DUAL APPROACHES TO UTILITY*

Martin Browning**

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** McMaster University.

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ABSTRACT

This paper surveys some basic results on the representation of consumer preferences using duality methods. It has been prepared for inclusion in the forthcoming *Handbook of Utility Theory*. It is aimed at graduate students who have a solid background in economics but have not seen too much about this topic. The main purpose of the paper is to show how learning something about duality methods can make utility analysis much easier. To emphasise this, an optimal tax example is given at the beginning of the survey.

The usual representations are discussed: the cost function; the indirect utility function; the distance function and the profit function. Examples of the use of each are given. The emphasis everywhere is on how quick and easy many duality methods are; relatively little attention is paid to mathematical detail.

CONTENTS

1. INTRODUCTION

Most analyses of consumer allocation problems start from the direct utility function. My feelings about this are captured well by the following commentary on the most common opening in chess:

1. **P-K4** ......Already white is in complete disarray.

There are lots of different ways of describing a convex set; consequently there are lots of different ways of representing convex preferences. Although the usual way to do this is by the direct utility function, this is by no means the most useful representation in most contexts.

In this chapter I shall define several ways of representing preferences and give examples of their use. This list is not meant to be exhaustive; indeed the primary 'message' of this survey is that the economic problem at hand may require the construction of some new representation that facilitates the analysis. Ideally the economic problem should define how we do the analysis rather than what we can do with conventional tools defining what we actually do.

The notion of duality is not very well defined in economics[1] but it may loosely be given as the art of describing preferences (or technology) to make the analysis easier. An alternative characterisation is given by Gorman (1976): "duality is about the choice of independent variables in terms of which one defines a theory". For example, analyses that start with the direct utility function take quantities as the 'independent variables' but in the

[1] This contrasts with mathematics where the notion of a dual space and a duality is very precisely defined.

usual choice situation they are the objects of choice (that is, the 'dependent variables'). This leads to all sorts of complications and concerns with horrendous objects like bordered Hessians which may have been of interest to 19th. century mathematicians but are surely not of interest to modern economists. All of this is avoided if we take as our independent variables those variables that are parametric to the agent; for example, prices and income (strictly, total expenditure).

To give some idea of how we use duality methods I present a simple example in section 2. This examines when a tax on a particular good will not lead to any dead-weight loss. Although the example has some intrinsic interest I present it mainly as an example of how choosing the 'correct' independent variables makes life easier.

As the first paragraph in this introduction suggests, I think that the best formal way to think of duality is to think of different ways of describing a convex set (in particular, an upper contour set). To emphasise this, in section 3 I present a short mathematical digression that illustrates most of the important points about the mathematics of duality. This section is largely optional; it may add to the understanding of the later results but it can be skipped by readers who are impatient of mathematical underpinnings. On the other hand, the fact that many results that are useful in duality theory (as used in economics) already exist in the convex analysis literature should not be ignored; there are limits to how many times we wish to re-invent any particular wheel.

After section 3 I shall dispense almost entirely with formalities and simply state results in a relatively rough and ready form. This is how it should be when we use duality methods; we should not get bogged down in worries about continuity or what is happening at boundaries[2]. This is not to say that such issues are unimportant but rather that they can usually be left for tidying up if the analysis goes anywhere. In keeping with this I shall

[2] As Diewert (1982) remarks "continuity complexities appear to be the only difficult concepts associated with duality theory". I would add that they are also the least important.

also assume that all prices and quantities are positive unless otherwise stated.

The duality literature is venerable (dating back, at least, to Hotelling in 1932 and, perhaps, to Antonelli in 1886) and large. There are several good surveys of this literature (see, for example, Diewert (1982), or Blackorby, Primont and Russell (1978)). Deaton and Muellbauer (1980) also provide a very accessible introduction to the area. There is also a recent text by Cornes (Cornes (1992)) which is highly recommended. Even more highly recommended is Gorman (1976) which gives a tour of some of the main landmarks of duality (and separability) theory by one of the most important contributors to the literature. Given this wealth of previous discussion, I shall devote very little space to the historical roots of the ideas presented here (so that the references omit most of the important innovators in duality theory); neither shall I give many formal proofs. Rather the intent is to give a fairly informal guide to what can be done; this is in line with my belief suggested above that duality is about thinking about how to make the analysis of a specific problem easier.

2. AN OPTIMAL TAX EXAMPLE

It is well established that lump sum taxes are at least as good as commodity taxes for raising revenue (where 'at least as good as' refers to Parteo dominance). An interesting question is: when is a lump sum tax no better than a tax on a particular commodity? Put another way: when does a tax on a particular good not impose any dead weight loss? It turns out that this can be answered completely and simply using the cost function representation.

