EVOLVING ASPIRATIONS AND COOPERATION

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A B S T R A C T

A model of "satisficing" behavior in the repeated Prisoners Dilemma is studied. Each player has an *aspiration* at each date, and takes an action. [S]he switches from the action played in the previous period only if the achieved payoff fell below the aspiration level (with a probability that depends on the shortfall). Aspirations are updated in each period, according to payoff experience in the previous period. In addition, aspirations are subjected to random perturbations around the going level, with a small "tremble" probability. For sufficiently slow updating of aspirations, and small tremble probability, it is shown that in the long run both players cooperate most of the time.

KEYWORDS: Cooperation; Aspirations; Learning.

1 Introduction

Consider a Prisoner's Dilemma played repeatedly by two 'satisficing' players. Each player has an *aspiration* at each date, and takes an action. [S]he switches from the action played in the previous period only if the achieved payoff fell below the aspiration level (with a probability that depends on the shortfall). Aspirations are updated in each period, depending on the divergence of achieved payoffs from aspirations in the previous period.

The aspiration-based process is characterized by two specific features: (i) *inertia*: every action is repeated with at least a certain probability bounded away from zero; and (ii) *experimentation*: with a small probability, players' aspiration levels experience small random trembles around the "going" aspiration, thus preventing them from being perpetually "satisfied" with any given action. We examine the long run outcomes that are induced by vanishingly small tremble probabilities. This paper therefore builds on the work of Bendor, Mookherjee and Ray [1992], in which a model of "consistent" aspirations-based learning was introduced.[5]

We make precise and prove the following result (Theorem 1 below):

If the speed of updating aspiration levels is sufficiently slow, then the outcome in the long run involves both players cooperating "most of the time".

The model therefore describes an adaptive learning process where individuals not only cooperate, but play strictly dominated strategies of the stage game for "most of the time". While players may (and occasionally do) profit by deviating from cooperative behavior, the dynamics of the process ultimately lead back to mutual cooperation.

Section 2 introduces the model, and some of its preliminary properties. Section 3 presents the main results. Section 4 contains an informal discussion of some of the results. Section 5 discusses related literature. All proofs are in Section 6.

2 The Model

Consider a Prisoner's Dilemma:

	C	D
C	(σ, σ)	$(0, \theta)$
D	$(\theta, 0)$	(δ, δ)

where $\theta > \sigma > \delta > 0$.

[5] That paper did not consider the updating of aspirations within a game. For discussion of this and related literature, see Section 5.

Player 1's state at date t is given by an action $A_t \in \{C, D\}$, and an aspiration level α_t, which is a real number. The corresponding objects for player 2 are given by B_t and β_t. A *state* is the pair made up of player 1's state and player 2's state.

The state at date t determines payoffs π_t^1 and π_t^2 for the two players. We now describe the *updating* of each player's state.

We describe the rule followed by player 1; an analogous rule is followed by player 2. There are two features. First, *actions are updated* as follows: if $\pi_t^1 \geq \alpha_t$, then player 1 is *satisfied*, and $A_t = A_{t+1}$. Otherwise player 1 is *disappointed*, so [s]he switches action ($A_{t+1} \neq A_t$) with probability $1 - p$, where p is an indicator of *inertia*. It is assumed that p is a nonincreasing function of the extent of disappointment ($\alpha_t - \pi_t^1$), satisfying:

1. $p = 1$ if $\alpha_t - \pi_t^1 \leq 0$, and
2. $p \in (\tilde{p}, 1)$ otherwise, for some $\tilde{p} \in (0, 1)$.
3. p is continuous and the rate at which it falls is bounded, i.e., for all $x > 0$, $1 - p(x) \leq Mx$ for some $M < \infty$.[6]

Figure 1 describes p. In words, for any given positive degree of disappointment, the player will switch his action with positive probability. However, the probability of *not* switching is bounded away from zero.

Second, with respect to the *updating of aspirations*, it is convenient to first consider the case without any "trembles".

2.1 The Model without Trembles

Aspirations are updated as an average of the aspiration level and the achieved payoff at the previous play. For player 1, this yields

$$\alpha_{t+1} = \lambda \alpha_t + (1 - \lambda)\pi_t^1, \tag{1}$$

where $\lambda \in (0, 1)$ may be thought of as a *persistence* parameter, assumed equal for both players for simplicity. A parallel equation applies to player 2.

The updating rules (without trembles) define a Markov process over the state space $\{C, D\}^2 \times I\!R^2$. This process will be denoted P, and will be referred to as the *untrembled process*.

Given any action pair (A, B), let the corresponding *pure strategy state* (pss) refer to the state where this action pair is played with aspiration levels exactly equal to the

[6] This last set of conditions is inessential for our main result, though employed in the proof. As will be discussed in Section 4, modified arguments apply in the case where the inertial probability is discontinuous at the point of zero disappointment, *i.e.*, is bounded away from 1 as well as 0 for any positive disappointment.

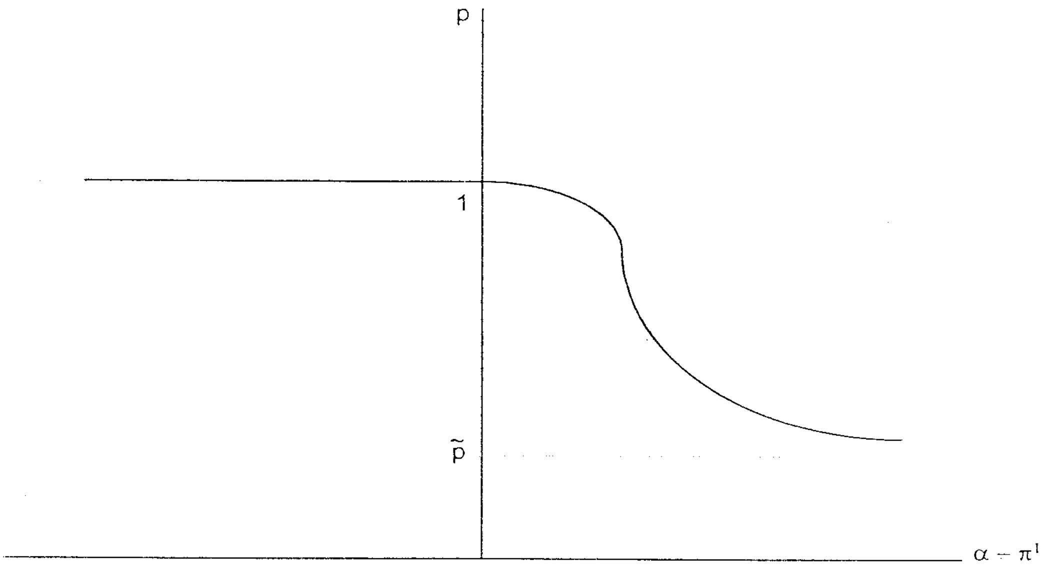

Figure 1: THE FUNCTION p.

achieved payoffs: $\alpha = \pi^1(A, B), \beta = \pi^2(A, B)$. It is clear that every pss is an absorbing state of the untrembled process: if players are satisfied with the payoffs they receive in an ongoing action pair, they have no reason to alter their actions or aspirations. Indeed, it is for this very reason that it is necessary to explore the robustness of any absorbing state by admitting the possibility of perturbations.

Before proceeding to the case of trembles, however, it is useful to settle a preliminary question: does the untrembled process always converge to some pss?

PROPOSITION 1 *From any given initial state, the untrembled process P converges almost surely to some pure strategy state.*

This result is discussed further in Section 4.

2.2 The Model with Trembles

While the untrembled process always converges to some pss, one suspects that some of these may not be robust to small perturbations in a player's state. To model such phenomena, we introduce trembles in the formation of aspirations.[7]

[7]Starting with any pss, a small upward push to a player's aspiration level will cause that player to be disappointed, and hence induced to experiment with other actions. An alternative modeling approach

With probability $1-\eta$, aspirations are formed according to the deterministic rule (1), while with the remaining probability η, the updated deterministic aspiration α is perturbed according to some density $g(.,\alpha)$. Assume, again for simplicity, that η is the same for both players.

Informally, we would like small perturbations on either side of α to be possible, but at the same time, uninteresting technical complexities would be introduced by allowing aspirations to wander too far from the payoffs of the game, and we want to avoid these. In what follows, we impose appropriate restrictions.

Assume, then, that there exists some compact interval Λ which contains all feasible payoffs such that for each $\alpha \in \Lambda$, the support of $g(.|\alpha)$ is contained in Λ, and moreover, that $g(\alpha'|\alpha) > 0$ for all α' in some nondegenerate interval around α (relative to Λ). Furthermore, suppose that g is continuous as a function of α.

Finally, assume that all initial aspiration vectors lie in the compact region Λ^2, and that all perturbations are independent over time and across players.[8]

Denote the resulting stochastic process by P^η. A standard theorem (see, e.g., Meyn and Tweedie (1993), Theorem 16.2.5) guarantees that the process has a well defined long run outcome:

PROPOSITION 2 *For $\eta > 0$, the process P^η converges (strongly) to a unique limit distribution Π^η, irrespective of the initial state.*

3 Main Results

By Proposition 2, the introduction of trembles serves to single out a unique (though probabilistic) long-run outcome. Obviously, one is interested in the nature of the long run distribution Π^η when the tremble probability η is close to zero, since this is likely to yield a selection from the multiple long-run limits of the untrembled process.

Some preliminary steps are needed before we can state such a result precisely. To begin with, one needs to ensure that the sequence of long run distributions Π^η has a well-defined limit as η goes to zero.

