = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ selects a consumption set S as a function of p. The payoff of each firm is its profit and the payoff of the consumer is his surplus. Formally, this can be described as a strategic game of $n+1$ players $0, 1, \dots, n$. The strategy set of every $i \epsilon N$ is R_+ and the strategy set of the consumer is $S_0 = \{\underset{\sim}{S} | \underset{\sim}{S} : R_+^n \to 2^N\}$. The payoff function for $i \in N$ is given by

$$h_i(p, \underset{\sim}{S}) = \begin{cases} p_i - c_i & i \in \underset{\sim}{S}(p), \\ 0 & i \notin \underset{\sim}{S}(p), \end{cases}$$

where $\underset{\sim}{S}(p)$ is the consumption set of the consumer corresponding to $p \in R_+^n$. The payoff of the consumer is given by

$$h_0(p, \underset{\sim}{S}) = v(\underset{\sim}{S}(p)) - \Sigma_{\underset{\sim}{S}(p)} p_k.$$

Denote by NE the set of all pure strategy subgame perfect equilibrium points of the above game.

Let $(p, \underset{\sim}{S})$ be in NE. Then p is called an NE-price vector, $S = \underset{\sim}{S}(p)$ an NE-consumption set and (p, S) an NE-outcome.

Definitions

(1) v is *monotonic* iff $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. It is strictly monotonic if $v(S) < v(T)$ whenever $S \subset T \subseteq N$.

The monotonicity of v implies that the willingness to pay of a consumer increases for larger consumption sets.

(2) v is *convex* iff

$$v(S+i) - v(S) \leq v(T+i) - v(T)$$

whenever $S \subseteq T \subseteq N - i$.

The convexity of v implies that the amount a consumer is willing to pay for an additional product increases with the number of products he is consuming. This reflects a kind of complementarity among products.

(3) v is *concave* iff

$$v(S+i) - v(S) \geq v(T+i) - v(T)$$

whenever $S \subseteq T \subseteq N - i$.

The concavity of v implies that the amount a consumer is willing to pay for an additional product decreases with the number of products he is consuming. This reflects a kind of substitutability among products.

Remark: It is easy to verify that v is convex iff

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T),$$

for all two subsets S and T of N.

(4) v is *additive* iff $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \phi$.

Definition. The *core of* v is denoted by cv and is defined as follows:

$$cv = \{p \in R^n | \Sigma_S p_k \geq v(S), \forall S \subset N, \text{ and } \Sigma_N p_k = v(N)\}.$$

Example 1. A single product Bertrand competition. Let $N = \{1, 2, \ldots, n\}$, and suppose that

$$v(S) = \begin{cases} 1 & S \neq \phi, \\ 0 & S = \phi. \end{cases}$$

This is the case where the firms produce the same good as each other and the consumer consumes one unit at most. Suppose that $c = (c_1, \ldots c_n)$ where $c_1 \leq c_2 \leq \ldots \leq c_n$ and suppose that $c_1 < c_2$. If $c_1 > 1$ then obviously no trade takes place (i.e. $S = \phi$). If $c_1 \leq 1$ then the equilibrium consumption set is $S = \{1\}$. A vector $p = (p_1, \ldots, p_n)$ which satisfies the following condition is an equilibrium price vector:

$$p_1 = \min(c_2, 1), \quad p_2 = c_2 \text{ and } p_k \geq c_2 \text{ for all } 3 \leq k \leq n.$$

Example 2. Let $N = \{1, 2, 3\}$. Suppose that firm 1 produces a product which contains two ingredients A and B. Firm 2 produces a product which contains the two ingredients A and C and firm 3 produces a product which contains the two ingredients B and C. The consumer needs to consume all three ingredients A, B and C. This is described by the value function

$$v(S) = \begin{cases} 1 & |S| \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that $c = 0$. Because of the perfect competition between the three firms the only equilibrium price vector which supports a nonempty consumption set is $p = (0, 0, 0)$. Hence, the consumer obtains the entire surplus. The equilibrium price vector supports any consumption set S s.t. $|S| \geq 2$. Note that $p = (0, 0, p_3)$ together with $S = \{1, 2\}$ is not an equilibrium price for $p_3 > 0$ since 1 (or 2) is better off raising its price slightly. Also $p = (1/2, 1/2, 1/2)$ together with $S = \{1, 2\}$ is not an equilibrium price, since 3 is better off slightly reducing its price to become a seller. Finally, observe that $((2, 2, 2), \phi)$ is also an equilibrium outcome.

Example 3. A market of two righthand gloves and one lefthand glove. Here

$N = \{1,2,3\}$ and

$$v(S) = \begin{cases} 1 & 3 \in S \text{ and } |S| \geq 2, \\ 0 & \text{otherwise} \end{cases}$$

(player 3 is the lefthand glove producer). Let $c = 0$. Then the only equilibrium price which supports the consumption set N is $p = (0,0,1)$. This price is the unique element of the core of v. The consumption set $\{1,3\}$ is supported by the prices $p = (0,\alpha,1)$ where $\alpha \geq 0$ and $\{2,3\}$ is supported by $p = (\alpha,0,1), \alpha \geq 0$. Note that in every equilibrium, firm 3 exploits the entire willingness to pay of the consumer leaving him with a zero surplus.

Example 4. A market of four firms, the first two producing righthand gloves and the other two producing lefthand gloves. The consumer consumes just one pair of gloves. Let $A = \{1,2\}$ and $B = \{3,4\}$. Then v is defined as follows:

$$v(S) = \begin{cases} 1 & S \cap A \neq \phi \text{ and } S \cap B \neq \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Every equilibrium outcome with nonempty consumption set is of the form (p,S) where $S \subseteq N, S \cap A \neq \phi, S \cap B \neq \phi$, and $p = 0$. Note that firms 1 and 2 (and similarly 3 and 4) are engaged in a perfect competition since the consumer needs just one of them. This competition drives the relevant prices to zero leaving the entire surplus to the consumer.

Example 5. Let $N = \{1,2,3,4\}, A = \{1,2\}$ and $B = \{3,4\}$ and let $c = 0$. Define for $0 < a \leq 1$

$$v(S) = \begin{cases} 1 & S \cap A \neq \phi \text{ and } S \cap B \neq \phi \text{ and } S \neq N, \\ 1+a & S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Let $c = 0$. Then the equilibrium outcomes with nonempty consumption set are (p,N), where $p = (x,x,a-x,a-x)$ and $0 \leq x \leq a$.

The firms in this example are less competitive since the consumer prefers two pairs of gloves on one pair. Thus, there is a "room" for every firm. The consumer obtains $1-a$ and the firms obtain a total of $2a$. If $a = 1$ the firm extracts the entire surplus. As a decreases the competition increases and the benefit of the consumer also increases. When a approaches zero the consumer obtains the entire surplus.

Example 6. Let $N = \{1,2,3,4\}$, let $c = 0$ and suppose that

$$v(S) = \begin{cases} 1 & S \supseteq \{1,2\}, S \neq N, \\ 1.5 & S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Then v is convex and monotonic. There is a multiplicity of equilibria, and the set of equilibrium prices which support the consumption set N is cv, the core of v. For instance if $p = (1/2, 1/2, 1/4, 1/4)$ then (p, N) is an equilibrium outcome. Since every $p \in cv$ satisfies $\Sigma_N p_k = v(N)$, the firms extract in every equilibrium point the entire surplus. Note that $((1/3, 1/3, 1/4, 1/4), N)$ is not an equilibrium outcome. Since firm 1 could raise the price up to 2/3 without losing its market. Observe that if $p = (1/2, 1/2, 1/2, 1/2)$ then $(p, \{1,2\})$ is an equilibrium outcome. Actually, $(p, \{1,2\})$ is an equilibrium outcome iff $p_1 + p_2 = 1$ and either $p_3 \geq 1/2$ or $p_4 \geq 1/2$ (or both)[1].

Remark: Note that if $v(i) > c_i$ for some $i \in N$ then (p, ϕ) cannot be obtained in equilibrium; i will always be better off reducing its price p_i below $v(i)$ but above c_i to become a seller and to extract positive profits.

