TEMPORARY EQUILIBRIUM WITH LEARNING: THE STABILITY OF RANDOM WALK BELIEFS[*]

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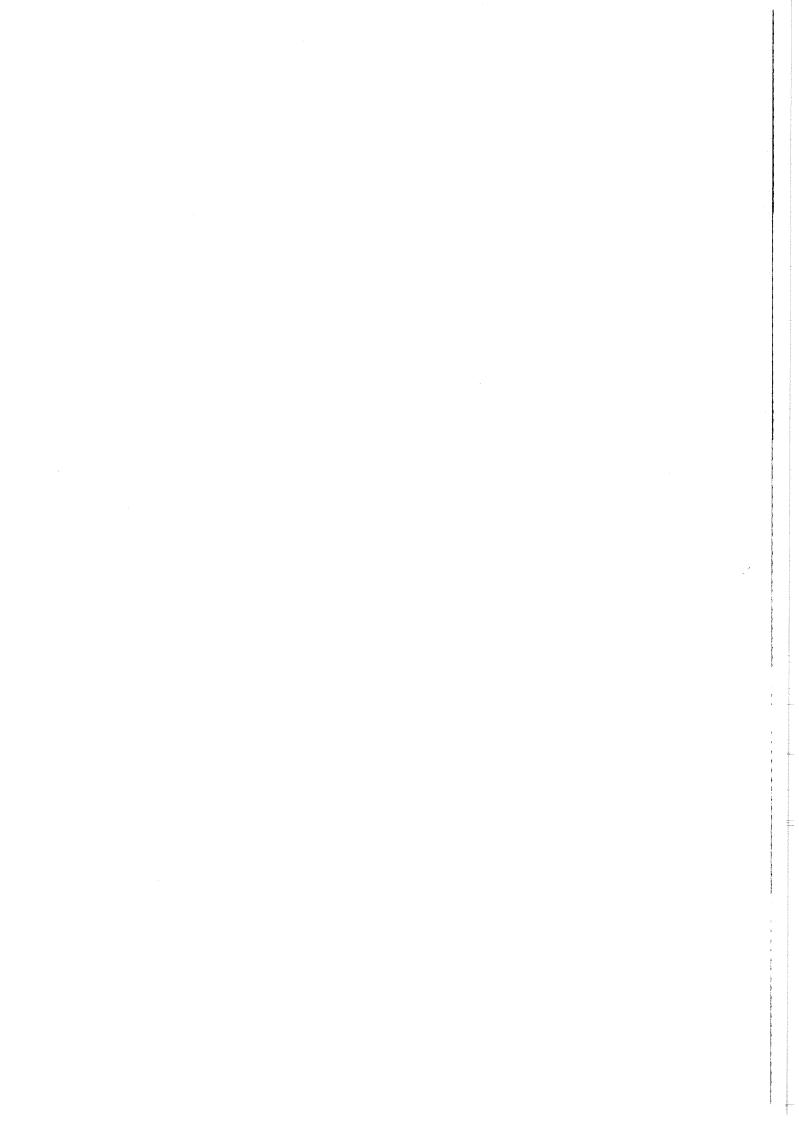[**] Indian Statistical Institute, New Delhi and University of Alicante.

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ABSTRACT

This paper examines the stability of deterministic steady-states with a one dimensional state-variable and a smooth, recursive updating rule. It is shown that the only possibly stable steady states are those associated with random walk beliefs, provided there is motion on a center manifold, which is the case when a key parameter is non-zero. In the extant literature, there is no motion on the center manifold (the parameter is zero), a consequence of the specific assumption that the expected value of the state variable next period determines its current value. The stability properties are seen to be robust with respect to small misspecifications in the agents fixed perception of the steady state.

KEY WORDS: Learning, Stability, Random Walk

This paper continues the local stability analysis of deterministic steady states [1,2 and 3] in a set up where the state variable is one dimensional and where the agents learning process is summarised by a smooth, recursive updating rule.

In the formulation of this and the earlier papers, agents beliefs about the dynamics are summarised by a stochastic linear model. The agents are assumed to recognize the steady state whose stability is being assessed. The agent's model is thus specified in deviations of the state variable from its steady state value. The agents learn about the parameter (denoted β) which they believe governs the dynamics outside the steady state by updating the parameter estimates using a smooth recursive rule. Given this structure, the dynamical system with learning has a unit eigenvalue at the steady state. Thus there exists a center manifold associated with the steady state, which must be considered while evaluating the local stability of the steady state.

Previous studies [1,2 and 3] have analysed stability under the specific assumption that the expected value of the state variable next period determines its current value, an assumption which puts restrictions on the underlying microeconomic structure. Under this assumption, there exists a manifold of equilibria of the learning system corresponding to the steady state being considered, and the partial derivative with respect to β, denoted b, of the map that determines the current value of the state variable as a function of the past values of the state variable and the β estimates, is 0. This manifold of equilibria coincides with the center manifold referred to above, and this has an important effect on the stability, namely, that since the center manifold is a set of equilibria, there is no motion on the center manifold. There one gets the intuitively appealing result that if the agents use sufficiently low values of $|\beta|$ (corresponding to the belief that the steady state is locally stable), one gets local stability and if, on the contrary, agents use high estimates (corresponding to beliefs that the system may be divergent), then one gets local instability.