PROPOSITION 3 *The sequence of distributions Π^η converges weakly to a distribution Π^* on E as $\eta \downarrow 0$.*

Proposition 3 is a corollary of a general theorem on the long run behavior of Markov processes subjected to small stochastic perturbations, which also provides a precise characterization of the limiting distribution. Because this result may be of wider interest than

would directly allow experimentation with different actions. We suspect that the results would be the same in such an approach.

[8] As usual, a similar definition holds for player 2. It is immaterial to the argument whether the function g is the same for both players.

the specific application studied here, we provide a self-contained statement of this general theorem in Section 5 (Theorem 2). Theorem 2 also yields the following description of the limit distribution Π^*.

Use Q_i to denote the one step transition probability of the stochastic process, conditional on the situation where *only* player i's aspiration is subjected to a tremble. Let Q denote $\frac{1}{2}(Q_1 + Q_2)$. We may interpret this as the transition rule when exactly one player trembles, with both players being equally likely to tremble.

Let R denote P^∞, the infinite step transition rule in the untrembled process. This is well-defined by Proposition 1. Finally, let QR denote the composition of Q and R. In words, the process QR refers to the effect of subjecting exactly one player (chosen randomly) to a tremble in her aspirations, followed by the untrembled process thereafter for ever.

PROPOSITION 4 *The limiting distribution Π^* is the unique invariant distribution of the process QR.*

By Proposition 1, the untrembled process converges to a pss. It follows that an invariant distribution of QR must be concentrated on the pss's. Proposition 4 says that QR has a unique invariant distribution, which is precisely the limit of the invariant distributions corresponding to vanishing tremble probabilities.

In words, the selected long run outcome can be obtained as the unique long run outcome of an "artificial" Markov process defined only over the four pss's, with the transition probability between pss's obtained as follows: Starting with the former pss, subject one player chosen randomly to a single tremble in aspiration to obtain a new state, from which the untrembled process is left to operate thereafter to arrive eventually at some pss.

Why do we need a precise characterization of the limit of the long run distributions, unlike previous authors? The reason is that the long run distribution will *not* generally be concentrated on a single pss. This is in sharp contrast with random matching contexts considered by Kandori, Mailath and Rob (1993) and Young (1993), where the corresponding process singles out a unique limit state.

In the informal discussion, which we relegate to Section 4, we attempt to provide a clearer intuitive explanation of this observation.

We are now in a position to state our main result.

THEOREM 1 *The weight $\Pi^*(C, C)$ placed by the limiting distribution Π^* on the mutual cooperation pure strategy state is close to 1, for persistence parameter λ sufficiently close to 1. Formally,*

$$\lim_{\lambda \to 1} \Pi^*(C, C) = 1.$$

4 Informal Discussion

The assertions thus far may be summarized as follows. First, the untrembled joint process of aspirations and actions always converges to a pure strategy steady state. Second, the trembled process is ergodic and converges to a unique invariant distribution. Third, these invariant distributions (viewed as functions of the tremble) themselves settle down to a "limit" invariant distribution. Finally — and this is the main result — the limit invariant distribution places almost all weight on the cooperative outcome, provided that the persistence parameter is sufficiently close to unity. We discuss these informally in turn.

Begin with the convergence of the untrembled process. Our formal proof takes the easiest route towards establishing convergence, by exploiting a degree of inertia in the model that is built in by assumption. Specifically, given any state, there can be an infinite run on the current action pair, which would cause aspirations to converge to the corresponding payoffs.

We illustrate this by starting with the (C, D) action vector, with 2's aspiration lying above the cooperative payoff, while player 1's aspiration does not exceed this payoff. What is the probability of an infinite run on (C, D) thereafter in the untrembled process, which would result in convergence to the (C, D) pss? Along such a path, player 2 would have no cause to switch away from D, so what is needed is for player 1 to stick with C perpetually despite being disappointed at every stage. At stage t, player 1's aspiration and hence disappointment level would be $\lambda^t \sigma$. Hence the probability of converging to (C, D) is $\prod_{t=1}^{\infty} p(\lambda^t \sigma)$, which is positive if and only if

$$\sum_{t=1}^{\infty} [1 - p(\lambda^t \sigma)] < \infty.$$

a condition which is satisfied, since the left hand side equals

$$\sum_{t=1}^{\infty} [p(0) - p(\lambda^t \sigma)] \leq \frac{M\lambda\sigma}{1-\lambda}.$$

A similar argument can be given for the possibility of an infinite run on any action pair, and any initial set of aspirations, as detailed in the proof of Proposition 1. Note that no assumption has been made concerning the speed at which aspirations are updated. This is an implication of our assumptions concerning inertia.

Howver, it must be emphasized that while this is the easiest way to prove Proposition 1, the result also holds under weaker assumptions concerning the inertia function p, providing we restrict aspirations to not be updated too rapidly. For instance, if the inertia probability p is discontinuous at 0, so players switch with a probability bounded away from zero whenever disappointed, the infinite runs used above to secure convergence

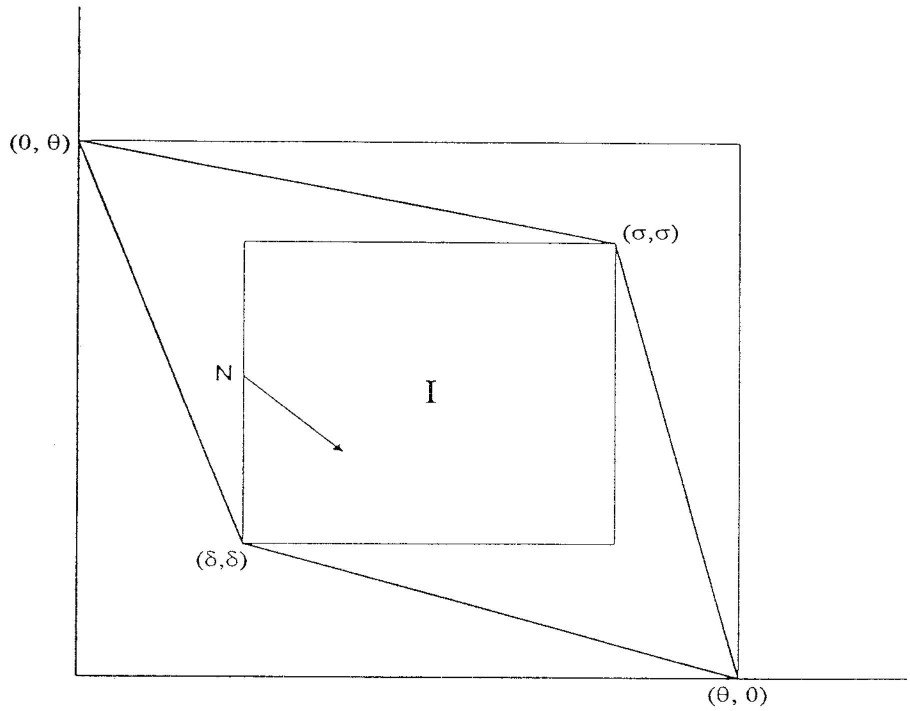

Figure 2: ILLUSTRATION OF THEOREM 1.

are no longer possible. Here is a sketch of an alternative argument for convergence for the case where λ is close to 1. Suppose, contrary to the assertion of the proposition, that the untrembled process does not converge. Then it can be shown that the process must wander infinitely often through the interior of the rectangle I depicted in Figure 2. Now concentrate on this rectangle. Observe that while aspirations lie within this rectangle, there is a positive probability that (C, C) will be played soon thereafter. To see this, suppose first that (D, D) is played. Then both players will be disappointed so with positive probability both will switch to (C, C). If (D, C) is played, player 1 will be happy while player 2 will be disappointed, precipitating (D, D) next period with positive probability, whereupon the previous argument applies. A completely parallel argument holds for the action pair (C, D).

All that remains is to bound this probability away from zero, which can be done by applying the argument to a suitably chosen compact subrectangle of I. Now couple this observation with the infinite recurrence of I to establish convergence to (C, C). This contradicts the assumption that the untrembled process does not converge, and we are done.

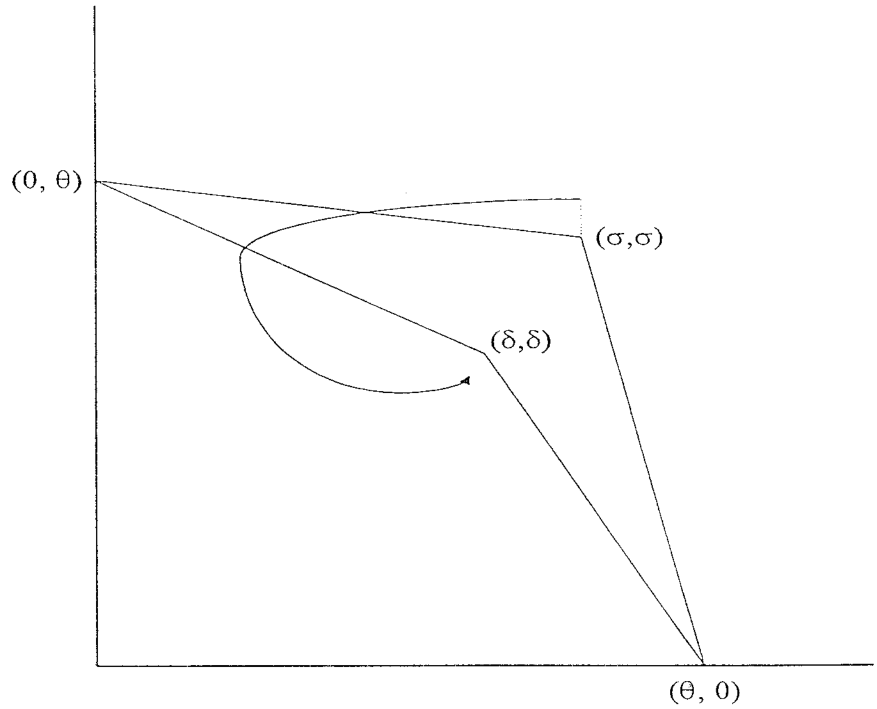

Figure 3: POSSIBLE CONVERGENCE TO (D, D) FROM PERTURBED (C, C).

