UNIFIED TREATMENT OF THE PROBLEM OF EXISTENCE OF MAXIMAL ELEMENTS IN BINARY RELATIONS. A CHARACTERIZATION.

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UNIFIED TREATMENT OF THE PROBLEM OF EXISTENCE OF MAXIMAL ELEMENTS IN BINARY RELATIONS. A CHARACTERIZATION

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A B S T R A C T

The aim of this paper is twofold. On the one hand to present by means of a unique statement an existence result which covers both ways (convexity and acyclicity) of analyzing the problem of existence of maximal elements of non transitive binary relations. And on the other hand, to introduce the concept of an abstract convexity structure, which we call mc-spaces, that generalizes the notion of usual convexity. It is presented as a powerful tool which allows many problems which have only been analyzed (previously) under convexity conditions to be solved.

Keywords: Maximal Elements, Fixed point, Abstract Convexity.

1.- INTRODUCTION

The notion of preference relation or utility function is a basic concept in economics, in particular in consumer theory. When a consumer is faced with the problem of choosing a bundle of products, in the end he will look for the bundle which maximizes his preference relation from those which he can afford. The problem, then, of looking for sufficient conditions which ensure the existence of maximal elements of a binary relation is one of the most important problems in economic theory.

Continuity and convexity conditions on the upper and lower contour sets ($U(x) = \{y \in X \mid yPx\}$; $U^{-1}(x) = \{y \in X \mid xPy\}$) or continuity and transitivity conditions of the preference relation (**P**) are usually required. Some of these, notably (the transitivity condition (Luce, 1956; Starr, 1969)) have been criticized as being strongly unrealistic.

The purpose of dropping the transitivity condition (especially the transitivity of the indifference) has involved the problem of existence of maximal elements in two different and independent ways. On the one hand, in considering weaker relations and on the other hand in considering conditions instead of transitive ones.

If we relax the transitivity, acyclic binary relations (a binary relation **P** is acyclic if $x_1 \mathbf{P} x_2$, $x_2 \mathbf{P} x_3$, ... , $x_{n-1} \mathbf{P} x_n$, then not $x_n \mathbf{P} x_1$) can be considered. In this line, there are several results that give us sufficient conditions to obtain maximal elements such as Bergstrom (1975) or Walker's (1977) results. In the second approach, the results are mainly based on convexity conditions on the set and on the upper contour sets (U(x)). In this case, classical fixed point results (Brouwer, Browder, ...) or non empty intersection results (Kanaster-Kuratowski-Mazurkiewicz type) are usually applied to obtain most of the results, therefore convexity conditions are required on some mapping and on the set where it is defined (see Border, 1985). In this approach, we should mention the results obtained by Fan (1961), Sonnenschein (1971), Yannelis and Prabhakar (1983), Tian (1993), among others.

The aim of this paper is twofold. First of all to present a result which covers both ways of analyzing the problem of existence of maximal elements (convexity and acyclicity) by means of only one statement. Therefore it will represent an unification of the different treatments of the problem of existence of maximal elements in non ordered binary relations. And secondly to remark that the notion of abstract convexity (mc-spaces) which will be introduced, is presented as a powerful tool to analyze not only the problem of existence of maximal elements, but also other important problems in economic analysis which use the convexity condition as a fundamental hipothesis of the model, such as the existence of economic equilibrium.

In order to obtain the main result of the paper, we will consider two different kinds of hypothesis: Topological conditions (used by many authors, for example Sonnenschein (1971), Walker (1977), Tian (1993)), and a Convexity Condition. This Convexity Condition coincides with the irreflexivity-convexity condition used by Sonnenschein (1971) or Yannelis and Prabhakar (1983) when the usual convexity context is considered. Furthermore it will be proved that any acyclic binary relation defined on a topological space allow us to define a particular case of this mc-structure which verifies our Convexity Condition. Therefore this Convexity Condition defined in the context of mc-spaces will allow us to cover acyclicity and convexity at the same time.

The paper is organized in the following way. In Section 2 an abstract convexity structure (mc-spaces) is introduced containing as a particular case the notion of usual convexity. A fixed point result in the context of mc-spaces which generalizes Browder's selection and fixed point Theorem is given in Section 3. The main result which generalizes Sonnenschein and Walker's results, and their consequences are concentrated on in Section 4. And finally an appendix with the proofs of the Theorems closes the paper.

2.- ABSTRACT CONVEXITY

The notion of abstract convexity can be seen as a generalization of the notion of usual convexity based on properties that convex sets have. Hence, by an <u>abstract convexity</u> (Kay and Womble, 1971) on a set X we mean a family $\mathfrak{C} = \{A_i\}_{i\in I}$, of subsets of X, stable under arbitrary intersections ($\cap_{i\in J} A_i \in \mathfrak{C}$, $J\subset I$) and which contains the empty and the total set ($\emptyset$, $X \in \mathfrak{C}$).

The abstract convexity which will be introduced in this paper is based on the idea of substituting the segment which joins any pair of points (or the convex hull of a finite set of points) by an arc, path (or a set) which plays their role. In particular the idea is to associate to any finite family of points, a family of functions whose composition is continuous. The image of this composition generates a set associated to the finite family of points in the similar way in which the usual convex hull operator associates a set to each finite family of points.

Formally the definition of mc-spaces is as follows:

Definition 1. A topological space X is an <u>mc-space</u> if for any nonempty finite subset $A=\{a_0, a_1,\ldots,a_n\}$ of X, there exists a family of elements $\{b_0, b_1,\ldots,b_n\}$ in X and a family of functions,

$$P_i^A : X \times [0,1] \longrightarrow X \qquad i= 0,1, \ldots, n$$

such that

1. $P_i^A(x,0) = x,$ $\quad P_i^A(x, 1) = b_i \quad \forall x\in X.$

2. The following function

$$G_A : [0,1]^n \longrightarrow X$$

given by

$$G_A(t_0,t_1,\dots,t_{n-1}) = P_0^A\left(..P_{n-1}^A\left(P_n^A\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_0\right)$$

is a continuous function.

Henceforth, if X is an mc-space, we say that X has an <u>mc-structure</u>.