Let us first develop the standard proof that a lump sum tax is at least as good as a commodity tax. Suppose the tax authority wishes to raise revenue from a single agent. Let us suppose that two instruments are available: a lump sum tax (that is, taking some income) or a tax (t) on good 1. Since we are going to make prices ($\mathbf{p}$) and utility (u) parametric for the agent the natural representation is the cost function:

$$c(\mathbf{p},u) = \min_{\mathbf{q}} \{\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u\}.$$

where $\mathbf{q}$ is a vector of quantities and $\upsilon(.)$ is the direct utility function[3]. I shall discuss this representation at greater length below; for now all we need to note about $c(.)$ is that (under suitable regularity conditions) it is concave and increasing in $\mathbf{p}$ and the derivative of $c(.)$ with respect to p_i is equal to the compensated (Hicksian) demand for good i.

Given fixed $(\mathbf{p},u)$ define:

$$\Psi(t) = c(p_1+t,\mathbf{p}_{-1},u) - c(\mathbf{p},u)$$

(where $\mathbf{p}_{-1}$ is the vector of prices of all goods other than good 1). Thus $\Psi(.)$ is the amount of money the agent needs to compensate her for the commodity tax. The function $\Psi(.)$ is concave, increasing and has $\Psi(0) = 0$. Moreover, the compensated demand for good 1 with the tax t is given by the derivative of $\Psi(t) = \Psi'(t)$. From this we have that the tax revenue that the tax authority receives is $\Psi'(t)t$. Since $\Psi(.)$ is concave, increasing and goes through the origin we have:

$$\Psi'(t)t \leq \Psi(t)$$

Thus starting from a commodity tax of t and no lump sum tax we could raise at least as much revenue and leave the agent indifferent by setting $t = 0$ and imposing a lump sum tax of $\Psi(t)$. This completes the demonstration that lump sum taxes are at least as good as commodity taxes.

Now consider when lump sum taxes are no better than a tax on good 1. For this to hold we require that the inequality above be an equality which implies that $\Psi(t)$ is linear; this in turn requires that $c(\mathbf{p},u)$ be linear in p_1:

$$c(\mathbf{p},u) = a(\mathbf{p}_{-1},u) + b(\mathbf{p}_{-1},u)p_1$$

One thing to note about this condition is that it is necessary and sufficient for a commodity tax on good 1 to be 'first best'. There are much stronger sufficient conditions that apply to the direct utility representation. For example, if preferences are Leontief then a tax on any good is 'first best'. This illustrates a theme that runs through the duality literature - whilst

[3] I assume here that the minimum is attained (that is, we do not use 'inf'). This places implicit restrictions on $\upsilon(.)$; I shall return to this in section 4 below.

direct methods (that is, methods that start from the direct utility function) can often be used to give sufficient conditions for some desired property we generally need to resort to duality methods to give full characterisations.

The point of this exercise is to show that if we set up the problem in the 'right' way at the start then the analysis can be relatively trivial. I turn now to the formalities of setting a problem up in the 'right' way; that is, to duality methods.

3. MATHEMATICAL BACKGROUND

A function $f: \mathbb{R}^n \to \bar{\mathbb{R}}$ (where $\bar{\mathbb{R}}$ is the extended real line) is convex if its epigraph (the volume above the graph) is a convex set. One way to describe a particular convex function is to characterise all the hyperplanes (lines if n = 1) that 'support' the graph of the function. The equation for the tangent at any point $(\mathbf{x}, f(\mathbf{x}))$ is given by $(\mathbf{x}'\mathbf{y} - f^*(\mathbf{y}))$ where $f^*: \mathbb{R}^n \to \bar{\mathbb{R}}$ is defined by:

$$f^*(\mathbf{y}) = \sup_{\mathbf{x}} \{\mathbf{x}'\mathbf{y} - f(\mathbf{x})\}$$

Thus as we change y we map out the graph of f(x)[4]: knowing f^* is equivalent to knowing f and *vice versa* (see Rockafellar (1970) for a very thorough account of convex analysis).

Given any (not necessarily convex) function f(.) we can define $f^*(.)$ as above; this new function is known as the (convex) conjugate of f(.). For our purposes the two most important facts about conjugates are:

Fact 1: For any f(.), $f^*(.)$ is convex.

Fact 2: If f(.) is convex[5] then $f^{**} = f$ (where f^{**} is the conjugate of the conjugate).

From Fact 1 we see that no matter what function we start with (convex or not) the conjugate is convex. The second Fact establishes that conjugacy induces a one-one relationship between f and f^* if f(.) is convex. Starting from either function we can recover the other by taking conjugates; if you like, f and f^* are 'dual' representations of the same structure. We shall see analogs of

[4] The reader is encouraged to sketch the diagram that goes with (1).
[5] and f(.) satisfies some regularity conditions, see Rockafellar (1970), Section 12.

these facts emerging again and again in the economics below. If f(.) is not convex then we only have $f^{***}(\mathbf{y}) = f^{*}(\mathbf{y})$. To see that f(.) and $f^{**}(.)$ can be different if f is not convex note that if $f(x) = x^3$ then $f^{*}(y) = +\infty$ and $f^{**}(x) = -\infty$.

