REDISTRIBUTION AND INDIVIDUAL CHARACTERISTICS*

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Abstract

I study a model where personal income is a function of two different groups of individual characteristics, called "irrelevant" and "relevant" respectively. The distinction between these two groups is that society has taken the prior decision that the influence of traits from the first group needs to be moderated by any fair redistribution mechanism while differences in income due to traits from the second group must be preserved. I present two solutions that satisfy several intuitive properties of fairness and I use these properties to characterize both of them.

KEYWORDS: Redistribution mechanisms, compensation.

1. Introduction

Personal income is the result of many factors. Some of them may be thought about as being the consequence of conscious decisions of the individuals -think, for example, of the number of hours that a person decides to work- while others are the result of circumstances on which individuals have no influence at all. For instance, a person who was born in a rich family is very likely to have a higher income than another person who was born in a poor suburb.

The key idea in this distinction is that of *responsibility.* Some factors as effort, number of hours at work, etc. may be put under individuals' responsibility while others, as parents' wealth may not. I use different names for these different factors. The first ones are called "relevant" traits with the intended meaning that they are relevant to determine personal income. The traits of the second group are termed "irrelevant". Even though this distinction can be a matter of discussion, I assume that society has already decided which individual traits are included in each group. In particular, there is a prior decision on which is the set of traits that persons are not responsible for.

The ethical argument to make such distinction is that differences in income attributed to irrelevant traits are not admissible and thus any fair redistribution mechanism must aim at balancing the effect of these differences. But, on the other side, when differences are due to relevant traits that were chosen by the individuals (a person working twice as much as another) these differences must be preserved.

The issue of responsibility is the central point in the works of Arneson [1], Cohen [4], Roemer [9] and Fleurbaey [6] among others. A somewhat different approach is that of Kranich [8] where instead of income there is a production function and the output must be divided among the agents. Moreover, the amount of labor supplied by the individuals is the relevant factor while preferences and skills are the irrelevant ones.

Bossert [2] and Bossert and Fleurbaey [3] develop a simple model trying to gather these features. They present several redistribution mechanisms and they arrive at some characterization results by using two groups of properties. The first group leans on the intuition that irrelevant traits should be *fully* compensated among individuals. The second one tries to capture the intuition that inequality in relevant traits must be reflected in post-tax income.

It would take too much space the description of all these properties and so the reader is referred to the references. I will present some of the properties later but only when needed.

In this paper I follow the research line of Bossert and Fleurbaey [3] and I present new properties that deal with the question of what kind of changes I want to require to the *entire profile* of post-tax income. The main properties I use are *Lorenz equivalence under irrelevant changes* and a weaker version of it. The idea is that if we consider that some of the individual traits are indeed irrelevant to the income distribution, a natural axiom that we must call for any redistribution mechanism is that *the proportion that represents the income of any individual with respect to total income should be independent of all changes in the particular profile of irrelevant characteristics.* I prove none of the solutions studied by Bossert and Fleurbaey [3] fulfills this property. Moreover, I prove that by requiring two other very weak properties I can characterize two different redistribution mechanisms. Both of them belong to a family that includes all the solutions dividing total income among agents according to fixed proportions that are determined by the relevant traits and a reference vector of irrelevant traits that is different for each one of the two solutions.

Section 2 presents the model. Section 3 introduces some properties and first results. Section 4 contains the main results. Finally, Section 5 concludes with some remarks.

2. Preliminaries

The model I am going to develop is the same than in Bossert [2] and Bossert and Fleurbaey [3]. $N = \{1, 2, ..., n\}$ is the set of individuals in the society. Agents are denoted by $i \in N$. Individuals have r traits or characteristics that are deemed "relevant" and s "irrelevant" traits, where r and s are natural numbers greater or equal than one. The relevant traits of agent $i \in N$ are $a_i^R \in \Re^r$ and the irrelevant traits are $a_i^S \in \Re^s$. Then, agent i is described by the vector $a_i = (a_i^R, a_i^S) \in \Re^{r+s}$. A *characteristics profile* is $\mathbf{a} = (a_1, a_2, ..., a_n) \in \Re^{n(r+s)}$. $\mathbf{a}$ can be partitioned into a profile of relevant characteristics $\mathbf{a}^R = (a_1^R, a_2^R, ..., a_n^R) \in \Re^{nr}$ and a profile of irrelevant characteristics $\mathbf{a}^S = (a_1^S, a_2^S, ..., a_n^S) \in \Re^{ns}$.