Proposition 1. (p, N) is an NE-outcome iff $p \geq c$ and

(i) *Consumer Optimality.* $v(N) - \Sigma_N p_k \geq v(S) - \Sigma_S p_k$ for all $S \subseteq N$;

(ii) *Firm Optimality.* For every $j \in N$ there is $S_j \subset N$ *s.t.* $j \notin S_j$ and

$$v(N) - \Sigma_N p_k = v(S_j) - \Sigma_{S_j} p_k.$$

Note that the set S_j in condition (ii) may be empty. In this case $v(N) - \Sigma_N p_k = 0$ and the firms extract the entire surplus.

[1] Suppose that $\bar{S} = \{1, 2\}$ and $p_3 < 1/2$ and $p_4 < 1/2$. Then firm 3 which obtains zero could set a price p_3 s.t. $0 < p_3 < 1/2 - p_4$, to induce the consumer to switch to $\bar{S} = N$ and to obtain positive share of the surplus. In this case firm 3 will obtain positive payoff.

Proof. Suppose that (p, N) is an NE-outcome. Condition (i) is implied by the subgame perfection requirement. Suppose next that (ii) does not hold. Then by (i) for some $j \in N$ and for every $S_j \subset N$ *s.t.* $j \notin S_j$,

$$v(N) - \Sigma_N p_k > v(S_j) - \Sigma_{S_j} p_k. \qquad (1)$$

Thus firm j is better off charging a price $p_j + \epsilon$ provided that $\epsilon > 0$ is sufficiently small s.t. (1) holds for all $S_j \not\ni j$. The consumer observing the price vector $(p_{-j}, p_j + \epsilon)$ will again choose the consumption set N.

Conversely, if conditions (i) and (ii) hold then (p, N) is an NE-outcome since N is a best choice for the consumer and no firm has the incentive to either reduce or increase its price. □

Proposition 1'. Suppose that $S \neq N$. Then (p, S) is an NE-equilibrium outcome iff $p_k \geq c_k$ for every $k \in S$ and

(i) $v(S) - \Sigma_S p_k \geq v(T) - \Sigma_T p_k$ for all $T \subseteq N$;

(ii) for every $j \in S$, there is $S_j \subset N$ s.t. $j \notin S_j$ and

$$v(S) - \Sigma_S p_k = v(S_j) - \Sigma_{S_j} p_k;$$

(iii) for every $i \notin S$ and for every $T \ni i$,

$$v(S) - \Sigma_S p_k \geq v(T) - \Sigma_{T \setminus i} p_k - c_i.$$

The proof of Proposition 1' is similar to that of Proposition 1. Observe that condition (iii) guarantees that no firm outside S benefits from a price reduction, since S remains a best choice for the consumer even if i reduces the price to its marginal cost level.

To motivate a certain restriction of the class NE let us consider Example 2 above. We wish to eliminate the equilibrium outcome $((p_1, p_2, p_3), \phi)$ where $p_i \geq 1, i = 1, 2, 3$. The "problem" with this type of outcome is that all three firms charge unreasonably high prices so that no individual firm can benefit from a price reduction of its product only. To rule out this type of equilibrium outcome we restrict the analysis to a certain subset NE^* of NE, defined below.

Let p be in $\mathbb{R}^n$ and let $S \subseteq N$. Denote by p^S the element in $\mathbb{R}^n$ s.t.

$$p_i^S = \begin{cases} p_i & i \in S, \\ 0 & i \notin S. \end{cases}$$

Definition. For every triple (N, v, c) define

$$NE^* = \left\{ (p, \underset{\sim}{S}) \in NE | p \geq c \text{ and } ((p^{N\setminus T}, c^T), \underset{\sim}{S}) \in NE \text{ for every } T \subset N \setminus \underset{\sim}{S}(p) \right\}$$

That is, $(p, \underset{\sim}{S})$ is in NE^* if it is a subgame perfect equilibrium of the game and it remains so even if some of the nonproducing firms (outside S) set marginal cost prices.

Proposition 1". (1) Suppose that $S \neq N$. Then (p, S) is an NE^*-outcome iff $p \geq c$, conditions (i) and (ii) of Proposition 1' hold, and (iii) for every $A \subseteq N \setminus S$ and for every $T \supseteq A$

$$v(S) - \Sigma_S p_k \geq v(T) - \Sigma_{T \setminus A} p_k - \Sigma_A c_k$$

(2) If $S = N$ then (p, S) is an NE^*-outcome iff it is an NE-outcome.

Note that $p = (1/2, 1/2, 1/2, 1/2)$ together with $S = \{1, 2\}$ is not an equilibrium price in NE^* for Example 6, above. If firms 3 and 4 jointly reduce their prices so that $p_3 + p_4 < 1/2$ then the best choice of the consumer is to purchase all four products.

Proof of Proposition 1". Suppose first that $(p, \underset{\sim}{S}) \in NE^*$ and let us prove that the three conditions hold. The first two conditions hold since $(p, \underset{\sim}{S}) \in NE$. Let $A \subseteq N \setminus S$ and let $T \supseteq A$. Since $(q, \underset{\sim}{S}) \in NE$ where $q = (p^S, c^{N \setminus S})$ we have by condition (i) that

$$v(S) - \Sigma_S q_k \geq v(T) - \Sigma_T q_k.$$

Since $q \leq p$ and $q^A = c^A$

$$v(S) - \Sigma_S p_k \geq v(T) - \Sigma_{T \setminus A} q_k - \Sigma_A q_k \geq v(T) - \Sigma_{T \setminus A} p_k - \Sigma_A c_k$$

and condition (iii) holds for $(p, \underset{\sim}{S})$.

Suppose next that the three conditions (i), (ii), and (iii) hold for $(p, \underset{\sim}{S})$ where $p \geq c$ and let us prove that $(p, \underset{\sim}{S}) \in NE^*$. It is sufficient to prove that $((p^S, c^{N\setminus S}), \underset{\sim}{S})$ is in NE. By Proposition 1' it is sufficient to prove that $((p^S, c^{N\setminus S}), \underset{\sim}{S})$ satisfies conditions (i) and (ii). Let us start with (ii). Let $q = (p^S, c^{N\setminus S})$ and let $j \in S$. Since condition (ii) is satisfied for (p, S), there exists $S_j \not\ni j$ s.t.

$$v(S) - \Sigma_S p_k = v(S_j) - \Sigma_{S_j} p_k \leq v(S_j) - \Sigma_{S_j \cap S} p_k - \Sigma_{S_j \setminus S} c_k$$

Applying condition (iii) for (p, S) when $T = S_j$ and $A = S_j \setminus S$ we have

$$v(S) - \Sigma_S p_k \geq v(S_j) - \Sigma_{S_j \cap S} p_k - \Sigma_{S_j \setminus S} c_k.$$

From the last two inequalities we have

$$v(S) - \Sigma_S q_k = v(S) - \Sigma_S p_k = v(S_j) - \Sigma_{S_j \cap S} p_k - \Sigma_{S_j \setminus S} c_k = v(S_j) - \Sigma_{S_j} q_k,$$

and condition (ii) is satisfied for (q, S).

Next, suppose to the contrary that condition (i) is not satisfied for (q, S). Then there is $T \subseteq N$ s.t.

$$v(S) - \Sigma_S p_k = v(S) - \Sigma_S q_k < v(T) - \Sigma_T q_k = v(T) - \Sigma_{T \cap S} p_k - \Sigma_{T \setminus S} c_k.$$

Therefore

$$v(S) < v(T) + \Sigma_{S \setminus T} p_k - \Sigma_{T \setminus S} c_k. \qquad (*)$$

Applying condition (iii) to (p, S) we have for every $A \subseteq N \setminus S$ and for every $\hat{T} \supseteq A$

$$v(S) - \Sigma_S p_k \geq v(\hat{T}) - \Sigma_{\hat{T} \setminus A} p_k - \Sigma_A c_k.$$

In particular if $A = T \setminus S$ and $\hat{T} = T$

$$v(S) - \Sigma_S p_k \geq v(T) - \Sigma_{T \cap S} p_k - \Sigma_{T \setminus S} c_k,$$

contradicting $(*)$. □

Proposition 2. (1) If (p, S) is an NE^*-outcome then $S \neq \phi$, unless $v(T) - \Sigma_T c_k \leq 0$ for every $T \subseteq N$.