If f(.) is convex and differentiable then a natural way to derive the conjugate is to solve the first order conditions (FOC):

$$\mathbf{y} = \nabla f(\hat{\mathbf{x}})$$

and then to substitute for $\hat{\mathbf{x}}$ in (1).[6] Although this is very natural for economists we can rarely invert the FOC so that this is not usually a practical route to deriving conjugates. This is exactly analogous to deriving closed form expressions for demand functions (by analogy, $\hat{\mathbf{x}}$ in this case) from the direct utility function ($f(\mathbf{x})$): we can only do it for a very restricted class of utility functions.

Although the conjugate is the most common dual operator for convex functions it is not the only one. Almost as widely used is the polar of f(.):

$$f^{0}(\mathbf{y}) = \max_{\mathbf{x}} \{\mathbf{x}'\mathbf{y} \mid f(\mathbf{x}) = 1\}.$$

The polarity operator induces a one-one relationship in the class of functions f(.) such that f(.) is a positively linear homogeneous function with f(0) = 0 and f(.) positive and finite everywhere else. Although this might seem to be a class of functions that is rather restricted we shall see below that it is an important set in duality theory.

4. THE COST AND INDIRECT UTILITY FUNCTIONS

The usual way to represent preferences is by the utility function $\upsilon(\mathbf{q})$. In keeping with tradition we shall start from this. In section 2 we introduced the **cost function** representation (sometimes termed the **expenditure function**) using the direct representation:

$$c(\mathbf{p},u) = \min_{\mathbf{q}} \{\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u\}. \quad (1)$$

The following assumption on the direct utility function is sufficient to

[6] This procedure is known as the Legendre transform.

ensure that the minimum in (1) is attained for every $\mathbf{p}$[7]:

(U1) $\upsilon(.)$ is continuous.

Given this assumption we have the following properties for the cost function (proofs can be found in any of the references given at the end of section 1):

(C1) $c(.,u)$ is non-decreasing in $\mathbf{p}$;

(C2) $c(.,u)$ is linear homogeneous in $\mathbf{p}$;

(C3) $c(.,u)$ is concave in $\mathbf{p}$.

The statement of properties here brings out an important feature of dual representations that we shall see many more times: the dual does not necessarily 'inherit' properties from the direct representation in any obvious way. Thus the cost function is concave whether or not the direct utility function is quasi-concave. This is an analogue of Fact 1 in Section 3. As another example, differentiability of the cost function is neither necessary nor sufficient for differentiability of the direct utility function. A final item to note is that the cost function uses the linear function $\mathbf{p}'\mathbf{q}$ to describe preferences; this does not imply that the agent actually faces a linear budget constraint. The cost function description of preferences is entirely independent of the environment the agent faces. As always, we must keep the description of preferences and constraints completely separate until we are ready to bring them together in the analysis of behaviour. Thus cost function representations can be useful even when agents face non-linear prices.

In the derivation above we started with the direct utility function and defined the cost function. We can also go the other way; indeed, this is the whole point of duality. Thus given any function $c(\mathbf{p},u)$ we can define a new function:

$$\upsilon^*(\mathbf{q}) = \min_u \{u \mid \mathbf{p}'\mathbf{q} \geq c(\mathbf{p},u) \text{ for all } \mathbf{p}\} \tag{2}$$

If $c(\mathbf{p},u)$ satisfies (C1) to (C3) then $\upsilon^*(\mathbf{q})$ is a 'well behaved' utility function. Furthermore, if preferences are convex (and monotone) then this utility function is the 'original' utility function $\upsilon(\mathbf{q})$; this is the analogue

7 so long as u is chosen to be in the range of $\upsilon(.)$. I shall assume this condition in all that follows.

of Fact 2 in section 3. We shall formalise this as a proposition with proof. We first specify some assumptions on the direct utility function:

(U2) $\upsilon(\mathbf{q})$ is increasing in $\mathbf{q}$

(U3) $\upsilon(\mathbf{q})$ is quasi-concave.

Proposition. Given (U1)-(U3) the function $\upsilon^*(\mathbf{q})$ defined in (2) is the original utility function $\upsilon(\mathbf{q})$ used to construct $c(\mathbf{p},u)$ in (1).

Proof The proof proceeds in two parts; the first part does not need (U1)-(U3).