Remark 1. Note that if X is a convex subset of a topological vector space and we consider functions $P_i^A(x,t) = (1-t)\,x + ta_i$, then they define an mc-structure on X. In this case, $b_i=a_i$, and functions $P_i^A(x,t)$ represent the segment joining a_i and x when $t\in[0,1]$. Therefore mc-spaces are extensions of convex sets. Moreover, the image of the composition $G_A([0,1]^n)$ in this particular case represents the usual convex hull of $A=\{a_0,a_1,\dots,a_n\}$ (C{A}) because

$$P_{n-1}^A(P_n^A(x,1),t_{n-1}) = P_{n-1}^A(a_n,\ t_{n-1}) = t_{n-1}a_{n-1} + (1-t_{n-1})a_n$$

$$P_{n-2}^A\left(P_{n-1}^A\left(P_n^A(x,1),t_{n-1}\right),t_{n-2}\right) = t_{n-2}a_{n-2}+(1-t_{n-2})\left(t_{n-1}a_{n-1}+(1-t_{n-1})a_n\right)$$

in general

$$G_A(t_0,\dots,t_{n-1}) = P_0^A\left(..P_{n-1}^A\left(P_n^A\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_0\right) =$$

$$t_0a_0 + \sum_{j=1}^{n} a_jt_j\left(\prod_{i=0}^{j-1}(1-t_i)\right) = \sum_{i=0}^{n} a_i\alpha_i$$

where α_i are continuous functions depending on $(t_0,\ldots,t_{n-1})$ such that $\sum\alpha_i=1$.(In the previous expression, we have considered $t_n=1$ in order to simplify it).

In general in this structure for any nonempty finite subset of X ($A = \{a_0, \ldots ,a_n\}$), for each element $a_i \in A$ and for each $x \in X$, it is possible to define a function $P^A_{i[x]}: [0, 1] \longrightarrow X$, $\left(P^A_{i[x]}(t) = P^A_i(x,t)\right)$ satisfying that

$$P^A_{i[x]}(0) = x \qquad \text{and} \qquad P^A_{i[x]}(1) = b_i.$$

If $P^A_{i[x]}$ is continuous, then it represents a path which joins x and b_i. Furthermore, if b_i is equal to a_i, $P^A_i(x,[0,1])$ represents a continuous path which joins x and a_i. These paths depend, in a sense, on the points which are considered, as well as the finite subset A which contains them. So, the nature of these paths can be very different.

Function G_A can be interpreted as follows:

the point $P^A_{n-1}(b_n, \lambda_{n-1}) = p_{n-1}$, represents a point of the path which joins b_n with b_{n-1}, $P^A_{n-2}(p_{n-1},\lambda_{n-2}) = p_{n-2}$ is a point of the path which joins p_{n-1} with b_{n-2}, etc.

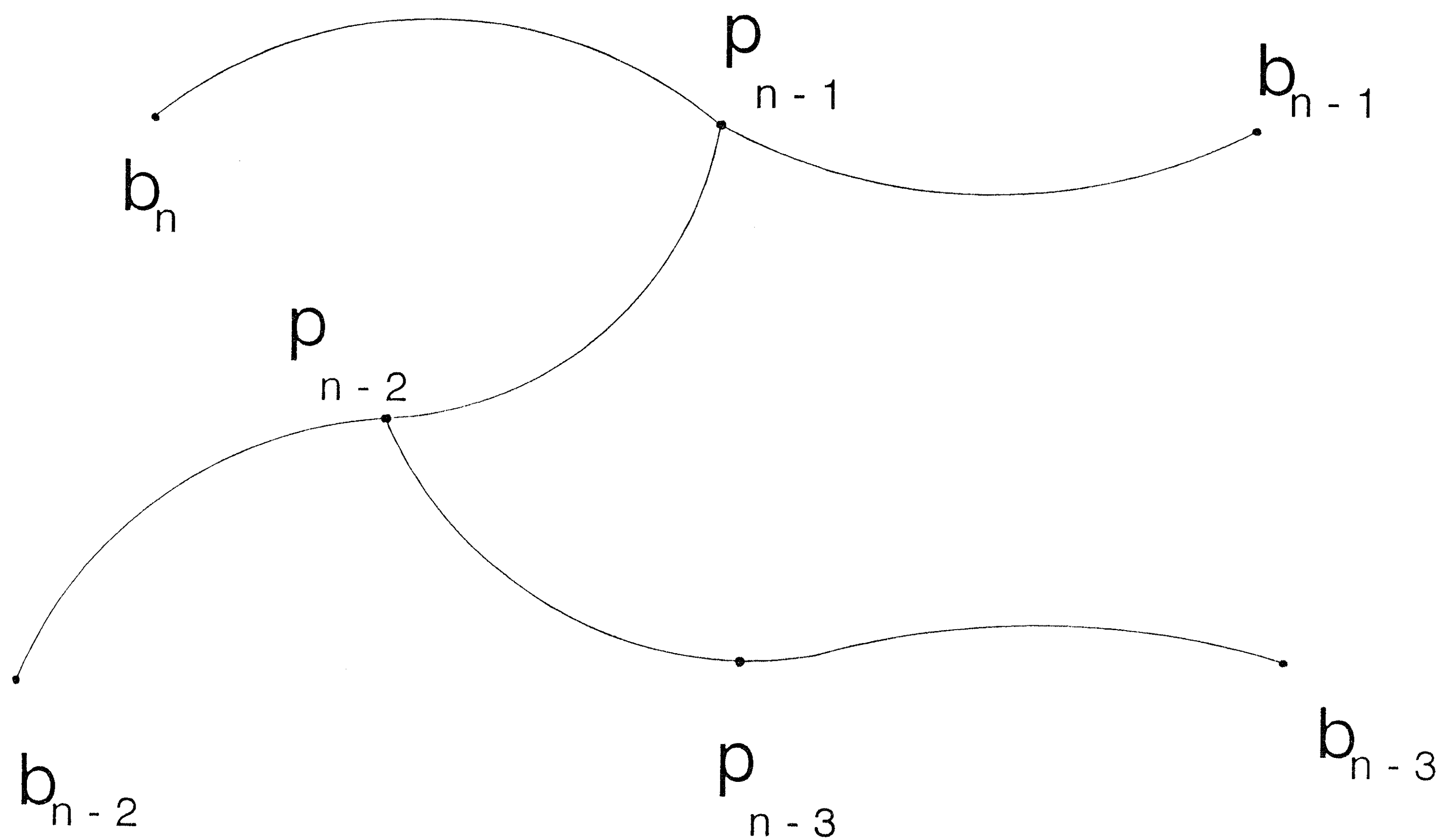

FIGURE 1

So if we want to ensure that the composition of functions P_i^A is continuous, we need to ask for the continuity of functions $P_{i[z]}^A : [0, 1] \longrightarrow X$ in any point "z" which belongs to the path joining b_{i+1} and p_{i+2}, with $i = 0,\dots,n-2$.