The observation that the trembled process is ergodic is a standard one, familiar by now in the literature on stochastic evolution. What is of interest is that *unlike* the results in that literature, it is perfectly possible for the limit invariant distribution (as trembles vanish) to place weight on more than one pure strategy state. In part this comes from the kind of inertia-based arguments provided above. But it is important to observe, yet again that inertia is only sufficient and not necessary to generate these features. For instance, a "long aspiration cycle" can be triggered by a perturbation from the (C, C) pss, leading to ultimate convergence (with positive probability) to the (D, D) pss from below (Figure 3 depicts such a path for a particular parametric specification, albeit somewhat cryptically, by the curved arrow). These possibilities necessitate a more detailed analysis. To get a handle on which pure strategy state is likely to receive the lion's share of probability weight, we must deduce a formula for the limit invariant distribution. This is done in Theorem 2, which is stated and proved below.

This theorem is the building block on which our proof of the main result (Theorem 1 is based. We show that while the limiting distribution assigns some weight to all pss's, the weight placed on the (C, D), (D, C) or (D, D) pss's must become arbitrarily small when aspirations adjust sufficiently slowly: the process spends "most of its time" in the vicinity of the (C, C) outcome. In this sense, mutual cooperation *is* the unique long run outcome.

The proof of the theorem is long and involved, requiring an assessment of the limiting probabilities of transiting from the (C, C) pss to the other pss's (in the process QR), relative to transitions in the reverse direction, as λ converges to 1. It is shown that the probability of transiting from any of the other pss's to the (C, C) pss is bounded away from zero for all values of λ close to 1, whereas the probabilities of the reverse transitions converge to zero as $\lambda \to 1$.

The main reason for this is that transitions following a single perturbation of the (C, C) pss to the other pss's must *necessarily* require one or both players to not switch their actions for long stretches of time, even if they are disappointed. As aspirations adjust more and more slowly, the bouts of inertia required become indefinitely large, and therefore increasingly improbable.[9]

On the other hand, the transition to the (C, C) pss, can be shown to be inertia-free relative to the reverse transitions described above. Showing that this is so involves some long and delicate calculations, embedded largely in Lemma 3. For instance, starting with the (D, D) pss, when player 2 experiences a positive tremble on his/her aspirations, aspirations move to a point like N in Figure 2. From that point onwards, we show that aspirations tend to drift back, with large probability, into the interior of the rectangle I, and that this argument can be made *without* any reliance on inertia, so that it survives even when the persistence parameter is close to unity. Intuitively, the argument is simple. Imagine a point just to the left of N in Figure 2. Then player 1 has aspirations low enough so that he is satisfied with his choice of the D action irrespective of what player 2 does. Player 2, dissatisfied, will switch back between C and D. This induces a drift of aspirations to the southeast, pointing into the interior of I. What is needed is a similar result for the point N *itself* and points even slightly to the right of it, which necessitates a complicated analysis.

Once the process drifts back into I, we are in a familiar realm. We have already argued (see above) that from this point on, convergence to (C, C) occurs with positive probability, in a way that does not rely on inertia.

Note in conclusion that even for small tremble probabilities and speed of aspiration updating, the process does *not* converge to mutual cooperation in the long run. Cooperation simply becomes statistically dominant. Players perpetually oscillate between different action pairs, as their aspirations are occasionally subjected to trembles in different directions, with effects that last beyond the trembles. In particular, these trembles

[9] If the inertia function p were to be discontinuous at 0, then the probability of converging to any other pss from the cooperative pss following one tremble is obviously zero, so the result is straightforward in that case.

cause them to experiment with actions different from those used in the recent past. For instance, starting with mutual cooperation, such trembles induce experimentation with defection, which leads to transitory gains from exploiting the cooperative partner. However, the partner is dissatisfied in such situations, and will switch to defection as well, which serves to "punish" the initial defector. A period of mutual defection then ensues, which tends to disappoint *both* players, inducing an eventual return to the cooperative phase. The players play a dominated strategy often not owing to a feature of the learning rule which causes them to stick to an inferior action in a single person environment, but rather owing to the nature of interaction between the decision rules chosen by the two players.

5 Related Literature

The model of learning considered in this paper presumes a limited form of rationality, where players need not know the structure of the game, or the opponents' previous actions; nor do they have to be able to solve maximization problems. In the terminology of Selten (1991), such models represent "stimulus" or "reinforcement" rather than "belief" learning. They originated in the mathematical psychology literature (see Bush and Mosteller (1955)), and have received a certain degree of support in laboratory experiment situations involving human subjects (see Suppes and Atkinson (1960), Selten and Stoecker (1986), Roth and Erev (1993), Mookherjee and Sopher (1994, 1995)). Other recent explorations of such models of learning include Binmore and Samuelson (1993) and Börgers and Sarin (1994), who both explore the relationship with 'replicator dynamics' models, and Gilboa and Schmeidler (1992, 1993) who develop an axiomatic "case-based decision theory" where players satisfice relative to aspiration levels that are based on past experience.

The structure of interaction between players in our model does not correspond to random matching of pairs selected from a certain population, as in the 'evolutionary' literature – cf., for example, Binmore and Samuelson (1993), Börgers and Sarin (1994), Kandori, Mailath and Rob (1993) or Young (1993). In our context, a *given* pair of agents plays the game repeatedly over time. This stands in contrast with some of the well-known models whose concern is to provide an evolutionary basis for the rise of cooperation, e.g., Robson (1990), Fudenberg and Maskin (1990), or Binmore and Samuelson (1992). They consider situations where the *repeated* Prisoner's Dilemma is recurrently being played between pairs of individuals randomly selected from the general population. In a sense, their objective is to select among multiple equilibria of the underlying supergame by embedding it in a wider intertemporal framework. Instead, our approach remains within the the scenario of a single indefinite repetition of the stage game, singling out the *stage* outcome which happens to be played most of the time in the long run.

Bendor, Mookherjee and Ray (1992) is a precursor to this paper. In that paper, aspirations were assumed to be fixed, and models of reinforcement learning that led to long run outcomes consistent with aspiration levels in general two player repeated games were analyzed. Such an equilibrium notion is appropriate for characterizing long run outcomes in contexts of evolving aspirations where the latter converge. This paper extends this model to include an explicit process of evolving aspirations as well as of strategies of players, but in the context of a specific game, the Prisoner's Dilemma. Aspirations do turn out to converge, and the long run outcome is essentially cooperative, thus vindicating the solution concept used in that paper.

Kim (1995) and Pazgal (1995) both apply the Gilboa-Schmeidler case-based theory to games of coordination or the Prisoners Dilemma. They allow aspirations to evolve simultaneously with the strategies selected by players in a context of repeated interaction, and provide conditions under which long run outcomes entail cooperation. These conditions entail initial aspiration levels lying in prespecified ranges: for instance, Pazgal needs to assume that they are sufficiently high relative to the cooperative payoffs for both players, while Kim assumes that they lie slightly below the cooperative payoffs for both players. Our model in contrast predicts cooperation in the long run, irrespective of initial conditions. Another important difference is that Kim and Pazgal both assume that aspirations average *maximal* experienced payoffs in past plays, whereas we assume they average the actual experienced payoffs. Hence their theory imparts a certain additional degree of "ambitiousness" to players, which helps in ensuring convergence to cooperative outcomes.

6 Concluding Comments

The learning dynamics studied in this paper is admittedly stylized and the context to which it is applied undoubtedly special. However, we want to conclude the main body of the paper by discussing its robustness in the face of alternative specifications, outlining as well a variety of different possible extensions.

As amply discussed, the essential role of trembles in our framework is that of liberating the process from "fragile" pss's. And, as it turns out, only the (C, C) pss proves to be resilient enough to display any significant long-run weight, as the adjustment of aspirations becomes very gradual. In this light, it seems clear that a wide variety of alternative perturbations of the process would tend to produce the same results. Consider, for example, a model where aspirations always adjust deterministically according to (1) but, occasionally (i.e., with some small probability η) players adjust their actions arbitrarily (that is, *not* according to the function p). This would be a formulation more in line with the approach of recent evolutionary literature (where players are assumed to "experiment") and would seem to lead to results which are qualitatively the same as described here. It would still be true that a transition from the pss (C, C) to any of the other three pps's will be a very unlikely event after one tremble, if λ is close to one. Reciprocally, the converse transitions would still require very little inertia (and, therefore, would be much more likely), after one single tremble. The logic underlying our analysis would seem to be essentially unaffected by this alternative specification.

Another variation on the model pertains to the postulated law of motion on aspirations. The geometric formulation considered here could be replaced with

$$\alpha_{t+1} = \frac{1}{t+1} \sum_{\tau=0}^{t} \pi_{\tau}^{1}, \tag{2}$$

that is, a simple average of past payoffs. In a heuristic sense, (2) could be interpreted as a limiting case of our framework when $\lambda \to 1$.[10] Whether this new feature also leads to similar conclusions is an open question. The main issue here is that, in this alternative context, *any* transition becomes progressively more lengthy as time proceeds. This could conceivably lead to non-ergodic behavior despite the fact that the *relative* likelihoods of the different transitions across pss's would display the same features as described in our model.

