EXISTENCE OF MAXIMAL ELEMENTS IN A BINARY RELATION RELAXING THE CONVEXITY CONDITION*

Juan-Vicente Llinares**

WP-AD 95-10

* I would like to express my gratitude to C. Herrero, J.E. Martínez Legaz, J.E. Peris, M.C. Sánchez and B. Subiza for their helpful comments. Any remaining errors are my exclusive responsibility. Financial support from DGICYT under project PS 90-0156 and Instituto de Cultura Juan Gil-Albert is also acknowledged.

** University of Alicante.

Editor: **Instituto Valenciano de Investigaciones Económicas, S.A.**
Primera Edición Mayo 1995.
ISBN: 84-482-0989-3
Depósito Legal: V-1987-1995
Impreso por Copisteria Sanchis, S.L.,
Quart, 121-bajo, 46008-Valencia.
Printed in Spain.

EXISTENCE OF MAXIMAL ELEMENTS IN A BINARY RELATION RELAXING THE CONVEXITY CONDITION

Juan-Vicente LLinares

A B S T R A C T

In this paper we relax the convexity condition in some classical results on the existence of maximal elements in binary relations in order to generalize them. To do this, we replace the linear segments in the usual convexity with a family of previously fixed paths joining up each two points. From these paths, we introduce a family of sets which generalizes the usual convex sets, and in this context we extend Sonnenschein's theorem on the existence of maximal elements and Browder's theorem on the existence of continuous selection and fixed point to correspondences.

KEYWORDS: Maximal elements, Fixed points, non-Convexity.

0. INTRODUCTION.

When an agent is faced with the problem of choosing a bundle of products, in the end, he will look for the bundle which maximizes his preference relation from those which he can afford. If we consider that preference relations can be represented by continuous utility functions, then the existence of maximal elements on compact subsets is ensured (by applying Weierstrass' theorem). Considering utility functions is very restrictive, mainly because transitivity of the indifference is an assumption much criticized as being strongly unrealistic (see Luce (1956), Starr (1969), etc). It is for this reason that the problem of the existence of maximal elements in non transitive binary relations is a very intersting one.

There are two different and independent approaches to be used in showing nonemptyness of the set of maximal elements without assuming transitivity of preferences. On the one hand acyclicity conditions instead of transitivity ones are considered (Bergstrom, 1975; Walker, 1977, see Border, 1985) and on the other, binary relations which verify some convexity conditions on the contour sets are used. In this line there are some known results, such as those of Fan's Lemma (see Border (1985)), Sonnenschein (1971), Yannelis and Prabhakar (1983).

Although convexity assumption can sometimes be justified as a "natural" requirement, many authors have criticized this assumption from both the experimental and formal point of view. Starr (1969) criticizes the

requirement of convex preferences since "it postulates away all forms of indivisibility and a class of relations which one might call anticomplementarity (those in which there are two different goods and the simultaneous use of both of them yields less satisfaction to the consumer than would the use of one of them)".

In this paper we present a new point of view in order to weaken the notion of convexity. In particular, we introduce a generalized convexity structure called *K-convex continuous structure* which is based on the idea of replacing the linear segments which appear in the usual convexity with a family of previously fixed paths. From these paths we define a family of sets (called *K-convex sets*) which generalizes the notion of usual convex sets (in a natural way, a subset A will be K-convex if the path joining up each two points in A is contained in A), and in this framework we obtain the results. Furthermore, in this case neither a vectorial structure nor a finite dimension on the spaces are necessary. So our results will cover situations without convexity and/or with non finite dimensionality.

The paper is organized as follows, in Section I we introduce the K-convex continuous structure. By using this structure, in Section II we present a result on the existence of maximal elements in binary relations which generalizes Sonnenschein's Theorem. To obtain this generalization we use a Lemma which extends Browder's selection and fixed point Theorem to the context of K-convex spaces. We finish this Section by showing an example in which our results can be applied (and Sonnenschein's and other aforementioned results can not). The proofs are presented in a final appendix.

I. K-CONVEX SPACES.

Definition I.1. Let X be a topological space; a *K-convex continuous structure* on X is defined by a continuous function

$$K: X \times X \times [0,1] \longrightarrow X$$

such that

$$K(x,y,0) = x \quad \text{and} \quad K(x,y,1) = y \qquad \forall\, x,y \in X.$$

Under these conditions, we call the pair (X, K) a *K-convex space.*

Remark. Note that for any pair of points x, y in a K-convex space X, we can define the function, $K_{xy}:[0,1] \longrightarrow X$; $K_{xy}(t) = K(x,y,t)$ such that K_{xy} is continuous and $K_{xy}(0) = x$, $K_{xy}(1) = y$. So function K_{xy} can be interpreted as a continuous path joining x and y. Furthermore, it is verified that if we consider points which are close together (x' close to x, y' close to y), then the path which joins x and y and the path which joins x' and y' are also close together.

In a K-convex space, we can consider a family of subsets of X which represents the generalization of convex sets, which are exactly those which are stable under function K.

Definition I.2. We say that a non-empty subset A of a K-convex space X, is a *K-convex set* if

$$\forall x,y \in A \qquad K(x,y,[0,1]) \subseteq A.$$

This means that, for any pair of points in A, the path which joins them up, K(x,y,[0,1]), is contained in A. Moreover, it is not difficult to prove that the arbitrary intersection of K-convex sets is also a K-convex set, which in turn makes it possible to define the extension of the notion of convex hull.

Definition I.3. For any subset A of a K-convex space X, we will give the name *K-convex hull* of A, to the K-convex set given by:

$$C_K(A) = \cap \{B: A \subseteq B, B \text{ K-convex set}\}.$$

It is obvious that in a convex set X it is always possible to define a K-convex continuous structure by using the function

$$K(x,y,t) = (1-t)x + ty.$$

In this case the K-convex sets coincide with the convex subsets of X, and the K-convex hull of any subset $A \subset X$ is the usual convex hull.