The set of all possible characteristics vector is $\Omega = \Omega_R \times \Omega_S$ where $\Omega_R \subseteq \Re^r$, $\Omega_S \subseteq \Re^s$ and $\Omega_R, \Omega_S \neq \emptyset$.

I can define reference vectors for both relevant and irrelevant characteristics as $\tilde{a}^R \in \Omega_R$ and $\tilde{a}^S \in \Omega_S$.

I define an *income function* f as a continuous mapping assigning to each vector in Ω a non-negative real number representing the pre-tax income of the individuals described by that vector, that is:

$$f : \Omega \rightarrow \Re_+, \;\; a = (a^R, a^S) \longmapsto f(a).$$

It is interesting to remark that the income function is the same for all individuals and that all variables that may influence the income level are summarized in the vector a. In particular, (pre-tax) income is not affected by taxes. This seems to be a strong assumption, but I may argue that for those whose income comes mainly from labor, changes in the tax scale (except when they are really large) do not affect labor supply and therefore they do not affect pre-tax income. A *redistribution mechanism* is a mapping $F : \Omega^n \rightarrow \Re^n$, $\mathbf{a} \mapsto F(\mathbf{a})$, such that:

$$\sum_{i=1}^{n} F_i(\mathbf{a}) = \sum_{i=1}^{n} f(a_i), \;\; \forall \mathbf{a} \in \Omega^n.$$

My aim is to find mechanisms such that i's post-tax income be a reflection of i's relevant traits but not of i's irrelevant traits.

A redistribution mechanism is *anonymous* if and only if the following holds:

$$\forall \mathbf{a} \in \Omega^n,\ \forall i,j \in N,\ a_i = a_j \Longrightarrow F_i(\mathbf{a}) = F_j(\mathbf{a}).$$

Most of the results in Bossert [2] are established for the particular domain of additively separable income functions. I introduce it in the following definition.

Definition 2.1. *The income function f is additively separable in R and S if and only if there exist functions $g : \Omega_R \mapsto \Re$ and $h : \Omega_S \mapsto \Re$ such that*

$$f(a) = g(a^R) + h(a^S),\quad \forall a \in \Omega.$$

In this domain the following solution F^0 is characterized:

$$F_k^0(\mathbf{a}) := g(a_k^R) + \frac{1}{n}\sum_{i=1}^{n} h(a_i^S),\quad \forall \mathbf{a} \in \Omega^n,\ \forall k \in N.$$

This solution gives to each individual all the income generated by her relevant traits and a lump sum equal to the average income due to irrelevant traits. The problem is that separability is a very strong restriction. In particular, it implies that the influence of relevant traits on income is independent of irrelevant traits, a rather counterintuitive event.

In the following I will not assume separability, unless otherwise stated.

3. Properties and first results

As I said in the introduction, the aim of any redistribution mechanism must be to compensate differences in income due to irrelevant traits while preserving differences due to relevant traits. The following two properties try to capture these ideas:

Property 3.1. <u>*Equal income for equal R (EIER)*</u>*: For all* $\mathbf{a} \in \Omega^n$, $\forall i,j \in N$,

$$a_i^R = a_j^R \Rightarrow F_i(\mathbf{a}) = F_j(\mathbf{a}).$$

EIER simply says that if two persons have the same relevant traits, their post-tax income must be identical. Notice that this is the traditional principle of *horizontal equity*: those individual whose relevant traits are identical must be treated in the same way. That is, the redistribution mechanism must provide a *full* compensation to counteract the effect of differences in irrelevant traits. Fleurbaey [5] considered this property in a more abstract context. See also Iturbe-Ormaetxe and Nieto [7].

Property 3.2. *Equal transfer for equal S (ETES): For all* $\mathbf{a} \in \Omega^n$, $\forall i, j \in N$,

$$a_i^S = a_j^S \Rightarrow F_i(\mathbf{a}) - F_j(\mathbf{a}) = f(a_i) - f(a_j).$$

This property says that whenever differences in income are exclusively due to relevant traits, these differences must be maintained in post-tax income.

These two properties gather very interesting and intuitive ideas on how any redistribution mechanism must work. Unfortunately, Fleurbaey [6] proved that for $n \geq 4$, additive separability is a necessary condition for any mechanism satisfying EIER and ETES simultaneously. This impossibility result points out an interesting line of research consisting in weakening one axiom or the other stated above in order to find redistribution mechanism that are not necessarily defined only for that particular domain of income functions.