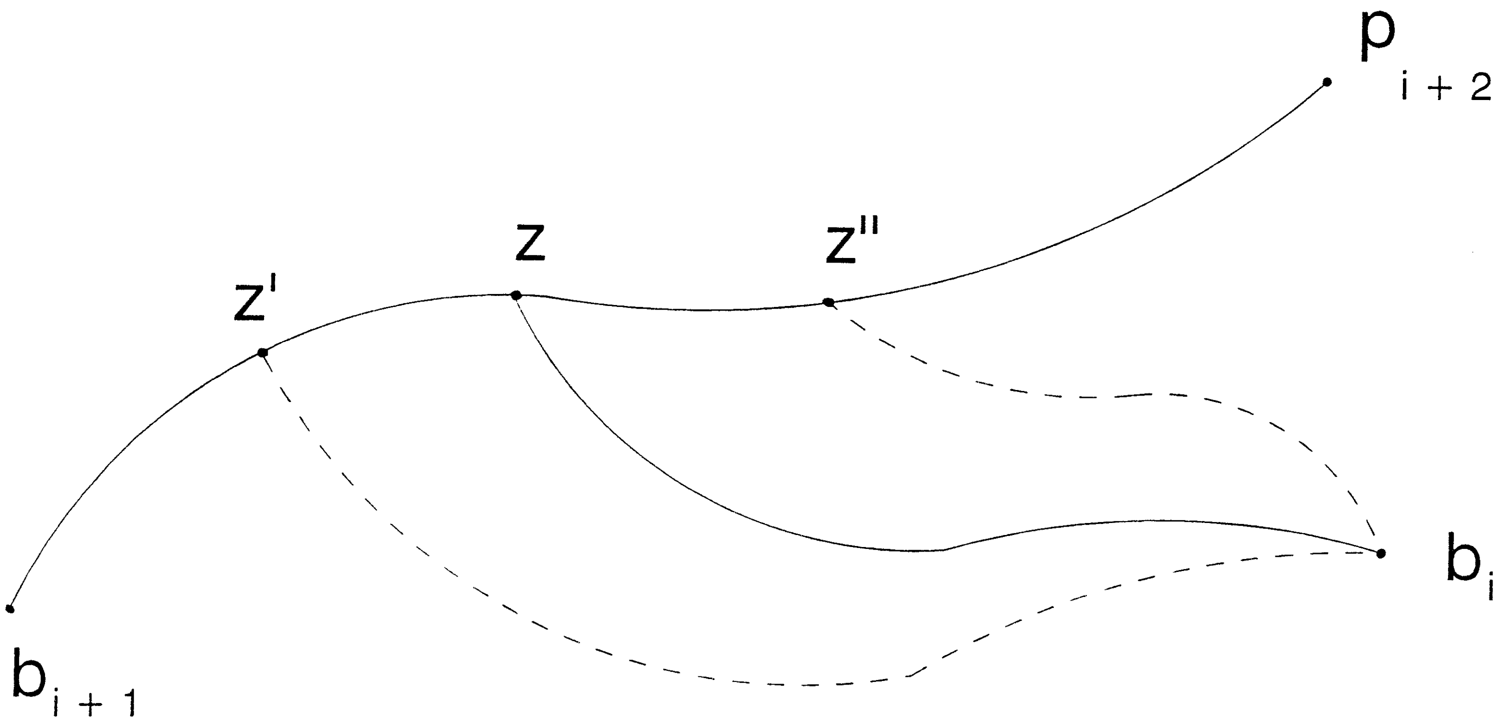

FIGURE 2

Finally note that if $t_i = 1$, then any t_j such that $j>i$, does not affect the function G_A (since $P_i^A(x,1) = b_i \;\; \forall x \in X$). Moreover if $t_i = 0$, then b_i will not appear in this path.

From an mc-structure it is always possible to define an abstract convexity given by the family of sets which are stable under function G_A. To define this abstract convexity we need some previous concepts.

Definition 2. Let X be an mc-space and Z a subset of X. $\forall A \subseteq X$, A finite, such that $A \cap Z \neq \emptyset$, $A \cap Z = \{a_0, a_1, .., a_n\}$, we define the restriction of function G_A to Z as follows:

$$G_{A|Z} : [0, 1]^n \longrightarrow X$$

$$G_{A|Z}(t) = P_0^A(\ldots P_{n-1}^A(P_n^A(a_n, 1), t_{n-1})\ldots.), t_0)$$

where P_i^A are the functions associated to the elements a_i in A which belong to Z.

From this notion we define mc-sets (which are an extension of convex sets) in the following way.

Definition 3. A subset Z of an mc-space X is an <u>mc-set</u> if and only if it is verified

$$\forall A \subseteq X,\ A \text{ finite},\ A \cap Z \neq \emptyset \qquad G_{A|Z}([0,1]^n) \subseteq Z$$

Notice the paralelism between usual convex sets and mc-sets: as the image of function G_A can be interpreted in some sense as the "convex hull" of set A, then a subset Z will be an mc-set if it contains the "convex hulls" of every finite subset of Z $\left(G_{A|Z}([0,1]^n)\right)$ in a similar way to that of the usual convex case.

Since the family of mc-sets is stable under arbitrary intersections, it defines an abstract convexity on X. In this case, we can define an mc-hull operator (which is the extension of the convex hull operator) as

$$C_{mc}(A) = \bigcap \{B \mid A \subset B, \ \ B \text{ is an mc-set}\}$$

An example of a set where an mc-structure can be defined is the following: Let C be a convex set and X a set such that there exists an homeomorphism from X into C, $h: X \longrightarrow C$. In this particular case it is possible to define an mc-structure on X by means of function h as follows: For each $A = \{a_0,..,a_n\}$ subset of X we define functions P_i^A as

$$P_i^A: X \times [0,1] \longrightarrow X; \qquad P_i^A(x,t) = h^{-1}\Big((1-t)h(x) + th(a_i) \Big)$$

In this context a subset B of X is an mc-set if and only if h(B) is a convex subset of C.

The examples presented correspond either to convex sets (Remark 1.) or to situations where an homeomorphism from these sets into convex sets can be stated (Example above). The following example shows a non contractible set (and therefore not homeomorphic to a convex set) where an mc-structure can be defined:

Example 1. Let $X \subset \mathbb{R}^2$ be the following set,

$$X = \{ x \in \mathbb{R}^2 : 0 < a \leq \|x\| \leq b \ , \ a,b \in \mathbb{R} \}$$

Considering the complex representation,

$$x = \rho_x \, e^{i\alpha x} \quad , \; y = \rho_y e^{i\alpha y}$$

for any nonempty finite subset A of X, functions P_i^A can be defined as follows, $\forall y \in X$

$$P_y^A : X \times [0,1] \longrightarrow X; \qquad P_y^A(x,t) = \Big((1-t)\rho_x + t\rho_y \Big) e^{i((1-t)\alpha x + t\alpha y)}$$