This paper demonstrates that a very different picture emerges when this assumption is removed. Specifically, in Section 2, I develop the local stability analysis under

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the assumption that the key partial derivative b is non zero. I present in Appendix A a utility maximising model based on an overlapping generations economy to illustrate that non zero values of b arise robustly. Under this assumption, there is motion on the center manifold, and this drives the stability result summarised in Proposition 2.1. The proposition proves that the only possibly stable steady state is one where the agents use the β value 1. This follows from the analysis that shows that motion on the center manifold is locally convergent if and only if $\beta = 1$ at the steady state. Local stability now requires that the remaining eigenvalue of the system at the steady state have modulus less than one. This condition is satisfied if the effect of expectations on the economy at the steady state corresponding to $\beta = 1$ is small enough. On the other hand, if expectations matter significantly, this remaining eigenvalue exceeds 1 in modulus and one gets local instability. To summarize, the only steady states that are locally stable for a robust specification of economies are those where (i) agents beliefs at the steady state follow a random walk and (ii) where the effect of expectations on the economy at the steady state is not too large.

In contrast to the earlier results [1,2 and 3], where one got stability for an open set of $|\beta|$ estimates, allowing for the general possibility of motion on the center manifold gives a very different qualitative picture. The restriction on agents beliefs required for stability implied by the result of this paper is more severe. The earlier stability results required agents to believe sufficiently strongly that the steady state value of the system was stable, whereas here I show that for stability one requires 'singular beliefs'— that at the steady state (where the β estimate is 1), agents believe that all possible values of the state variable, and not just the known steady state value, be possible equilibria of the system. Note that values of the state variable other than the steady state value are *not* rest points of the learning system and that for small perturbations of the state variable from its steady state value (and/or of the β estimates from 1), there are dynamics induced by learning, which, provided the influence of expectations is not too large, converge to the steady state where the β estimate is 1.

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It remains to be seen to what extent the result of this and the earlier papers referred to above hold when agents are no longer subjectively certain about the steady state of the system, and try to learn it by reformulating their regression model so as to include an intercept term as well. A complete analysis of this more general situation is quite complex and is not attempted here. However, some preliminary observations that are of relevance are noted in Appendix B. First, Lemma B.1 states that one may get convergence to the random walk model even if the agents fixed perception of the steady state differs (unlike the case studied in Section 2) from the steady state value associated with the random walk model, provided the difference is small enough. This shows that the stability results obtained in Section 2 are robust with respect to small misspecifications in the agents (fixed) perception of the steady state, and points towards the subsequent analysis of stability when agents learn about the steady state as well. Lemma B.2 describes the similarity in the structure of the stability issue in this new framework with the one studied in Section 2. Here too, the dynamics with learning possesses a unit eigenvalue, which requires one to consider a one dimensional center manifold in evaluating the local stability of the random walk model. When the expected value of the state variable next period determines its current value, this center manifold coincides with a set of equilibria, whose stability properties are easily deduced. For the more general case, one has to carry out a center manifold reduction to ascertain the stability issue as in Section 2, but this is not attempted here.

1 The Model

1A Expectation Formation :

Agent's beliefs about the dynamics will be assumed to be summarised by a model of the form (as in [1,2 and 3])

$$x_{t+1} = \beta x_t + \epsilon_{t+1} \tag{1.1}$$

where ϵ_t is a white noise process with bounded support and a continuous density.

The model (1.1) has been expressed in deviations of the state-variable from a constant term, where the constant represents the agents perception of the steady-state of the economic system. As in [1,2 and 3], the constant is never revised along the learning process, only the β estimates are revised. The state variable is $y_t = \bar{y} + x_t$, where $\bar{y}$ is the perceived steady state (whose precise value is determined by steady state considerations, [see Definition 2] for the analysis of section 2) and x_t the deviation from it. At time t, agents will be assumed to use information upto $t - 1$ (as is standard in the literature) to generate their distribution of y_{t+1}. Thus

$$y_{t+1} = \bar{y} + \beta_{t-1}^2 x_{t-1} + \beta_{t-1}\epsilon_t + \epsilon_{t+1} \tag{1.2}$$

Given the probability distribution of y_{t+1} (where y_{t+1} is bounded away from zero) generated by (1.2), y_t is determined from the temporary equilibrium map. Agents now use the 'realization' x_t to update their estimate of β by taking a convex combination of β_{t-1} and $\frac{x_t}{x_{t-1}}$ according to the rule (where $m > 0$)

$$\beta_t = \frac{\beta_{t-1}}{1 + mx_{t-1}^2} + \left(\frac{mx_{t-1}^2}{1 + mx_{t-1}^2}\right)\frac{x_t}{x_{t-1}} \tag{1.3}$$

The rule (1.3) is smooth though the initial estimate of β may be determined by initial conditions (lagged values of x) in a discontinous manner. Alternative smooth updating rules are provided by Recursive Least Squares and Bayesian updating.[1]

1B Temporary Equilibrium Map and Steady States :

Existing literature has considered the formulation where y_t depends on y_{t+1}^e, which is the expected value of the state variable next period conditional on information through t-1. For the sake of illustration (but this is not necessary for the analysis of Section 2), consider the more general formulation where y_t depends on $E[y_{t+1}^{\alpha-1}], \alpha > 0, \alpha \neq 1$ and where this dependence is summarised by the smooth

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[1]In the case of Recursive Least Squares, the constant m in (1.3) needs to be updated over time. This causes the updating rule to become two dimensional, which complicates the computations of the subsequent analysis considerably, without changing the result qualitatively.

temporary equilibrium map defined implicitly as

$$T(y_t, E[y_{t+1}^{\alpha-1}]) = 0 \qquad (1.4)$$

Appendix A presents an example of a simple pure exchange OLG economy with money (where α is the coefficient of relative risk aversion of the utility function of the agents) that yields a formulation of the form (1.4). Assuming $\alpha = 2$ gives $T(y_t, y_{t+1}^e) = 0$, which is the special case that has been considered in the literature thus far [1,2,3 and 5]. This particular choice of α has a considerable simplifying effect on the eventual analysis of the stability of deterministic steady states which will be spelt out subsequently.