(2) (p, N) is an NE^*-outcome iff it is an NE-outcome.

Proof. (1) Let $T \subseteq N$ s.t. $v(T) - \Sigma_T c_k > 0$. Suppose to the contrary that (p, ϕ) is an equilibrium outcome in NE^*. By condition (iii) of Proposition 1" applying to the case where $T = A$

$$0 = v(\phi) - \Sigma_\phi p_k \geq v(T) - \Sigma_T c_k > 0,$$

contradiction.

(2) Follows directly by the second part of Proposition 1". □

From now on we will restrict our analysis to equilibrium points in NE^* only. Hence, in the sequel whenever we refer to equilibrium outcomes we mean outcomes of equilibrium points in NE^*.

3. The Zero Cost Case

In this section we assume that the unit cost of production is zero for all firms in N. The consumer's value function can take positive as well as negative values (unless otherwise is specified). This generalization plays an important role in the analysis of the non-zero cost case in Section 4, below.

Proposition 3. For every value function v, there exists an equilibrium in NE^*. Furthermore, $\bar{S}$ is an equilibrium consumption set if and only if $\bar{S} \in \text{argmax}_{S \subseteq N} v(S)$.

Proposition 3 asserts that in the zero cost case any equilibrium consumption set is efficient as it maximizes the social surplus. This result is generalized to the non-zero cost case in Proposition 9, below.

Proof. Let

$$\bar{S} \in \text{argmax}_{S \subseteq N} v(S).$$

Denote

$$Y_0 = \{p \in \mathbb{R}^n_+ | p_i = 0, \ \forall i \notin \bar{S}\}$$

and

$$Y_1 = \{p \in Y_0 | \Sigma_{\bar{S} \setminus S} p_k \leq v(\bar{S}) - v(S) \ , \ \forall S \subseteq N\}$$

The set Y_1 is nonempty since $0 \in Y_1$. It is bounded since for every $p \in Y_1$ $p_k \leq v(\bar{S}) - v(\bar{S} \setminus k)$ for every $k \in \bar{S}$ and $p_k = 0$ for every $k \notin \bar{S}$.

Also observe that Y_1 is closed and hence compact. Thus Y_1 contains an element, p, which is maximal with respect to the lexicographic order on Y_1. We claim that $(p, \bar{S})$ is an NE^*-outcome. Since $p \in Y_0$ it is sufficient to prove that $(p, \bar{S})$ is an NE-outcome. Since $p \in Y_1$, condition (i) of Proposition 1' holds. As for condition (ii) of Proposition 1', suppose to the contrary that there exists $j \in \bar{S}$ s.t. $v(\bar{S}) - \Sigma_{\bar{S}} p_k > v(T) - \Sigma_T p_k$ for every $T \subseteq N \setminus j$. Let

$$\epsilon = \min_{T \subseteq N \setminus j} \{v(\bar{S}) - v(T) - \Sigma_{\bar{S} \setminus T} p_k\}.$$

Then $\epsilon > 0$. Let $q \in \mathbb{R}^n_+$ be defined as follows

$$q_k = \begin{cases} p_k & k \neq j \\ p_j + \epsilon & k = j. \end{cases}$$

Since p is a maximal element of $Y_1, q \notin Y_1$. On the otherhand we will show that $\Sigma_{\bar{S} \setminus S} q_k \leq v(\bar{S}) - v(S)$ for all $S \subseteq N$ and hence $q \in Y_1$, contradiction. If $S \not\ni j$ then $\Sigma_{\bar{S} \setminus S} q_k \leq v(\bar{S}) - v(S)$ since $q_{-j} = p_{-j}$ and $p \in Y_1$. Suppose next that $S \ni j$. By the definition of ϵ

$$v(\bar{S}) - \Sigma_{\bar{S}} p_k \geq v(S) - \Sigma_S p_k + \epsilon.$$

Hence,

$$\Sigma_{\bar{S}\setminus S} q_k = \Sigma_{\bar{S}\setminus S} p_k + \epsilon \leq v(\bar{S}) - v(S),$$

as claimed. Therefore $(p, \bar{S})$ is an NE^*-outcome.

To complete the proof of Proposition 3 let us show that if $(p, \bar{S})$ is an NE^*-outcome then $\bar{S} \in \text{argmax}_{S\subseteq N} v(S)$. Indeed, suppose that $(p, \bar{S})$ is an NE^*-outcome. Then $(\bar{S}, p^{\bar{S}})$ is an NE^*-outcome. Hence, by condition (i) of Proposition 1"

$$v(\bar{S}) - \Sigma_{\bar{S}} p_k \geq v(T) - \Sigma_{T\cap\bar{S}} p_k$$

for every $T \subseteq N$. Therefore

$$v(\bar{S}) - v(T) \geq \Sigma_{\bar{S}\setminus T} p_k \geq 0.$$

and $v(\bar{S}) \geq v(T)$ for every $T \subseteq N$. □

Corollary 1. Suppose that v is a monotonic value function. Then there exists an NE^*-outcome of the form (p, N). Furthermore, if v is strictly monotonic then N is the only NE^*-consumption set.

Next we define the notion of the T-core for any $T \subseteq N$. This definition is applied to any cost configuration $c = (c_1, \ldots, c_n)$ and not only to the case where $c = 0$.

Definition Let v be a value function and let $T \subseteq N$. The T-core of v is denoted by $c_T v$ and is defined by

$$c_T v = \{p \in \mathbb{R}^n | \Sigma_T p_k = v(T) \quad , \Sigma_{S\cap T} p_k \geq v(S) - \Sigma_{S\setminus T} c_k \text{ for every } S \subseteq N\}.$$

Note that the N-core of v coincides with the core of v.

Let us elaborate on the T-core concept. Suppose that the equilibrium set of sellers is T. Then, firms in $N\setminus T$ obtain zero and will be willing to join the set of sellers, essentially if they just cover their production costs. Hence, every subset S of T can actually achieve $u_T(S)$ where

$$u_T(S) = \max_{A\subseteq N\setminus T} [v(S \cup A) - \Sigma_A c_k].$$

It is easy to verify that the projection of $c_T v$ on T coincides with the core of u_T.

Let us return now to the case where $c = 0$.

The next proposition asserts that core payoffs form a subset of equilibrium payoffs. This is in sharp contrast to the Arrow-Debreu model of perfect competition where the reverse inclusion obtains. There is an essential difference between our model and theirs. Producers are price-takers in their case, who have no effect on the prices, whereas in ours they are price-makers.

In the event that the value function is convex, our model gives an equivalence result in line with Arrow-Debreu with a continuum of agents: core and competitive payoffs coincide (see Proposition 5 below); otherwise the equivalence generally breaks down (see Example 7 below).

Proposition 4. Let v be a value function and let $\bar{S} \in \text{argmax}_{S \subseteq N} v(S)$. If $p \in c_{\bar{S}}v \cap \mathbb{R}^n_+$ then $(p, \bar{S})$ is an NE^*-outcome.

Proof. Since $p \in c_{\bar{S}}v, v(\bar{S}) = \Sigma_{\bar{S}} p_k$ and $v(T) \leq \Sigma_{T \cap \bar{S}} p_k$ for all $T \subseteq N$. Since $p \in \mathbb{R}^n_+$

$$v(\bar{S}) - \Sigma_{\bar{S}} p_k = 0 \geq v(T) - \Sigma_{T \cap \bar{S}} p_k \geq v(T) - \Sigma_T p_k$$

and condition (i) of Proposition 1" holds. Since $v(\bar{S}) - \Sigma_{\bar{S}} p_i = 0$ condition (ii) of Proposition 1" holds for $S_j = \phi$, for every $j \in \bar{S}$. Finally, let $A \subseteq N \backslash \bar{S}$ and let $A \subseteq T \subseteq N$. Since $p \in \mathbb{R}^n_+$

$$v(T) - \Sigma_{T \backslash A} p_k \leq v(T) - \Sigma_{(T \backslash A) \cap \bar{S}} p_k = v(T) - \Sigma_{T \cap \bar{S}} p_k \leq 0 = v(\bar{S}) - \Sigma_{\bar{S}} p_k.$$

Consequently, condition (iii) holds as well and thus $(p, \bar{S})$ is an NE^*-outcome.