Let us now outline some interesting extensions of the game studied here. We shall focus on three of them: (a) coordination games; (b) random-matching population contexts; (c) richer intertemporal strategies. The extent to which we feel we understand each of these extensions is decreasing in the order mentioned.

First, it is not difficult to see that the same asymmetry between upward and downward movements in aspiration levels would appear in strict coordination games (i.e., symmetric games where every main-diagonal payoff is positive, all others being equal to zero). If the game has a unique equilibrium which dominates all others, it can be shown that the long-run outcome of a process analogous to the one described in this paper would coincide with such an equilibrium.

A second extension would involve a finite-population context where players are randomly matched every period to play a game, say a Prisoner's Dilemma. Even though we have not worked out the details of such a model, we conjecture that the cooperative outcome (i.e., all players choosing C) would still be the unique long-run outcome for λ close to one. The reason is that, in this context, the destabilizing effect of one tremble would still be the same as before, once we are allowed to specify the particular outcome (or chain of outcomes) of the matching mechanism. Since every matching outcome has some positive probability, so has any finite chain of them required for the perturbation to operate in the desired direction. Combining this ideas with an analogous role for

[10] Of course, one could not carry out this limit operation literally since this would amount to freezing the aspiration level at its initial value.

inertia still present when λ is large, the logic used above would seem applicable to the population context outlined.

Finally, it might be of interest to allow for players who may rely on the intertemporal nature of the interaction and use history-dependent rules. In this context, players could end up using certain rules not only due to their "direct" payoff performance but also, indirectly, because of the behavior they induce in others. Again, one would postulate reinforcing mechanisms of the type described on the set of possible rules, the learning process being occasionally perturbed by stochastic trembles. A preliminary issue which would arise in this respect pertains to the specification of the space of rules to be considered. Initially, one might consider simply addding to the "flat" (history-independent) rules, a simple reactive one (for example, in the Prisoner's Dilemma, a strategy of the Tit-For-Tat variety). In this framework, it may be possible to shed some light on the difficult issue of whether sophisticated rules may either arise or, at least, play a crucial role in facilitating the emergence of simple cooperative behavior.

7 Proofs

Proof of Proposition 1. Given initial aspirations (α_0, β_0), aspirations at all later dates will be contained in the convex hull of (α_0, β_0) and the four (pure) payoff points of the game. Let this convex hull be denoted C and let the maximum aspirations for the two players in C be denoted by $\bar{\alpha}$ and $\bar{\beta}$ respectively.

Consider any state (α, β, A, B) with $(\alpha, \beta) \in C$. Let the payoffs generated by the action pair (A, B) be denoted (π^1, π^2). Consider the probability of an infinite run on this action pair, which is given by

$$h(\alpha, \beta) \equiv \prod_{t=1}^{\infty} p(\lambda^t(\alpha - \pi^1))p(\lambda^t(\beta - \pi^2)).$$

We claim that (i) $h(\alpha, \beta) > 0$ for every $(\alpha, \beta) \in C$, and (ii) h is nonincreasing in each argument.

To prove (i), it suffices to check that:

$$\sum_{t=1}^{\infty}[1 - l(\lambda^t(\alpha - \pi^1)), \lambda^t(\beta - \pi^2))] < \infty,$$

where $l(d_1, d_2)$ denotes $p(d_1)p(d_2)$. This condition is satisfied because the left hand side is bounded above by

$$M\frac{\lambda}{1-\lambda}\{|\alpha - \pi^1| + |\beta - \pi^2|\},$$

thus establishing (i). Claim (ii) follows directly from the fact that p is nonincreasing

Given (i) and (ii), we see that for every $(\alpha, \beta) \in C$,

$$h(\alpha, \beta) \geq h(\bar{\alpha}, \bar{b}) > 0.$$

Let $\epsilon > 0$ denote the minimum value of $h(\bar{\alpha}, \bar{\beta})$ across all possible initial action pairs. It follows that at every date the probability of converging to the pure strategy state corresponding to the ongoing action pair is at least ϵ, thereby completing the proof. ∎

We now prepare for the statement and proof of Theorem 2. Let $\Delta(E)$ denote the set of probability measures on a compact state space E, endowed with the Borel σ-algebra. For any transition probability Q on E and any measure $\mu \in \Delta(E)$, define $\mu.Q \equiv \int_E \mu(x)Q(x,.)$. Two transition probabilities P and Q naturally induce a third PQ through composition: $PQ(x,.) \equiv P(x,.).Q$. This permits us to define m-step transition probabilities iteratively: $P^m(x,.) = P^{m-1}(x,.).P$, where P^0 is the degenerate probability on x.

Given a real valued function f on E, define $Pf \equiv \int_E P(x,dy)f(y)$.

A measure Π on E is *invariant with respect to* P if $\Pi.P = \Pi$.

Perturb the transition probability P in the following manner: define for $\eta \in (0,1)$,

$$\begin{aligned} P^\eta &= (1-\phi(\eta))P + \phi(\eta)Q^\eta \\ Q^\eta &= (1-\psi(\eta))Q + \psi(\eta)Q_1^\eta \end{aligned}$$

where $\phi(\eta) \to 0, \psi(\eta) \to 0, 0 < \phi(\eta) < 1, 0 \leq \psi(\eta) < 1$ as $\eta \to 0$

THEOREM 2 *Assume that*

[a] For each $x \in E$, $\frac{1}{T+1}\sum_{t=0}^{T} P^t(x,.)$ converges weakly to a probability measure $R(x,.)$.

[b] Q has the strong Feller property: Qf is continuous for all bounded measurable f on E.

[c] Q is open set irreducible, i.e., for all open sets U and all $x \in E$, $\sum_{n=1}^{\infty} Q^n(x,U) > 0$; and

[d] QR has a unique invariant measure Π^*.

Then P^η has a unique invariant measure Π^η, which converges weakly to Π^* as $\eta \downarrow 0$.

Proof. Given any $\eta > 0$, properties [b] and [c] imply that P^η is a T-chain. Applying Theorem 16.2.5 in Meyn and Tweedie (1993), P^η is uniformly ergodic, and has a unique invariant measure Π^η. Then

$$\Pi^\eta.[(1-\phi(\eta))P + \phi(\eta)Q^\eta] = \Pi^\eta,$$

implying that

$$\Pi^\eta[\phi(\eta)Q^\eta] = \Pi^\eta - (1-\phi(\eta))\Pi^\eta P.$$

Given any bounded continuous function f, apply the probability measures above to $P^m f$:

$$\Pi^\eta.[\phi(\eta)Q^\eta]P^m f = \Pi^\eta.P^m f - (1-\phi(\eta))\Pi^\eta.P^{m+1} f$$

Multiplying by $(1-\phi(\eta))^t$ and summing over $t = 0, \ldots, T$ we obtain

$$\Pi^\eta.(\phi(\eta)Q^\eta)\{\sum_{t=0}^{T}(1-\phi(\eta))^t P^t f\} = \Pi^\eta f - (1-\phi(\eta))^{T+1}\Pi^\eta.P^{T+1} f.$$

Taking $T \to \infty$, and using $\sup_x |\Pi^\eta.P^{T+1} f(x)| \leq \sup_x |f(x)|$, we get

$$\Pi^\eta.\{Q^\eta g^\eta\} = \Pi^\eta f, \tag{3}$$

where

$$g^\eta(x) = \phi(\eta)\sum_{t=0}^{\infty}(1-\phi(\eta))^t P^t f(x).$$

Note that $\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T} P^t f(x) = Rf(x)$ implies $g^\eta(x) \to Rf(x)$ as $\eta \downarrow 0$. Now

$$\Pi^\eta.Q^\eta g^\eta = (1-\psi(\eta))\Pi^\eta.Qg^\eta + \psi(\eta)\Pi^\eta.Q_1^\eta g^\eta. \tag{4}$$

Since $|g^\eta(x)| \leq \sup_y |f(y)| \equiv M < \infty$, we also have $|\Pi^\eta.Q_1^\eta g^\eta| \leq M$. As Q is strong Feller, it follows that Qg^η and QRf are continuous. Hence

$$\sup_x |Qg^\eta(x) - QRf(x)| \to 0$$

as $\eta \downarrow 0$. Thus, if η_n is any sequence with $\eta_n \downarrow 0$ with $\Pi^{\eta_n} \to \hat{\Pi}$ in the topology of weak convergence,

$$\Pi^{\eta_n}.Qg^{\eta_n} \to \hat{\Pi}.QRf. \tag{5}$$

(4) and (5) together imply that

$$\lim_n \Pi^{\eta_n}.Q^{\eta_n} g^{\eta_n} = \hat{\Pi}.QRf \tag{6}$$

Combining (3) and (6),

$$\hat{\Pi}.QRf = \hat{\Pi}f.$$

It then follows from [d] that $\hat{\Pi} = \Pi^*$, which completes the proof. ∎

Proof of Propositions 3 and 4. Define

$$\phi(\eta) \equiv \eta^2 + 2\eta(1-\eta),$$

$$\psi(\eta) \equiv \frac{\eta^2}{\eta^2 + 2\eta(1-\eta)}.$$

It is then evident that

$$P^\eta = [1 - \phi(\eta)]P + \phi(\eta)Q^\eta$$

where

$$Q^\eta \equiv [1 - \psi(\eta)]Q + \psi(\eta)Q^*,$$

Q^* denoting the transition probability when both players are subjected simultaneously to a tremble.

Assumption (a) of Theorem 2 is established by Proposition 1, while assumptions (b) and (c) are valid by construction. Hence it remains to check assumption (d), i.e., that the process QR has a unique invariant distribution. We know from Proposition 1 that every invariant distribution of QR must be concentrated on the four pss's. Hence it suffices to show that there exists a pss (the mutual defection pss) which can be reached with positive probability from every other pss in the QR process.