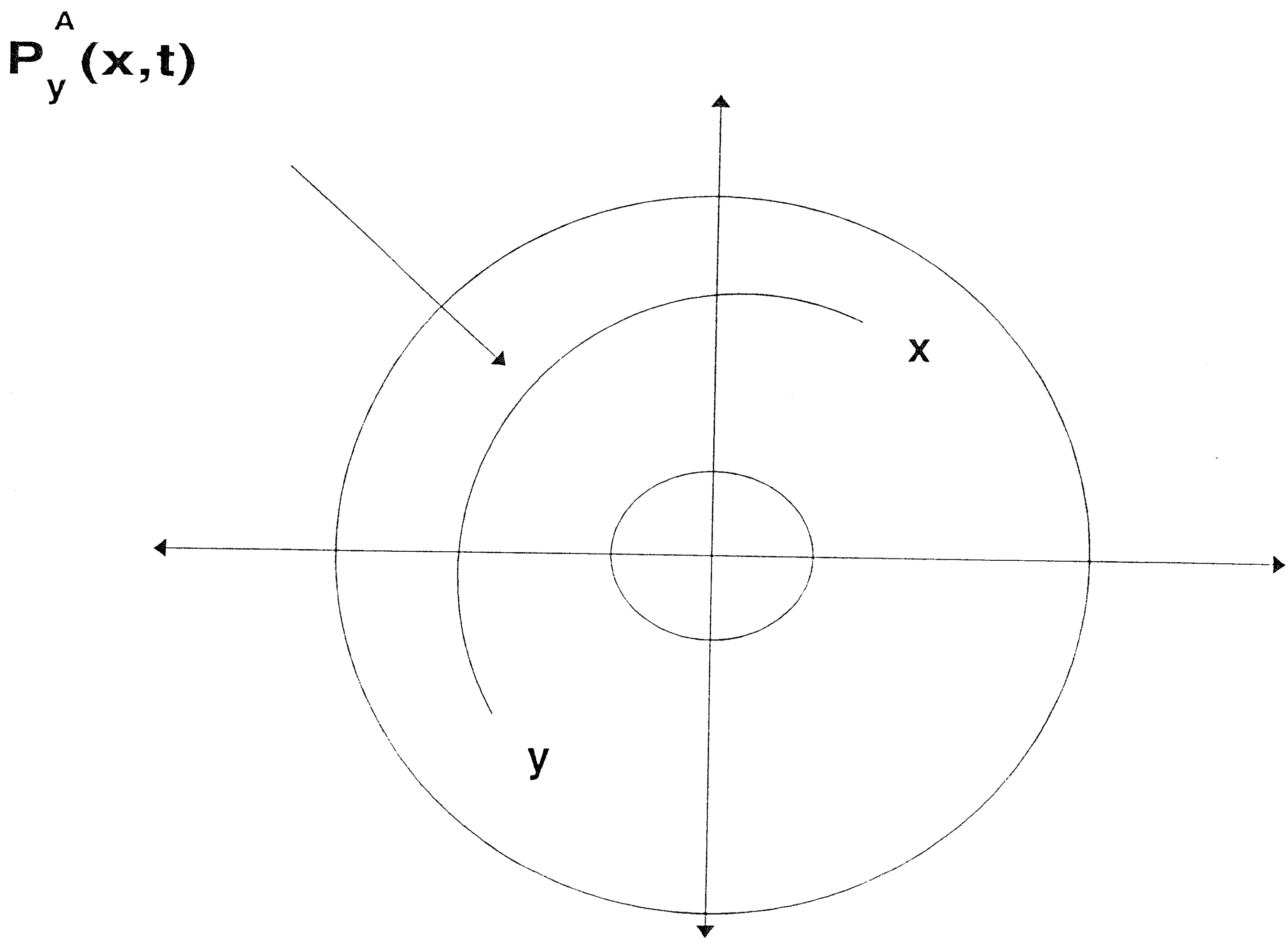

FIGURE 3

3.- A FIXED POINT RESULT

Fixed Point results are basic tools used to prove the existence of solutions to several problems in economics (Border, 1985; Villar, 1992). In particular, in this paper, we will use a fixed point result to prove the existence of maximal elements in binary relations under some weak conditions.

We will use the following Lemma in order to state the existence of a continuous selection and fixed point to correspondences with open inverse images in the context of mc-spaces.

Lemma 1: *Let X be a compact Hausdorff topological mc-space and* $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ *a nonempty valued correspondence such that if* $y \in \Gamma^{-1}(x)$*, then there exists some* $x' \in X$ *such that* $y \in int\Gamma^{-1}(x')$*. Then there exists a nonempty finite subset A of X, and a continuous function* $f: X \longrightarrow X$ *which verifies:*

i) $\exists x^* \in X$ *such that* $x^* = f(x^*)$.

ii) $f(x) \in G_{A|\Gamma(x)}([0,1]^m) \quad \forall x \in X$.

(Proof in the appendix).

From the previous Lemma, the following theorem is immediately obtained. It is an extension of Browder's Theorem (1967), [see Border (1985)].

Theorem 1: *Let X be a compact Hausdorff topological mc-space and* $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ *a correspondence with open inverse images and nonempty mc-set values. Then* Γ *has a continuous selection and a fixed point.*

Aa a consequence of this result we obtain Browder's Theorem,

Theorem 2: *[Browder, 1967; Yannelis and Prabhakar, 1983]. Let X be a compact Hausdorff topological vector space and $\Gamma: X \twoheadrightarrow X$ a correspondence with open inverse images and nonempty convex values. Then Γ has a continuous selection and a fixed point.*

4.- EXISTENCE OF MAXIMAL ELEMENTS

Next, we present an existence result of maximal elements in binary relations (not necesarily representable by utility functions), which constitutes the union point of the two focuses previously commented. This result, then, generalizes those which consider acyclic binary relations such as Bergstrom (1975) and Walker (1977), as well as those which consider usual convexity conditions as Fan (1961), Sonnenschein (1971), Yannelis and Prabhakar (1983), Tian (1993). In this context, an element x^* is a maximal element for a binary relation P if there is no other element in X which is preferred to x^* (that is, $U(x^*) = \emptyset$, where $U(\cdot)$is the correspondence defined by means of the upper contour sets of a preference relation P).

In the result which is going to be presented, the conditions considered are stated in a similar way to those of Sonnenschein but by considering mc-spaces and mc-sets rather than usual convex sets.

The method used to prove this result is based on the fixed point result (Lemma 1.) presented in the previous section. In this way, it is pointed out that the fixed point technique covers these two different ways of analyzing the problem[1] of the existence of maximal elements.

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[1] In this line, Tian (1993) presents a result which considers the case of acyclic relations, but only in the context of convex sets and topological vector spaces. This result is also a consequence of our main Theorem which will be presented as follows.

In order to obtain the main result, we consider two different kind of conditions: <u>Continuity</u> and <u>Convexity</u>.

CONTINUITY CONDITION (T).

<u>If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int}\ U^{-1}(x')$.</u>

It could be argued in an analogous way, that by considering Tarafdar's condition (1992)[2] (which in this context is equivalent) same conclusion is obtained.

Convexity Condition is the analogue in the usual convex case to that considered by Sonnenschein (1971), Yannelis and Prabhakar (1983) etc, but in the context of mc-spaces which has been presented.

CONVEXITY CONDITION (C).