□

Remark: It can be easily shown that if $p \in cv \cap \mathbb{R}^n_+$ then $p \in c_{\bar{S}}v \cap \mathbb{R}^n_+$. Thus, by Proposition 4 $(p, \bar{S})$ is an NE^*-outcome.

Next let us show that the converse of Proposition 4 does not hold.

Example 7. Let $N = \{1, 2, 3\}$ and let

$$v(S) = \begin{cases} 1 & |S| = 1 \text{ or } S = \{2,3\} \\ 2 & S = \{1,2\} \text{ or } S = \{1,3\} \\ 3 & S = N. \end{cases}$$

Here $\bar{S} = N$ and if $p = (2, 0, 0)$ then (p, N) is an NE^*-outcome. But $p \notin cv$ (although $cv \neq \phi$).

Note that the set of equilibria price vectors is also not contained in the ε-core[2] even when the ε-core is non-empty. Applying Wooders and Zame (1984) to our framework, for every $\varepsilon > 0$ and for sufficiently large markets the ε-core is non-empty. Consider an extension of Example 2 above.

Example 2'. There are n firms, each producing one product with zero cost. Let

$$v(S) = \begin{cases} 1 & |S| > m, \\ 0 & 0 \leq |S| \leq m, \end{cases}$$

for some m, $1 \leq m < n$. Then, in every NE^*-equilibrium $p_i = 0$ for every n and for all $i = 1, \ldots, n$. But for every $\varepsilon > 0$ the zero price vector is not in the ε-core since it is not efficient.

[2]Let $\varepsilon > 0$. Then $p \in \mathbb{R}^n$ is in the ε-core of v if and only if $\sum_N p_i = v(N)$ and $\sum_S p_i \geq v(S) - \varepsilon|S|$ for every $S \subseteq N$.

In Engle and Scotchmer (1995) and in Wooders (1994) the concept of ε-core is modified to allow ε-core payoffs to be non Pareto-optimal[3]. Still, the equilibrium price of Example 2' may not be in the ε-core. Let $m = 1$, then for p to be in the ε-core it must be that for every i,

$$p_i \geq v(i) - \varepsilon.$$

But $v(i) = 1$ and $p_i = 0$ contradicts the last inequality as long as ε is smaller than 1.

Next we show that if v is convex then the converse of Proposition 4 holds.

Proposition 5. Suppose that v is a convex value function. Then $(p, \bar{S})$ is an NE^*-outcome if and only if $\bar{S} \in \text{argmax}_{S \subseteq N} v(S)$ and $p \in c_{\bar{S}} v \cap \mathbb{R}^n_+$. In particular, in every equilibrium the consumer's payoff is zero and the firms extract the entire surplus.

The convexity of the value function v reflects complementarities among goods and therefore it induces only weak competition among the firms. This enables the firms to extract the entire consumer surplus. For simplicity suppose that $\bar{S} = N$ (which is the case when the value function $v(S)$ is

[3]The only change in the definition of ε-core is that $\sum_N p_i \leq v(N)$ replaces $\sum_N p_i = v(N)$.

increasing in S). Then in equilibrium $\sum_N p_i = v(N)$, and by the Consumer Optimality condition (Condition (i) of Proposition 1) we also have $v(T) - \sum_T p_i \leq v(N) - \sum_N p_i = 0$ for every $T \subseteq N$. Thus, $\sum_T p_i \geq v(T)$ for all T, and therefore p is in the core of v. The difficult part is to establish that the firms indeed extract the total surplus when v is convex.

Proof. By Propositions 3 and 4 it is sufficient to prove that if $\bar{S} \in \text{argmax}_{S \subseteq N} v(S)$ and if $(p, \bar{S})$ is an NE^*-outcome then $p \in c_{\bar{S}} v \cap \mathbb{R}^n_+$.

For every $j \in \bar{S}$, let $S_j \not\ni j$ be a minimal set with respect to inclusion which satisfies

$$v(\bar{S}) - \Sigma_{\bar{S}} p_i = v(S_j) - \Sigma_{S_j} p_i.$$

We will prove that $S_j = \phi$ for all $j \in \bar{S}$. Suppose to the contrary that $S_j \neq \phi$. Let $k \in S_j$. Since $(p, \bar{S})$ is an NE^*-outcome, by condition (i) of Proposition 1'' and by the convexity of v,

$$\begin{aligned} v(\bar{S}) - \Sigma_{\bar{S}} p_i &\geq v(S_j \cup S_k) - \Sigma_{S_j \cup S_k} p_i \\ &\geq v(S_j) + v(S_k) - v(S_j \cap S_k) \\ &- \Sigma_{S_j} p_i - \Sigma_{S_k} p_i + \Sigma_{S_j \cap S_k} p_i \\ &= 2(v(\bar{S}) - \Sigma_{\bar{S}} p_i) - [v(S_j \cap S_k) - \Sigma_{S_j \cap S_k} p_i], \end{aligned}$$

where $S_k \subseteq N \backslash k$ satisfies Condition (ii) of Proposition 1'' if $k \in \bar{S}$ and $S_k = \bar{S}$

otherwise.

Therefore,

$$v(S_j \cap S_k) - \Sigma_{S_j \cap S_k} p_i \geq v(\bar{S}) - \Sigma_{\bar{S}} p_i.$$

This is a contradiction to the minimality of S_j with respect to inclusion, unless $S_j \cap S_k = S_j$. But this is impossible since $k \in S_j$ and $k \notin S_k$. This contradiction implies that $S_j = \phi$, as claimed. Therefore, by condition (iii) of Proposition 1'' applied to $A = T \backslash \bar{S}$ we have for all $T \subseteq N$

$$0 = v(\bar{S}) - \Sigma_{\bar{S}} p_i \geq v(T) - \Sigma_{T \backslash A} p_i = v(T) - \Sigma_{T \cap \bar{S}} p_i.$$

Hence, $p \in c_{\bar{S}} v$. □

Note that Propositions 3 and 5 imply that $c_{\bar{S}} v \cap \mathbb{R}^n_+$ is nonempty for every convex function v.

Our next goal is to characterize equilibrium points of concave value functions. This is done in Proposition 6 below. To state the proposition, we need the following:

Definition. Let v be a value function. Define its dual function w by $w(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$.

Lemma 1 If (p, N) is an NE-outcome then $\sum_S p_k \leq w(S)$ for all $S \subseteq N$.

Proof. Suppose that (p, N) is an NE-outcome. Then by Proposition 1

$$v(N) - \Sigma_N p_k \geq v(N \backslash S) - \Sigma_{N \backslash S} p_k, \quad \forall S \subseteq N.$$

Hence,

$$v(N) - v(N \backslash S) \geq \Sigma_N p_k - \Sigma_{N \backslash S} p_k.$$

Therefore,

$$w(S) \geq \Sigma_S p_k,$$

for all $S \subseteq N$. □

Let

$$w^* = (w(1), \ldots, w(n))$$

where $w(i) = v(N) - v(N \backslash i), \quad \forall i \in N.$

Proposition 6. Suppose that v is concave and $w^* \in R^n_+$. Then, (p, S) is an NE^*-outcome if and only if $p_i = w(i)$ for all $i \in S$ and $v(S) = v(N)$.

The concavity assumption of the value function v implies that the more the consumer consumes, the less he is willing to pay for an additional product. If the products are all perfectly substituted, then this means a diminishing marginal utility of consumption. This is a standard assumption which usually results in price equals marginal utility. It is therefore not too surprising that the equilibrium prices for the concave case are the marginal values of the products.