In particular I investigate the response of the income distribution under a change in the profile of irrelevant traits, $\mathbf{a}^S$. If I consider that $\mathbf{a}^S = (a_1^S, ...a_n^S) \in \Re^{ns}$ are indeed irrelevant characteristics to the income distribution, a natural axiom that I must ask for any sensible redistribution mechanism is that the post-tax income distribution should be independent of all changes in the profile $\mathbf{a}^S$. Having this intention in mind I present the following previous definition.

Definition 3.3. *Lorenz curve. Let* $u \in \Re^n$. $u^* \in \Re^n$ *is the vector obtained from* u *by rearranging the coordinates of* u *in increasing order. The Lorenz curve of* u *will be the vector*

$$L(u) = (u_1^*, u_1^* + u_2^*, ..., u_1^* + u_2^* + ... + u_n^*) \text{ or } [L(u)]_k = \sum_{i=1}^{k} u_i^*.$$

One can think about the following requirement: when there is a change from $\mathbf{a}$ to $\hat{\mathbf{a}}$ such that only some of the irrelevant characteristics have changed, the Lorenz curve of both $F(\mathbf{a})$ and $F(\hat{\mathbf{a}})$ should be the same. But this is much more than I need because total income may have changed or perhaps the units that are used to measure income are different from $\mathbf{a}$ to $\hat{\mathbf{a}}$. In fact I propose that the Lorenz curves of the two income distributions have to be related by a constant factor.

Property 3.4. *Lorenz equivalence under irrelevant changes (LEIC): For all* $\mathbf{a}, \hat{\mathbf{a}} \in \Omega^n$ *with* $a_k^R = \hat{a}_k^R$ *for all* $k \in N \Rightarrow \exists \varphi \in \Re_+$ *such that* $L(F(\mathbf{a})) = \varphi L(F(\hat{\mathbf{a}}))$.

To illustrate this first property suppose that $F(\mathbf{a})$ is measured in dollars and $F(\hat{\mathbf{a}})$ in pesetas, then φ is the conversion factor between both currencies. I will also present a weaker version of LEIC by requiring the same relation between the Lorenz curves but only when total income is the same before and after the change in the profile of irrelevant traits.

Property 3.5. *Weak Lorenz equivalence under irrelevant changes (WLEIC): For all* $\mathbf{a}, \hat{\mathbf{a}} \in \Omega^n$ *with* $a_k^R = \hat{a}_k^R$ *for all* $k \in N$ *and* $\sum_{i=1}^n f(a_i) = \sum_{i=1}^n f(\hat{a}_i) \Rightarrow \exists \varphi \in \Re_+$ *such that* $L(F(\mathbf{a})) = \varphi L(F(\hat{\mathbf{a}}))$[1].

Though these are interesting properties they do not prevent from unpleasant events like the following one.

Example 3.6. *Let* $n = 2$, $\mathbf{a} = (a_1, a_2)$ *and* $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2)$ *such that* $a_1^R = \hat{a}_1^R$, $a_2^R = \hat{a}_2^R$. *Suppose that* $F_1(\mathbf{a}) = 10$, $F_2(\mathbf{a}) = 0$ *and* $F_1(\hat{\mathbf{a}}) = 0$, $F_2(\hat{\mathbf{a}}) = 10$. *The post-tax income distribution satisfies property LEIC (and then WLEIC) for* $\varphi = 1$, *but the income distribution has reversed.*

This example shows that a change in the irrelevant characteristics, even if LEIC holds, can affect dramatically the post-tax income distribution. In order to avoid this problem I need to impose an additional requirement on the way the income distribution responds to this kind of change.

[1] It is immediate to check that $\varphi = 1$.

Property 3.7. *Ranking independence of irrelevant changes (RIIC): For all* $\mathbf{a}, \hat{\mathbf{a}} \in \Omega$ *with* $a_k^R = \hat{a}_k^R$ *for all* $k \in N \Rightarrow$ *For all pair* $i, j \in N$, $F_i(\mathbf{a}) \leq F_j(\mathbf{a}) \Leftrightarrow F_i(\hat{\mathbf{a}}) \leq F_j(\hat{\mathbf{a}})$.

According to RIIC, if I arrange individuals according to their post-tax income, this order must be the same for both **a** and **â** defined as above.

When I impose LEIC and RIIC together (or alternatively WLEIC and RIIC), I am forcing the redistribution mechanism to behave in a very particular way when the profile of irrelevant traits change. I present this result in the following lemma.