<u>Let X be an mc-space, and let $U(\cdot)$ be a correspondence (upper contour sets). Then, $\forall x \in X$ and $\forall A \subset X$, A finite, $A \cap U(x) \neq \emptyset$ it is verified $x \notin G_{A|U(x)}([0,1]^m)$.</u>

2 $\forall x \in X$, $U^{-1}(x)$ contains an open subset O_x which fulfills the condition that $\bigcup_{x \in X} O_x = X$.

Let us notice that in the usual convex case, Convexity Condition is as follows: functions P_i^A which define function G_A are defined as segments joining pairs of points, that is, $P_i^A(x,t) = (1-t)x + ta_i$, where $A=\{a_0,\ldots,a_n\}$ in this case it is verified that (see Remark 1.)

$$G_A(t_0,\ldots,t_{n-1}) = \sum_{i=0}^{n} a_i\alpha_i \qquad \text{with } \sum\alpha_i = 1 \quad \alpha_i \geq 0$$

so, $G_A([0,1]^n)=C(\{a_0,\ldots,a_n\})=C(A)$, therefore

$$x \notin G_{A|U(x)}([0,1]^m) = C\{a_i:\ a_i\in A\cap U(x)\}$$

for any finite subset A. Then the convexity condition in this case is reduced to $x \notin C\{U(x)\}$ $(\forall x\in X)$ (which is the irreflexivity-convexity considered by Sonnenschein (1971)).

By making use of these conditions we present the main result on the existence of maximal elements.

Theorem 3: *Let X be a compact Hausdorff topological mc-space and let P be a binary relation defined on X, verifying*

T) Continuity Condition.

C) Convexity Condition.

Then the set of maximal elements, $\{x^*:\ U(x^*)=\emptyset\}$, is nonempty and compact.

(Proof in the appendix).

As a consequence of this Theorem, we obtain Sonnenschein's (1971) and Walker's (1977) results among others, by considering an appropiate mc-structure. To obtain Sonnenschein's result we consider the paths P_i^A as linear segments joining pairs of points, then the mc-structure coincides with the usual convexity and the irreflexivity-convexity condition considered by Sonnenschein ($\forall x \in X$, $x \notin C(U(x))$ is our Convexity Condition.

Theorem 4: *[Sonnenschein, 1971]. Let X be a compact convex subset of $\mathbb{R}^n$ and let P be a binary relation defined on X, such that it verifies the Continuity Condition and* $\forall x \in X \qquad x \notin C(U(x))$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, *is nonempty and compact.*

The next Lemma shows how from an acyclic binary relation defined on a topological space, it is possible to define an mc-structure on X such that the upper contour sets verify the Convexity Condition, hence Walker's Theorem will be a particular case of Theorem 3.

Lemma 2: *Let X be a topological space, and P an acyclic binary relation defined on X. Then there exists an mc-structure on X such that Convexity Condition is verified.*

(Proof in the appendix).

From this lemma and Theorem 3. we obtain Bergstrom (1975) and Walker's (1977) results,

Theorem 5: *[Bergstrom, 1975; Walker, 1977]. Let X be a topological space, and let P be a binary relation on X, such that it verifies:*

1. *P is an acyclic binary relation.*
2. *$U^{-1}(x)$ are open sets $\forall\ x \in X$.*

Then every compact subset of X has a P-maximal element.

So, it is worth remarking that Theorem 3. yields unified treatment to analyze the existence problem of maximal elements in preference relations when either acyclic binary relations or convexity conditions are considered. Thus, this result allows us to gather most of the results obtained until now by means of these two different ways under a unique statement which generalizes and extends them.

Tian's results (1993), are a particular case of Theorem 3. is that their conditions (Transfer SS-convex) imply our Convexity condition by considering an appropiate mc-structure. In this line, we can also obtain a characterization result about the existence of maximal elements, that is, a converse of the Theorem 3. also holds.

Theorem 6: *Let X be a compact Hausdorff topological space and let P be a binary relation defined on X, verifying the Continuity Condition.*

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact if and only if X is an mc-space which fulfills the Convexity Condition.
(Proof in the appendix)

The following example is based on the euclidean distance and shows a simple situation of non acyclic and nonconvex preferences (non contractible upper contour sets) in which Sonnenschein's and Walker's results (Theorem 4. and 5.) cannot be applied. However this example is covered by Theorem 3.

Example 2. Let X be the following subset of $\mathbb{R}^2$,

$$X = \{ (x,y)\in\mathbb{R}^2: 1\leq\|(x,y)\| \leq 2 \}$$

Let us consider the following subsets of X.

$$B = \{ (x,y)\in\mathbb{R}^2: \|(x,y)\| = 2, 0\geq x\geq -2, y\geq 0 \}.$$

$$A = \{ (x,y)\in\mathbb{R}^2: \|(x,y)\| = 2\} \setminus B$$

The preference relation (P) is defined on X as follows:

$\forall b \in B, \forall x \in X\setminus B$	$b \, P \, x$
$\forall a \in A, x \in X\setminus A\cup B\cup\{y^*\}$	$a \, P \, x$
$x^*=(1.5,0), y^*=(2,0)$	$x^* \, P \, y^*$
$\forall x,y \in X\setminus A\cup B\cup\{x^*,y^*\}$	$x \, P \, y \iff \|x\| > \|y\|$
$x \in \{x^*,y^*\}, \forall z \in X\setminus\{x^*,y^*\}$	$x \, P \, z \iff \|x\| > \|z\|$
$x \in \{x^*,y^*\}, \forall z \in X\setminus\{x^*,y^*\}$	$z \, P \, x \iff \|z\| > \|x\|$