Proof. Step 1. Let us show that (w^*, N) is an NE^*-outcome. By part (2) of Proposition 1" it is sufficient to prove that conditions (i) and (ii) of Proposition 1 hold. Let $S \subseteq N$ and suppose that $S = \{i_1, \dots, i_k\}$. Then,

$$v(N)-v(N\backslash S) = v(N)-v(N\backslash i_1)+\Sigma_{j=1}^{k-1}[v(N\backslash\{i_1 \dots i_j\})-v(N\backslash\{i_1, \dots, i_{j+1}\})]$$

with the convention that $\Sigma_{j=1}^{0} = 0$. By the concavity of v,

$$v(N\backslash\{i_1 \dots i_j\}) - v(N\backslash\{i_1, \dots, i_{j+1}) \geq v(N) - v(N\backslash i_{j+1}).$$

Hence,

$$v(N) - v(N\backslash S) \geq \Sigma_{j=0}^{k-1}[v(N) - v(N\backslash i_{j+1})] = \Sigma_S w(k).$$

Thus,

$$v(N) - \Sigma_N w(k) \geq v(N\backslash S) - \Sigma_{N\backslash S} w(k), \quad \forall S \subseteq N,$$

and condition (i) of Proposition 1 holds. Condition (ii) of Proposition 1 holds for the sets $S_j = N\backslash j$, for all $j \in N$ since

$$v(N) - \Sigma_N w(k) = v(N\backslash j) - \Sigma_{N\backslash j} w(k).$$

Step 2. Let us show that if (p, N) is an NE^*-outcome, then $p_j = w(j)$ for all $j \in N$. To this end we need the following Lemma.

Lemma 2 Suppose that v is concave. Let $p \in R^n$ be s.t. $p \leq w^*$; then $v(N) - \Sigma_N p_k \geq v(S) - \Sigma_S p_k$ for all $S \subseteq N$. Furthermore, if $p_j < w(j)$ then $v(N) - \Sigma_N p_k > v(S) - \Sigma_S p_k$ for all $S \subseteq N\backslash j$.

Proof of Lemma 2. By induction on n. If $|N| = 1$ then the Lemma clearly holds. Suppose that it holds whenever $|N| = n - 1$. Let $N = \{1, \dots, n\}$ and let $j \in N$. Then,

$$v(N) - \Sigma_N p_k \geq v(N) - \Sigma_{N\backslash j} p_k - w(j) = v(N\backslash j) - \Sigma_{N\backslash j} p_k. \quad (2)$$

Let $N' = N\backslash j$. Then $|N'| = n - 1$ and by the concavity of v

$$p_k \leq v(N) - v(N\backslash k) \leq v(N') - v(N'\backslash k),$$

for all $k \in N'$. Hence, by the induction hypothesis (applied to N')

$$v(N') - \Sigma_{N'} p_k \geq v(S) - \Sigma_S p_k, \quad \forall S \subseteq N'.$$

This together with (2) implies that

$$v(N) - \Sigma_N p_k \geq v(S) - \Sigma_S p_k, \quad \forall S \subseteq N'. \quad (3)$$

Since this is true for every $j \in N$ then (3) holds for all $S \subseteq N$.

Finally, if $p_j < w(j)$ then (2) holds as strict inequality and hence (3) holds as strict inequality. □

Suppose that (p, N) is an NE^*-outcome. By Lemma 1 $p_j \leq w(j)$ for all $j \in N$. Suppose that for some $j \in N$ $p_j < w(j)$. By Lemma 2, the consumer strictly prefers N to every $S \subseteq N\backslash j$ when the price is p. Hence, j has the incentive to raise its price, contradiction. Hence, $p_j = w(j)$ and Step 2 is complete.

Step 3. Let us prove that if (p, S) is an NE^*-outcome then (p^S, N) is also an NE^*-outcome and $v(S) = v(N)$. Indeed, by the definition of NE^* (p^S, S) is an NE^*-outcome. Hence applying condition (i) of Proposition 1" to (p^S, S) we have

$$v(N) - \Sigma_S p_k^S \geq v(N) - \Sigma_N p_k^S = v(N) - \Sigma_S p_k^S. \tag{4}$$

Since v is monotonic (as $w^* \in \mathbb{R}_+^n$)

$$v(N) - \Sigma_S p_k^S \geq v(S) - \Sigma_S p_k^S. \tag{5}$$

Therefore, by (4) and (5)

$$v(N) - \Sigma_N p_k^S = v(S) - \Sigma_S p_k^S \geq v(T) - \Sigma_T p_k^S, \tag{6}$$

for all $T \subseteq N$. Hence (p^S, N) satisfies condition (i) of Proposition 1". By (6) it follows that (p^S, N) satisfies condition (ii) as well. Consequently (p^S, N)

is an NE^*-outcome. Also, since $\Sigma_N p_k^S = \Sigma_S p_k^S$ it follows by (6) that $v(S) = v(N)$ and Step 3 is complete.

We are now ready to complete the proof of Proposition 6. Suppose first that (p, S) is an NE^*-outcome. Then by Step 3 (p^S, N) is also an NE^*-outcome and $v(S) = v(N)$. Hence by Step 2 $p^S = w^*$.

Finally, suppose that $p \in \mathbb{R}_+^n, S \subseteq N$, $v(S) = v(N)$ and $p_i = w(i), i \in S$. Let us show that (p, S) is an NE^*-outcome. By the concavity and by the monotonicity of v, for each $j \in N\backslash S$

$$0 = v(N) - v(S) \geq v(S + j) - v(S) \geq v(N) - v(N\backslash j) \geq 0.$$

Consequently,

$$w(j) = v(N) - v(N\backslash j) = 0.$$

Thus, $p^S = w^*$ and by Step 1 (p^S, N) is an NE^*-outcome. Hence for all $T \subseteq N$

$$v(S) - \Sigma_S p_k = v(N) - \Sigma_S p_k^S \geq v(T) - \Sigma_T p_k^S$$

and condition (i) of Proposition 1" holds for (p^S, S). Condition (ii) holds for $S_i = N\backslash i$. Therefore (p^S, S) is an NE^*-outcome and (p, S) is an NE^*-outcome. □

Corollary 3. If v is concave but not additive and if $w^* \in \mathbb{R}^n_+$ then the consumer obtains a positive payoff.

The assumption that $w^* \in \mathbb{R}^n_+$ (which for concave functions v is equivalent to the monotonicity of v) is crucial for the result obtained in Proposition 6.

Example 8. Let $N = \{1,2,3\}$, $c = 0$, and suppose that

$$v(S) = \begin{cases} 1 & |S| = 1 \\ 1.5 & S = \{1,2\} \\ 1 & \text{otherwise} . \end{cases}$$

Then v is concave and $w^* = (0,0,-\frac{1}{2})$. It is easy to verify that if $(p,\bar{S})$ is an NE^*-outcome then $\bar{S} = \{1,2\}$ and $p = (p_1,p_2,0)$ where $p_1 + p_2 = \frac{1}{2}$ and $p_i \geq 0, i = 1,2$. Therefore $p \neq w^*$. Observe that $\bar{p}_1 = \bar{p}_2 = \frac{1}{2}$ are the marginal contributions of 1 and 2 respectively to $\bar{S}$. However, $\bar{p} = (\frac{1}{2},\frac{1}{2},0)$ together with $\bar{S}$ does not form an NE^*-equilibrium outcome since $\{3\}$ is the consumer's best choice under $\bar{p}$.

Definition A monotonic function v is $0-1$ if for all $S \subseteq N, v(S) = 0$ or $v(S) = 1$ and $v(N) = 1$. A consumption set is said to be an *admissible blend* if $v(S) = 1$. A commodity $i \in N$ is *essential* if it belongs to every admissible blend.

Proposition 7. Let v be a $0-1$ function. Let T be the set of essential commodities in N. If $T \neq \phi$, then (p,S) is an NE^*-outcome iff $v(S) = 1, \Sigma_T p_k = 1$ and $p_j = 0$ for all $j \in N \backslash T$. If $T = \phi$, then (p,S) is an equilibrium outcome iff $v(S) = 1$ and $p = 0$.

Note that if $T \neq \phi$, then the set of NE^*-prices coincides with the core of v. Also note that Proposition 7 does not hold for equilibrium points in NE.

Example 9. Let $N = \{1,2,3,4\}$ and let

$$v(S) = \begin{cases} 1 & S \supseteq \{1,2\} \text{ and } |S| \geq 3, \\ 0 & \text{otherwise} . \end{cases}$$

The set of essential commodities is $T = \{1,2\}$. Let $S = \{3,4\}$ and let $p = (1\frac{1}{2},1\frac{1}{2},0,0)$. Then (p,S) is an NE-outcome (though $p \notin cv$).