Lemma 3.8. *a) A redistribution mechanism* F *satisfies LEIC and RIIC if and only if for all* $\mathbf{a}, \hat{\mathbf{a}} \in \Omega^n$ *such that* $a_k^R = \hat{a}_k^R$ *for all* $k \in N$*, it holds that* $\frac{F_i(\mathbf{a})}{F_j(\mathbf{a})} = \frac{F_i(\hat{\mathbf{a}})}{F_j(\hat{\mathbf{a}})}$ *for all* $i, j \in N$*. b) A redistribution mechanism* F *satisfies WLEIC and RIIC if and only if for all* $\mathbf{a}, \hat{\mathbf{a}} \in \Omega^n$ *such that* $a_k^R = \hat{a}_k^R$ *for all* $k \in N$ *and* $\sum_{i=1}^n f(a_i) = \sum_{i=1}^n f(\hat{a}_i)$*, it holds that* $F_i(\mathbf{a}) = F_i(\hat{\mathbf{a}})$ *for all* $i \in N$.

Proof. a) Recall that $F_i^*(\mathbf{a})$ is the post-tax income of the individual whose position is the i-th when I arrange these incomes in increasing order. I know that:

$$L(F(\mathbf{a})) = (F_1^*(\mathbf{a}), F_1^*(\mathbf{a}) + F_2^*(\mathbf{a}), ..., \sum_{i=1}^n F_i^*(\mathbf{a}))$$

and

$$L(F(\hat{\mathbf{a}})) = (F_1^*(\hat{\mathbf{a}}), F_1^*(\hat{\mathbf{a}}) + F_2^*(\hat{\mathbf{a}}), ..., \sum_{i=1}^n F_i^*(\hat{\mathbf{a}})).$$

By LEIC, $\exists \varphi \in \Re_+$ such that $L(F(\mathbf{a})) = \varphi L(F(\hat{\mathbf{a}}))$ and then $F_1^*(\mathbf{a}) = \varphi F_1^*(\hat{\mathbf{a}})$, $F_1^*(\mathbf{a}) + F_2^*(\mathbf{a}) = \varphi(F_1^*(\hat{\mathbf{a}}) + F_2^*(\hat{\mathbf{a}}))$ and thus $F_2^*(\mathbf{a}) = \varphi F_2^*(\hat{\mathbf{a}})$, and so on. According to RIIC the i-th individual in $F^*(\mathbf{a})$ is the same than in $F^*(\hat{\mathbf{a}})$ and then $F_i(\mathbf{a}) = \varphi F_i(\hat{\mathbf{a}})$ for all $i \in N$ which in turn implies that $\frac{F_i(\mathbf{a})}{F_j(\mathbf{a})} = \frac{F_i(\hat{\mathbf{a}})}{F_j(\hat{\mathbf{a}})}$ for all $i, j \in N$. The converse is immediate. It is also very easy to check that $\varphi = \frac{\sum_{i=1}^n f(a_i)}{\sum_{i=1}^n f(\hat{a}_i)}$.
b) Immediate because $\varphi = 1$. ∎

Lemma 3.8 allows me to clarify the relation between WLEIC and RIIC (and LEIC and RIIC) and the first group of properties in Bossert and Fleurbaey [3]. In particular, I show that WLEIC and RIIC (and thus LEIC and RIIC), under anonymity, imply property EIER.

Proposition 3.9. *Let F satisfy WLEIC and RIIC and suppose that it is also an anonymous mechanism. Then it satisfies EIER.*

Proof. [2] Take any $\mathbf{a} = (a_1, ..., a_n) \in \Omega^n$ such that $a_i^R = a_j^R$ for some pair $i, j \in N$. Construct $\hat{\mathbf{a}}$ by making $\hat{a}_k^R = a_k^R$ for all $k \in N$, $\hat{a}_k^S = a_k^S$ for $k \neq i, j$, $\hat{a}_i^S = a_j^S$ and $\hat{a}_j^S = a_i^S$. It is obvious that $\sum_{k=1}^n f(a_k) = \sum_{k=1}^n f(\hat{a}_k)$. By lemma 3.8, WLEIC and RIIC imply that for all $k \in N$, $F_k(\mathbf{a}) = F_k(\hat{\mathbf{a}})$ and in particular, $F_i(\mathbf{a}) = F_i(\hat{\mathbf{a}})$ and $F_j(\mathbf{a}) = F_j(\hat{\mathbf{a}})$. Finally, since $a_i = \hat{a}_j$ and $a_j = \hat{a}_i$ by anonymity it holds that $F_i(\mathbf{a}) = F_j(\mathbf{a})$.