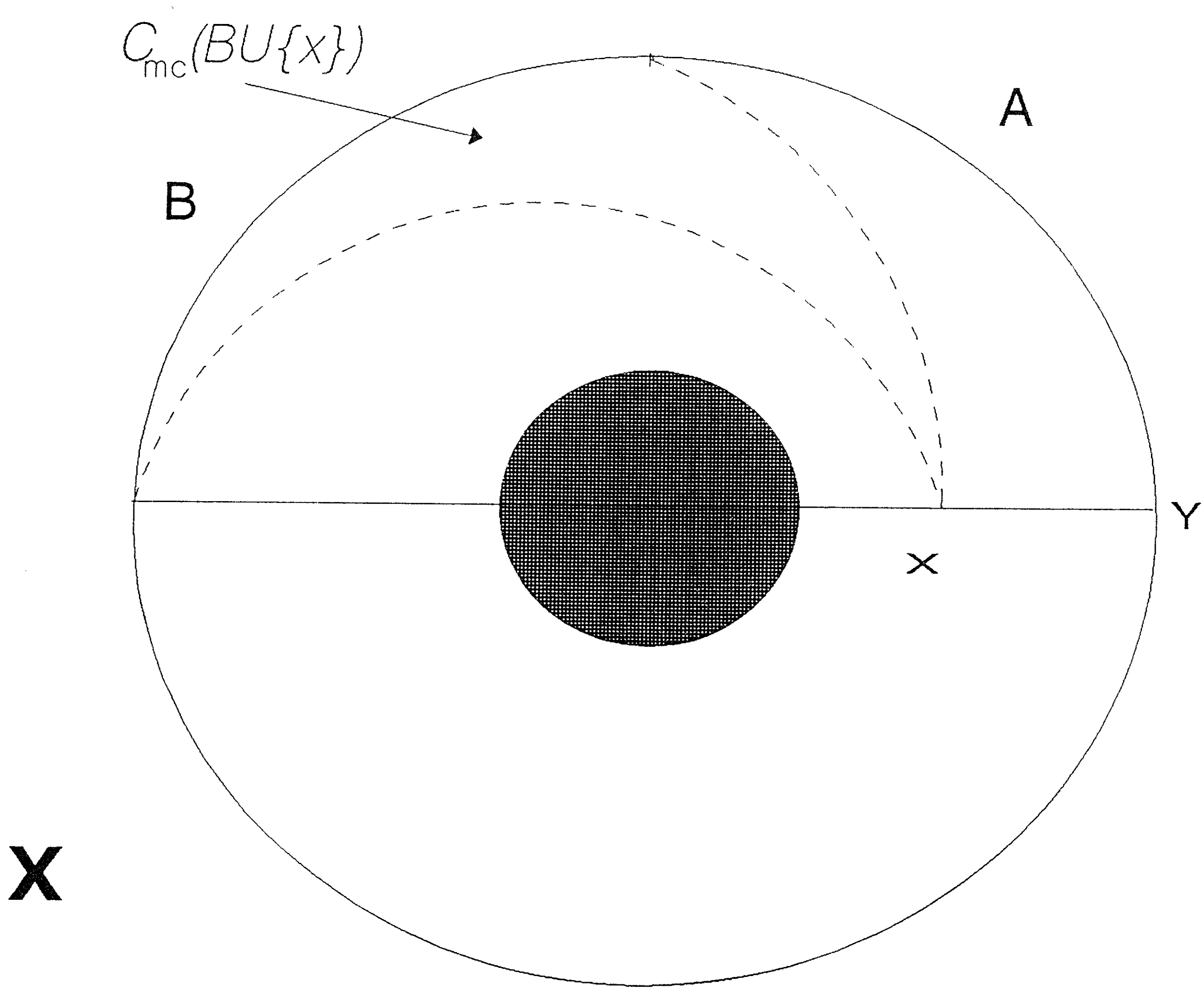
This preference relation verifies every condition in Theorem 3. as is shown inmediately below.

It is possible to define an mc-structure in which the upper contour sets (U(x)) are mc-sets (semicircular rings in this case). This structure is defined as in example 1.

In order to see that Convexity Condition of Theorem 3. is verified, note that if x is different from x* and y*, it is fulfilled obviously from the definition of preference relation P. Since U(x*) is an mc-set,

$$x^* \notin C_{mc}(U(x^*)) = U(x^*) \geq G_{A|U(x^*)}([0,1]^m).$$

And $y^* \notin C_{mc}(U(y^*))$, because $U(y^*)=B \cup \{x^*\}$ and the mc-hull of this set does not contain the point y* as it is shown in the following graph,



hence $y^* \notin C_{mc}(U(y^*)) \supseteq G_{A|U(x^*)}([0,1]^m)$, and the Convexity Condition is also verified in this case.

Finally it is not difficult to see that continuity condition is fulfilled and it can be concluded from Theorem 3. that the set of maximal elements is nonempty.

Note that this is in fact a non acyclic binary relation because there is a cycle y^* P (1.75, 0) P x^* P y^*. Therefore results for acyclic binary relations can not be applied.

APPENDIX

Proof of Lemma 1. As $\Gamma(x) \neq \emptyset$, for each $x \in X$, then for each x there exists $y \in \Gamma(x)$, so $x \in \Gamma^{-1}(y)$. Thus, $\{\Gamma^{-1}(y) : y \in X\}$ covers X, so from the hipothesis

If $y \in \Gamma^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int}\,\Gamma^{-1}(x')$

we obtain $\{\text{int}\Gamma^{-1}(y) : y \in X\}$ is an open cover of X. Since X is a compact set, then there exists a finite subcover $\{\ \text{int}\ \Gamma^{-1}(y_i) : i=0, \ldots, n\ \}$ and a continuous finite partition of unity subordinate to this subcovering,

$$\{\psi_i\}_{i=0}^{n}, \quad \psi_i(x) \geq 0\ , \quad \sum \psi_i(x) = 1\ , \quad \psi_i(x)>0 \Rightarrow x \in \text{int}\Gamma^{-1}(y_i)$$

Let be $J(x)=\{\ i : \psi_i(x) > 0\ \}$, then we have

$$y_i \in \Gamma(x) \quad \forall\ i \in J(x) \qquad (1)$$

If we take $A = \{y_0, y_1, \ldots, y_n\}$; since X is an mc-space, then there exist functions $P_i^A: X\times[0,1] \longrightarrow X$, such that $P_i^A(x,0) = x$ and $P_i^A(x,1) = b_i$ in such a way that $G_A:[0,1]^n \longrightarrow X$ is a continuous function.

a. Construction of the selection.

From the partition of unity, we define the following family of functions

$$t_i(x) = \begin{cases} 0 & \text{if } \psi_i(x) = 0 \\ \dfrac{\psi_i(x)}{\sum_{j=i}^{n} \psi_j(x)} & \text{if } \psi_i(x) \neq 0 \end{cases} \qquad (i = 0,1,\ldots,n-1)$$

so, function f is defined as follows

$$f(x) = G_A(t_0(x), \ldots , t_{n-1}(x)) =$$

$$P^A_0\left(..P^A_{n-2}\left(P^A_{n-1}\left(P^A_n(y_n,1),t_{n-1}(x)\right),t_{n-2}(x)\right),\ldots,t_0(x)\right)$$

Note that if $\psi_n(x) = 0$, and $\psi_{n-1}(x) > 0$, then $t_{n-1}(x) = 1$, therefore

$$P^A_{n-1}(b_n,t_{n-1}(x)) = P^A_{n-1}(b_n,1) = b_{n-1}$$

that is, b_n is not in the path defined by G_A. By applying the same reasoning repeatedly, if

$$\psi_n(x) = \psi_{n-1}(x) = 0 \quad \text{and} \quad \psi_{n-2}(x) > 0$$

then we have that $t_{n-1}(x) = 0$ and $t_{n-2}(x) = 1$, so

$$P^A_{n-2}(P^A_{n-1}(b_n,t_{n-1}(x)),t_{n-2}(x)) = P^A_{n-2}(b_n,1) = b_{n-2}$$

Therefore to construct the selection f we only need points b_i such that $y_i \in \Gamma(x)$. Hence f(x) would be contained in the image of the $G_{A|\Gamma(x)}([0,1]^m)$

$$f(x) \in G_{A|\Gamma(x)}([0,1]^m)$$

b. Continuity of selection f.