The set of NE^*-prices in this example is $\{p \in \mathbb{R}^4_+ | p_1 + p_2 = 1, p_3 = p_4 = 0\}$.

Proof of Proposition 7. Let (p,S) be an NE^*-outcome and let T be the set of essential commodities in N. Suppose first that $T \neq \phi$. By definition (p^S,S) is also an NE^*-outcome. Let $j \in T$ by condition (ii) of Proposition 1" there exists $S_j \subset N$ s.t. $j \notin S_j$ and

$$0 \leq v(S) - \Sigma_S p_k = v(S_j) - \Sigma_{S_j} p_k.$$

Since $j \in T\setminus S_j$ then $v(S_j) = 0$. Hence $\Sigma_{S_j} p_k = 0$ must hold and therefore $v(S) - \Sigma_S p_k = 0$. Next, let $i \notin T$. Then there exists a subset $\hat{S}$ of N s.t. $i \notin \hat{S}$ and $v(\hat{S}) = 1$ (this follows by the definition of essentiality). By condition (i) of Proposition 1''

$$0 = v(S) - \Sigma_S p_k = v(S) - \Sigma_S p_k^s \geq v(\hat{S}) - \Sigma_{\hat{S}} p_k^s = v(\hat{S}) - \Sigma_{S \cap \hat{S}} p_k. \quad (7)$$

Since $v(\hat{S}) = 1 \geq v(S)$ it follows that (4) holds as equality, $v(S) = 1$ and $\Sigma_{S\cap\hat{S}} p_k = \Sigma_S p_k = 1$. Therefore $p_j = 0$ for all $j \in S\setminus\hat{S}$. But $v(S) = 1$ implies that $S \supset T$ and hence $i \in S\setminus\hat{S}$ and $p_i = 0$. We conclude that $\Sigma_T p_k = 1$ and $p_i = 0$ for all $i \notin T$.

Suppose next that $T = \phi$ and let $j \in N$. Since j is not essential there is a subset $\hat{S}$ of N s.t. $j \notin \hat{S}$ and $v(\hat{S}) = 1$. Similarly to (4)

$$v(S) - \Sigma_S p_k \geq v(\hat{S}) - \Sigma_{S\cap\hat{S}} p_k$$

and again it holds as equality. Thus, by condition (i) of Proposition 1''

$$v(\hat{S}) - \Sigma_{S\cap\hat{S}} p_k \geq v(N) - \Sigma_N p_k.$$

Since $v(N) = 1$ we have that $\Sigma_N p_k = \Sigma_{S\cap\hat{S}} p_k$. Consequently, $p_i = 0$ for all $i \in N\setminus(S\cap\hat{S})$. But $j \in N\setminus(S\cap\hat{S})$ and thus $p_j = 0$. This holds for every $j \in N$ hence $p = 0$. □

4. The Non-Zero Cost Case

Throughout this section we fix the set N of firms to be $N = \{1, 2, \ldots n\}$.

Let

$$\mathcal{F} = \{(v,c) | v : 2^N \to \mathbb{R}, v(\phi) = 0 \text{ and } c \in \mathbb{R}^n_+\}.$$

Every element (v,c) of $\mathcal{F}$ is identified with the model consisting of (i) A set N of firms and one consumer, (ii) A vector $c = (c_1 \ldots c_n)$ of firms' marginal costs and (iii) A value function v of the consumer. Let

$$\mathcal{F}_0 = \{(v,c) | (v,c) \in \mathcal{F} \text{ and } c = 0\}.$$

Notice that in Section 3 we dealt only with models in $\mathcal{F}_0$.

The next proposition relates equilibrium outcomes of elements in $\mathcal{F}$ to those in $\mathcal{F}_0$. Let $(v,c) \in \mathcal{F}$ and let $v - c$ be defined for all $S \subseteq N$ by

$$(v-c)(S) = v(S) - \Sigma_S c_k.$$

Proposition 8. Let $(v,0) \in \mathcal{F}_0$. Then (p,S) is an NE^*-outcome of $(v,0)$ if and only if $(p+c, S)$ an NE^*-outcome of $(v+c, c)$.

Proposition 8 asserts that it is sufficient to find the equilibrium outcomes of the zero cost-models in $\mathcal{F}_0$. To find the equilibrium outcomes of a general model (v,c) in $\mathcal{F}$ one finds first the equilibrium outcomes of $(v-c, 0)$. If

(p, S) is an equilibrium outcome of $(v - c, 0)$ then $(p + c, S)$ is an equilibrium outcome of (v, c).

Proof. It is sufficient to prove that (p, S) satisfies condition (x) of Proposition 1" for $(v, 0)$ iff $(p + c, S)$ satisfies the same condition for $(v + c, c)$, where $x = i, ii, iii$. As for condition (i) of Proposition 1" for every $T \subseteq N$

$$v(S) - \Sigma_S p_k \geq v(T) - \Sigma_T p_k \iff v(S) + \Sigma_S c_k - (\Sigma_S p_k + \Sigma_S c_k) \geq$$

$$v(T) + \Sigma_T c_k - (\Sigma_T p_k + \Sigma_T c_k)$$

$$\iff (v + c)(S) - \Sigma_S(p_k + c_k) \geq (v + c)(T) - \Sigma_T(p_k + c_k).$$

The equivalence relation for the other two conditions (ii) and (iii) of Proposition 1" holds similarly. □

Using Proposition 8 we can now generalize the results obtained in Section 3 to the non zero cost case.

Proposition 9. For every $(v, c) \in \mathcal{F}$ there exists an equilibrium in NE^*. Furthermore, $\bar{S}$ is an NE^*-consumption set if and only if $\bar{S} \in \text{argmax}_{S \subseteq N}[v(S) - \Sigma_S c_k]$.

Proof. By Proposition 3 $(v - c, 0)$ has an equilibrium in NE^* with an outcome of the form $(\bar{p}, \bar{S})$ where $\bar{S} \in \text{argmax}_{S \subseteq N}[v(S) - \Sigma_S c_k]$. By Proposition 8 (v, c) has an equilibrium in NE^* with an outcome of the form $(\bar{p} + c, \bar{S})$.

Furthermore, by Proposition 3 every NE^*-consumption set of $(v - c, 0)$ is one which maximizes $v(S) - \Sigma_S c_k$ over $S \subseteq N$. Hence, by Proposition 8 this is also the property of every equilibrium consumption set of (v, c). □

Proposition 10. Let $(v, c) \in \mathcal{F}$. Suppose that v is concave and $w^* \geq c$. Then (p, S) is an NE^*-outcome for (v, c) iff $p_i = w(i)$ for all $i \in S$ and $v(S) = v(N)$.

Proof. Follows by Propositions 6 and 8.

Example 10. Let $N = \{1, 2, 3\}$ and let $c = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Suppose that

$$v(S) = \begin{cases} 1 & |S| = 1 \\ 3/2 & S = \{1, 2\} \\ 2 & S = \{1, 3\} \text{ or } S = \{2, 3\} \\ 9/4 & S = N. \end{cases}$$

Observe that v is concave and hence so is $v - c$.

$$(v - c)(S) = \begin{cases} 3/4 & S = \{1\} \text{ or } S = \{2\} \\ 1/2 & S = \{3\} \\ 1 & S = \{1, 2\} \\ 5/4 & S = \{1, 3\} \text{ or } S = \{2, 3\} \\ 5/4 & S = N. \end{cases}$$

Note that $v - c$ is monotonic and concave hence by Proposition 6 the only NE^*-outcome for $(v - c, 0)$ is (p, N) where $p = (0, 0, 1/4)$. That is, p is the marginal contribution vector which corresponds to $v - c$. By Proposition 9 the only NE^*-outcome for (v, c) is $(p + c, N)$ where $p + c = (1/4, 1/4, 3/4)$. Finally,

Proposition 11. Let $(v, c) \in \mathcal{F}$ where v is a convex value function. Then $(p, \bar{S})$ is an NE^*-outcome if and only if $\bar{S} \in \text{argmax}_{S \subseteq N}[v(S) - \Sigma_S c_i], p \in c_{\bar{S}}v$ and $p \geq c$.