Another example whereby a K-convex continuous structure can be defined in a natural way is given by *star-shaped sets*; a subset X of a linear space is called *a star-shaped set* if there is some $a \in X$ (the center of the star) such that

$$tx + (1-t)a \in X \qquad \forall\, x \in X,\ \forall t \in [0,1]$$

In this case, function K can be defined by

$$K(x,y,t) = \begin{cases} (1-2t)x + 2ta & t \in [0,\ 0'5] \\ (2-2t)a + (2t-1)y & t \in [0'5,\ 1] \end{cases}$$

A necessary condition for a set A to be a K-convex set is that it contain the center of the star.

Another example is given by the sets which are homeomorph to a convex set; if h: $X \longrightarrow C$ is the homeomorphism, then we can define

$$K(x,y,t) = h^{-1}\Big((1-t)h(x) + th(y)\Big).$$

In this case a subset B is a K-convex set if and only if h(B) is a convex subset of C.

The next proposition states the conditions which have to be required in order that it be possible for a K-convex continuous structure to be defined on a set X. For this, we will need the notion of contractible set, which is defined as follows:

Definition I.4. A topological space X is *contractible* if there is a point x^* in X and a continuous function $H:X\times[0,1] \longrightarrow X$ such that for each x in X it is verified $H(x,1) = x^*$ and $H(x,0) = x$.

Proposition I.1. Let X be a subset of a topological space, then it is possible to define a K-convex continuous structure on X if and only if X is a contractible set.

(Proof in the appendix).

It is important to note that although the contractibility condition and the condition of having a K-convex continuous structure are equivalent, this

does not mean that the K-convex subsets of X coincide with the contractible subsets. This is due to the fact that the family of contractible sets is not stable under arbitrary intersections, while the family of K-convex sets verifies this property. Hence the family of K-convex sets defined by function K is given by some of the contractible subsets of X (since it is true that any K-convex set is contractible). For instance, in the example of star-shaped sets, the K-convex sets coincide with the contractible ones containing the center of the star.

II. EXISTENCE OF MAXIMAL ELEMENTS.

In this section we are going to present a generalization of Sonnenschein's Theorem when convexity is dropped, and K-convexity, (which is a weaker condition) is assumed.

From a binary relation **P** defined on X, the *upper* and *lower contour sets* of an element $x \in X$ are defined as usual:

$$U(x) = \{y \in X : y \mathbf{P} x\}$$

$$U^{-1}(x) = \{y \in X : x \mathbf{P} y\}.$$

An element $x^* \in X$ is maximal for a binary relation **P** if there is no other element which is preferred to it, that is, if $U(x^*) = \emptyset$.

To prove the extension of Sonnenschein's theorem, we will make use of the following Lemma which is the generalization of Browder's Theorem on the existence of a continuous selection and a fixed point to correspondences with open inverse images in the context of K-convex spaces.

Lemma II.1. Let X be a compact Hausdorff topological space with a K-convex continuous structure and let U: $X \longrightarrow\!\!\!\!\!\rightarrow X$ be a nonempty valued correspondence such that,

$$\text{if} \quad y \in U^{-1}(x), \ \exists\ x' \in X \quad \text{with} \quad y \in \text{int}U^{-1}(x').$$

Then there exists a continuous function $f:X \longrightarrow X$ which verifies:

i) $\exists\ x^* \in X$ such that $x^* = f(x^*)$.

ii) $f(x) \in C_K(U(x)) \quad \forall x \in X$.

(Proof in the appendix).

The next result is the extension of Sonnenschein's Theorem (1971) as well as Yannnelis and Prabhakar's result (1983) on the existence of maximal elements in the context of K-convex spaces.

Theorem II.2. Let X be a compact Hausdorff topological space with a K-convex continuous structure. Let **P** be a binary relation on X satisfying:

1. $x \notin C_K(U(x))$ for all $x \in X$.
2. If $y \in U^{-1}(x), \ \exists\ x' \in X$ with $y \in \text{int}U^{-1}(x')$.

Then the set of maximal elements $\{x^* : U(x^*) = \varnothing\}$ is nonempty and compact.

Proof.

If $U(x) \neq \varnothing \quad \forall x \in X$, then from Lemma II.1, we deduce that there exists a continuous function f with a fixed point, which verifies $f(x) \in C_K(U(x))$; in particular, $x^* = f(x^*) \in C_K(U(x^*))$ which is a contradiction with 1. Therefore, the set of maximals is nonempty.

We can prove that the set $\{x^* : U(x^*) = \varnothing\}$ is closed, by seeing that its complement is open. If $w \notin \{x^* : U(x^*) = \varnothing\}$ then $U(w) \neq \varnothing$. Let $y \in U(w)$;

therefore $w \in U^{-1}(y)$ and by hypothesis there is some $y' \in X$ such that

$$w \in \text{int}U^{-1}(y')$$

so $\text{int}U^{-1}(y')$ is an open set which contains w. To prove that it is contained in $X/\{x^* : U(x^*) = \varnothing\}$, let $z \in \text{int}U^{-1}(y')$; then $y' \in U(z)$ and $U(z) \neq \varnothing$.

Consequently it is obtained that the set of maximals is closed and therefore compact.

■

Notice that in the previous Theorem, we did not impose any condition on the dimensionality or linearity of the reference space. So, as a consequence of this result we obtain:

Corollary II.3. *[Sonnenschein, 1971].* Let X be a compact convex subset of $\mathbb{R}^n$ and let P be a binary relation defined on X, such that:

1. $\forall x \in X \quad x \notin C(U(x))$.
2. If $y \in U^{-1}(x)$, $\exists x' \in X$ with $y \in \text{int}U^{-1}(x')$.

Then the set of maximal elements $\{x^* : U(x^*) = \varnothing\}$ is nonempty and compact.

And in the context of infinite dimensional spaces,

Corollary II.4. *[Yannelis and Prabhakar, 1983].* Let X be a compact convex subset of a linear topological space and let $U: X \longrightarrow\!\!\!\!\rightarrow X$ be a correspondence which verifies:

1. $\forall x \in X \quad x \notin C(U(x))$.
2. $\forall x \in X \quad U^{-1}(x)$ is an open set in X.

Then the set of maximal elements $\{x^* : U(x^*) = \varnothing\}$ is nonempty.

Furthermore, as a consequence of Lemma II.1, we obtain the generalization of Fan (1961), and Browder's (1968) Theorems on the existence of continuous selection and fixed point to convex valued correspondences.