(i) For arbitrary $\tilde{\mathbf{q}}$ we have:

$$\begin{aligned}\upsilon^*(\tilde{\mathbf{q}}) &= \min_u\{u \mid \mathbf{p}\tilde{\mathbf{q}} \geq \min_{\mathbf{q}}(\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u),\ \forall\mathbf{p}\} \\ &= \min_u\{u \mid [\mathbf{p}\tilde{\mathbf{q}} \geq \min_{\mathbf{q}}(\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq \upsilon(\tilde{\mathbf{q}}))\ \forall\mathbf{p}] \text{ and } \\ &\qquad [\upsilon(\tilde{\mathbf{q}}) = u]\} \\ &\geq \min_u\{u \mid \upsilon(\tilde{\mathbf{q}}) = u\} \\ &= \upsilon(\tilde{\mathbf{q}}).\end{aligned}$$

(ii) By (U1) to (U3) for any $\tilde{\mathbf{q}}$ we can find $\tilde{\mathbf{p}}$ which gives:

$$\tilde{\mathbf{q}} = \operatorname{argmin}_{\mathbf{q}}(\tilde{\mathbf{p}}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u).$$

Thus we have:

$$\begin{aligned}\upsilon^*(\tilde{\mathbf{q}}) &= \min_u\{u \mid \mathbf{p}\tilde{\mathbf{q}} \geq \min_{\mathbf{q}}(\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u),\ \forall\mathbf{p}\} \\ &\leq \min_u\{u \mid \tilde{\mathbf{p}}\tilde{\mathbf{q}} \geq \min_{\mathbf{q}}(\tilde{\mathbf{p}}'\mathbf{q} \mid \upsilon(\mathbf{q}) \geq u)\} \\ &= \upsilon(\tilde{\mathbf{q}}).\end{aligned}$$

Combining the two inequalities, we have:

$$\upsilon^*(\mathbf{q}) = \upsilon(\mathbf{q}).$$ QED

From the proof we see that $\upsilon^*(\mathbf{q}) \geq \upsilon(\mathbf{q})$ whatever the original preferences. The monotonicity and quasi-concavity of the utility function was needed to prove the reverse inequality. These assumptions allow us to pick out any point on any indifference curve.

The other commonly used dual representation is the **indirect utility function** which is defined as:

$$V(\mathbf{p},x) = \max_{\mathbf{q}} \{\upsilon(\mathbf{q}) \mid \mathbf{p}'\mathbf{q} = x\} \tag{3}$$

Thus $V(\mathbf{p},x)$ gives the maximum utility level attainable for an agent with income x who faces prices $\mathbf{p}$. An alternative derivation using the cost function defines $V(\mathbf{p},x)$ implicitly by:

$$c(\mathbf{p},V(\mathbf{p},x)) = x \tag{3'}$$

Thus the indirect utility function is the inverse of the cost function. This leads to another 'duality' between these two representations:

$$V(\mathbf{p},c(\mathbf{p},u)) \equiv u \tag{4}$$

Although the cost and indirect utility function are closely connected their properties are very different, as can be seen by comparing the following with (C1) to (C3) above:

The indirect utility function satisfies:

(I1) $V(.)$ is increasing in x and non-increasing in $\mathbf{p}$;

(I2) $V(.)$ is zero homogeneous in $(\mathbf{p},x)$;

(I3) $V(.)$ is quasi-convex in $(\mathbf{p},x)$.

Given the zero homogeneity we can also define the **normalised indirect utility function** $\Psi(\mathbf{p}^*) = V(\mathbf{p}x^{-1},1)$. The negative of this function looks very much like a direct utility function that is defined on (normalised) prices, $\mathbf{p}^*$, rather than quantities.

The widespread use of the cost and indirect utility function derives from the fact that we can derive demand functions from them by simple differentiation. This is in marked contrast to the direct representation which requires us to solve systems of non-linear equations to find demand functions. Since we very often want to move backwards and forwards from preferences to demands this gives dual representations a decisive advantages in many contexts.

To derive demand functions from the cost function, use the envelope theorem through (1) above:

$$\text{Shephard's Lemma} \qquad \frac{\partial c}{\partial p_k}(\mathbf{p},u) = q_k = h^k(\mathbf{p},u) \tag{5}$$

Since these demands condition on prices and utility they are known as compensated (or Hicksian) demands. To derive uncompensated (or Marshallian) demands we need to substitute for u in $c(\mathbf{p},u) = x$ which gives $u = V(\mathbf{p},x)$. Substituting in (5) we have:

$$q_k = h^k(\mathbf{p},V(\mathbf{p},x)) = f^k(\mathbf{p},x) \tag{6}$$

Alternatively, we can start from the indirect utility function:

$$\text{Roy's Identity} \qquad q_k = f^k(\mathbf{p},x) = \frac{\partial V(\mathbf{p},x)}{\partial p_k} \Big/ \frac{\partial V(\mathbf{p},x)}{\partial x} \tag{7}$$

As a second illustration of the power of dual arguments (the first was the tax example given in section 2), consider the following simple derivation of the Slutsky equation (see Cook (1972) and McKenzie (1957)). Taking derivatives of both sides of $q_j = f^j(\mathbf{p},c(\mathbf{p},u))$ with respect to p_k we have:

$$\frac{\partial q_j}{\partial p_k}(\mathbf{p},u) = \frac{\partial f^j}{\partial p_k}(\mathbf{p},x) + \frac{\partial f^j}{\partial x}(\mathbf{p},x)\,\frac{\partial c}{\partial p_k}(\mathbf{p},u) = \frac{\partial q^j}{\partial p_k}(\mathbf{p},x) + \frac{\partial q^j}{\partial x}q_k \tag{8}$$

This one line derivation of the Slutsky equation compares dramatically with older derivations that use bordered Hessians; on its own it might justify the study of duality methods. Once we have the compensated price response in terms of observable responses we might go further and ask what conditions are implied by the existence of a utility function. This is simply a matter of drawing out the implications of (C1) to (C3) for the response on the right hand side of (8). These are adding up; zero homogeneity; symmetry and "negativity" (the Hessian of $c(\mathbf{p},u)$ is negative semi-definite for fixed u).