For this it suffices to check that the action pair (D, D) will be played at some date, when we start at any pss and subject the aspiration of one player to a tremble. The reason is that once this happens, the reasoning of Proposition 1 implies that with positive probability there will be an infinite run on (D, D) thereafter, causing convergence to the (D, D) pss. If we start with the (D, D) pss then this is obvious, owing to inertia. And if we start with any other pss, then one small upward tremble to one player's aspiration will cause (C, D) or (D, C) followed by (D, D) to be played with some probability. ∎

For the proof of Theorem 1, we need some additional notation. If π_s denotes the payoff to player 1 in any period s, then

$$\alpha_{s+1} = \lambda\alpha_s + (1 - \lambda)\pi_s,$$

so that

$$|\alpha_s - \alpha_{s+1}| = (1 - \lambda)|\alpha_s - \pi_s| \leq (1 - \lambda)W,$$

where W is the maximum conceivable value between aspirations and payoffs (the width of the compact interval Λ to which aspirations belong).

Let T be the minimum number of periods that need to elapse before aspirations at time $T + 2$ are different from aspirations at time 0 by an amount not less than a. Then it is clear that $T \geq T(\lambda, a)$, where $T(\lambda, a)$ is the smallest integer such that

$$(1 - \lambda)[T(\lambda, a) + 2]W \geq a.$$

It follows that

$$(1 - \lambda)T(\lambda, a) \geq \frac{a}{W} - 2(1 - \lambda). \tag{7}$$

Note that if $a > 0$, then $T(\lambda, a) > 0$ for all λ sufficiently close to unity, and indeed, that $T(\lambda, a) \to \infty$ as $\lambda \to 1$. This construct will be used at various stages below.

For any $(\hat{a}, \hat{b}) \gg 0$ such that $\delta + \hat{a} < \sigma - \hat{a}$ and $\delta + \hat{b} < \sigma - \hat{b}$, let $I(\hat{a}, \hat{b})$ be the rectangle defined by $[\delta + \hat{a}, \sigma - \hat{a}] \times [\delta + \hat{b}, \sigma - \hat{b}]$.

LEMMA 1 *Consider any* $(\hat{a}, \hat{b}) \gg 0$ *such that* $\delta + \hat{a} < \sigma - \hat{a}$ *and* $\delta + \hat{b} < \sigma - \hat{b}$. *Then given any* $\epsilon > 0$, *there exists* $\lambda_1 \in (0,1)$ *such that*

$$\text{Prob}^{\lambda}((\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C,C)) | (\alpha_T, \beta_T) \in I(\hat{a}, \hat{b})) > 1 - \epsilon$$

for all T *and all* $\lambda \in (\lambda_1, 1)$.

Proof. Fix any T with $(\alpha_T, \beta_T) \in I(\hat{a}, \hat{b}))$. Let $T^*(\lambda)$ be the minimum number of periods t such that $(\alpha_{T+t+2}, \beta_{T+t+2}) \notin I(\frac{\hat{a}}{2}, \frac{\hat{b}}{2})$. Observe that if $a \equiv \frac{1}{2}\min\{\hat{a}, \hat{b}\}$, then that $T^*(\lambda) \geq T(\lambda, a)$ (see (7)). It follows that a lower bound for $T^*(\lambda)$ can be found that is independent of T.

We observe, next, that there is $\zeta > 0$ (independent of T and λ) such that as long as $t = T+2, T+3, \ldots, T(\lambda, a) + T$, $s_t = (C,C)$ with probability at least ζ. To see this, suppose first that (C,C) is played in period $t-2$. In that case (C,C) is played in period t with probability one. If (D,D) is played in period $t-2$, then the probability is easily seen to be at least $p(\frac{\hat{a}}{2})p(\frac{\hat{b}}{2})$. If at date $t-2$, (D,C) is played, then the conditional probability of playing (D,D) at $t-1$ is at least $[1-p(\delta)]$ (because player 1 will stick for sure to D, and player 2 will switch with probability at least $[1-p(\delta)]$). Thereafter, a switch to (C,C) in period t occurs with probability at least $p(\frac{\hat{a}}{2})p(\frac{\hat{b}}{2})$. The conditional probability in this case is therefore at least $[1-p(\delta)]p(\frac{\hat{a}}{2})p(\frac{\hat{b}}{2})$. Finally, the argument for (C,D) is symmetric.

Thus in all cases, the conditional probability of playing (C,C) at date t is bounded below by the positive number $\zeta \equiv [1-p(\delta)]p(\frac{\hat{a}}{2})p(\frac{\hat{b}}{2})$.

We may now compute a lower bound on the probability of (C,C) being played at least once in $T(\lambda, a) - 2$ periods. The probability that (C,C) will *never* be played during this stretch is clearly bounded above by $(1-\zeta)^{T(\lambda,a)-2}$. Thus the required lower bound is given by $1 - (1-\zeta)^{T(\lambda,a)-2}$. Choose λ_1 such that

$$1 - (1-\zeta)^{T(\lambda_1,a)-2} \geq 1 - \epsilon.$$

We complete the proof by showing that once (C,C) is played during these periods, $(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C,C))$ as $t \to \infty$. To see this, simply observe that whenever (C,C) is played during this time, say at date s, $(\alpha_s, \beta_s) \leq (\sigma - \frac{\hat{b}}{2}, \sigma - \frac{\hat{b}}{2})$ by the construction of $T(\lambda, a)$. It follows that once (C,C) is played, it will be played repeatedly thereafter, and we are done. ∎

LEMMA 2 *Let* $T(\lambda)$ *be a sequence of positive integers such that* $T(\lambda) \to \infty$ *as* $\lambda \to 1$. *For each* λ, *let* $\{X_t^{\lambda}, Z_t^{\lambda}\}$ *be a finite horizon stochastic process with terminal date* $T(\lambda)$, *such that* X_t *takes values only in* $\{0,1\}$ *(there is no restriction on* Z_t*).*

Suppose that for each λ, X_0^λ is equal to a constant $i(\lambda)$ (which can take the values 0 or 1). Use the notation h_t to denote t-histories for each $t \geq 1$, and the notation $L(h_t)$ to denote the value of X_{t-1}^λ for every t-history.

Suppose that for every $\lambda \in (0,1)$,

$$\text{Prob}(X_t^\lambda = 1|h_t) \leq u < 1,$$

whenever $L(h_t) = 0$, while

$$\text{Prob}(X_t^\lambda = 1|h_t) \leq v < 1,$$

if $L(h_t) = 1$. Then for every $\nu > 0$,

$$\text{Prob}\{\frac{1}{T(\lambda)+1}\sum_{t=0}^{T(\lambda)} X_t^\lambda \leq \frac{u+\nu}{1-v+u}\} \to 1 \text{ as } \lambda \to 1. \tag{8}$$

Proof. We begin with a coupling result that will be needed for the proof.

CLAIM. Suppose $\{U_0, U_1, U_2, \ldots, U_T\}$ is a finite sequence of random variables that assume values in $\{0,1\}$. Suppose that $U_0 = i$ (where i is either 0 or 1), and that for $t \geq 1$,

$$\text{Prob}(U_t = 1|h_t) = p_t(h_t) \leq u < 1,$$

if $L(h_t) = 0$, and

$$\text{Prob}(U_t = 1|h_t) = p_t(h_t) \leq v < 1,$$

if $L(h_t) = 1$.

Let $\{V_1, V_2, \ldots, V_T\}$ be a (finite-horizon) Markov Chain with values in 0, 1 such that $V_0 = i$ as well and for all $t \geq 0$

$$\text{Prob}(V_{t+1} = 1|V_t = 0) = u \qquad \text{Prob}(V_{t+1} = 0|V_t = 0) = 1 - u$$

$$\text{Prob}(V_{t+1} = 1|V_t = 1) = v \qquad \text{Prob}(V_{t+1} = 0|V_t = 0) = 1 - v$$

Then for all x,

$$\text{Prob}(U_1 + U_2 + \ldots + U_T \leq x) \geq \text{Prob}(V_1 + V_2 + \ldots + V_T \leq x) \tag{9}$$

We prove this using a standard coupling argument. Define $q(0) = u$ and $q(1) = v$. Take a probability space $\Omega = [0,1]^T$ with the Borel σ-field on Ω, and product Lebesgue measure.

For $x = (x_1, x_2, \ldots, x_T) \in \Omega$, let $\tilde{U}_0(x) = i$. Recursively, for $t \geq 1$, having defined $\tilde{U}_0, \ldots, \tilde{U}_{t-1}$; $\tilde{V}_1, \ldots, \tilde{V}_{t-1}$, let $\xi_t(x) \equiv p_t(\tilde{U}_0(x), \tilde{U}_2(x), \ldots, \tilde{U}_{t-1}(x))$ and $\eta_t(x) = q(\tilde{V}_{t-1}(x))$. Now define $\tilde{U}_t, \tilde{V}_t$ by

$$\tilde{U}_t(x) = 1_{\{x_t \leq \xi_t(x)\}}$$

and

$$\tilde{V}_t(x) = 1_{\{x_t \le \eta_t(x)\}}.$$

Observe now that for all $t = 0, 1, \ldots, T$, and each $x \in \Omega$,

$$\tilde{U}_t(x) \le \tilde{V}_t(x). \tag{10}$$

To see this, proceed inductively, noting that the statement is trivially true at $t = 0$. Recursively, if the statement is true at some $t-1$, note that by the construction of U, V and q,

$$\xi_t(x) \le \eta_t(x).$$

The required inequality (10) at date t follows right away from this observation.