Theorem II.5. Let X be a compact Hausdorff topological space with a K-convex continuous structure and let $U: X \longrightarrow\!\!\!\!\rightarrow X$ be a correspondence with nonempty K-convex values such that,

$$\text{if} \quad y \in U^{-1}(x), \ \exists\, x' \in X \quad \text{with} \quad y \in \text{int}U^{-1}(x').$$

Then U has a continuous selection $f: X \longrightarrow X$, and a fixed point:

$$f(x) \in U(x) \qquad \forall\, x \in X \quad \text{and}$$
$$\exists\, x^* \in X : x^* \in U(x^*).$$

Proof.

By applying Lemma II.1, there exists a continuous function f such that $f(x) \in C_K(U(x)) \quad \forall\, x \in X$, and $x^* \in X$ exists such that $x^* = f(x^*)$. As U has K-convex values, then $C_K(U(x)) = U(x) \quad \forall\, x \in X$, which completes the proof.

■

From the previous Theorem, the following is inmediately obtained

Corollary II.6. *[Browder, 1968].* Let C be a nonempty compact convex subset of a topological vector space. Let $\Gamma: C \longrightarrow\!\!\!\!\!\rightarrow C$ be a nonempty convex valued correspondence with open inverse images. Then Γ has a continuous selection and a fixed point.

To finish we will show an example where Sonnenschein (1971) or Yannelis and Prabhakar's (1983) results of existence of maximal elements cannot be applied and in which the existence of maximal elements can be obtained by applying Theorem II.2.

Example. Let X be the following subset of $\mathbb{R}^2$,

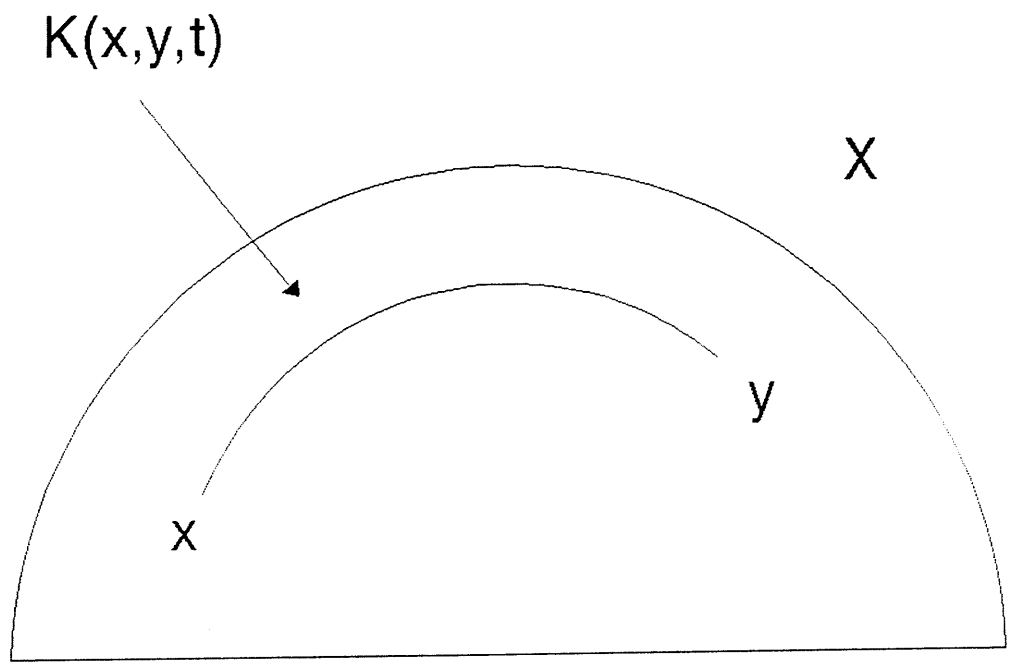
$$X = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\| \leq 1,\ y \geq 0 \}$$

and consider the K-convex continuous structure defined by

$$K: X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = \left((1-t)\rho_x + t\rho_y\right) e^{i((1-t)\alpha x + t\alpha y)}$$

where $x = \rho_x e^{i\alpha x}$, $y = \rho_y e^{i\alpha y}$ is the complex representation of the elements in the plane



GRAPHIC 1.

Let us consider now the following subsets of X:

$$A = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\| = 1,\ x < 0,\ y > 0 \}.$$

$$B = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\| = 1,\ x \geq 0,\ y \geq 0 \}.$$

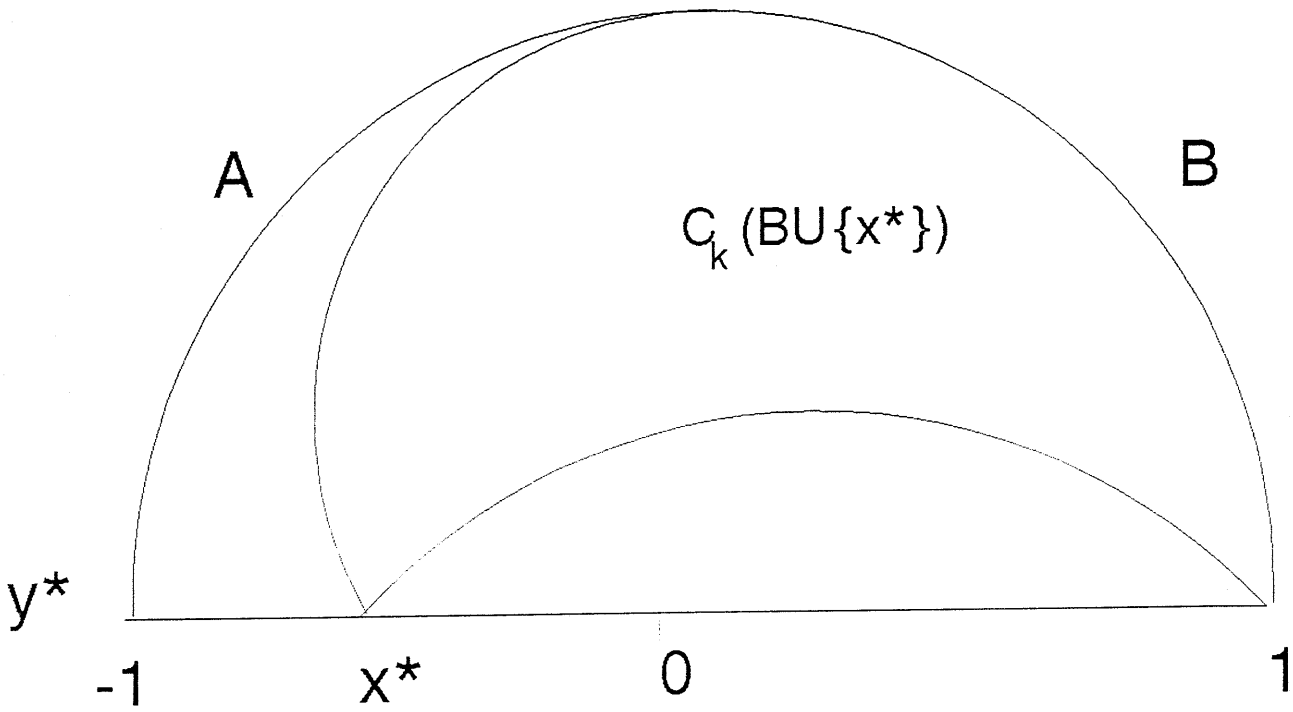
The preference relation P is defined on X as follows:

$\forall b \in B,\ \forall x \in X \setminus B$	$b\ P\ x$
$\forall a \in A,\ x \in X \setminus A \cup B \cup \{y^*\}$	$a\ P\ x$
$x^*=(-1/2,0),\ y^*=(-1,0)$	$x^*\ P\ y^*$
$\forall x,y \in X \setminus A \cup B \cup \{x^*,y^*\}$	$x\ P\ y \iff \|x\| > \|y\|$

$$x \in \{x^*,y^*\},\ \forall z \in X\backslash\{x^*,y^*\} \qquad x\ P\ z \iff \|x\| > \|z\|$$

$$x \in \{x^*,y^*\},\ \forall z \in X\backslash\{x^*,y^*\} \qquad z\ P\ x \iff \|z\| > \|x\|$$

In order to see that condition 1 of Theorem II.2 is verified, note that if x is different from x^* and y^*, it is obviously fulfilled from the definition of the preference relation P. Since $U(x^*)$ is a K-convex set, $x^* \notin C_K(U(x^*)) = U(x^*)$. And, $y^* \notin C_K(U(y^*))$, because $U(y^*) = B \cup \{x^*\}$ and the K-convex hull of this set does not contain the point y^*



GRAPHIC 2.

Finally it is not difficult to see that condition 2 is also verified and it can be concluded, from Theorem II.2 that a maximal element exists.

Note that this is in fact a non acyclic binary relation, and as such results for acyclic binary relations can not be applied.

APPENDIX.

Proof Proposition I.1.

Let K be the function which defines the K-convex continuous structure. For any fixed $a \in X$, the following function can be considered,

$$H : X \times [0, 1] \longrightarrow X \; ; \qquad H(x,t) = K(x,a,t)$$

It is a continuous function since K is continuous and furthermore it verifies that $H(x, 0) = x$, $H(x, 1) = a$, so X is a contractible set.

Conversely, if X is a contractible set then there exists a continuous function H which satisfies the previous assumptions, and from which it is possible to define the following function K,

$$K(x,y,t) = \begin{cases} H(x,2t) & t \in [0, 0.5] \\ H(y,2-2t) & t \in [0.5, 1] \end{cases}$$

which defines a K-convex continuous structure on X.

■

Proof of Lemma II.1.

As $U(x) \neq \emptyset$, for each $x \in X$, then there is $y \in U(x)$, so $x \in U^{-1}(y)$. Thus, $\{U^{-1}(y) : y \in X\}$ covers X and from hypothesis we obtain that $\{\text{int}U^{-1}(y) : y \in X\}$ is an open cover of X. Since X is a compact set, then

there is a finite subcover $\{ \text{int } U^{-1}(y_i) : i=0, \ldots,n \}$ and a continuous finite partition of unity subordinate to this subcovering,

$$\{\psi_i\}_{i=0}^{n}, \quad \psi_i(x) \geq 0 \ , \ \sum \psi_i(x) = 1 \ , \ \psi_i(x) > 0 \Rightarrow \ x \in \text{int}U^{-1}(y_i)$$

If we define $J(x) = \{ i : \psi_i(x) > 0 \}$, then $y_i \in U(x) \ \ \forall \ i \in J(x)$ and

$$C_K(\{y_i : i \in J(x)\}) \subseteq C_K(U(x))$$

As $C_K(\{y_i : i \in J(x)\})$ is a K-convex set, we can ensure that for every y_i , $i \in J(x)$ and any point p in $C_K(\{y_i : i \in J(x)\})$, the path which joins them up will be contained in $C_K(\{y_i : i \in J(x)\})$. So, from these arcs, the selection of the correspondence is obtained by composing them.

a. Construction of the continuous selection f.

From the finite partition of unity, we define the following family of functions,

$$t_i(x) = \begin{cases} 0 & \text{if } \psi_i(x) = 0 \\ \dfrac{\psi_i(x)}{\sum_{j=i}^{n} \psi_j(x)} & \text{if } \psi_i(x) \neq 0 \end{cases} \qquad i=0,1,\ldots,n-1$$

If we call $h_{n-1} = y_n$, then both h_{n-1} and y_{n-1} belong to U(x) (whenever $\psi_n(x) > 0$, $\psi_{n-1}(x) > 0$); therefore the path $K(h_{n-1}, y_{n-1}, [0,1])$ joining these points (which we call g_{n-1}) is contained in $C_K(\{y_i : i \in J(x)\})$ since it is a K-convex set. By computing g_{n-1} at $t_{n-1}(x)$, we obtain the point

$$h_{n-2} = g_{n-1}(t_{n-1}(x)) = K(h_{n-1}, y_{n-1}, t_{n-1}(x))$$

and by construction $h_{n-2} \in C_K(\{y_i : i \in J(x)\})$. By repeating the argument with the path which joins h_{n-2} and y_{n-2} (which we will call g_{n-2}) and computing it in $t_{n-2}(x)$, we obtain,