5. THE DISTANCE AND PROFIT FUNCTIONS

Although the cost and indirect utility functions are the most common alternatives to the direct representation they are by no means the only ones. In this section we introduce two other representations that have been quite widely used: the distance function and the profit function. Both of these representations had their origin in the analysis of producer behaviour but both have turned out to be useful in consumer theory as well. Amongst other places, discussions of both representations in the consumer context can be found in Gorman (1976) and Cornes (1992). For more extended discussions see Deaton (1979) and Browning, Deaton and Irish (1985) for the distance and profit functions respectively.

The distance function is not strictly a dual representation since it is defined on quantities. It can be defined using either the direct utility function or the cost function:

$$d(\mathbf{q},u) = \max_{\delta} \{\delta \mid \upsilon(\delta^{-1}\mathbf{q}) \geq u\} = \min_{\mathbf{p}} \{\mathbf{p}'\mathbf{q} \mid c(\mathbf{p},u) = 1\} \tag{9}$$

(thus, $d(\mathbf{q},.)$ is the polar of $c(\mathbf{p},.)$). Since $c(\mathbf{p},.)$ is concave in $\mathbf{p}$ we can also move back the other way:

$$c(\mathbf{p},u) = \min_{\mathbf{q}} \{\mathbf{p}'\mathbf{q} \mid d(\mathbf{q},u) = 1\} \tag{10}$$

so that the cost and distance function are very closely related. To move from the distance function to the direct utility function we simply define $\upsilon(\mathbf{q})$ implicitly by $d(\mathbf{q},\upsilon(\mathbf{q})) = 1$.

Given the close relationship between the cost and distance functions it will come as no surprise that they have similar properties (compare these properties with (C1) to (C3) above):

(D1) $d(.,u)$ is non-decreasing in $\mathbf{q}$;

(D2) $d(.,u)$ is linear homogeneous in $\mathbf{q}$;

(D3) $d(.,u)$ is concave in $\mathbf{q}$.

Indeed the 'duality' between the cost function and the distance function is so pervasive that we can adopt a slogan: if anything is true of the cost function for prices then it will generally be true of the distance function for quantities. To illustrate, suppose that the cost function has the following separability structure:

$$c(\mathbf{p}^1,\mathbf{p}^2,\ldots\mathbf{p}^T,u) = \chi(\phi^1(\mathbf{p}^1,u),\phi^2(\mathbf{p}^2,u),\ldots\phi^T(\mathbf{p}^T,u),u)$$

where $(\mathbf{p}^1,\mathbf{p}^2,\ldots\mathbf{p}^T)$ is a partition of the price vector $\mathbf{p}$. This restriction on preferences is usually known as quasi-separability; it is independent of the more familiar (weak) separability of the direct utility function. It is straightforward to show that this structure on the cost function implies a similar structure for the distance function:

$$d(\mathbf{q}^1,\mathbf{q}^2,\ldots\mathbf{q}^T,u) = \chi(\sigma^1(\mathbf{q}^1,u),\sigma^2(\mathbf{q}^2,u),\ldots\sigma^T(\mathbf{q}^T,u),u)$$

where $(\mathbf{q}^1,\mathbf{q}^2,\ldots\mathbf{q}^T)$ is the corresponding partition of quantities. Moreover, each $\phi^t(\mathbf{p},u)$ has $\sigma^t(\mathbf{q},u)$ as its associated distance function.

To illustrate the use of the distance function, consider the need for a quantity index (see Deaton (1979) or Diewert (1980) for a full discussion). In the contest of consumer theory, a quantity index is simply a function that takes two quantity vectors $\mathbf{q}^0$ (the base year quantity vector) and $\mathbf{q}^1$ (the current year quantity vector) and returns a value that tells us how much 'real consumption' has risen. There are many possible candidates. For example, one candidate would be $\upsilon(\mathbf{q}^1)/\upsilon(\mathbf{q}^0)$. Although this index has the virtue that the index is greater than one if the bundle $\mathbf{q}^1$ is preferred to the bundle $\mathbf{q}^0$ it suffers from a number of flaws. The most important of these is that unless $\upsilon(.)$ is linear homogeneous it fails the *proportionality* test: if $\mathbf{q}^1 = k\mathbf{q}^0$ for some scalar k then the index should equal k. This suggests the following index:

$$Q(\mathbf{q}^0,\mathbf{q}^1,u) = \frac{d(\mathbf{q}^1,u)}{d(\mathbf{q}^0,u)} \qquad (11)$$

This is known as the Malmquist quantity index.