Finally, observe that using product Lebesgue measure, the distribution of $(\tilde{U}_0, \tilde{U}_1, \ldots, \tilde{U}_T)$ is the same as the distribution of $(U_0, U_1, \ldots U_T)$, and also that the distribution of $(\tilde{V}_0, \tilde{V}_1, \ldots, \tilde{V}_T)$ is the same as the distribution of $(V_0, V_1, \ldots V_T)$.

Now the required conclusion (9) follows immediately, because the corresponding statement for $(\tilde{U}_0, \ldots \tilde{U}_T)$, $(\tilde{V}_0, \ldots, \tilde{V}_T)$ is a direct consequence of (10).

With the Claim in hand, we return to the proof of the lemma. It will be sufficient to establish the lemma for the case where $i(\lambda)$ is a constant i, independent of λ. The general case then follows easily from a subsequence argument.

Let $\{V_t\}$ be the Markov chain constructed in the Claim. Then interpreting X_t^λ as U_t, (9) implies immediately that

$$\text{Prob}\left(\frac{1}{T(\lambda)+1}\sum_{t=0}^{T(\lambda)} X_t^\lambda \le \frac{u+\nu}{1-v+u}\right) \ge \text{Prob}\left(\frac{1}{T(\lambda)+1}\sum_{t=0}^{T(\lambda)} V_t \le \frac{u+\nu}{1-v+u}\right).$$

By the strong law of large numbers for Markov chains, the RHS above converges to 1, and we are done. ∎

LEMMA 3 *There exist positive numbers a and b with $\delta + b < \sigma - b$, with the property that for any $\epsilon > 0$, there exists $\lambda_2 \in (0,1)$ such that*

$$\text{Prob}\{(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C, C)) | (\alpha_T, \beta_T) \in N\} \ge 1 - \epsilon \tag{11}$$

for every T and $\lambda \in (\lambda_2, 1)$, where $N \equiv [\delta - a, \delta + a] \times [\delta + 2b, \sigma - 2b]$

Proof. We begin the proof by choosing the numbers a and b, and defining some other values that will be needed in the argument.

Start by choosing $b > 0$ such that $0 < \delta - 2b < \delta + 2b < \sigma - 2b$, and define the rectangle M_1 by

$$M_1 \equiv \{(\alpha, \beta) | \alpha \in [\delta - b, \delta + b], \beta \in [\delta + 2b, \sigma - 2b]\}.$$

Consider the process commencing from some aspiration vector $(\alpha_T, \beta_T) \in M_1$ at date T, and any given vector of actions. For each λ, note that a lower bound on the the minimum number of periods after which $(\alpha_{T+t+2}, \beta_{T+t+2})$ fails to lie within the larger rectangle, M_2, defined by

$$M_2 \equiv \{(\alpha, \beta) | \alpha \in [\delta - 2b, \delta + 2b], \beta \in [\delta + b, \sigma - b]\}$$

is given by $T(\lambda, b)$ (see (7)).

Clearly, there is an interval $[\bar{\lambda}, 1]$ such that for every λ in this interval, $T(\lambda, b) > 0$. *For the rest of the argument, λ will be taken to lie in this interval.*

Define $K \equiv \frac{b}{W} - 2(1 - \bar{\lambda}) > 0$. It follows then that for all $\lambda \in (\bar{\lambda}, 1)$,

$$(1 - \lambda)T(\lambda, b) \geq K > 0. \tag{12}$$

Next, fix $a \in (0, b]$ and define the following quantities:

$$\begin{aligned} d' &\equiv 1 - p(a) > 0 \\ e' &\equiv p(2b)[1 - p(b)] > 0 \\ i' &\equiv p(\delta - 2b) < 1. \end{aligned} \tag{13}$$

Observe that the inequalities above hold because $p(x) \in (0, 1)$ whenever $x > 0$.

We will impose a further restriction on the choice of a. It must satisfy the additional restriction that

$$[\theta - (\delta + a)] \left(e'\chi - \frac{d'}{1 - i' + d'} \right) - a(1 - e'\chi) - \frac{(\delta + a)d'}{1 - i' + d'} > \frac{2a}{K} \tag{14}$$

where K is given by (12), and χ is a strictly positive number satisfying the equality

$$\chi = \min\{p(2b)p(\sigma - \delta - b), p(\delta + b)\} - \hat{\epsilon} \tag{15}$$

for some small but positive $\hat{\epsilon}$.

Let us check to see that this can be done. Certainly χ can be chosen as in (15). Having done so, note that the RHS of (14) goes to 0 as $a \to 0$, while the LHS of (14) converges to the positive quantity

$$[\theta - \delta]e'\chi$$

(to see this, use (13)). By the continuity of both sides of (14) in a at $a = 0$, it follows that there exists a small but strictly positive such that both $\delta + a < \sigma - a$ and (17) holds.

We complete our construction by noting that there exists $\epsilon' > 0$ such that if we define

$$\begin{aligned} d &\equiv 1 - p(a) + \epsilon' \\ e &\equiv p(2b)[1 - p(b)] - \epsilon' \\ i &\equiv p(\delta - 2b) - \epsilon', \end{aligned} \tag{16}$$

then d, e and i all lie strictly between 0 and 1, and moreover,

$$[\theta - (\delta + a)]\left(e\chi - \frac{d}{1 - i + d}\right) - a(1 - e\chi) - \frac{(\delta + a)d}{1 - i + d} \geq \frac{2a}{K}. \tag{17}$$

In what follows, a, b, d, e, i, $\hat{\epsilon}$, and ϵ' are fixed by these considerations, irrespective of the value of $\lambda \in [\bar{\lambda}, 1)$.

Let $W_1 \equiv \{(C,C)$ is played for some $T \leq t \leq T(\lambda, b) + T$, and $\alpha_k \leq \delta + a$ for $T \leq k \leq t\}$. Note by the construction of $T(\lambda, b)$ that $(\alpha_t, \beta_t) \leq (\sigma - b, \sigma - b)$ for all $t \in \{0, 1, \ldots, T(\lambda)\}$. Consequently, if W_1 occurs, (C,C) will be played repeatedly thereafter, and $(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C,C))$ for sure. Thus

$$\text{Prob}^\lambda\{(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C,C)) | W_1\} = 1. \tag{18}$$

In what follows, we suppose, then, that the event W_1 does not occur.

Denote by S^λ the event $\{(\alpha_t, \beta_t) \in I(a,b)$ for some $T \leq t \leq T(\lambda, b) + T\}$.

Claim. *There exists a function $g(\lambda)$ on $(\bar{\lambda}, 1)$, independent of T, with $g(\lambda) \to 1$ as $\lambda \to 1$, such that*

$$\text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_T) \in N, \sim W_1) \geq g(\lambda) \tag{19}$$

To establish this claim, let W_2 be the event $\{\alpha_t \geq \delta + a$ for some $t = T, T + 1, \ldots, T(\lambda, b) + T - 1\}$. Note that by the definition of conditional probabilities,

$$\begin{aligned}\text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_T) \in N, \sim W_1) &= \text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_t) \in N, W_2, \sim W_1)\text{Prob}^\lambda(W_2) \\ &+ \text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_T) \in N, \sim W_2, \sim W_1)\text{Prob}^\lambda(\sim W_2).\end{aligned}$$

First evaluate $\text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_t) \in N, W_2, \sim W_1)$. Note that if W_2 occurs (at some date t), then by a property of $T(\lambda, b)$, $\alpha_t \leq \delta + 2b \leq \sigma - 2b \leq \sigma - a$, by the choices of a and b. So $\alpha_t \in [\delta + a, \sigma - a]$. Also, by construction of $T(\lambda, b)$, it must be the case that $\beta_t \in [\delta + b, \sigma - b]$, so that $(\alpha_t, \beta_t) \in I(a,b)$. This shows that

$$\text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_T) \in N, W_2, \sim W_1) = 1. \tag{20}$$

It remains to evaluate the conditional probability

$$\text{Prob}^\lambda(S^\lambda | (\alpha_T, \beta_T) \in N, \sim W_2, \sim W_1)$$

for each T.

It will be useful in what follows to concentrate on the action plays in each of the periods $T, T + 1, \ldots, T(\lambda, b) + T$. One way of doing this is to note that the overall stochastic process, conditional on some initial action s_T, $(\alpha_T, \beta_T) \in N$ and the event $\sim W$, defines a stochastic process (which in general will be non-Markovian) on the *actions* s_t played in periods $t = T, T+1, \ldots, T(\lambda, b) + T$. This process can be described, given the initial conditions, by a sequence of functions (one for each date), describing the probability of each action pair at date t conditional on the entire history of actions h_t up to that date. At t, let $P^\lambda(., h_t)$ denote this function. With slight abuse of notation, we will use P^λ to denote the probability of various events as well, conditional or otherwise.

Let μ denote the fraction of occurrences of (D, D), and γ the fraction of occurrences of (C, D), for the dates $T, T+1, \ldots, T(\lambda, b) + T$. Of course, μ and γ are random variables for each λ.

Recalling the definitions in (16) and the definition of χ in (15), consider the event Z described by

$$\mu \leq 1 - e\chi \tag{21}$$

and

$$\gamma \leq \frac{d}{1 - i + d}. \tag{22}$$

SUBCLAIM. *There exists a function* $g(\lambda)$ *on* $(\bar{\lambda}, 1)$, *independent of* T, *with* $g(\lambda) \rightarrow 1$ *as* $\lambda \rightarrow 1$ *such that*

$$\text{Prob}^\lambda(Z | (\alpha_T, \beta_T) \in N, \sim W_2, \sim W_1) \geq g(\lambda). \tag{23}$$

The argument up to the third paragraph following (27) is concerned with the proof of this Subclaim.