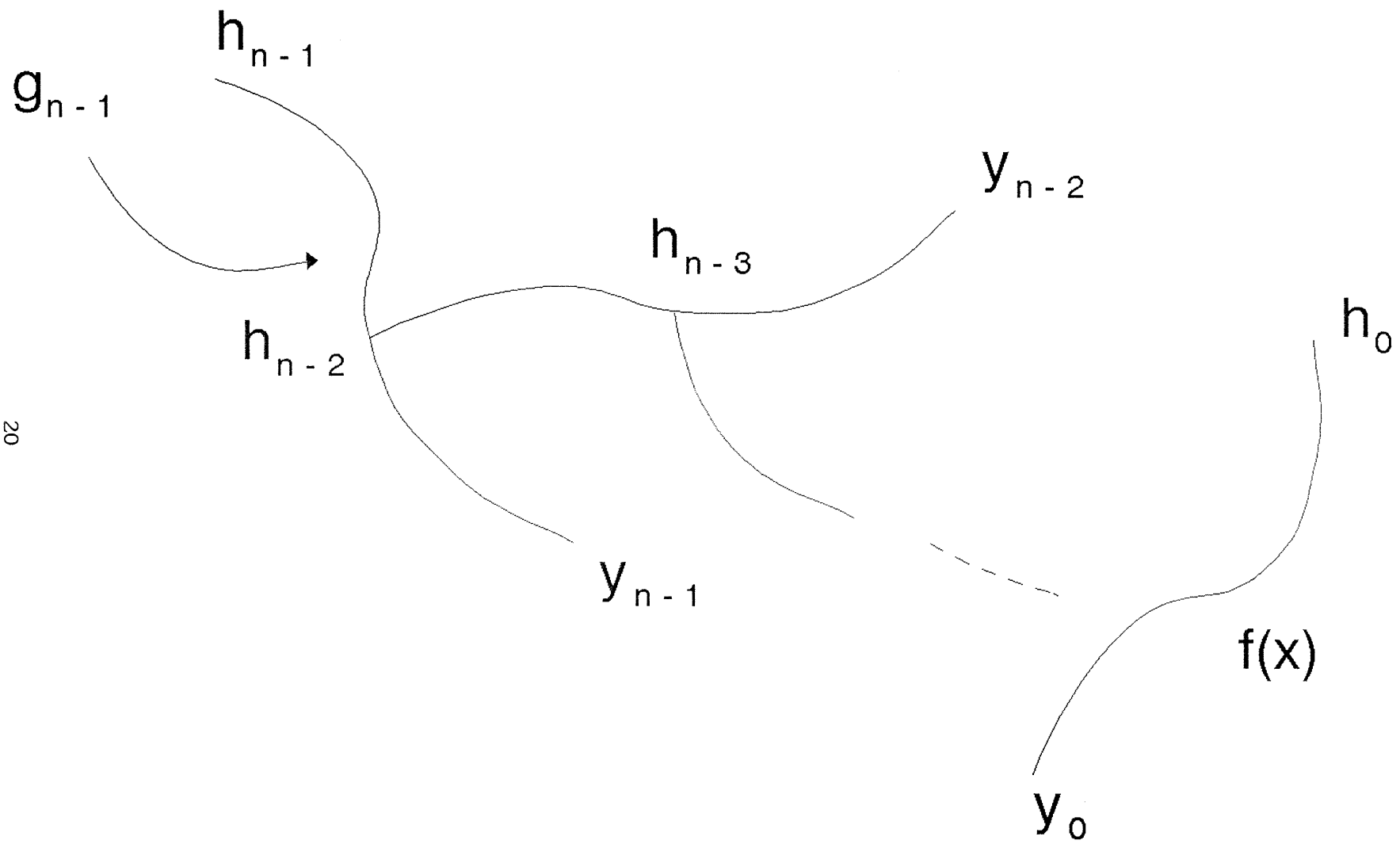
$$h_{n-3} = g_{n-2}(t_{n-2}(x)) = K(h_{n-2}, y_{n-2}, t_{n-2}(x)).$$

We repeat this reasoning until we obtain the element h_0 which will be linked to y_0 by means of the path g_0. Finally we define

$$f(x) = g_0(\psi_0(x)) = K(h_0, y_0, t_0(x))$$

that is,

$$f(x) = K\Big(..K\Big(K\Big(K(y_n, y_{n-1}, t_{n-1}(x)), y_{n-2}, t_{n-2}(x)\Big), y_{n-3}, t_{n-3}(x)\Big)...\Big)$$



GRAPHIC 3.

As a result of the way in which f is defined, it is immediate that $f(x) \in C_K(\{y_i : i \in J(x)\})$, $\forall x \in X$, since the "relevant" paths are contained in $C_K(\{y_i : i \in J(x)\})$ [notice that if $\psi_i(x) = 0$ for some i, then we have that $t_i(x) = 0$, so

$$K(h_i, y_i, t_i(x)) = K(h_i, y_i, 0) = h_i$$

and y_i will not appear in the construction of function f; hence to construct selection f we only use the points y_i such that $i \in J(x)$].

b. Continuity of selection f.

Selection f can be rewritten as the following composition

$$f(x)=\bar{K}(\mathcal{T}(\Psi(x)))$$

where $\bar{K}: [0,1]^n \longrightarrow X$ is defined by

$$\bar{K}(t_0,..,t_{n-1})=K\left(..K\left(K\left(y_n,y_{n-1},t_{n-1}\right),y_{n-2},t_{n-2}\right),..\right),y_0,t_0\right)$$

$\Psi : X \longrightarrow \Delta_n$

$$\Psi(x) = (\psi_0(x),\psi_1(x),\ldots,\psi_n(x))$$

and

$\mathcal{T} : \Delta_n \longrightarrow \mathbb{R}^n$

$$\mathcal{T}_i(z) = \begin{cases} 0 & \text{if} \quad z_i = 0 \\ \dfrac{z_i}{\sum_{j=i}^{n} z_j} & \text{if} \quad z_i \neq 0 \end{cases} \qquad i=0,\ldots,n-1$$

In order to prove the continuity of f at any point x, we shall first prove that $\bar{K}\circ\mathcal{T} : \Delta_n \longrightarrow X$ is a continuous function. If this is true then the continuity of f will be obtained inmediately (since f is a composition of continuous functions, $\bar{K}\circ\mathcal{T}$ and Ψ).