Proof. Follows by Propositions 5 and 8. □

5. Conclusions and Discussion

The paper provides a very simple model of price competition in a multiple product oligopoly market. The products are of a very general nature, but we do need some special assumptions. Every firm produces one good with a constant marginal cost. All consumers are identical and each consumes either zero or one unit of each one of the products. The firms are engaged in price competition in the first stage and consumers make their consumption decisions in the second stage. We have shown that a pure strategy equilibrium always exists and it is efficient as the total social surplus is maximized. If

the willingness to pay of a consumer as a function of his consumption set is convex, then an equivalence result is obtained: the core of the game generated by the willingness to pay function coincides with the set of equilibrium prices. In this case, the prevailing competition is weak and firms extract the entire surplus. If the willingness to pay function is concave, the equilibrium price of a product is its marginal contribution to the total willingness to pay. In this case, the consumers obtain a positive share of the surplus. If the products are perfect substitutes the standard Bertrand equilibrium results, and consumers extract the entire surplus.

An important and nontrivial extension of the above model is to the case where firms produce more than one product. Proposition 1" above which characterizes the NE^*-equilibrium outcomes fails to hold in the multiproduct firms case. More specifically, conditions (ii) and (iii) are no longer valid. Suppose that a certain firm, say 1, produces two substitute products X and Y and suppose that the market prices of all products are such that the consumer consumes X and not Y. Then firm 1 may possibly benefit from raising the price of its product X if this will cause the consumer to switch from X to Y and if the difference between the price of Y and its production cost exceeds that of X. This rules out condition (ii) of Proposition 1". Similarly, it may pay firm 1 not to reduce the price of Y if it will cause the consumer to switch from X to Y and if the difference between the price of X and its production cost exceeds that of Y. Hence condition (iii) of Proposition 1" is no longer valid.

Let us provide a simple example to demonstrate how drastically the equilibrium changes when firms produce more than one good. Consider Example 2 above, with one change. The two products, 1 and 2 are produced by firm 1, and product 3 is produced by firm 2 (firm 3 produces nothing). The value function does not change and it is

$$v(S) = \begin{cases} 1 & |S| \geq 2, \\ 0 & \text{otherwise} . \end{cases}$$

Also, assume that $c = 0$. The equilibrium outcomes are $p^* = ((1-\alpha, \beta), \alpha)$ together with $S^* = \{1,3\}$ or $p^* = ((\beta, 1-\alpha), \alpha)$ together with $S^* = \{2,3\}$ where $0 \leq \alpha \leq \frac{1}{3}$ and $\beta \geq 1-\alpha$. The two firms therefore extract the entire surplus. This is the opposite of Example 2 above. There the price vector is $p = (0,0,0)$ and the consumer extracts the entire surplus.

We strongly believe in the existence of a pure strategy equilibrium in the case where firms produce multiple products. However we are less sure of whether or not the equilibrium consumption set maximizes the social surplus

as in our model.

Our model deals with one consumer only (or equivalently with multiple identical consumers). A natural extension of the model is to the case where there are consumers of different types. Suppose that there are two consumers with value functions v_1 and v_2. If the two consumers view the product quite differently, a pure strategy equilibrium (with positive production) may not exist.

Example 11. Suppose that $N = \{1,2,3\}, c = 0$ and

$$v_1(S) = \begin{cases} 1 & |S| \geq 2, \\ 0 & \text{otherwise} . \end{cases}$$

and

$$v_2(S) = \begin{cases} 1 & |S| \geq 2, \text{ and } 3 \in S \\ 0 & \text{otherwise} . \end{cases}$$

For consumer 1, any two products are perfect substitutes, if he has the third product. For consumer 2, the two products 1 and 2 are perfect substitutes, while 1 and 3 (or 2 and 3) are perfect complements. It can be verified that there is no pure strategy equilibrium (with nonempty consumption sets). Note that $p = (0,0,1)$ together with $S_1 = \{1,2\}$ and $S_2 = \{2,3\}$ (or $S_2 = \{1,3\}$) is not an equilibrium outcome since firm 1 is better off by raising its price. If $p = (0,0,0), S_1 = \{1,2\}$ and $S_2 = \{2,3\}$ then firm 3 is better off by reducing its price. If $p = (0,1,1), S_1 = \{1,3\}$ and $S_2 = \{1,3\}$ then firm 2 is better off by raising its price.

REFERENCES

[1] W.J., Baumol, E. Barley, E. and D. Willing, Weak Invisible Hand Theorems on the Sustainability of Multiproduct Natural Monopoly, *American Economic Review,* **67** (1977), 350-365.

[2] W.J., Baumol, J.C. Panzar and R.D. Willig, "Contestable Markets and the Theory of Industry Structure", Harcourt Brace Jovanovich, San Diego, 1982.

[3] C. d'Aspremont, J. Gabszewicz and J.F. Thisse, On Hotelling's Stability in Competition, *Econometrica,* **17** (1979), 1145-1151.

[4] A. Dixit and J. Stiglitz, Monopolistic Competition and Optimum Product Diversity, *American Economic Review,* **67** (1977), 297-308.

[5] G. Engle and S. Scotchmer, The Core and the Hedonic Core: Equivalence and Comparative Statics, *Journal of Mathematical Economics*, (1995), forthcoming.

[6] J. Gabszewicz and J.-F. Thisse, Price Competition, Quality and Income Disparities, *Journal of Economic Theory,* **20** (1979), 340-359.

[7] J. Gabszewicz and J.-F. Thisse, Entry (and Exit) in a Differentiated Industry, *Journal of Economic Theory*, **22** (1980), 327-338.

[8] H. Hotelling, "Stability in Competition", *Economic Journal,* **39** (1929), 41-57.

[9] P. Milgrom and J. Robert, Rationalizability, Learning and Equilibrium in Games with Strategic Complimentalities, *Econometrica,* **58** (1990), 1255-1278.

[10] P. Milgrom and C. Shannon, Monotone Comparative Statics, **62** (1994), 1, 157-180.

[11] L. Mirman, Y. Tauman and I. Zang, "Monopoly and Sustainable Prices as a Nash Equilibrium in Contestable Markets", Issues in Contemporary Microeconomics, MacMillan Company, London, 328-339, 1985.

[12] S. Salop, Monopolistic Competition with Outside Goods, *Bell Journal of Economics*, **10** (1979), 141-156.

[13] A. Shaked and J. Sutton, Relaxing Price Competition through Product Differentiation, *Review of Economic Studies*, **49** (1982), 3-13.

[14] A. Shaked and J. Sutton, Natural Oligopolies, *Econometrica*, **51** (1983), 1469-1484.

[15] C. Shapiro, "Theories of Oligopoly Behavior", Handbook of Industrial Organization, Vol. 1, edited by R. Schmalensee and R. Willig, North Holland, Chapter 6, 329-414, 1989.

[16] M. Spence, Product Selection, Fixed Costs and Monopolistic Competition, *Review of Economic Studies*, **43** (1976), 217-235.

[17] M. Wooders and W. Zame, Approximate Cores of Large Games, *Econometrica*, **52** (1984), 1327-1350.

[18] M. Wooders, "The Attribute Core, Core Convergence, and Small Group Effectiveness; The Effects of Property Rights Assignment on the Attribute Core", to appear in Essays in Honor of Martin Shubik, P. Dubey and J. Geanokoplos, eds. 1994.

PUBLISHED ISSUES*

WP-AD 93-01 "Introspection and Equilibrium Selection in 2x2 Matrix Games"
G. Olcina, A. Urbano. May 1993.

WP-AD 93-02 "Credible Implementation"
B. Chakravorti, L. Corchón, S. Wilkie. May 1993.

WP-AD 93-03 "A Characterization of the Extended Claim-Egalitarian Solution"
M.C. Marco. May 1993.

WP-AD 93-04 "Industrial Dynamics, Path-Dependence and Technological Change"
F. Vega-Redondo. July 1993.

WP-AD 93-05 "Shaping Long-Run Expectations in Problems of Coordination"
F. Vega-Redondo. July 1993.

WP-AD 93-06 "On the Generic Impossibility of Truthful Behavior: A Simple Approach"
C. Beviá, L.C. Corchón. July 1993.