Although the Malmquist index has many attractive features it does have the drawback that it depends on a reference level of utility. Clearly this will not be the case if and only if $d(\mathbf{q},u) = \varphi(\mathbf{q})u$. This in turn implies that preferences are homothetic (in which case all sensible definitions of quantity indices coincide). Generally, then, the Malmquist quantity index depends on the reference level of utility. Two obvious candidates are $u = \upsilon(\mathbf{q}^0)$ and $u = \upsilon(\mathbf{q}^1)$ which give the Laspeyres-Malmquist and Paasche-Malmquist quantity indices respectively.

The other representation we present in this section is the profit function[8]. This function is familiar from producer theory:

$$\pi(\mathbf{p},r) = \max_{\mathbf{q}} \{r\upsilon(\mathbf{q}) - \mathbf{p}'\mathbf{q}\} = \max_{u} \{ru - c(\mathbf{p},u)\} \tag{12}$$

Thus the profit function gives the 'profit' if the agent faces 'input' prices $\mathbf{p}$ and 'sells' utility at a price r[9]. The interpretation of the parameter r follows directly from the first order conditions of either of the maximisation problems in (12):

$$r = \frac{p_i}{\upsilon_i(\mathbf{q})} \text{ (for all i)} = c_u(\mathbf{p},u) \tag{13}$$

Thus r is the inverse of the marginal utility of money (the Lagrange multiplier in the usual direct maximisation problem) or the marginal cost (or price) of utility.

The 'price' of utility defined in (13) depends on the normalisation of the direct utility function so that the profit function representation also depends on the normalisation. This is in contrast to other dual representations. Indeed, we require that the direct utility function in (12) be strictly concave to ensure that the profit function is useful. Fairly obviously if $\upsilon(.)$ exhibits 'increasing returns' then $\pi(.)$ can only take on values $\{0,+\infty\}$ which is not very useful. Since not all convex preferences that are representable by a utility function admit of a concave representation this restricts the domain of this dual representation somewhat. On the other hand, not too much should be made of this since weak conditions do imply that we can find a concave representation (particularly if the direct representation has an additive direct representation).

The profit function is convex and linear homogeneous in $(\mathbf{p},r)$ and

8 Actually, for reasons that will become clear below it would be easier to work with the negative of the profit function (the 'net loss' function) but the use of the profit function is too well entrenched to be worth trying to alter the usual approach.

9 In the language of convex analysis minus the profit function with r = 1 is the (concave) conjugate of the direct representation. Thus the distance and profit functions represent the two most widely used 'duals' of the direct utility function in convex analysis. Note as well that the profit function is the (convex) conjugate of the cost function on u.

increasing in r and decreasing in $\mathbf{p}$. Just as we can derive Hicksian (or constant utility compensated) demands as derivatives of the cost function, so we can derive Frisch (or constant marginal utility of money) demand functions as (the negative of) the partials of the profit function with respect to prices. Taking the envelope theorem through (12) we have:

$$-\frac{\partial \pi}{\partial p_k}(\mathbf{p},r) = q_k = \xi^k(\mathbf{p},r) \qquad (14)$$

Thus it is easy to derive Frisch demands from the profit function. Why, though, should we interested in Frisch demands? The answer is that they arise very naturally in the study of additive preferences.

The profit function is useful because it 'inherits' additivity from the direct representation. That is, if the direct utility function is additive then the profit function will also be additive. This inheritance of structure from the direct representation is by no means automatic. Although, for example, separability of the direct utility function implies restrictions on the cost function, it does not imply that the cost function is itself separable.

The best illustration of the usefulness of the profit function is in the analysis of intertemporal allocation where we often assume (intertemporal) additivity. Let $\mathbf{q} = (\mathbf{q}^1,\mathbf{q}^2,\ldots\mathbf{q}^T)$ be a vector of n-vectors of quantities in periods $t = 1,2,\ldots T$. Let $\mathbf{p} = (\mathbf{p}^1,\mathbf{p}^2,\ldots\mathbf{p}^T)$ be the corresponding vector of vectors of discounted prices (where all prices are discounted to period 1 using a fixed nominal interest rate). We have:

$$\upsilon(\mathbf{q}) = \Sigma_t \psi^t(\mathbf{q}^t) \quad \text{if and only if} \quad \pi(\mathbf{p},r) = \Sigma_t \lambda^t(\mathbf{p}^t,r) \qquad (15)$$

Thus the additivity of the direct utility function is equivalent to the additivity of the profit function in prices. Moreover we have:

$$\lambda^t(\mathbf{p},r) = \max_{\mathbf{q}} \{r\psi^t(\mathbf{q}) - \mathbf{p}'\mathbf{q}\} \qquad (16)$$

so that the sub-profit function is period t is dual to the sub-utility function in t. Since the normalisation of the latter is given by the additivity (up to an affine transformation) this makes the dependence of the profit function on the normalisation undisturbing.