To analyze the continuity of function $\bar{K}\circ\mathcal{T}$ at any point $z \in \Delta_n$ it is important to note that if $z_n > 0$ then

$$\mathcal{T}_i(z) = \frac{z_i}{\sum_{j=i}^{n} z_j} \qquad i=0,1,\ldots,n-1$$

are continuous functions, since they are quotients of continuous functions whose denominators are not zero.

In the other case, $\mathcal{T}_i(z)$ could not be continuous (when its denominator is zero, that is, when z_k are zero for all k = i, ... ,n).

In the first case, the continuity is immediate since $\bar{K}\circ\mathcal{T}$ is a composition of continuous functions and therefore continuous.

In the second case, if we define

$$j = \max \{i : z_i > 0\}$$

then $z_{j+1} = 0, \ldots, z_n = 0$, hence

$$\mathcal{T}_{j+1}(z) = 0, \ \ldots\ , \ \mathcal{T}_{n-1}(z) = 0 \quad \text{and} \quad \mathcal{T}_{j}(z) = 1$$

Furthermore $\mathcal{T}_r$ (r = 0,...,j) are continuous functions at z because their denominators are not zero. As a result of the way in which function $\bar{K}$ Has been defined, it is verified that

$$\bar{K}(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z)) = \bar{K}(\mathcal{T}_0(z),..,1,0,..,0) =$$
$$= K\left[..K\left(K\left(y_n,y_{n-1},0\right),y_{n-2},0\right),..y_j,\ 1\right),\ldots\Big),y_0,\mathcal{T}_0(z)\right]$$

and since K(a,b,1) = b, $\forall$ a $\in$ X, then

$$K\left(..K\left(K\left(y_n,y_{n-1},\mathcal{T}_{n-1}(z)\right),y_{n-2},\mathcal{T}_{n-2}(z)\right),..y_j,1\right) = y_j$$

and it is independent of the values of $\mathcal{T}_{n-1}(z)$, $\mathcal{T}_{n-2}(z)$, ..., $\mathcal{T}_{j+1}(z)$; that is,

$$K\left(..K\left(K\left(y_n,y_{n-1},\lambda_{n-1}\right),y_{n-2},\lambda_{n-2}\right),..y_j,\ 1\right) = y_j$$

$$\forall\ \lambda_{n-1},\ \ldots,\ \lambda_{j+1} \in [0,1]$$

so, $\forall\ \lambda_{n-1},\ \ldots,\ \lambda_{j+1} \in [0,1]$

$$\bar{K}(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z)) = \bar{K}(\mathcal{T}_0(z),..,1,0,..,0) =$$
$$= \bar{K}(\mathcal{T}_0(z),..,1,\ \lambda_{j+1},\ \ldots,\ \lambda_{n-1})$$

In order to simplify the notation, we denote $T = (\mathcal{T}_0(z),..,1)$ and $\lambda = (\lambda_{j+1},\ldots,\lambda_{n-1})$; thus $\bar{K}(\mathcal{T}_0(z),..,1,\lambda_{j+1},..,\lambda_{n-1}) = \bar{K}(T,\lambda)$ $\forall\lambda\in[0,1]^m$ (m=n-j-1).

In order to obtain that function $\bar{K}\circ\mathcal{T}$ is continuous, we will prove that[1]

$$\forall z \in \Delta_n,\ \forall\ W \in N(\bar{K}\circ\mathcal{T}(z))\ ,\ \exists\ V' \in N(z) \quad \text{such that} \quad \bar{K}\circ\mathcal{T}(V') \subseteq W$$

By applying that $\bar{K}\circ\mathcal{T}(z) = \bar{K}(T,\lambda)$ $\forall\lambda\in[0,1]^m$ and that $\bar{K}$ is a continuous function, we have that

$$\forall W \in N(\bar{K}(T,\ \lambda)),\quad \exists V_T^{\lambda} \times V_{\lambda} \in N((T,\lambda)):\quad \bar{K}(V_T^{\lambda} \times V_{\lambda}) \subseteq W \qquad [1]$$

Moreover, since the family of open neighborhoods V_{λ} when $\lambda \in [0,1]^m$ is a covering of $[0,1]^m$, which is a compact subset, we know that there exists a finite subcovering

$$[0,1]^m = \cup\ \{V_{\lambda i}\ :\ i=1,\ldots,p\}$$

Hence, if we take $V_T^{\lambda i}$, $\forall i=1,..,p$, and we consider

$$V_T = \cap\ \{V_T^{\lambda i}:\ \forall i=1,..,p\},$$

---

1 $N(A)$ denotes the family of neighborhoods of A.

then V_T is a neighborhood of $T = (\mathcal{T}_0(z),..,\mathcal{T}_{j-1}(z),1)$ and we can rewrite

$$V_T^{\lambda i} = V_{T0}^{\lambda i} \times \ldots \times V_{Tj}^{\lambda i} \quad \text{where} \quad V_{Tk}^{\lambda i} \in N(\mathcal{T}_k(z)),$$

hence $V_T = V_{T0} \times \ldots \times V_{Tj}$ where $V_{Tk} = \cap \{ V_{Tk}^{\lambda i} : i=1,..,p\}$ $k = 0, \ldots, j.$

As V_{Tk} is a neighborhood of $\mathcal{T}_k(z)$, since it has been defined as a finite intersection of neighborhoods of $\mathcal{T}_k(z)$, and functions $\mathcal{T}_k$ are continuous at z, $\forall$ k = 0,...,j, there are neighborhoods U_k of z such that $\mathcal{T}_k(U_k) \subset V_{Tk}$.