DEFINITION 1: The state variable with learning at time t is the vector $(y_t, (\bar{y}, \beta_t))$ where y_t is the original state variable and $(\bar{y}, \beta_t)$ the vector of beliefs about the constant and the parameter β.

DEFINITION 2: A steady state is a vector $(\bar{y}(\bar{\beta}), (\bar{y}, \bar{\beta}))$ where the market clearing steady state value of y coincides with the agents perceived steady state, i.e $\bar{y}(\bar{\beta}) = \bar{y}$.

Thus, $\bar{y}(\bar{\beta})$ satisfies (from (1.2) and (1.4)) and the fact that $x_t = 0 \forall t$

$$T(\bar{y}(\bar{\beta}), E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-1}]) = 0 \qquad (1.5)$$

Since at a steady state $x_t = 0 \forall t$, it follows from (1.3) that β remains at $\bar{\beta} \forall t$. This verifies that steady states defined above are indeed equilibria of the dynamical system defined by (1.2), (1.3) and (1.4).

1C: Local Dynamical System with Learning

The prinicipal purpose of the paper is to analyse the local stability of a steady state $(\bar{y}(\bar{\beta}), (\bar{y}, \bar{\beta}))$ where the dynamics are defined by (1.2),(1.3) and (1.4).

Let n_1, n_2 be the partial derivatives of (1.4) w.r.t to its arguments respectively evaluated at the steady state $(\bar{y}(\bar{\beta}), (\bar{y}, \bar{\beta}))$. Assuming $n_1 \neq 0, n_2 \neq 0$ one gets a local solution to (1.4) of the form

$$y_t = G(E[y_{t+1}^{\alpha-1}]) \qquad (1.6)$$

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Using (1.2) in (1.6) and subtracting the constant term $\bar{y}(\bar{\beta})$ from both sides of the equation, (1.6) is written in deviations as

$$x_t = F(x_{t-1}, \beta_{t-1}) \tag{1.7}$$

(1.7) is combined with (1.3) to define the local dynamical system with learning. The local stability of equilibria of the form $(0, \bar{\beta})$ are to be assessed.

It is verified by direct inspection that the partial derivatives of (1.3) with respect to x and β evaluated at $(0, \bar{\beta})$ are 0 and 1 respectively. The partial derivative of F at $(0, \bar{\beta})$ with respect to x is denoted

$$a = \frac{-n_2}{n_1}(\alpha - 1)E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-2}\bar{\beta}^2] \tag{1.8}$$

and the partial derivative at the equilibrium with respect to β is denoted

$$b = \frac{-n_2}{n_1}(\alpha - 1)E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-2}\epsilon_t]. \tag{1.9}$$

Given these derivatives, one obtains:

(1.10) . . . . . The eigenvalues of the Jacobian of the system (1.3), (1.7) evaluated at an equilibrium $(0, \bar{\beta})$ are $\lambda_1 = a$ and $\lambda_2 = 1$. Thus there exists a one dimensional center manifold corresponding to the unit root which must be considered while evaluating the local asymptotic stability of $(0, \bar{\beta})$.

One can now examine the consequence of the assumption that is common to earlier studies [1,2 and 3], namely that $E[y_{t+1}]$ enters the temporary equilibrium map in (1.6), i.e $\alpha = 2$. In that case $b = 0$ and also $0 = F(0, \beta) \forall \beta$, which illustrates that there is a whole one dimensional manifold of equilibria of the form $(0, \beta)$. This set of equilibria in fact coincides with the center manifold. Thus there is no motion on the center manifold, since it is a set of equilibria. Stability of an element of the equilibrium set is now determined by the modulus of the remaining eigenvalue $\lambda_1 = a$ [3, section 4]. One can verify from the formula of a above that one gets $|a| < 1$ (and hence local stability) when β is small enough and conversely $|a| > 1$

(local instability) for sufficiently large β values. The conclusion of the discussion is that the β estimates affect stability issue only through their effect on λ_1. This is the simplifying effect of setting $\alpha = 2$, since it suppresses the possibility of motion on the center manifold; but this motion must be carefully taken into account for cases where $b \neq 0$.

In the next section I analyse the local stability of equilibria of the form $(0, \bar{\beta})$ for the general case $b \neq 0$ without reference to the specific microeconomic structure underlying the equilibrium whose stability is being assessed.

2 Stability

Consider the dynamical system

$$x_{t+1} = F(x_t, \beta_t) \tag{2.1}$$

$$\beta_{t+1} = \frac{\beta_t}{1 + mx_t^2} + \left(\frac{mx_t^2}{1 + mx_t^2}\right)\frac{F(x_t, \beta_t)}{x_t} \tag{2.2}$$

where F is smooth and $m > 0$, which possesses $(0, \bar{\beta})$ as an equilibrium.

The principal proposition of the paper is the following characterisation of a locally stable equilibrium:

Proposition 2.1:*Let $(0, \bar{\beta})$ be an equilibrium of (2.1)-(2.2) such that $b \neq 0$ where $b = F'_{\beta}(0, \bar{\beta})$ and $a \neq 0$, $a \neq 1$ where $a = F'_x(0, \bar{\beta})$. Then $(0, \bar{\beta})$ is locally asymptotically stable if and only if the following conditions hold:*
(1) $\bar{\beta} = 1$
(2) $|a| < 1$. This condition is satisfied if $|\frac{n_2}{n_1}|$, which measures the effect of expectations on the economy, is small enough and is violated if the effect of expectations on the economy is large enough.