Begin by computing $P^\lambda((D, D), h_t)$ for each λ, $t \geq 1$ and h_t.

Let $s(h_t)$ denote the last action vector, i.e., the action at date $t - 1$, under the t-history h_t. If $s(h_t) = (D, D)$, then the probability that both players continue with D is at least $p(2b)p(\sigma - \delta - b)$ (since player 1's aspiration cannot exceed $\delta + 2b$ and player 2's aspiration cannot exceed $\sigma - b$, by the construction of $T(\lambda, b)$). If $s(h_t) = (C, D)$, then by a similar argument, the probability of moving to (D, D) the next period is at least $p(\delta - 2b)$. If $s(h_t) = (D, C)$, the probability of moving to (D, D) is at least $p(\delta + b)$. By invoking (15), we see, therefore, that

$$P^\lambda((D, D), h_t) \geq \chi + \hat{\epsilon}$$

for all h_t and all λ.

Let E^λ be the event

$$\{\frac{1}{T(\lambda, b) + 1} \sum_{t=T}^{T(\lambda, b)+T} 1_{(D,D)} \geq \chi\},$$

where the notation 1_s denotes the indicator function of the action vector s. Note that χ is independent of T. Now apply Lemma 2, with $X_t = 0$ whenever (D, D) is played, $X_t = 1$ if anything else is played, with Z_t set equal to some constant, with $u = v = 1 - \chi$, and $\nu = \hat{\epsilon}$. We may deduce that there exists a function $g_1(\lambda)$ (with $g_1(\lambda) \to 1$ as $\lambda \to 1$) such that

[I] $P^\lambda(E^\lambda) \geq g_1(\lambda)$.

Next, consider the probability $P^\lambda((D,C),h_t)$ for histories such that $s(h_t) = (D,D)$. For (D,C) to follow (D,D), player 1 must stay at D while player 2 switches. Because $\alpha_t \leq \delta + 2b$, player 1 stays with probability at least $p(2b)$, while because $\beta_t \geq \delta + b$, player 2 switches with probability at least $[1-p(b)]$. Consequently, recalling (16), we see that

$$P^\lambda((D,C),h_t) \geq e + \epsilon' \tag{24}$$

for all $t \geq 1$, all λ, and all h_t with $s(h_t) = (D,D)$.

Now, consider the event

$$F^\lambda \equiv \{ \sum_{t=T}^{T(\lambda,b)+T} 1_{(D,C)} \geq e\chi T(\lambda,b)\},$$

where χ is the constant used to define E^λ, and a particular conditional probability $P^\lambda(F^\lambda|E^\lambda)$. Because the event E^λ occurs in the conditioning, the number of occurrences of (D,D) is at least as big as $\chi T(\lambda,b)$ for each λ.

Use this information to construct a stochastic process as follows. Each occurrence of (D,D) is to be treated as a "date". The number of dates will be taken to be the greatest integer not exceeding $\chi T(\lambda,b)$: this is $T(\lambda)$ in Lemma 2. Throw away all information after this date. Let X_t be the following random variable that describes the action vector immediately following the tth realization of (D,D): $X = 0$ if (D,C) occurs, and $X = 1$ otherwise. Let Z_t be a list of all the action vectors that follow the tth occurrence of (D,D), up to the $t+1$ th occurrence of (D,D). [If (D,D) is immediately followed by another (D,D), then set Z equal to some arbitrary constant.] This process fits all the conditions of Lemma 2, if both u and v are identified with $1-e-\epsilon'$. Applying the lemma, we may conclude that there exists a function $g_2(\lambda)$ with $g_2(\lambda) \to 1$ as $\lambda \to 1$ such that

[II] $P^\lambda(F^\lambda|E^\lambda) \geq g_2(\lambda)$

Again, note that g_2 is independent of T, since e is.

Next, consider the probability $P^\lambda((C,D),h_t)$. First suppose that $s(h_t) = (D,D)$. For (C,D) to follow (D,D), player 1 must switch to C while player 2 stays at D. Because we are conditioning on $\sim W$ and so in particular on $\sim W_2$, we have $\alpha_t \leq \delta + a$. It follows that the transition occurs with probability *no more* than $[1-p(a)]$. It follows (recall (16)) that

$$P^\lambda((C,D),h_t) \leq 1 - p(a) = d - \epsilon' \tag{25}$$

for all $t \geq 1$, all λ, and all h_t with $s(h_t) = (D,D)$.

Next, suppose that $s(h_t) \neq (D, D)$. In this case, $P^\lambda((C, D), h_t)$ can only be positive when (C, D) itself was played last (remember that we are conditioning on $\sim W$). This requires that player 1 stick to his previous action, which will occur with probability at most $p(\delta - 2b)$. Using (16), it follows that

$$P^\lambda((C, D), h_t) < i + \epsilon' \tag{26}$$

for all $t \geq 1$, all λ, and all h_t with $L(h_t) \neq (D, D)$.

Consider, then, the event

$$G^\lambda \equiv \{ \sum_{t=T}^{T(\lambda)+T} 1_{(C,D)} \leq \frac{d}{1 - i + d} \}.$$

We claim that there exists a function $g_3(\lambda)$, independent of T, with $g_4(\lambda) \to 1$ as $\lambda \to 1$ such that

[III] $P^\lambda(G^\lambda) \geq g_3(\lambda)$.

Proving this claim requires the application of Lemma 2 yet again. Start with the (unconditional) event G^λ. Define a stochastic process as in Lemma 2 with $T(\lambda) = T(\lambda, b)$, with $X_t = 1$ if the action vector at time $t+T$ is (C, D) (and 0 otherwise), with $u = d - \epsilon'$ and $v = i + \epsilon'$, and with $\nu = \epsilon'$. Take Z_t to be some constant for all t. Now the lemma applies, so we see that there exists a function $g_3(\lambda)$ with the required properties.

We may now combine observations [I]–[III]. The point is to recognize that all these three hold, then it must be the case that

$$P^\lambda(G^\lambda \cap F^\lambda \cap E^\lambda) \geq g(\lambda), \tag{27}$$

for some function $g(\lambda)$ that is independent of T and with the property that $g(\lambda) \to 1$ as $\lambda \to 1$.

To complete the proof of (23), we unravel what the event $G^\lambda \cap F^\lambda \cap E^\lambda$ implies for the values of μ and γ, which, it will be recalled, are the fractions of (D, D)'s and (C, D)'s respectively during the dates $T, T+1, \ldots, T(\lambda, b) + T$.

Let κ denote the fraction of (D, C)'s during this period. Note, first, that $\mu + \kappa \leq 1$, while under the event F^λ, $\kappa \geq e\chi$. Combining these two observations, we may conclude that $\mu \leq 1 - e\chi$. This shows that (21) must hold under the events E^λ and F^λ.

Next, note that under the event G^λ, $\gamma \leq \frac{d}{1-i+d}$, which is, of course, (22).

The observations in the last two paragraphs, coupled with (27), establish (23), and the proof of the Subclaim is complete.

Suppose, then, that the conditional event described by (23) does in fact occur. Let us find a lower bound on the change in player 1's aspirations as a result of this event. Recall that $\alpha_{t+1} = \lambda\alpha_t + (1 - \lambda)\pi_t$, where π_t is the payoff at date t, so that

$$\alpha_{t+1} - \alpha_t = (1 - \lambda)(\pi_t - \alpha_t) \tag{28}$$

Recalling that $\alpha_t \leq \delta + a$ for $t = 0, 1, \ldots T(\lambda, b) - 1$ (i.e., the event W_2 does not occur), we see that the RHS of (28) is bounded below by $(1-\lambda)[\theta - (\delta - a)]$ when the action (D, C) is played, by $-(1-\lambda)a$ when the action (D, D) is played, and by $-(1-\lambda)(\delta + a)$ when the action (C, D) is played. Using (21) and (22), we may therefore compute the *total* rightward drift over $T(\lambda, b)$ periods as

$$\begin{aligned}
& T(\lambda,b)(1-\lambda)\{[\theta - (\delta - a)][1 - \gamma - \mu] - \mu a - \gamma(\delta + a)\} \\
\geq\ & T(\lambda,b)(1-\lambda)\{[\theta - (\delta - a)]\left(e\chi - \frac{d}{1-i+d}\right) - a(1 - e\chi) - \frac{(\delta + a)d}{1-i}\} \\
\geq\ & T(\lambda,b)(1-\lambda)\frac{2a}{K} \geq 2a,
\end{aligned}$$

using (12) and (17). We conclude, then, that under the event Z,

$$\alpha_{T(\lambda,b)} \geq \delta + a, \tag{29}$$

while, by the construction of $T(\lambda, b)$,

$$\beta_{T(\lambda)} \geq \delta + b. \tag{30}$$

From (29) and (30), it follows right away that

$$\text{Prob}^{\lambda}(S^{\lambda} | (\alpha_T, \beta_T) \in N, \sim W_2, \sim W_1) \geq g(\lambda), \tag{31}$$

where $g(\lambda)$ was introduced in (27).

By combining (20) and (31), and defining $g(\lambda) \equiv \min\{\hat{g}(\lambda), \tilde{g}(\lambda)\}$, we obtain (19). Recalling (19), we may conclude that

$$\text{Prob}^{\lambda}(S^{\lambda} | (\alpha_T, \beta_T) \in N, \sim W_1) \geq g(\lambda) \tag{32}$$

for some function $g(\lambda)$, independent of T, such that $g(\lambda) \to 1$ as $\lambda \to 1$. This completes the proof of the Claim.