Selection f can be written as the following composition $f(x)=G_A(\mathcal{T}(\Psi(x)))$ where

$$\Psi : X \longrightarrow \Delta_n : \qquad \Psi(x) = (\psi_0(x),\psi_1(x),\dots,\psi_n(x))$$

$\mathcal{T} : \Delta_n \longrightarrow \mathbb{R}^n$:

$$\mathcal{T}_i(z) = \begin{cases} 0 & \text{if } z_i = 0 \\ \dfrac{z_i}{\sum_{j=i}^{n} z_j} & \text{if } z_i \neq 0 \end{cases} \qquad (i=0,\dots,n-1)$$

$$\mathcal{T}(\Psi(x)) = (t_0(x),t_1(x),\dots,t_{n-1}(x))$$

$$f(x) = G_A(\mathcal{T}(\Psi(x))) = G_A(t_0(x),t_1(x),\dots,t_{n-1}(x))$$

In order to prove the continuity of $f=G_A(\mathcal{T}(\Psi))$ at any point x, firstly we are going to prove that $G_A \circ \mathcal{T}: \Delta_n \longrightarrow X$ is a continuous function. If this is true then the continuity of f would be inmediately obtained (since f is a composition of continuous functions : $G_A \circ \mathcal{T}$ and Ψ).

To analyze the continuity of function $G_A \circ \mathcal{T}$ at any point $z \in \Delta_n$ it is important to note that if $z_n > 0$ then

$$\mathcal{T}_i(z) = \frac{z_i}{\sum_{j=i}^{n} z_j}$$

is a continuous function, since it is a quotient of the continuous function whose denominator is not null.

In an other case, $\mathcal{T}_i(z)$ could not be continuous (when its denominator is zero, that is when z_k are zero for all k = i, ... ,n-1).

In the first case, the continuity is not a problem: since $G_A \circ \mathcal{T}$ is a composition of continuous functions and therefore continuous.

In the second case, we define

$$j = \max \{i : z_i > 0\}$$

then $z_{j+1} = 0, \ldots, z_n = 0$, hence

$$\mathcal{T}_{j+1}(z) = 0, \ldots, \mathcal{T}_n(z) = 0 \quad \text{and} \quad \mathcal{T}_j(z) = 1$$

$$\text{because} \qquad \mathcal{T}_j(z) = \frac{z_j}{z_j + z_{j+1} + \ldots + z_n} = \frac{z_j}{z_j} = 1$$

Furthermore $\mathcal{T}_a$ $(a=0,\ldots,j)$ are continuous functions at z because their denominators are nonzero, ($z_j>0$ and $z_k \geq 0$ $\forall$ $k\neq j$, therefore $\sum_{K=a}^{n} z_k > 0$, $\forall k=0,\ldots,j$)

By definition of G_A, it is verified that

$$G_A(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z))=G_A(\mathcal{T}_0(z),..,1,0,..,0)=$$

$$P^A_0\left(..P^A_{n-2}\left(P^A_{n-1}\left(P^A_n(y_n,1),0\right),0\right),\ldots,1\right),\ldots,\mathcal{T}_0(z)\right)$$

and since $P^A_j(a,1) = b_j$, $\forall$ $a \in X$, then the part of the function

$$P^A_j\left(..P^A_{n-2}\left(P^A_{n-1}\left(P^A_n\left(y_n,1\right),\mathcal{T}_{n-1}(z)\right),\mathcal{T}_{n-2}(z)\right),\ldots,1\right) = b_j$$

and it is independent of the values of $\mathcal{T}_{n-1}(z)$, $\mathcal{T}_{n-2}(z)$, ..., $\mathcal{T}_{j+1}(z)$ that is,

$$P^A_j\left(..P^A_{n-2}\left(P^A_{n-1}\left(P^A_n\left(y_n,1\right),\lambda_{n-1}\right),\lambda_{n-2}\right),...,1\right) = b_j \qquad \forall\ \lambda_{n-1},...,\lambda_{j+1}\in[0,1]$$

so,

$$G_A(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z)) = G_A(\mathcal{T}_0(z),..,1,0,..,0) =$$

$$= G_A(\mathcal{T}_0(z),..,1,\ \lambda_{j+1},\ ...,\ \lambda_{n-1}), \qquad \forall\ \lambda_{n-1},\ ...,\ \lambda_{j+1} \in [0,1]$$

To simplify, we call $T=(\mathcal{T}_0(z),..,1)$ and $\lambda = (\lambda_{j+1},...,\lambda_{n-1})$, thus $G_A(\mathcal{T}_0(z),..,1,\lambda_{j+1},..,\lambda_{n-1}) = G_A(T,\ \lambda)$ $\forall\lambda\in[0,1]^m$ $(m=n-j-1)$.

In order to show that function $G_A\circ\mathcal{T}$ is continuous, we are going to prove that[3]

$$\forall z \in \Delta_n,\ \forall W \in N(G_A\circ\mathcal{T}(z))\ ,\ \exists\ V' \in N(z)\ :\qquad G_A\circ\mathcal{T}(V') \subseteq W$$

By applying that $G_A\circ\mathcal{T}(z)=G_A(T,\lambda)$ $\forall\lambda\in[0,1]^m$ and that G_A is a continuous function, we have that

$$\forall W \in N(G_A(T,\ \lambda)),\ \ \exists V^\lambda_T \times V_\lambda \in N((T,\lambda)):\quad G_A(V^\lambda_T \times V_\lambda) \subseteq W \qquad (2)$$

---

3 N(A) denotes the family of neighborhoods of A.

Moreover, since the family of open neighborhoods V_{λ} when $\lambda \in [0,1]^m$ is a covering of $[0,1]^m$, which is a compact subset, we know that there exists a finite recovering which will be denoted as follows

$$[0,1]^m = \cup \{V_{\lambda i} : i=1,\ldots,p\}$$

Hence, if we take $V_T^{\lambda i}$, $\forall i=1,..,p$, and we consider

$$V_T = \cap \{V_T^{\lambda i} : \forall i=1,..,p\},$$

then V_T is a neighborhood of T. But by considering that

$$T=(\mathcal{T}_0(z),..,\mathcal{T}_{j-1}(z),1)$$

we can rewrite $V_T^{\lambda i} = V_{T0}^{\lambda i} \times \ldots \times V_{Tj}^{\lambda i}$ where $V_{Tk}^{\lambda i} \in N(\mathcal{T}_k(z))$,

hence $V_T = V_{T0} \times \ldots \times V_{Tj}$ where $V_{Tk} = \cap \{ V_{Tk}^{\lambda i} : i=1,..,p\}$ $k = 0, \ldots, j$.

Hence, V_{Tk} is a neighborhood of $\mathcal{T}_k(z)$ since it has been defined as a finite intersection of neighborhoods of $\mathcal{T}_k(z)$. Moreover, these functions $\mathcal{T}_k$ are continuous at z $\forall k=0,\ldots,j$, so, neighborhoods U_k of z exist such that $\mathcal{T}_k(U_k) \subset V_{Tk}$.