Proof: To analyse the local stability of $(0, \bar{\beta})$, one has to carry out a center manifold reduction, which requires that the system be transformed so that the equilibrium is translated to $(0,0)$ and the linear part is in Block Diagonal form.

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Observe that (2.2) can be written as:

$$\beta_{t+1} = \beta_t + \frac{m}{1+mx_t^2}[-\beta_t x_t^2 + x_t F(x_t, \beta_t)] \dots \dots \dots \dots \dots \dots \quad (2.2')$$

Replacing F by its Taylor series at $(0, \bar{\beta})$ and using (2.2') one gets

$$\begin{pmatrix} x_{t+1} \\ \beta_{t+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\beta} \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_t \\ \hat{\beta}_t \end{pmatrix} + R(\hat{x}_t, \hat{\beta}_t)$$

where $R(\hat{x}_t, \hat{\beta}_t) = \begin{pmatrix} c\hat{x}_t^2 + d\hat{x}_t\hat{\beta}_t + e\hat{\beta}_t^2 + O[(\hat{x}_t, \hat{\beta}_t)^3] \\ \frac{m}{1+m\hat{x}_t^2}[-\hat{x}_t^2(\bar{\beta} + \hat{\beta}_t) + \hat{x}_t(a\hat{x}_t + b\hat{\beta}_t + O[(\hat{x}_t, \hat{\beta}_t)^2])] \end{pmatrix}$

and where $(\hat{x}_t, \hat{\beta}_t)$ are deviations of (x, β) from $(0, \bar{\beta})$. The particular form of (2.2') is such that one need not Taylor expand to separate the linear part from the higher order terms. One could of course use the Taylor expansion and conduct the exercise in an identical manner as below. (The only difference would be that in (2.8) below, one would replace $T(w)$ by m).

Rearranging, one obtains:

$$\begin{pmatrix} \hat{x}_{t+1} \\ \hat{\beta}_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_t \\ \hat{\beta}_t \end{pmatrix} + \begin{pmatrix} c\hat{x}_t^2 + d\hat{x}_t\hat{\beta}_t + e\hat{\beta}_t^2 + O[(\hat{x}_t, \hat{\beta}_t)^3] \\ \frac{m}{1+m\hat{x}_t^2}[-\hat{x}_t^2(\bar{\beta} + \hat{\beta}_t) + \hat{x}_t(a\hat{x}_t + b\hat{\beta}_t + O[(\hat{x}_t, \hat{\beta}_t)^2])] \end{pmatrix}$$

which is of the form

$$\Gamma_{t+1} = B\Gamma_t + C(\Gamma_t)$$

where

$$\Gamma = \begin{pmatrix} \hat{x} \\ \hat{\beta} \end{pmatrix} \text{ and } B = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

I know make the coordinate change $\Gamma = PZ$ so as to bring B into the block diagonal form. The columns of P are composed of the eigenvectors of B. These are $(1,0)$ and $(p, 1)$ where $p = \frac{b}{1-a}$. Thus one gets

$$Z_{t+1} = P^{-1}BPZ_t + P^{-1}[C(PZ_t)] \qquad (2.3)$$

Letting $Z = \begin{pmatrix} v \\ w \end{pmatrix}$ and using (2.3) one finally gets:

$$\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} r(v_t, w_t) - pq(v_t, w_t) \\ q(v_t, w_t) \end{pmatrix} \qquad (2.4)$$

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where

$r(v, w) = c[v + pw]^2 + d[v + pw]w + ew^2 + O[(v, w)^3]$ and

$q(v, w) = \frac{m}{1+m[v+pw]^2}[-[v + pw]^2(\bar{\beta} + w) + [v + pw](a[v + pw] + bw + c[v + pw]^2 + d[v + pw]w + ew^2 + O[(v, w)^3])]$.

The system (2.4) is finally in a form such that one can apply center manifold theory to it, from which one has:

(i) [4, Theorem 2.1.4] There exists a local center manifold $v = h(w), h(0) = 0, Dh(0) = 0$ s.t the dynamics on the center manifold for w small are governed by

$$w_{t+1} = J(w_t) = w_t + q(h(w_t), w_t) \tag{2.5}$$

(ii) [4, 2.1.53] $h(w)$ satisfies:

$$N[h(w)] = h(w + q(h(w), w)) - ah(w) - g(h(w), w) = 0 \tag{2.6}$$

where $g = r - pq$.

(iii) Finally [4, Theorem 2.1.6], one can in fact assess the stability of 0 under (2.5) by assuming

$$h(w) = kw^2 + lw^3 + O(w^4) \tag{2.7}$$

and using (2.6) to identify k, l. For the present proof, it will become clear that knowledge of k is in fact sufficient to determine the stability of (2.5) for small w.

Inserting (2.7) in (2.5), one gets

$J(w) = w + T(w)(kw^2 + lw^3 + O(w^4) + pw)^2[-(\bar{\beta} + w)] + T(w)(kw^2 + lw^3 + O(w^4) + pw)[a(kw^2 + lw^3 + O(w^4) + pw) + bw + c(kw^2 + lw^3 + O(w^4) + pw)^2 + d(kw^2 + lw^3 + O(w^4) + pw)w + ew^2 + O(w^3)]$

where

$T(w) = \frac{m}{1+m[kw^2+lw^3+O(w^4)+pw]^2} > 0$ and $\lim T(w) = m$ as $w \to 0$.