Finally, if we denote

$$V' = \cap\{U_k : k = 0,\ldots,j\}$$

then V' is a neighborhood of z, and it is verified that

$$\forall\ w \in V',\ (\mathcal{T}_0(w),..,\ \mathcal{T}_j(w)) \in V_{T0} \times \ldots \times V_{Tj} = V_T \subset V_T^{\lambda i} \quad \forall i=1,..,p$$

For the rest of the indexes (k=j+1,...,n) it is verified that

$$(\mathcal{T}_{j+1}(w),\ \ldots,\ \mathcal{T}_{n-1}(w)) \in [0,1]^m = \bigcup \{V_{\lambda i} : i=1,\ldots,p\},$$

so there is an index i_0 such that

$$(\mathcal{T}_{j+1}(w),\ \ldots,\ \mathcal{T}_{n-1}(w)) \in V_{\lambda i0},\ i_0 \in \{1,\ldots,p\}$$

Thus we can ensure that

$$(\mathcal{T}_0(w),..,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),..,\mathcal{T}_{n-1}(w)) \in V_T \times V_{\lambda i0} \subset V_T^{\lambda i0} \times V_{\lambda i0}$$

and from [1], we can conclude that for any $w \in V'$

$$\bar{K}(\mathcal{T}_0(w),..,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),..,\mathcal{T}_{n-1}(w)) \subset W$$

so function f is continuous.

c. Fixed point existence.

Consider now the function $g = \Psi \circ \bar{K} \circ \mathcal{T}: \Delta_n \longrightarrow \Delta_n$. Since Ψ and $\bar{K}\circ\mathcal{T}$ are continuous, it is a continuous function from a convex compact set into itself; so Brouwer's Theorem can be applied and we have

$$\exists\ x_o \in \Delta_n : \quad g(x_o) = x_o$$

Therefore, $\bar{K}\circ\mathcal{T}(g(x_o)) = \bar{K}\circ\mathcal{T}(x_o)$ and, if we call $x^* = \bar{K}\circ\mathcal{T}(x_o)$, we obtain that $\bar{K}\circ\mathcal{T}\circ(\Psi\circ\bar{K}\circ\mathcal{T})(x_o) = \bar{K}\circ\mathcal{T}(x_o)$

$$f(x^*) = x^*,$$

that is, f has a fixed point which is also a fixed point to the correspondence.

■

REFERENCES

K.C. Border, Fixed point theorems with applications to economics and game theory (Cambridge University press, 1985).

Browder, F.E, Fixed point theory of multivalued mappings in topological vector spaces, Math. Ann. (1968), 283-301.

K.Fan, A Generalization of Tychonoff's Fixed Point Theorem, Mathematische Annalen 142 (1961), 305-310.

R.D. Luce, Semiorder and a Theory of Utility Discrimination, Econometrica 24 (1956), 178-191.

H. Sonnenschein, Demand Theory without transitive preferences with applications to the theory of competitive equilibrium, in: J.S. Chipman, L. Hurwicz, M.K. Richter and H.F. Sonnenschein eds., Preferences Utility and Demand (New York: Harcourt Brace Javanovich, 1971) Chap. 10.

R.M. Starr, Quasi-Equilibria in Markets with non-convex Preferences, Econometrica 37 (1969), 25-38.

N. Yannelis and N. Prabhakar , Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12 (1983), 233-245.

PUBLISHED ISSUES*

WP-AD 91-01 "A Characterization of Acyclic Preferences on Countable Sets"
C. Herrero, B. Subiza. May 1991.

WP-AD 91-02 "First-Best, Second-Best and Principal-Agent Problems"
J. Lopez-Cuñat, J.A. Silva. May 1991.

WP-AD 91-03 "Market Equilibrium with Nonconvex Technologies"
A. Villar. May 1991.

WP-AD 91-04 "A Note on Tax Evasion"
L.C. Corchón. June 1991.

WP-AD 91-05 "Oligopolistic Competition Among Groups"
L.C. Corchón. June 1991.

WP-AD 91-06 "Mixed Pricing in Oligopoly with Consumer Switching Costs"
A.J. Padilla. June 1991.

WP-AD 91-07 "Duopoly Experimentation: Cournot and Bertrand Competition"
M.D. Alepuz, A. Urbano. December 1991.

WP-AD 91-08 "Competition and Culture in the Evolution of Economic Behavior: A Simple Example"
F. Vega-Redondo. December 1991.

WP-AD 91-09 "Fixed Price and Quality Signals"
L.C. Corchón. December 1991.

WP-AD 91-10 "Technological Change and Market Structure: An Evolutionary Approach"
F. Vega-Redondo. December 1991.

WP-AD 91-11 "A 'Classical' General Equilibrium Model"
A. Villar. December 1991.

WP-AD 91-12 "Robust Implementation under Alternative Information Structures"
L.C. Corchón, I. Ortuño. December 1991.

WP-AD 92-01 "Inspections in Models of Adverse Selection"
I. Ortuño. May 1992.

WP-AD 92-02 "A Note on the Equal-Loss Principle for Bargaining Problems"
C. Herrero, M.C. Marco. May 1992.

WP-AD 92-03 "Numerical Representation of Partial Orderings"
C. Herrero, B. Subiza. July 1992.

WP-AD 92-04 "Differentiability of the Value Function in Stochastic Models"
A.M. Gallego. July 1992.

---

* Please contact IVIE's Publications Department to obtain a list of publications previous to 1991.

WP-AD 92-05 "Individually Rational Equal Loss Principle for Bargaining Problems"
C. Herrero, M.C. Marco. November 1992.

WP-AD 92-06 "On the Non-Cooperative Foundations of Cooperative Bargaining"
L.C. Corchón, K. Ritzberger. November 1992.

WP-AD 92-07 "Maximal Elements of Non Necessarily Acyclic Binary Relations"
J.E. Peris, B. Subiza. December 1992.

WP-AD 92-08 "Non-Bayesian Learning Under Imprecise Perceptions"
F. Vega-Redondo. December 1992.

WP-AD 92-09 "Distribution of Income and Aggregation of Demand"
F. Marhuenda. December 1992.

WP-AD 92-10 "Multilevel Evolution in Games"
J. Canals, F. Vega-Redondo. December 1992.

WP-AD 93-01 "Introspection and Equilibrium Selection in 2x2 Matrix Games"
G. Olcina, A. Urbano. May 1993.

WP-AD 93-02 "Credible Implementation"
B. Chakravorti, L. Corchón, S. Wilkie. May 1993.