Without anonymity this result is not true. To check it, take the following solution for all $\mathbf{a} \in \Omega^n$:

$$F_i(\mathbf{a}) = \begin{cases} \sum_{j=1}^n f(a_j) & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}$$

It does satisfy WLEIC and RIIC but not EIER. ∎

Bossert and Fleurbaey [3] also present two properties that are stronger (under anonymity) than EIER. The strongest one is the so called *Group solidarity in S (GSS)* that can be stated as follows:

Property 3.10. Group solidarity in S (GSS): *For all $\mathbf{a}, \hat{\mathbf{a}} \in \Omega^n$ such that for all $k \in N$, $a_k^R = \hat{a}_k^R \Rightarrow \exists \psi \in \Re$ such that $F_i(\hat{\mathbf{a}}) = F_i(\mathbf{a}) + \psi$ for all $i \in N$.*

As LEIC and WLEIC, this property imposes a restriction on the entire profile of post-tax income. In particular it implies that any change in total income due exclusively to a change in the profile of irrelevant traits must be shared *equally* by all members of the society. On the other hand, LEIC and RIIC requires that the change in total income must be shared *proportionally* to the post-tax income that

[2] I am indebted to Mark Fleurbaey for this shortened version of the proof.

people had before the change. It seems that this is more closer to what intuition suggests than what GSS requires. In the case of GSS the income distribution turns out to be completely disturbed in terms of the Lorenz curve. Moreover, it may happen that some individual i end up by receiving a negative amount of post-tax income $F_i(\hat{\mathbf{a}})$.

To demand properties LEIC and RIIC is somehow a strong requirement. In fact, none of the solutions provided by Bossert and Fleurbaey [3] satisfy both of them. This is very easy to check because all of them collapse in the solution F^0 when the income function is separable and, as the next example shows, this solution is not compatible with our two properties.

Example 3.11. *Let $n = 2$, $r = s = 1$ and $a_1 = (1,1)$, $a_2 = (3,1)$. Take $f(a_i^R, a_i^S) = a_i^R + a_i^S$ for $i = 1,2$. Then $F_1^0(\mathbf{a}) = 2$ and $F_2^0(\mathbf{a}) = 4$. Now, $\hat{a}_1 = (1,2)$ and $\hat{a}_2 = (3,2)$. Then $F_1^0(\hat{\mathbf{a}}) = 3$ and $F_2^0(\hat{\mathbf{a}}) = 5$. It is immediate to see that there exists no $\varphi \in \Re_+$ such that $2 = 3\varphi$ and $4 = 5\varphi$.*

On the other hand, the reader can verify easily that F^0 does satisfy WLEIC and RIIC.

4. Main Results

Next I present a new family of solutions. First, recall that $\tilde{a}^S \in \Omega^S$ denotes a reference vector of irrelevant characteristics. Now, the family of solutions $F^{\tilde{a}^S}$ is defined by:

$$F_k^{\tilde{a}^S}(\mathbf{a}) := \frac{f(a_k^R, \tilde{a}^S)}{\sum_{j=1}^n f(a_j^R, \tilde{a}^S)} \sum_{i=1}^n f(a_i).$$

$F^{\tilde{a}^S}$ gives to every individual a fraction of total income and this fraction depends only on $\mathbf{a}^R$ and the reference vector $\tilde{a}^S$. Now, by taking different values for $\tilde{a}^S$ I can arrive at different solutions. In particular I am interested in anonymous solutions. I focus on the two following ones.

First, suppose that for any $\mathbf{a} \in \Omega^n$, I take a value for $\tilde{a}^S$ such that it is completely independent of the particular profile $\mathbf{a}$. I call this reference vector

$\overline{\alpha}^S$, and the solution that I get *Proportional Solution with Exogenous Reference Vector*, F^{PEX}.

Second, I can make endogenous the selection of $\tilde{a}^S$ by means of the following procedure. Take $\tilde{a}^S = \alpha^S(\mathbf{a})$ such that $\sum_{i=1}^n f(a_i^R, a_i^S) = \sum_{i=1}^n f(a_i^R, \alpha^S(\mathbf{a}))$. I call this reference vector $\alpha^S(\mathbf{a})$ to emphasize that it is a function of the particular profile $\mathbf{a}$. The solution that I get is called *Proportional Solution with Endogenous Reference Vector*, F^{PEN}. It has a very simple structure, because $F_i^{PEN}(\mathbf{a}) = f(a_i^R, \alpha^S(\mathbf{a}))$ for all $i \in N$. Notice that $\alpha^S(\mathbf{a})$ is like an average of the irrelevant traits across the population.