Rearranging, one gets

$J(w) = w + T(w)[-p^2\bar{\beta} + ap^2 + bp]w^2 + T(w)[-2pk\bar{\beta} - p^2 + 2apk + bk + cp^3 + dp^2 +$

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$ep]w^3 + T(w)O(w^4)$

which simplifies to:

$$J(w) = w + T(w)p^2(1 - \bar{\beta})w^2 + T(w)[S(k)]w^3 + O(w^4) \qquad (2.8)$$

where

$$S(k) = [-2pk\bar{\beta} - p^2 + 2apk + bk + cp^3 + dp^2 + ep].$$

The stability of 0 under (2.5) is identical to the stability of 0 under (2.8). Two cases need to be considered separately.

Case 1: When $\bar{\beta} \neq 1$, the quadratic term $T(w)p^2(1 - \bar{\beta})w^2$ governs the dynamics for w small. Hence there is no stability.

Case 11: When $\bar{\beta} = 1$, the quadratic term vanishes and one is left with

$$J(w) = w + T(w)[S(k)]w^3 + O(w^4) \qquad (2.9)$$

The sign of $S(k)$ is crucial in determining stability since $T(w) > 0$. Thus one needs to identify k from (2.6). To do so, differentiate (2.6) twice to obtain

$N'' = h''(w + q(h(w), w))[1 + Dq_w + Dq_v h'(w)]^2 + h'(w + q(h(w), w))D[1 + Dq_w + Dq_v h'(w)] - ah''(w) - Dg_v(h(w), w)h''(w) - h'(w)[Dg_{vw} + Dg_{vv}h'(w)] - h'(w)Dg_{vw} - Dg_{ww}(h(w), w) = 0.$

Using the fact that $h(0) = 0$, $h'(0) = 0$ and $Dg_v(0,0) = 0$, one gets

$h''(0) = \frac{Dg_{ww}(0,0)}{1-a}$.

$Dg_{ww} = Dr_{ww} - pDq_{ww}$

$Dr_{ww}(0,0) = 2[cp^2 + pd + e]$

$Dq_{ww}(0,0) = 2p^2[1 - \bar{\beta}].$

Finally, $k = \frac{h''(0)}{2} = \frac{cp^2+pd+e}{1-a}$ and thus

$S(k) = [(cp^2 + dp + e)(\frac{-2p+2ap+b}{1-a} + p) - p^2]$, which reduces to $-p^2$.

Thus (2.9) becomes

$$J(w) = w + T(w)[-p^2]w^3 + O(w^4) \qquad (2.10)$$

and one gets stability for small w.

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Thus the dynamics on the center manifold are locally stable iff $\bar{\beta} = 1$ which proves part (a) of the proposition. The local stability of $(0,0)$ under (2.4) now depends on the eigenvalue $\lambda_1 = a$. When $|a| < 1$ one gets stability and when $|a| > 1$, instability. The influence of expectations on the parameter a is obvious from the formula of a.∎

Remark 2.1: Given the existence of an eigenvalue that equals one in the system being considered, and the assumption that $b \neq 0$, a necessary condition for local asymptotic stability is that the reduced dynamics on the invariant center manifold be locally convergent. This obtains iff $\bar{\beta} = 1$; i.e. the only possibly stable steady states are those where the configuration of beliefs follows a random walk, since with $\bar{\beta} = 1$, the agents model (1.1) becomes

$$x_{t+1} = x_t + \epsilon_{t+1} \tag{2.11}$$

Thus for small deviations of x from 0 and small perturbations of the parameter estimates from 1, the learning dynamics converge to $x = 0$ and $\beta = 1$, provided the influence of expectations on the economy at the steady state is small enough.

Remark 2.2: I clarify here that one may get the stability of random walk beliefs in the earlier formulations [1,2 and 3], provided the influence of expectations is small enough. However, local stability in that framework only guarantees that for small perturbations, the system with learning converges to $(0, \beta)$ where β is some value close to 1. One cannot guarantee that agents will learn exactly the β value 1, but only that the parameter estimates will converge to some value close to it. But the same would be true for any β value provided the influence of expectations is small enough. In particular the dynamics may converge to the steady state with the limiting belief about the parameter satisfying $|\beta| > 1$, a potentially contradictory result since with the limiting β estimates agents would expect the system to diverge from the steady state. This is excluded in the present formulation, since as shown above, the β value 1 is singled out uniquely by considerations of local stability.

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3 Conclusion

To conclude, the result presented in this paper refers to the formulation wherein agents recognize the steady-state of the system and learn about the parameter which governs the dynamics outside the steady state using a smooth updating rule. The only possibly stable steady states are those associated with random walk beliefs (Proposition 2.1). The analysis pertains to cases where a key parameter is non zero and the result is in sharp contrast to the earlier stability results (Remarks 2.1 and 2.2). It is observed in Appendix B (Lemma B.1) that the stability result is robust with respect to small misspecifications of the steady state in the agents model. Lemma B.2 indicates the possibility (without analysing the system completely) of obtaining similar results for the random walk model in a more general set up where agents try to learn the steady state as well.

4 Appendix A

I present a specification of an OLG set up that yields a temporary equilibrium map of the form (1.4).

Consider a pure exchange overlapping generations economy with two period lived agents, one perishable good each time period and money. At time t, trade takes place between one young and one old agent.

The utility function of a young agent born at time t is:

$$U(c_t, c_{t+1}) = \frac{c_t^{1-\alpha}}{1-\alpha} + \frac{c_{t+1}^{1-\alpha}}{1-\alpha}, \alpha > 0, \alpha \neq 1.$$

The agents have an endowment of ω in the first period of their life and nothing in the second. There are M units of money in the economy. Let k denote $\frac{M}{\omega}$.