The importance of Frisch demands in intertemporal analysis has been recognised for some time (see, for example, Heckman (1974)). Primarily this is because if we take the Frisch demands for good k in period t:

$$q_k^t = \xi^{kt}(\mathbf{p}^t, r) = -\frac{\partial \lambda^t}{\partial p_k}(\mathbf{p}^t, r) \tag{17}$$

then the variable r is a 'sufficient statistic' for all extra-period information (see MaCurdy (1981)). Thus we do not need to know much about what has happened in the past or what the agent believes about the future to predict demands. This enormous parsimony in information that the econometrician needs is, however, bought at a price since we need to deal with the unobservable marginal cost of utility.

The use of the profit function as a potential function for Frisch demands was introduced in Browning, Deaton and Irish (1985). As an example of the use to which we can put this, consider the econometric requirements to estimate the parameters of $\xi^{kt}(.)$ in (17). To overcome the unobservability of the marginal costs of utility we need to parameterise $\xi^{kt}(\mathbf{p}, r)$ in such a way that we can 'difference' away r. That is, we need to write $\xi^{kt}(\mathbf{p}, r)$ as an additive function of $\mathbf{p}$ and r (where we drop the t superscript for convenience): $\xi^{k}(\mathbf{p}, r) = \zeta^{k}(\mathbf{p}) + \tau_k(r)$. In Browning *et al* (1985) it is shown that this additive-in-r form implies that the within period profit function takes the form:

$$\lambda(\mathbf{p}, r) = \alpha r + \phi(\mathbf{p}) + \Sigma_k\ \mu_k p_k \ln\left(\frac{p_k}{r}\right) \tag{18}$$

Once we have the characterisation of preferences that give additive-in-r Frisch functions we can then determine exactly what extra restrictions are imposed on preferences by this assumption. As it turns out, (18) is quite restrictive.

For the purposes of this survey the details of the derivations of the additive-in-r Frisch demands above are unimportant. Two general features are, however, of critical importance. The first of these is that it would be impossible to characterise the preferences that give rise to additive-in-r Frisch demands without using the profit function representation. It is impossible to go 'backwards' through the derivation of Frisch demands from the direct representation (that is, invert on the solution of non-linear equations and differentiation) to the direct utility function. The second general remark is that once we have gone back to the original preferences then we have to 'go forward' to the demands once again to see what additional restrictions the additive-in-r assumption imposes. These are by no means obvious from just looking at the demands. Indeed, without being able to go back to some representation of preferences it is doubtful whether we could *fully* determine exactly what extra restrictions are involved.

6. MIXED REPRESENTATIONS

Up until now we have considered only representations that depend just on prices or quantities (and some measure of welfare). In many contexts, however, it is useful to have representations that depend on the prices of some goods and the quantities of others. This goes back to the remarks by Gorman quoted at the beginning of this survey to the effect that we want to set up the analysis in terms of variables that are exogenous to the agent being considered. To illustrate, suppose that an agent can buy private goods $\mathbf{q}$ at market prices $\mathbf{p}$ and also receives publicly provided (private or public) goods $\mathbf{z}$, where the goods in $\mathbf{z}$ and $\mathbf{q}$ do not overlap (that is, the agent cannot buy any goods that the government provides). Let the direct utility function be $\upsilon(\mathbf{q},\mathbf{z})$. Now define the conditional (or restricted) cost function:

$$c^{*}(\mathbf{p},\mathbf{z},u) = \min_{\mathbf{q}} \{\mathbf{p}'\mathbf{q} \mid \upsilon(\mathbf{q},\mathbf{z}) \geq u\}. \tag{19}$$

This representation has the properties of a conventional cost function in $(\mathbf{p},u)$ and looks like (the negative of) a utility function in $\mathbf{z}$.

Although the inclusion of $\mathbf{z}$ above was motivated by the example of a publicly provided good, $\mathbf{z}$ can be a vector of any conditioning variables that affect preferences. One particularly important case is that for which $\mathbf{q}$ is a basket of market goods and $\mathbf{z}$ includes the labour supplies of the members of the household; if labour supply is constrained in some way then labour supply is the exogenous variable rather than the wage[10]. Alternatively **q** could be market goods that are adjustable in the short run whilst **z** might be market goods that can only be changed in the long run. A wider interpretation for **z** is also possible; for example, **z** could include the number of children in the household and the number of bathrooms in the dwelling the household occupies. In the case where **z** are goods in the usual sense[11] we can put a good deal more structure on this framework. Let **r** be a vector with the same dimension as z. If the **z** goods are available in the market at prices **r** then we can define the rationed cost function:

$$\tilde{c}^*(\mathbf{p},\mathbf{z},\mathbf{r},u) = c^*(\mathbf{p},\mathbf{z},u) + \mathbf{r}'\mathbf{z} \tag{20}$$

Thus the rationed[12] cost function is the conditional cost function plus the cost of the vector **z** at prices **r**.