This second solution has the problem that for some income functions $\alpha^S(\mathbf{a})$ may not be unique. I can think in two ways of avoiding this problem. First, I can assume $s = 1$. This assures that $\alpha^S(\mathbf{a})$ will be unique. Second, instead of doing $s = 1$, I can look for suitable restrictions on the domain of admissible income functions such that, even though $\alpha^S(\mathbf{a})$ is not necessarily unique, the solution is *essentially unique* in the sense that every individual receives the same post-tax income for all possible values of $\alpha^S(\mathbf{a})$. For example, one possibility is to require the following property : for all income function f, it can be separated into two functions $g : \Omega_R \mapsto \Re$ and $h : \Omega_S \mapsto \Re$ such that

$$f(a) = g(a^R)h(a^S), \quad \forall a \in \Omega.$$

This means that f is (multiplicatively) separable. It includes, in particular, all Cobb-Douglas type functions.

Both solutions F^{PEX} and F^{PEN} are very easy to compute because they only need information about the values that the income function takes at the vectors $(a_j^R, \tilde{a}^S)$. The first solution is uniquely characterized by means of three properties: LEIC, RIIC and ETRS. This last property belongs to the second group of properties studied in Bossert and Fleurbaey [3]. On the other hand, the second solution is uniquely characterized by WLEIC, RIIC and ETUS, another property of that group.

Property 4.1. *Equal transfer for uniform S (ETUS): For all* $\mathbf{a} \in \Omega^n$ *such that* $a_i^S = a_j^S$ *for all* $i, j \in N \implies F_i(\mathbf{a}) = f(a_i)$ *for all* $i \in N$.

Property 4.2. Equal transfer for reference S (ETRS): *For all* $\mathbf{a} \in \Omega^n$, $a_i^S = \tilde{a}^S$ *for all* $i \in N \implies F_i(\mathbf{a}) = f(a_i)$ *for all* $i \in N$.

ETUS says that whenever all individuals' irrelevant traits are equal there is no need to perform any redistribution. ETRS requires the same but only when all irrelevant traits are equal to a given reference level. It can be seen that ETES $\Rightarrow$ ETUS $\Rightarrow$ ETRS. ETES appeared in section 3.

The following lemma will be used in the first characterization result.

Lemma 4.3. *For any* $\mathbf{a} \in \Omega^n$, *write* $F_i(\mathbf{a}) = k_i \sum_{i=1}^n f(a_i)$. *A redistribution mechanism* F *satisfies LEIC and RIIC if and only if for all* $\mathbf{a} \in \Omega^n$ *and for all* $i \in N$, k_i *is independent of the particular profile* $\mathbf{a}^S$.

Proof. Take any $\mathbf{a} \in \Omega^n$. I can always find n numbers k_i (with $k_i \geq 0$ for all $i \in N$ and $\sum_{i=1}^n k_i = 1$) such that for all $i \in N$ I can write $F_i(\mathbf{a}) = k_i \sum_{i=1}^n f(a_i)$. If I choose $\hat{\mathbf{a}} \in \Omega^n$ such that $a_k^R = \hat{a}_k^R$ for all $k \in N$, I can prove that $F_i(\hat{\mathbf{a}}) = k_i \sum_{i=1}^n f(\hat{a}_i)$ for all $i \in N$. The meaning of this is that k_i is independent of $\mathbf{a}^S$ for all $i \in N$. To do it, take any pair $\{i, j\}$. There exist $k_i, k_j \in \Re_+$ such that:

$$F_i(\mathbf{a}) = k_i \sum_{h=1}^n f(a_h^R, a_h^S),$$
$$F_j(\mathbf{a}) = k_j \sum_{h=1}^n f(a_h^R, a_h^S) \text{ with}$$
$$k_i + k_j + \sum_{h \neq i,j} k_h = 1.$$

For any $\hat{\mathbf{a}} \in \Omega^n$ as above, there also exist $\hat{k}_i, \hat{k}_j \in \Re_+$ such that:

$$F_i(\hat{\mathbf{a}}) = \hat{k}_i \sum_{h=1}^n f(a_h^R, \hat{a}_h^S),$$
$$F_j(\hat{\mathbf{a}}) = \hat{k}_j \sum_{h=1}^n f(a_h^R, \hat{a}_h^S) \text{ with}$$
$$\hat{k}_i + \hat{k}_j + \sum_{h \neq i,j} \hat{k}_h = 1.$$