The young agents are assumed to maximise expected utility and the their demand function is

$$c_t = p_t \omega [p_t + (p_t E[p_{t+1}^{\alpha-1}])^{\frac{1}{\alpha}}]^{-1}.$$

From market clearing, one obtains the following temporary equilibrium map

which is in implicit form:

$$E[p_{t+1}^{\alpha-1}] = \frac{k^\alpha}{p_t[1 - \frac{k}{p_t}]^\alpha} \tag{4.1}$$

One can verify that given $E[p_{t+1}^{\alpha-1}]$, there exists a unique temporary equilibrium at each date satisfying $p_t > k \forall t$. The above temporary equilibrium map is of the form (1.4).

Steady states of the form $(\bar{p}(\bar{\beta}), (\bar{p}(\bar{\beta}), \bar{\beta}))$ satisfy:

$$E[(\bar{p}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-1}] = \frac{k^\alpha}{\bar{p}(\bar{\beta})[1 - \frac{k}{\bar{p}(\bar{\beta})}]^\alpha}.$$

When $\alpha = 2$, $\bar{p}(\bar{\beta})$ is in fact independent of $\bar{\beta}$ and coincides with the Golden Rule price which equals $2k$.

For other values of α, $\bar{p}(\bar{\beta})$ varies with $\bar{\beta}$. For instance, if $\alpha = 3$, $\bar{p}(\bar{\beta})$ and $\bar{\beta}$ satisfy the relation:

$$\sigma_\epsilon^2(\bar{\beta}^2 + 1) = \bar{p}(\bar{\beta})^2 \left(\frac{k^3}{(\bar{p}(\bar{\beta}) - k)^3} - 1 \right) \tag{4.2}$$

Let $\hat{k}$ be the value of $\bar{p}(\bar{\beta})$ which solves the above relation for the minimum possible value of the left hand side, which is σ_ϵ^2. Then for all real $\bar{\beta}$, there exists $\bar{p}(\bar{\beta}) \in (k, \hat{k})$ satisfying (4.2). The existence of steady states can similarly be demonstrated for other values of α as well, by imposing a lower bound on $E[p_{t+1}^{\alpha-1}]$ and using the continuity of the right hand side of (4.1).

5   Appendix B

[5.A] I first examine the possibility of convergence to random walk beliefs when the equilibrium value of the state variable $(y_t = y^* \forall t)$ does not coincide with $\bar{y}$, the perceived steady state, which is held fixed in the agents model.

So consider a fixed value of $\bar{y}$, and now to identify the equilibrium, I keep β fixed at 1, so that agents beliefs are represented by the random walk model:

$$y_{t+1} = \bar{y} + x_{t-1} + \epsilon_t + \epsilon_{t+1} \tag{5.1}$$

or

$$y_{t+1} = y_{t-1} + \epsilon_t + \epsilon_{t+1} \tag{5.2}$$

Using the temporary equilibrium map (1.4), the equilibrium value of y is the solution (assumed to exist) to

$$T(y, E[(y + \epsilon_t + \epsilon_{t+1})^{\alpha-1}]) = 0 \tag{5.3}$$

The solution is denoted y^* and coincides with $\bar{y}(1)$, which is the solution to (1.5) with $\bar{\beta} = 1$.

The difference with the cases examined in sections 1 and 2 above is that $y^* = \bar{y}(1)$ is not necessarily equal to $\bar{y}$. In deviations, one gets $x^* = y^* - \bar{y}$. The ratio $\frac{x_{t+1}}{x_t}$ at the equilibrium x^* equals 1 $\forall t$. Given the rule (1.3), since at the equilibrium agents use $\beta = 1$, there is no updating and thus $(x^*, 1)$ is indeed an equilibrium of the system (5.4)-(5.5) below, where F^* is obtained from (1.6) (using (1.2)) and subtracting the constant $\bar{y}$:

$$x_t = F^*(x_{t-1}, \beta_{t-1}) \tag{5.4}$$

$$\beta_t = \frac{\beta_{t-1}}{1 + mx_{t-1}^2} + \left(\frac{mx_{t-1}^2}{1 + mx_{t-1}^2}\right)\frac{F^*(x_{t-1}, \beta_{t-1})}{x_{t-1}} \tag{5.5}$$

where F^* is the solution to the temporary equilibrium map when expressed in deviations from $\bar{y}$. The system (5.4)-(5.5) with $\bar{y} \neq \bar{y}(1)$ will be referred to as the misspecified system to represent the fact that equilibrium value of the state variable does not coincide with the agents perceived steady state. In case the perceived steady state $\bar{y}$ equals the equilibrium level $\bar{y}(1)$, the equilibrum deviation $x^* = 0$, F^* becomes F and one is back to the formulation of sections 1 and 2, where the equilibrium whose stability is being assessed is $(0, 1)$. The system is then referred to as the correctly specified system.

I show below that one may indeed get convergence to the random walk model under the misspecified system (5.4)-(5.5) as well, i.e $(x^*, 1)$ may be locally stable under the learning dynamics. The proposition below asserts that in fact, the misspecified system inherits the stability properties of the correctly specified system, provided the misspecification x^* is small.