The conditional cost function is not, of course, the only mixed representation that we can define. In some contexts the conditional indirect utility function or the conditional profit function (or even less common representations based on the prices of some goods and the quantities of others) makes the analysis more tractable. For example, it may be that once we condition on the stock of semi-durables, durables and housing then intertemporal preferences can be represented by an additive-over-time utility function on other goods (non-durables and services). If this is the case then the conditional profit function is the obvious representation to use.

Although we defined the conditional cost function in (19) in terms of the direct utility function, we could have started from the cost function (see Browning (1983) for all the pedantic details):

10 The use of the term exogenous here is meant to indicate that the agent ta labour supply as given. This does not necessarily imply that labour supply exogenous in the econometric sense (see Browning and Meghir (1991) for a discussion).

11 and many investigators would include leisure and children in this definition.

12 There is not much consensus in the literature about what to call these different representations. I prefer the terminology 'conditional' for $c^*(.)$ in (19) since **z** can be any variable that affects preferences and 'rationed' for this representation since it captures the idea that agents have to consume **z** and pay **r'z** for this bundle of goods.

$$c^*(\mathbf{p},\mathbf{z},u) = \max_{\mathbf{r}} \{c(\mathbf{p},\mathbf{r},u) - \mathbf{r}'\mathbf{z}\} \tag{21}$$

(so that $c^*(.)$ and $c(.)$ are (concave) conjugates on $\mathbf{r}/\mathbf{z}$). This implies that starting from the conditional cost function we can derive the (unconditional) cost function:

$$c(\mathbf{p},\mathbf{r},u) = \min_{\mathbf{z}} \{c^*(\mathbf{p},\mathbf{z},u) + \mathbf{r}'\mathbf{z}\} = \min_{\mathbf{z}} \tilde{c}^*(\mathbf{p},\mathbf{z},\mathbf{r},u) \tag{21'}$$

(note the change in sign in the first maximisation problem). The first order conditions for the first optimisation problem in (21') yield the important concept of virtual prices:

$$\hat{\mathbf{r}}(\mathbf{p},\mathbf{z},u) = -\nabla_{\mathbf{z}} c^*(\mathbf{p},\mathbf{z},u)$$

(here $\nabla_{\mathbf{z}}$ denotes the gradient vector with respect to z). The prices $\hat{\mathbf{r}}$ are the prices which would induce the agent to purchase $\mathbf{z}$ if she faces prices $\mathbf{p}$ for the other goods and has utility level u.

The use of mixed representations has turned out be fruitful in a wide range of applications; see Cornes (1992), chapter 7 for an insightful discussion[13] and further references. Here I shall only give a simple application to illustrate the use of mixed representations. Consider the policy of issuing food stamps to agents who are designated as being in straitened circumstances. Let r be the (competitive market) price of food and let $r\bar{z}$ be the value of the food stamp; that is, the agent can obtain the amount $\bar{z}$ of food at zero price. Any food purchased above the amount $\bar{z}$ must be bought at the market price r. We are interested in the following question: what level of extra income is equivalent to the food stamp for any agent.

Let $\mathbf{p}$ be the price of other market goods (and assume that there are no other conditioning goods) and let x be the income of the consumer net of the food stamp. Let u_0 be the utility level of the agent with no food stamp (= $V(\mathbf{p},r,x)$) and let u_1 be the utility level of an agent who receives the food stamp. It is clear that if the agent consumes at least $\bar{z}$ of food at utility level u_1 then the value of the food stamp to the agent is $r\bar{z}$ (= $c(\mathbf{p},r,u_1)-x$).

[13] Including a discussion of the analysis of allocation under uncertainty whic has been a relatively barren area for duality theory up until now.

Almost as obviously, the value to an agent who sets $z = \bar{z}$ is $c^*(\mathbf{p},\bar{z},u_1)-x$. A necessary and sufficient condition for $z = \bar{z}$ is that the virtual price $\hat{r}(\mathbf{p},\bar{z},u)$ is at least as great as the market price r. This is all rather trivial; the important point is that we can find all the values of interest from a knowledge of preferences. The simplest way to think of doing this is to first estimate the parameters of $c(\mathbf{p},z,u)$ on a sample of agents who do not receive food stamps. From this we can derive $c^*(\mathbf{p},r,u)$ and $\hat{r}(\mathbf{p},\bar{z},u)$ and consequently find the values we are interested in. This procedure promises an exact answer to the question posed at a modest informational cost; no approximations of any sort are needed.

7. CONCLUDING REMARKS

I don't have too much to say in conclusion. The primary 'message' of this survey is that there are lots of different ways of describing preferences and some are more suited to some substantive areas of interest than others. Consequently it is almost always worthwhile spending some time at the beginning of any analysis in thinking about how to think about the problem at hand. This is all very worthy, but of little practical use unless we have some alternative ways of thinking about particular problems. The methods outlined in this survey are the most common alternative methods in consumer theory. As such they are useful items for any microeconomists to have in her or his tool kit.

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