By lemma 3.8 it must be that for all i, j :

$$\frac{F_i(\mathbf{a})}{F_j(\mathbf{a})} = \frac{F_i(\hat{\mathbf{a}})}{F_j(\hat{\mathbf{a}})}$$

and then

$$\frac{k_i}{k_j} = \frac{\hat{k}_i}{\hat{k}_j}.$$

Moreover, it follows that $k_i = \hat{k}_i$ and $k_j = \hat{k}_j$ for all i, j. If this is not the case, for example $k_i < \hat{k}_i$, it must be that $k_j < \hat{k}_j$ for all $j \neq i$. But then $1 = \sum_{h=1}^{n} k_h < \sum_{h=1}^{n} \hat{k}_h = 1$, a contradiction. The converse is again immediate. ∎

Theorem 4.4. *A redistribution mechanism F satisfies LEIC, RIIC and ETRS if and only if $F = F^{PEX}$.*

Proof. First it is easy to check that F^{PEX} satisfies these three properties. On the other hand, for any $\mathbf{a} = ((a_1^R, a_1^S), ..., (a_n^R, a_n^S)) \in \Omega^n$, define $\overline{\mathbf{a}} = ((a_1^R, \overline{\alpha}^S), ..., (a_n^R, \overline{\alpha}^S))$. By lemma 4.3 if $F_i(\mathbf{a}) = k_i \sum_{j=1}^{n} f(a_j^R, a_j^S)$ then $F_i(\overline{\mathbf{a}}) = k_i \sum_{j=1}^{n} f(a_j^R, \overline{\alpha}^S)$ for all $i \in N$. But by ETRS it must be that $F_i(\overline{\mathbf{a}}) = f(a_i^R, \overline{\alpha}^S)$ for all $i \in N$. This implies that

$$k_i(\mathbf{a}) = \frac{f(a_i^R, \overline{\alpha}^S)}{\sum_{j=1}^{n} f(a_j^R, \overline{\alpha}^S)} \text{ for all } i \in N \text{ and for all } \mathbf{a} \in \Omega^n,$$

and the result follows. ∎

Theorem 4.5. *Suppose that F^{PEN} is essentially unique. A redistribution mechanism satisfies WLEIC, RIIC and ETUS if and only if $F = F^{PEN}$.*

Proof. By assumption F^{PEN} is well defined. Again, the reader can check that it also satisfies the three properties. For any $\mathbf{a} = ((a_1^R, a_1^S), ..., (a_n^R, a_n^S)) \in \Omega^n$ define $\hat{\mathbf{a}} = ((a_1^R, \alpha^S(\mathbf{a})), ..., (a_n^R, \alpha^S(\mathbf{a})))$ where $\alpha^S(\mathbf{a})$ is such that $\sum_{i=1}^{n} f(a_i) = \sum_{i=1}^{n} f(a_i^R, \alpha^S(\mathbf{a}))$. By ETUS, $F_i(\hat{\mathbf{a}}) = f(a_i^R, \alpha^S(\mathbf{a}))$ for all $i \in N$. By lemma 3.8, $F_i(\hat{\mathbf{a}}) = F_i(\mathbf{a})$ for all $i \in N$. Then, $F_i(\mathbf{a}) = f(a_i^R, \alpha^S(\mathbf{a}))$ for all $i \in N$. ∎

It is interesting to study the tightness of the properties that I use in the characterization results. In particular I want to see how large is the domain of solutions when I remove the axioms one by one.

First, consider theorem 4.4. If I drop ETRS it is easy to see that a much larger class of solutions is characterized. This is clear if I go back to lemma 4.3. I get the family consisting of all solutions such that for all $i \in N$, and for all

$\mathbf{a} \in \Omega^n$, $F_i(\mathbf{a}) = k_i \sum_{i=1}^n f(a_i)$ provided that the numbers k_i do not depend on the particular profile a^S. Just take any set of n numbers such that all of them are non-negative numbers adding up to unity. In particular this includes any set of n fixed numbers, that is, not depending even in the profile $\mathbf{a}^R$.

When I drop RIIC a different class of solutions is characterized. Let i^* be the individual such that $f(a_{i*}^R, \overline{\alpha}^S)$ takes up the i-th position when we rearrange the terms $f(a_i^R, \overline{\alpha}^S)$ in increasing order. Then, the solution we have is:

$$F_{i^*}(\mathbf{a}) = \frac{f(a_{i^*}^R, \overline{\alpha}^S)}{\sum_{h=1}^n f(a_h^R, \overline{\alpha}^S)} \sum_{i=1}^n f(a_i) \text{ for all } i \in N, \text{ and for all } \mathbf{a} \in \Omega^n.$$

Finally, dropping LEIC allows for a very large domain of solutions including some of the solutions studied by Bossert and Fleurbaey [3]. This stresses the fact that LEIC is the most crucial property in the theorem.