WP-AD 93-07 "Cournot Oligopoly with 'Almost' Identical Convex Costs"
N.S. Kukushkin. July 1993.

WP-AD 93-08 "Comparative Statics for Market Games: The Strong Concavity Case"
L.C. Corchón. July 1993.

WP-AD 93-09 "Numerical Representation of Acyclic Preferences"
B. Subiza. October 1993.

WP-AD 93-10 "Dual Approaches to Utility"
M. Browning. October 1993.

WP-AD 93-11 "On the Evolution of Cooperation in General Games of Common Interest"
F. Vega-Redondo. December 1993.

WP-AD 93-12 "Divisionalization in Markets with Heterogeneous Goods"
M. González-Maestre. December 1993.

WP-AD 93-13 "Endogenous Reference Points and the Adjusted Proportional Solution for Bargaining Problems with Claims"
C. Herrero. December 1993.

WP-AD 94-01 "Equal Split Guarantee Solution in Economies with Indivisible Goods Consistency and Population Monotonicity"
C. Beviá. March 1994.

WP-AD 94-02 "Expectations, Drift and Volatility in Evolutionary Games"
F. Vega-Redondo. March 1994.

* Please contact IVIE's Publications Department to obtain a list of publications previous to 1993.

WP-AD 94-03 "Expectations, Institutions and Growth"
F. Vega-Redondo. March 1994.

WP-AD 94-04 "A Demand Function for Pseudotransitive Preferences"
J.E. Peris, B. Subiza. March 1994.

WP-AD 94-05 "Fair Allocation in a General Model with Indivisible Goods"
C. Beviá. May 1994.

WP-AD 94-06 "Honesty Versus Progressiveness in Income Tax Enforcement Problems"
F. Marhuenda, I. Ortuño-Ortín. May 1994.

WP-AD 94-07 "Existence and Efficiency of Equilibrium in Economies with Increasing Returns to Scale: An Exposition"
A. Villar. May 1994.

WP-AD 94-08 "Stability of Mixed Equilibria in Interactions Between Two Populations"
A. Vasin. May 1994.

WP-AD 94-09 "Imperfectly Competitive Markets, Trade Unions and Inflation: Do Imperfectly Competitive Markets Transmit More Inflation Than Perfectly Competitive Ones? A Theoretical Appraisal"
L. Corchón. June 1994.

WP-AD 94-10 "On the Competitive Effects of Divisionalization"
L. Corchón, M. González-Maestre. June 1994.

WP-AD 94-11 "Efficient Solutions for Bargaining Problems with Claims"
M.C. Marco-Gil. June 1994.

WP-AD 94-12 "Existence and Optimality of Social Equilibrium with Many Convex and Nonconvex Firms"
A. Villar. July 1994.

WP-AD 94-13 "Revealed Preference Axioms for Rational Choice on Nonfinite Sets"
J.E. Peris, M.C. Sánchez, B. Subiza. July 1994.

WP-AD 94-14 "Market Learning and Price-Dispersion"
M.D. Alepuz, A. Urbano. July 1994.

WP-AD 94-15 "Bargaining with Reference Points - Bargaining with Claims: Egalitarian Solutions Reexamined"
C. Herrero. September 1994.

WP-AD 94-16 "The Importance of Fixed Costs in the Design of Trade Policies: An Exercise in the Theory of Second Best". L. Corchón, M. González-Maestre. September 1994.

WP-AD 94-17 "Computers, Productivity and Market Structure"
L. Corchón, S. Wilkie. October 1994.

WP-AD 94-18 "Fiscal Policy Restrictions in a Monetary System: The Case of Spain"
M.I. Escobedo, I. Mauleón. December 1994.

WP-AD 94-19 "Pareto Optimal Improvements for Sunspots: The Golden Rule as a Target for Stabilization"
S.K. Chattopadhyay. December 1994.

WP-AD 95-01 "Cost Monotonic Mechanisms"
M. Ginés, F. Marhuenda. March 1995.

WP-AD 95-02 "Implementation of the Walrasian Correspondence by Market Games"
L. Corchón, S. Wilkie. March 1995.

WP-AD 95-03 "Terms-of-Trade and the Current Account: A Two-Country/Two-Sector Growth Model"
M.D. Guilló. March 1995.

WP-AD 95-04 "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets"
J. Alcalde. March 1995.

WP-AD 95-05 "Implementation of Stable Solutions to Marriage Problems"
J. Alcalde. March 1995.

WP-AD 95-06 "Capabilities and Utilities"
C. Herrero. March 1995.

WP-AD 95-07 "Rational Choice on Nonfinite Sets by Means of Expansion-Contraction Axioms"
M.C. Sánchez. March 1995.

WP-AD 95-08 "Veto in Fixed Agenda Social Choice Correspondences"
M.C. Sánchez, J.E. Peris. March 1995.

WP-AD 95-09 "Temporary Equilibrium Dynamics with Bayesian Learning"
S. Chatterji. March 1995.

WP-AD 95-10 "Existence of Maximal Elements in a Binary Relation Relaxing the Convexity Condition"
J.V. Llinares. May 1995.

WP-AD 95-11 "Three Kinds of Utility Functions from the Measure Concept"
J.E. Peris, B. Subiza. May 1995.

WP-AD 95-12 "Classical Equilibrium with Increasing Returns"
A. Villar. May 1995.

WP-AD 95-13 "Bargaining with Claims in Economic Environments"
C. Herrero. May 1995.

WP-AD 95-14 "The Theory of Implementation when the Planner is a Player"
S. Baliga, L. Corchón, T. Sjöström. May 1995.

WP-AD 95-15 "Popular Support for Progressive Taxation"
F. Marhuenda, I. Ortuño. May 1995.

WP-AD 95-16 "Expanded Version of Regret Theory: Experimental Test"
R. Sirvent, J. Tomás. July 1995.

WP-AD 95-17 "Unified Treatment of the Problem of Existence of Maximal Elements in Binary Relations. A Characterization"
J.V. Llinares. July 1995.

WP-AD 95-18 "A Note on Stability of Best Reply and Gradient Systems with Applications to Imperfectly Competitive Models"
L.C. Corchón, A. Mas-Colell. July 1995.

WP-AD 95-19 "Redistribution and Individual Characteristics"
I. Iturbe-Ormaetxe. September 1995.

WP-AD 95-20 "A Mechanism for Meta-Bargaining Problems"
Mª. C. Marco, J. E. Peris, B. Subiza. September 1995.

WP-AD 95-21 "Signalling Games and Incentive Dominance"
G. Olcina, A. Urbano. September 1995.

WP-AD 95-22 "Multiple Adverse Selection"
J.M. López-Cuñat. December 1995.

WP-AD 95-23 "Ranking Social Decisions without Individual Preferences on the Basis of Opportunities"
C. Herrero, I. Iturbe, J. Nieto. December 1995.

WP-AD 95-24 "The Extended Claim-Egalitarian Solution across Cardinalities"
Mª C. Marco. December 1995.

WP-AD 95-25 "A Decent Proposal"
L. Corchón, C. Herrero. December 1995.

WP-AD 96-01 "A Spatial Model of Political Competition and Proportional Representation"
I. Ortuño. February 1996.

WP-AD 96-02 "Temporary Equilibrium with Learning: The Stability of Random Walk Beliefs"
S. Chatterji. February 1996.

WP-AD 96-03 "Marketing Cooperation for Differentiated Products"
M. Peitz. February 1996.

WP-AD 96-04 "Individual Rights and Collective Responsibility: The Rights-Egalitarian Solution"
C.Herrero, M. Maschler, A. Villar. April 1996.

WP-AD 96-05 "The Evolution of Walrasian Behavior"
F. Vega-Redondo. April 1996.

WP-AD 96-06 "Evolving Aspirations and Cooperation"
F. Vega-Redondo. April 1996.

WP-AD 96-07 "A Model of Multiproduct Price Competition"
Y. Tauman, A. Urbano, J. Watanabe. July 1996.

WP-AD 96-08 "Numerical Representation for Lower Quasi-Continuous Preferences"
J. E. Peris, B. Subiza. July 1996.

WP-AD 96-09 "Rationality of Bargaining Solutions"
M. C. Sánchez. July 1996.