To complete the proof of the lemma, observe, noting carefully the definition of W_1, that

$$\begin{aligned}
& \text{Prob}^{\lambda}\{(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C, C)) | (\alpha_T, \beta_T) \in N, \sim W_1\} \\
\geq\ & \sum_{S=T}^{T(\lambda,b)+T} \text{Prob}^{\lambda}\{(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C, C)) | (\alpha_S, \beta_S) \in I(a, b)\} \\
& \cdot \text{Prob}^{\lambda}((\alpha_S, \beta_S) \in I(a, b) | (\alpha_T, \beta_T) \in N, \sim W_1\}
\end{aligned} \tag{33}$$

Using Lemma 1, we may conclude that there exists a function $g'(\lambda)$, independent of S, with $g'(\lambda) \to 1$ as $\lambda \to 1$, such that

$$\text{Prob}^{\lambda}\{(\alpha_t, \beta_t, s_t) \to (\sigma, \sigma, (C, C)) | (\alpha_S, \beta_S) \in I(a, b)\} \geq g'(\lambda)$$

Using this observation in (33), we see that

$$\begin{aligned}&\text{Prob}^{\lambda}\{(\alpha_t,\beta_t,s_t)\to(\sigma,\sigma,(C,C))|(\alpha_T,\beta_T)\in N,\sim W_1\}\\ \geq\;& g'(\lambda)\sum_{S=T}^{T(\lambda,b)+T}\text{Prob}^{\lambda}((\alpha_S,\beta_S)\in I(a,b)|(\alpha_T,\beta_T)\in N,\sim W_1\},\end{aligned}$$

and now using (32),

$$\text{Prob}^{\lambda}\{(\alpha_t,\beta_t,s_t)\to(\sigma,\sigma,(C,C))|(\alpha_T,\beta_T)\in N,\sim W_1\}\geq g'(\lambda)g(\lambda), \tag{34}$$

the RHS of which converges to one as $\lambda\to 1$, uniformly in T. Combining (18) and (34), the proof of the lemma is complete. ■

LEMMA 4 *Let a and b be positive numbers satisfying $a<\min\{\delta,b\}$. For any $\epsilon>0$ there exists $\lambda_3\in(0,1)$ such that:*

$$\text{Prob}^{\lambda}[\alpha_T<a|J_t]<\epsilon. \tag{35}$$

for any $\lambda\in(\lambda_3,1)$, any event J_t which is a subset of the event that $\{\alpha_t>b,\beta_t\leq\theta\}$, and any pair of dates t,T satisfying $T>t$.

Proof. Suppose that the conditional event described in (35) occurs. Pick λ_3 to satisfy $[p(a)]^{T(\lambda_3,b^*-a)}<\epsilon$, where $b^*\equiv\min\{b,\delta\}>a$. Take any $\lambda\in(\lambda_3,1)$ and any $t,T>t$. Define intervening dates l and m as follows: l is the *last* date k between t and T such that $\alpha_k\geq b^*$, and m is the *first* date k when $\alpha_k<a$. Clearly, $m-l\geq T(\lambda,b^*-a)$, and $a<\alpha_k<b^*$ for all intervening dates. Note also that if $\beta_t\leq\theta$ then $\beta_k\leq\theta$ for all $k\geq t$, since θ is the highest attainable payoff in the game.

We claim that the action pair (C,D) must have been played successively between l and $m-1$. First, note that (C,D) must have been played at date l: any other action pair could not have lowered player 1's aspiration below b^* at $l+1$. Now $\alpha_{l+1}<\delta$, while $\beta_{l+1}\leq\theta$: so player 2 is satisfied at t_1, while 1 is disappointed. Hence only player 1 has an incentive to switch: (C,D) or (D,D) must be played at $l+1$. If player 1 does switch to a D at $l+1$, [s]he will obtain a payoff of δ, greater than his/her aspiration. Player 2 may or may not switch thereafter, but irrespective of this, player 1 will obtain a payoff of at least δ, and so must stick with D at least until m, since his/her aspiration is less than δ until then. Moreover, her aspiration must go up successively from $l+1$ onwards, so her aspiration cannot fall below a at m. Hence (C,D) must be repeated at $l+1$. The same argument can then be applied successively to all dates $l+2,\ldots,m-1$.

Hence player 1 must have persisted with C between dates l and m despite being disappointed by at least a at every date. The probability of this event is therefore at most $[p(a)]^{m-l}\leq\epsilon$. Since this bound is independent of l and m, the lemma follows.

LEMMA 5 *For any* $\epsilon^* > 0$, *there exists* $\lambda_4 \in (0,1)$ *such that*

$$\text{Prob}\{(\alpha_t, \beta_t, s_t) \to (\delta, \delta, (D, D)) | (\alpha_0, \beta_0) \geq (\sigma, \sigma)\} < \epsilon^* \tag{36}$$

for all $\lambda \in (\lambda_4, 1)$

Proof. Under the event $F \equiv \{\alpha_t, \beta_t, s_t) \to (\delta, \delta, (D, D)\}$, there must exist a date T such that $\beta_T < \delta + 2b$, where b is as given in Lemma 3 . Since $\beta_0 \geq \sigma$, there exists $\tilde{\lambda} < 1$ such that for any $\lambda \in (\tilde{\lambda}, 1)$, there must exist a first date k such that $\beta_k \in (\delta + 2b, \sigma - 2b)$ We assert that by permuting the two players if necessary, it must also be true that there exists a first date k in which simultaneously $\beta_k \in (\delta + 2b, \sigma - 2b)$ and $\alpha_k < \sigma - a$, where a is also given by Lemma 3. Let E_k denote this event.

Take $\epsilon = \frac{\epsilon^*}{3}$ and $\lambda_4 = \max\{\tilde{\lambda}, \lambda_1, \lambda_2, \lambda_3\}$, where λ_2, λ_3 are as given in Lemmas 3 and 4 respectively, corresponding to this ϵ and a, b, and λ_1 is as given in Lemma 1 corresponding to this ϵ and $I(a, 2b)$. Taking any $\lambda \in (\lambda_4, 1)$ and any k as defined above:

$$\begin{aligned}
& \text{Prob}^\lambda(F | E_k, (\alpha_0, \beta_0) \geq (\sigma, \sigma)) \\
= \; & \text{Prob}^\lambda(F | \alpha_k < \delta - a, \beta_k \in (\delta + 2b, \sigma - 2b))\text{Prob}^\lambda(\alpha_k < \delta - a | E_k, (\alpha_0, \beta_0) \geq (\sigma, \sigma)) \\
+ \; & \text{Prob}^\lambda(F | (\alpha_k, \beta_k) \in N)\text{Prob}^\lambda(\delta - a \leq \alpha_k \leq \delta + a | E_k, (\alpha_0, \beta_0) \geq (\sigma, \sigma)) \\
+ \; & \text{Prob}^\lambda(F | \delta + a < \alpha_k < \sigma - a, \delta + 2b < \beta_k < \sigma - 2b)\text{Prob}^\lambda(F | E_k, (\alpha_0, \beta_0) \geq (\sigma, \sigma)),
\end{aligned}$$

where N is defined in Lemma 3. The first term on the RHS of this expression is bounded above by

$$\text{Prob}^\lambda(\alpha_k < \delta - a | E_k, (\alpha_0, \beta_0) \geq (\sigma, \sigma))$$

Since the event we are conditioning on is a subset of the event $\{\alpha_0 > \delta \; \beta_0 \leq \theta\}$. Lemma 4 implies that this term is less than $\frac{\epsilon^*}{3}$ The second term is bounded above by

$$\text{Prob}^\lambda(F | (\alpha_k, \beta_k) \in N),$$

which is less than $\frac{\epsilon^*}{3}$ by virtue of Lemma 3. Finally, the third term is bounded above by

$$\text{Prob}^\lambda(F | (\alpha_k, \beta_k) \in I(a, 2b))$$

which is also less than $\frac{\epsilon^*}{3}$ by Lemma 1. Hence the sum of these terms is less than ϵ^*, thereby completing the proof. ∎

We are finally in a position to complete the

Proof of Theorem 1. Using Propositions 1 and 4, it suffices to show that in the 'untrembled' process, the probability of transiting from the (C, C) pss following a single tremble to any of the other three pss's converges to zero, while the probability of a reverse transition is bounded away from zero, as $\lambda \to 1$.

Suppose we start from the (C, D) pss, and player 1's aspiration experiences an upward tremble by $x > 0$, to take the aspiration vector to (x, θ). Now apply Lemma 3 (with $b = \frac{x}{2}$ and $a = \frac{x}{3}$) to infer that the probability of converging back to the (C, D) pss, converges to zero as $\lambda \rightarrow 1$. Moreover, the same Lemma also implies that (i) starting with the (C, D) pss followed with one tremble, the probability of converging to the (D, C) pss also converges to zero (since $\beta_0 = \theta > \delta$), and (ii) the same is true for the probability of converging to the (C, D) pss or (D, C) pss, when we start with either the (D, D) pss or (C, C) pss and subject aspirations of one player to a single tremble. This establishes that as $\lambda \rightarrow 1$, the weight placed by Π^* on either the (C, D) pss or (D, C) pss must converge to zero.

Hence as $\lambda \rightarrow 1$, the sum of the weights placed on the (C, C) pss and the (D, D) pss will converge to 1.

Now suppose we start from the (D, D) pss, and player 2's aspiration experiences an upward tremble from δ to $\delta + x$, while player 1's aspiration remains at δ. The argument of Lemma 3 can then be applied (selecting a value of b smaller than x) to infer that from this state, the probability of the untrembled process converging to the (C, C) pss is close to 1, for λ sufficiently close to 1.

Finally, Lemma 5 shows that the probability of transiting to the (D, D) pss following application of one tremble to the (C, C) pss, converges to 0 as $\lambda \rightarrow 1$. This completes the proof. ∎

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