WP-AD 93-03 "A Characterization of the Extended Claim-Egalitarian Solution"
M.C. Marco. May 1993.

WP-AD 93-04 "Industrial Dynamics, Path-Dependence and Technological Change"
F. Vega-Redondo. July 1993.

WP-AD 93-05 "Shaping Long-Run Expectations in Problems of Coordination"
F. Vega-Redondo. July 1993.

WP-AD 93-06 "On the Generic Impossibility of Truthful Behavior: A Simple Approach"
C. Beviá, L.C. Corchón. July 1993.

WP-AD 93-07 "Cournot Oligopoly with 'Almost' Identical Convex Costs"
N.S. Kukushkin. July 1993.

WP-AD 93-08 "Comparative Statics for Market Games: The Strong Concavity Case"
L.C. Corchón. July 1993.

WP-AD 93-09 "Numerical Representation of Acyclic Preferences"
B. Subiza. October 1993.

WP-AD 93-10 "Dual Approaches to Utility"
M. Browning. October 1993.

WP-AD 93-11 "On the Evolution of Cooperation in General Games of Common Interest"
F. Vega-Redondo. December 1993.

WP-AD 93-12 "Divisionalization in Markets with Heterogeneous Goods"
M. González-Maestre. December 1993.

WP-AD 93-13 "Endogenous Reference Points and the Adjusted Proportional Solution for Bargaining Problems with Claims"
C. Herrero. December 1993.

WP-AD 94-01 "Equal Split Guarantee Solution in Economies with Indivisible Goods Consistency and Population Monotonicity"
C. Beviá. March 1994.

WP-AD 94-02 "Expectations, Drift and Volatility in Evolutionary Games"
F. Vega-Redondo. March 1994.

WP-AD 94-03 "Expectations, Institutions and Growth"
F. Vega-Redondo. March 1994.

WP-AD 94-04 "A Demand Function for Pseudotransitive Preferences"
J.E. Peris, B. Subiza. March 1994.

WP-AD 94-05 "Fair Allocation in a General Model with Indivisible Goods"
C. Beviá. May 1994.

WP-AD 94-06 "Honesty Versus Progressiveness in Income Tax Enforcement Problems"
F. Marhuenda, I. Ortuño-Ortín. May 1994.

WP-AD 94-07 "Existence and Efficiency of Equilibrium in Economies with Increasing Returns to Scale: An Exposition"
A. Villar. May 1994.

WP-AD 94-08 "Stability of Mixed Equilibria in Interactions Between Two Populations"
A. Vasin. May 1994.

WP-AD 94-09 "Imperfectly Competitive Markets, Trade Unions and Inflation: Do Imperfectly Competitive Markets Transmit More Inflation Than Perfectly Competitive Ones? A Theoretical Appraisal"
L. Corchón. June 1994.

WP-AD 94-10 "On the Competitive Effects of Divisionalization"
L. Corchón, M. González-Maestre. June 1994.

WP-AD 94-11 "Efficient Solutions for Bargaining Problems with Claims"
M.C. Marco-Gil. June 1994.

WP-AD 94-12 "Existence and Optimality of Social Equilibrium with Many Convex and Nonconvex Firms"
A. Villar. July 1994.

WP-AD 94-13 "Revealed Preference Axioms for Rational Choice on Nonfinite Sets"
J.E. Peris, M.C. Sánchez, B. Subiza. July 1994.

WP-AD 94-14 "Market Learning and Price-Dispersion"
M.D. Alepuz, A. Urbano. July 1994.

WP-AD 94-15 "Bargaining with Reference Points - Bargaining with Claims: Egalitarian Solutions Reexamined"
C. Herrero. September 1994.

WP-AD 94-16 "The Importance of Fixed Costs in the Design of Trade Policies: An Exercise in the Theory of Second Best", L. Corchón, M. González-Maestre. September 1994.

WP-AD 94-17 "Computers, Productivity and Market Structure"
L. Corchón, S. Wilkie. October 1994.

WP-AD 94-18 "Fiscal Policy Restrictions in a Monetary System: The Case of Spain"
M.I. Escobedo, I. Mauleón. December 1994.

WP-AD 94-19 "Pareto Optimal Improvements for Sunspots: The Golden Rule as a Target for Stabilization"
S.K. Chattopadhyay. December 1994.

WP-AD 95-01 "Cost Monotonic Mechanisms"
M. Ginés, F. Marhuenda. March 1995.

WP-AD 95-02 "Implementation of the Walrasian Correspondence by Market Games"
L. Corchón, S. Wilkie. March 1995.

WP-AD 95-03 "Terms-of-Trade and the Current Account: A Two-Country/Two-Sector Growth Model"
M.D. Guilló. March 1995.

WP-AD 95-04 "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets"
J. Alcalde. March 1995.

WP-AD 95-05 "Implementation of Stable Solutions to Marriage Problems"
J. Alcalde. March 1995.

WP-AD 95-06 "Capabilities and Utilities"
C. Herrero. March 1995.

WP-AD 95-07 "Rational Choice on Nonfinite Sets by Means of Expansion-Contraction Axioms"
M.C. Sánchez. March 1995.

WP-AD 95-08 "Veto in Fixed Agenda Social Choice Correspondences"
M.C. Sánchez, J.E. Peris. March 1995.

WP-AD 95-09 "Temporary Equilibrium Dynamics with Bayesian Learning"
S. Chatterji. March 1995.

WP-AD 95-10 "Existence of Maximal Elements in a Binary Relation Relaxing the Convexity Condition"
J.V. Llinares. May 1995.

WP-AD 95-11 "Three Kinds of Utility Functions from the Measure Concept"
J.E. Peris, B. Subiza. May 1995.

WP-AD 95-12 "Classical Equilibrium with Increasing Returns"
A. Villar. May 1995.

WP-AD 95-13 "Bargaining with Claims in Economic Environments"
C. Herrero. May 1995.

WP-AD 95-14 "The Theory of Implementation when the Planner is a Player"
S. Baliga, L. Corchón, T. Sjöström. May 1995.

WP-AD 95-15 "Popular Support for Progressive Taxation"
F. Marhuenda, I. Ortuño. May 1995.