Lemma B.1: *Let $(0, 1)$ be locally asymptotically stable (unstable) under (2.1)-(2.2),*

i.e $|a| < (>)1$. *Then* $(x^*, 1)$ *is also locally stable (unstable) under (5.4)-(5.5) provided the degree of misspecification* $|x^*| = |y^* - \bar{y}|$ *is small enough.*

Proof: The partial derivatives of F^* with respect to x and β evaluated at $(x^*, 1)$ are denoted $a(x^*) = a$ and $b(x^*)$. The partial derivatives of (3.5) with respect to x and β evaluated at $(x^*, 1)$ are $\frac{mx^*[a-1]}{1+mx^{*2}}$ and $\frac{1+mb(x^*)x^*}{1+mx^{*2}}$. Let $\lambda_1(x^*)$ and $\lambda_2(x^*)$ be the eigenvalues of (3.4)-(3.5) at $(x^*, 1)$. By continuity, the partial derivatives and hence the eigenvalues, converge to those of the correctly specified system at $(0,1)$ as the misspecification goes to 0. Thus $\lambda_1(x^*) \to a$ and $\lambda_2(x^*) \to 1$. Thus if $|a| > 1$, one is bound to get instability for x^* small. When $|a| < 1$, one will get stability provided $\lambda_2(x^*) < 1$ for x^* small. The characteristic polynomial at $(x^*, 1)$ defines a parabola with its asymptotic branches going up. To verify that $0 < \lambda_2(x^*) < 1$ for x^* small, it therefore suffices to show that the characteristic polynomial evaluated at $\lambda = 1$ becomes positive for x^* small enough. The value of the characteristic polynomial thus evaluated simplifies to $q = \frac{mx^{*2}[1-a]}{1+mx^{*2}}$, which is positive provided $|a| < 1$, and one gets local stability. ∎

The above proposition demonstrates that the stability results for the random walk model obtained in Section 2 are in fact robust with respect to small misspecifications of the constant term from the steady state value of the state variable.

The interpretation of the robustness implied by Lemma B.1 is not immediate, since the fact that the dynamics converge (in the stable case) to a steady state which differs from the perceived one may appear to contradict the agents model. One may expect that, given that the learning dynamics do converge, the agents in the limit of the learning process recognize the misspecification in their prior estimate of the steady state and incorporate the persitent one sided deviation x^* to obtain a correct specification of the steady state. [2] The new configuration of beliefs will now be described by $(y^*, 1)$ and this will indeed be a locally stable configuration as discussed above.

However, the convergence possibility described in Lemma B.1 does seem to point

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[2] I owe this point to an anonymous referee.

towards an enquiry of the possibility of convergence to the random walk model in a set up where agents are no longer certain about the steady state, and try to learn the β parameter and the steady state by reformulating their model so as to include an intercept term.

[5.B] I therefore now examine the possibility of convergence to random walk beliefs in the case where agents try to learn the steady state as well by reformulating their regression model so as to include an intercept term.

The model used by the agents is now:

$$y_{t+1} = \gamma + \beta y_t + \epsilon_{t+1} \tag{5.6}$$

Since agents iterate twice on the model, one gets the distribution of y_{t+1} using

$$y_{t+1} = \gamma_{t-1} + \gamma_{t-1}\beta_{t-1} + \beta_{t-1}^2 y_{t-1} + \beta_{t-1}\epsilon_t + \epsilon_{t+1} \tag{5.7}$$

One gets the random walk model when agents believe that $\gamma = 0$ and $\beta = 1$. The issue is to assess whether starting with initial parameter estimates (beliefs) close to these, the learning dynamics converge to an equilibrium corresponding to these particular estimates.

To formulate the issue, I revert to the temporary equilibrium map (1.6), which using (5.7) is rewritten as:

$$y_t = W(y_{t-1}, \gamma_{t-1}, \beta_{t-1}) \tag{5.8}$$

When agents form expectations according to random walk beliefs, the steady state value of y is, as in sections 1, and 5A, $\bar{y}(1) = y^*$.

The updating formula used to learn about γ and β is almost identical to the simplified version of least squares updating used by Woodford [5] and is given by:

$$\begin{pmatrix} \gamma_{t+1} \\ \beta_{t+1} \end{pmatrix} = \begin{pmatrix} \gamma_t \\ \beta_t \end{pmatrix} + N^{-1} \begin{pmatrix} 1 & M_{1t} \\ M_{1t} & M_{2t} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ y_t \end{pmatrix} [y_{t+1} - (\gamma_t + \beta_t y_t)]$$

$$\begin{pmatrix} M_{1t+1} \\ M_{2t+1} \end{pmatrix} = \begin{pmatrix} M_{1t} \\ M_{2t} \end{pmatrix} + N^{-1} \begin{pmatrix} y_t - M_{1t} \\ y_t^2 - M_{2t} \end{pmatrix} \quad \dots \dots \dots (5.9)$$

M_{1t}, M_{2t} are proxy estimates of the moments $E(y), E(y^2)$ respectively using past observations $(y_{t-1}, ...)$. The only difference from Woodford's specification [5, Section 2] is that I replace $(t - 1)$ by some large (greater than 1) number N. For the dynamics to be well defined for all t it suffices (as observed by the author in [5]), that the starting values (M_{10}, M_{20}) satisfy

$M_{20} > M_{10}^2$ . . . . . . (5.10).

The complete dynamical system is defined by (C.3)-(C.4). The state variable at time t is $(y_t, \gamma_t, \beta_t, M_{1t}, M_{2t})$ and the equilibria whose stability are to be assessed are of the form $(y^*, 0, 1, M_{10}, M_{20})$ satisfying (5.10). Under this condition, the dynamics at an equilibrium are differentiable.

The following observation about the local dynamics is noted:

Lemma B.2: *Consider an equilibrium of the form $(y^*, 0, 1, M_{10}, M_{20})$ of the system (5.8)-(5.9), satisfying (5.10). Then the following are true:*

(a). The Jacobian of $(y^*, 0, 1, M_{10}, M_{20})$ has an eigenvalue $\lambda_1 = 1$. Thus there exists a one dimensional center manifold associated with $(y^*, 0, 1, M_{10}, M_{20})$, which must be considered while evaluating stability.

(b). There exist two eigenvalues $0 < \lambda_4 < 1$, $0 < \lambda_5 < 1$.