In the second theorem, if I drop ETUS, I have, among others, the solution F^{PEX}. It satisfies LEIC and RIIC, and thus WLEIC and RIIC. Dropping RIIC I have a result very similar to the one above when I dropped it in the first theorem. Finally, WLEIC plays again the crucial role in the theorem.

5. Final remarks

In this work I characterize two different redistribution mechanisms. The first one consists in a family of redistribution mechanisms, because for any value of $\overline{\alpha}^S$ there is a different solution. In spite of this, we can see that for some income functions the solution is independent of the particular value of $\overline{\alpha}^S$. This is the case again, when the income function f is (multiplicatively) separable.

The last point that I want to mention deals with the behavior of the solutions F^{PEX} and F^{PEN} when the income function is additively separable. In particular I would like to check wether they collapse or not in the solution F^0 defined in section 2. Recall that this solution is given by:

$$F_k^0(\mathbf{a}) := g(a_k^R) + \frac{1}{n} \sum_{i=1}^n h(a_i^S), \quad \forall \mathbf{a} \in \Omega^n, \ \forall k \in N.$$

I prove that F^{PEN} is the unique solution belonging to the family $F^{\tilde{a}^S}$ that coincides with F^0 when the income function is additively separable.

Proposition 5.1. *Suppose f is additively separable and it is not true that $g(a_i^R) = g(a_j^R)$ for all i, j. Then $F^{\tilde{a}^S} = F^0$ if and only if for all profile $\mathbf{a}^S = (a_1^S, \ldots. a_n^S) \in \Re^{ns}$, $h(\tilde{a}^S) = \frac{1}{n}\sum_{j=1}^n h(a_j^S)$.*

Proof. First, the "if" part. Under separability, for all $\mathbf{a} \in \Omega^n$ and for all $k \in N$:

$$\begin{aligned} F_k^{\tilde{a}^S}(\mathbf{a}) &:= \frac{g(a_k^R)+h(\tilde{a}^S)}{\sum_{j=1}^n \left[g(a_j^R)+h(\tilde{a}^S)\right]} \sum_{i=1}^n \left[g(a_i^R) + h(a_i^S)\right] = \\ &= \frac{g(a_k^R)+h(\tilde{a}^S)}{\sum_{j=1}^n g(a_j^R)+nh(\tilde{a}^S)} \sum_{i=1}^n \left[g(a_i^R) + h(a_i^S)\right] = \\ &= \frac{g(a_k^R)+h(\tilde{a}^S)}{\sum_{j=1}^n g(a_j^R)+\sum_{j=1}^n h(a_j^S)} \sum_{i=1}^n \left[g(a_i^R) + h(a_i^S)\right] = \\ &= g(a_k^R) + h(\tilde{a}^S) = g(a_k^R) + \frac{1}{n}\sum_{j=1}^n h(a_j^S) = F_k^0(\mathbf{a}). \end{aligned}$$

Now, the "only if" part. Suppose that for all $\mathbf{a} \in \Omega^n$, $F_k^{\tilde{a}^S}(\mathbf{a}) = F_k^0(\mathbf{a})$ for all $k \in N$. Then,

$$\frac{g(a_k^R) + h(\tilde{a}^S)}{\sum_{j=1}^n \left[g(a_j^R) + h(\tilde{a}^S)\right]} \sum_{i=1}^n (g(a_i^R) + h(a_i^S)) = g(a_k^R) + \frac{1}{n}\sum_{j=1}^n h(a_j^S) \quad \text{for all } k \in N.$$

Straightforward computations lead to:

$$\left[g(a_k^R) - \frac{\sum_{j=1}^n g(a_j^R)}{n}\right] \sum_{j=1}^n h(a_j^S) = \left[g(a_k^R) - \frac{\sum_{j=1}^n g(a_j^R)}{n}\right] nh(\tilde{a}^S) \quad \text{for all } k \in N.$$

By assumption, the terms in brackets are not cero for all $k \in N$. Then the desired result follows. ∎

Now it is immediate to check that the necessary and sufficient condition of the proposition, when the income function is additively separable, is exactly the definition of F^{PEN}.

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