Finally, on the one hand, if we denote

$$V' = \cap\{U_k : k = 0,\dots,j\}$$

then V' is a neighborhood of z, and it is verified that

$$\forall w\in V',\ (\mathcal{T}_0(w),\dots,\ \mathcal{T}_j(w)) \in V_{T0} \times\dots\times V_{Tj} = V_T \subset V_T^{\lambda i} \quad \forall i=1,\dots,p$$

On the other hand, the remaining indices (k=j+1,...,n) it is verified that $(\mathcal{T}_{j+1}(w),\ \dots,\ \mathcal{T}_{n-1}(w)) \in [0,1]^m = \bigcup\{V_{\lambda i} : i=1,\dots,p\}$, so there exists an index i_0 such that

$$(\mathcal{T}_{j+1}(w),\ \dots,\ \mathcal{T}_{n-1}(w)) \in V_{\lambda i0},\ i_0 \in \{1,\dots,p\}$$

Thus we can ensure that

$$(\mathcal{T}_0(w),\dots,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),\dots,\mathcal{T}_{n-1}(w)) \in V_T \times V_{\lambda i0} \subset V_T^{\lambda i0} \times V_{\lambda i0}$$

and since we have obtained, (2), that $G_A(V_T^\lambda \times V_\lambda) \subseteq W \quad \forall\lambda\in[0,1]^m$, we can conclude that for any $w \in V'$ it is verified that

$$G_A(\mathcal{T}_0(w),\dots,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),\dots,\mathcal{T}_{n-1}(w)) \subset W$$

c. Fixed point existence.

Consider now the function $g = \Psi\circ\Phi\colon \Delta_n \longrightarrow \Delta_n$, where $\Phi=G_A\circ\mathcal{T}$. Since Ψ and Φ are continuous, it is a continuous function from a convex compact set into itself, so Brouwer's Theorem can be applied and we have

$$\exists\ x_o \in \Delta_n : \quad g(x_o) = x_o$$

Therefore, $\Phi(g(x_o)) = \Phi(x_o)$ and, thus $f(\Phi((x_o)) = \Phi(x_o)$, so if we call $x^* = \Phi(x_o)$ we have obtained that

$$f(x^*) = x^*,$$

that is, f has a fixed point.

Q.E.D.

Proof of Theorem 3. Suppose $U(x) \neq \varnothing$, for each $x \in X$, then from Lemma 1., we deduce that there exists a continuous function f with a fixed point, $x^*=f(x^*)$, and that verifies that $f(x^*) \in G_{A|U(x^*)}([0,1]^m)$, hence it is a contradiction with the Convexity Condition, therefore the set of maximal elements is nonempty.

Furthermore, the set of maximal elements is a closed set because its complement is open. This is proved as follows if $w \notin \{x: U(x) = \varnothing\}$ then $U(w) \neq \varnothing$, therefore, there exists $y \in X$: $y \in U(w)$, that is $w \in U^{-1}(y)$ and by the Continuity Condition there exists $y' \in X$ such that

$$w \in \text{int}U^{-1}(y') \subset U^{-1}(y'),$$

thus if $z \in \text{int}U^{-1}(y')$ then $y' \in U(z)$, that is, $U(z) \neq \varnothing$ thus

$$w \in \mathrm{int}U^{-1}(y') \subset X-\{x:\ U(x)=\emptyset\}$$

Consequently it is obtained that $\{x:\ U(x) = \emptyset\}$ is a closed set and thus compact.

Q.E.D.

Proof of Lemma 2. As P is an acyclic binary relation, then it is verified that every finite subset $A = \{x_0, x_1,\dots,x_n\} \subset X$, has a maximal element (with P being acyclic, maximal elements always exist in finite sets). Then there is an element in A for example x_0 such that, $U(x_0) \cap A = \emptyset$.

It is possible to define the following mc-structure $i=0,1,\dots,n$

$$P^A_i: X\times[0,1] \longrightarrow X$$

$$P^A_i(x,0) = x \quad ; \qquad P^A_i(x,t) = x_0 \quad \text{if} \quad t\in(0,1]$$

where x_0 is one of the maximal elements of the set A. Then, composition G_A is given by

$$G_A:[0,1]^n \longrightarrow X$$

$$G_A(t_0,\dots,t_{n-1}) = P^A_0\left(..P^A_{n-2}\left(P^A_{n-1}(P^A_n(x_n,1),t_{n-1}),t_{n-2}\right),\dots,t_0\right)$$

since $P^A_n(x_n,1) = x_0$ in the end, composition G_A will be a constant function equal to x_0,

$$G_A(t_0,\dots,t_{n-1}) = x_0 \qquad \forall t_0,\dots,t_{n-1} \in [0,1]$$

We prove that this mc-structure verifies Convexity Condition, by contradiction.

If it is supposed that there exists a finite nonempty subset A of X such that $A \cap U(x) \neq \varnothing$ and which verifies

$$x \in G_{A|U(x)}([0,1]^m)$$

by construction of function G_A, x has to be a maximal element on A, that is, $A \cap U(x) = \varnothing$ which is a contradiction because $A \cap U(x) \neq \varnothing$. Hence, if P is an acyclic binary relation, then Convexity Condition in Theorem 3. is verified.

Q.E.D.

Proof of Theorem 6. From Theorem 3. it only remains to prove that if the set of maximal elements is nonempty, then X is an mc-space. Suppose the set of maximal elements of a relation **P** defined on X is non-empty, and let x* be one of the maximal elements, then we can define the following mc-structure on X:

For any non-empty finite subset A of X, $A=\{a_0, \ldots, a_n\}$

$$P_i^A : X \times [0, 1] \longrightarrow X$$

such that

$$P_i^A(x,t) = x \quad t \in [0,1) \quad \text{and} \quad P_i^A(x,1) = x^*$$

then the composition $G_A : [0,1]^n \longrightarrow X$ is

$$G_A(t_0, \ldots, t_{n-1}) =$$

$$P_0^A\left(..P_{n-2}^A\left(P_{n-1}^A\left(P_n^A(a_n,1),t_{n-1}\right),t_{n-2}\right),\ldots,t_0\right) = x^*$$

Hence X is an mc-space and furthermore it fulfills the Convexity Condition, since otherwise, if there exists a finite subset $\emptyset \neq A \subset X$, and there exists an element x such that $A \cap U(x) \neq \emptyset$ and

$$x \in G_{A|U(x)}([0,1]^m)$$

Therefore, as $G_{A|U(x)}([0,1]^m) = x^*$ we have that $x = x^*$ so,

$$x^* \in G_{A|U(x^*)}([0,1]^m)$$

but $U(x^*)=\emptyset$, so, $A \cap U(x^*)=\emptyset$ which is a contradiction.

Q.E.D.

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