(c). Now assume $M_{20} > y^{*2}$ and that M_{10} is close enough to y^* (which is guaranteed if the initial conditions $[y_{-1}, .., y_{-L}]$ are close enough to y^*). This ensures that (5.10) is satisfied. Then if $|\frac{n_2}{n_1}|$, which measures the effect of expectations on the economy, is small enough, there exist two eigenvalues λ_2 close to zero and, $0 < \lambda_3 < 1$. Finally

(d). If $\alpha = 2$, then the center manifold is a set of equilibria of (5.8)-(5.9). If condition (c) is satisfied, there exists an open set of initial conditions V containing $(y^*, 0, 1, M_{10}, M_{20})$ such that the learning trajectories generated within V stay in V and converge to some element of the center manifold.

Remark B.1: The qualitative nature of the stability problem is very similar to the case where agents learn just the β parameter as in Section 2. Here too there is a unit eigenvalue and hence a one dimensional center manifold. As discussed in Remark

2.2, when the expected value of the state variable enters the temporary equilibrium map (i.e $\alpha = 2$), local stability only guarantees that the dynamics converge to some element of the center manifold close to the equilibrium being considered. For other cases, there is the possibility of getting convergence to exactly random walk beliefs. However, to formally demonstrate this, one has to implement a center manifold reduction as in Section 2, which in the present case is quite complicated and is not undertaken here.

Proof: The Jacobian evaluated at an equilibrium has the form

$$\begin{pmatrix} W_y & W_\gamma & W_\beta & 0 & 0 \\ \gamma_y & \gamma_\gamma & \gamma_\beta & 0 & 0 \\ \beta_y & \beta_\gamma & \beta_\beta & 0 & 0 \\ N^{-1} & 0 & 0 & 1 - N^{-1} & 0 \\ 2N^{-1}y^* & 0 & 0 & 0 & 1 - N^{-1} \end{pmatrix}.$$

There are thus two eigenvalues denoted $\lambda_4 = \lambda_5 = 1 - N^{-1}$ which for $N > 1$ are inside the unit circle. The remaining eigenvalues are the roots of the characteristic polynomial associated with

$$\begin{pmatrix} W_y & W_\gamma & W_\beta \\ \gamma_y & \gamma_\gamma & \gamma_\beta \\ \beta_y & \beta_\gamma & \beta_\beta \end{pmatrix} \quad \cdots \cdots \cdots (5.11)$$

where the entries are the appropriate partial derivatives. These are

$W_\beta = \frac{-n_2}{n_1}(\alpha - 1)E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-2}(2y^* + \epsilon_t)],$

$W_\gamma = \frac{-n_2}{n_1}(\alpha - 1)E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-2}2],$

$W_y = \frac{-n_2}{n_1}(\alpha - 1)E[(\bar{y}(\bar{\beta}) + \bar{\beta}\epsilon_t + \epsilon_{t+1})^{\alpha-2}],$

$\gamma_\beta = [N(M_{20} - M_{10}^2)]^{-1}[M_{20} - M_{10}y^*][W_\beta - y^*],$

$\gamma_\gamma = 1 + [N(M_{20} - M_{10}^2)]^{-1}[M_{20} - M_{10}y^*][W_\gamma - 1],$

$\gamma_y = [N(M_{20} - M_{10}^2)]^{-1}[M_{20} - M_{10}y^*][W_y - 1],$

$\beta_\beta = 1 + [N(M_{20} - M_{10}^2)]^{-1}[-M_{10} + y^*][W_\beta - y^*],$

$\beta_\gamma = [N(M_{20} - M_{10}^2)]^{-1}[-M_{10} + y^*][W_\gamma - 1],$

$\beta_y = [N(M_{20} - M_{10}^2)]^{-1}[-M_{10} + y^*][W_y - 1].$

Using these derivatives, one can evaluate the characteristic polynomial associated

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with (5.11) and confirm that $\lambda = 1$ is an eigenvalue.

As an approximation to the case where the effect of expectations is small and M_{10} is close enough y^*, I evaluate the characteristic polynomial with $|\frac{n_2}{n_1}| = 0$, and $M_{10} = y^*$. Thus the partials of W are set equal to 0, $\beta_\gamma = \beta_y = 0$ and $\beta_\beta = 1$. With these values, the roots of the characteristic polynomial are 0, γ_γ (which for the present specification lies between 0 and 1) and 1. Thus by continuity, when the effect of expectations is small enough and M_{10} is close enough to y^*, λ_2 is close to 0, λ_3 is between 0 and 1.

Finally, when $\alpha = 2$, there exists a one dimensional manifold of equilibria, since all γ, β values satisfying $y^*(1 - \beta) = \gamma$ when combined with $y^*, (M_{10}, M_{20})$ are equilibria of the learning system. The local stability now follows in an identical fashion to the earlier case where agents did not learn the steady state [3, Section 4], since all other eigenvalues have been shown to be inside the unit circle. ∎

REFERENCES

1. Chatterji, S. (1994), Temporary Equilibrium Dynamics with Bayesian learning, forthcoming Journal of Economic Theory.

2. Grandmont, J.M. and G. Laroque (1991), Economic Dynamics with Learning : Some Instability Examples in *Equilibrium Theory and Applications*, Cambridge University Press, 1991.

3. Grandmont, J.M. (1995), Expectation formation and Stability of Large Socioeconomic Systems, forthcoming Econometrica.

4. Wiggins, S. (1990), Introduction to Applied nonlinear dynamical systems and chaos, Springer Verlag, TAM 2.

5. Woodford, M. (1991), Least Squares Learning and Inflation Dynamics: A Nonlinear, Stochastic Model, University of Chicago.

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