

COURNOT OLIGOPOLY WITH "ALMOST" IDENTICAL CONVEX COSTS*

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A B S T R A C T

M. McManus has shown that a Cournot equilibrium among identical quantity-setting firms with a convex cost function always exists, whatever the inverse demand function. Here the result is extended in two directions: Firstly, the firms may differ in their production capacity. Secondly, the theorem remains valid for a discrete version of the Cournot model where all the outputs are integers.

EN BLANCO

1. INTRODUCTION

The Cournot oligopoly model provides the subject for an extremely wide variety of research. This paper considers the model from a rather trivial standpoint: we shall chiefly be concerned with conditions for the existence of a Cournot equilibrium.

The commonest way to prove the existence of an equilibrium is to make such assumptions that the profit functions be concave (so Kakutani's theorem can be applied). Many textbooks simply assume the linearity of the inverse demand function over the interval where it is positive and the convexity of all cost functions. That approach has two drawbacks. From the economical viewpoint, it is not so easy to support the assumption on the (aggregate) demand function with plausible micro-models of the consumers' behaviour and income structure (and some plausible consumption modes certainly lead to non-concave demand). From the purely aesthetical viewpoint, the assumptions are grossly superfluous: one can choose between quite different tools to prove the existence of an equilibrium under them.

If the demand function is concave, then each firm's incremental revenue is decreasing in the total output of the competitors, so the best replies are non-increasing and the existence of an equilibrium can be

proved regardless of the cost functions (Novshek, 1985). For the duopoly case this fact directly follows from Tarski's theorem (Vives, 1990); therefore, an equilibrium exists even if one or both of the firms have (for some reasons) to choose from finite sets of feasible outputs. Whether Tarski's theorem could be applied to Novshek's general case remains unclear; also unclear are the prospects for deriving the conditions from a plausible model of the consumption sector.

An important result, truly based on Kakutani's theorem, was recently obtained by Grandmont (1992), who showed that a version of the heterogeneity assumption on the demand sector implies the concavity of each firm's revenue in its output (without Novshek's conditions being satisfied); in fact, he has also obtained the uniqueness of the equilibrium, but this topic is not to be discussed here.

An alternative method for proving the existence of an equilibrium, with a narrower but non-empty scope of applications, was suggested in Kukushkin (1993). For the Cournot model, this ("Lyapunov's") approach gives a simple algebraic proof of the existence of a Cournot equilibrium for any inverse demand function and any upper restrictions on the production capacities if all the cost functions are identical and linear, a bit strengthening the results of McManus (1962), and Roberts and Sonnenschein

(1977). Moreover, this result, like Novshek's one for duopoly, survives arbitrary discretization of the strategy sets, although for different reasons.

The present paper is mostly connected with another result of McManus (1964): the existence of a Cournot equilibrium for any demand in the case of identical firms with convex costs. We relax the identity assumption, allowing the firms to differ in size (but only in size), while retaining the convexity assumption. An equilibrium still exists for any (reasonable) demand. The result is derived from Tarski's theorem, so it survives the discretization of the sets of feasible outputs but the discretization must be the same for all the firms and "uniform".

The application of Tarski's theorem to proving the existence of an equilibrium, started by Topkis (1979), is usually restricted to "super-modular" games - or games with strategic complementarities - having many nice properties besides the sheer equilibrium existence (Milgrom and Roberts, 1990; Vives, 1990; Milgrom and Shannon, 1992). The present result may be the first example of Tarski's theorem working for a game outside the class. (Historically, it might be the first without qualifications as the proofs below just "purify" McManus's reasoning.) Interestingly, Tarski's

theorem is applied in an indirect way, not to the best reply correspondence.

I would like to argue that the identity of technology throughout an industry is a much more plausible assumption than the identity of all the firms. The existence of an equilibrium being secured, one may study e.g. how the equilibrium total output(s) depend on the characteristics of the demand sector and technology, on the number and sizes of the firms involved, etc. As is shown below, the dependencies in the domain under consideration agree with economic intuition. Thus the present results may be of some use to future textbooks' authors.

The paper is organized as follows. The next section contains necessary general definitions and notations. We consider Cournot models with possible finite upper restrictions on the outputs - equivalently, one may speak of infinite costs; the principal assumption is that all cost functions are convex, and any two of them coincide wherever both are finite.

Section 3 contains basic lemmas and the main existence theorem for McManus models. Section 4, some comparative statics results; in particular, it is shown that the entry of a new firm never leads to a decrease in equilibrium total output(s). As incremental costs are assumed to be

increasing, there is nothing surprising in the fact (cf. e.g. Szidarovszky and Yakowitz, 1982, Theorem 4); an important feature of the present result, however, is its independence of any assumptions on the inverse demand function.

In Section 5 a discrete version of the model is considered: all the outputs must be integers. Obviously, neither Kakutani's theorem nor any reasoning about jumps up or down is of any help now (any function on a discrete set is continuous by definition). Nonetheless, the main results of the previous sections remain valid; moreover, the proof modifications needed are not too drastic.

Sections 6 and 7 contain several negative results: generally speaking, neither symmetry without convexity, nor convexity (even linearity) without symmetry guarantees the existence of a Cournot equilibrium. Unfortunately, the results are insufficient to assert that no other list of cost functions could guarantee the existence of a Cournot equilibrium for any demand, however plausible the statement might seem. A difference in robustness w.r.t. discretization is shown between general convex and linear costs.

The final section summarizes the principal message of the paper and outlines some important open problems.

2.- GENERAL FORMULATION AND BASIC DEFINITIONS

A Cournot oligopoly model (with quantity-setting firms producing a homogeneous good) is described by a finite set N of firms, a non-negative cost function $C_i(\cdot)$ and an upper bound K_i ($0 \leq K_i \leq +\infty$) on the output for each firm $i \in N$, and a non-negative inverse demand function $P(\cdot)$ expressing the price on the market where the goods produced are to be sold as a function of the total supply.

The model defines a strategic game with strategy sets $X_i = [0, K_i]$ (if $K_i < +\infty$) or $X_i = [0, +\infty[$ (if $K_i = +\infty$), and utility functions

$$u_i(x) = P(\sum_{j \in N} x_j) \cdot x_i - C_i(x_i)$$

defined on $X = \times_{i \in N} X_i$. A Nash equilibrium of the game is called a Cournot equilibrium of the oligopoly model.

We will call an inverse demand function $P(\cdot)$ regular if it satisfies the following three assumptions:

- (i) $P(\cdot)$ is non-increasing;
- (ii) $P(\cdot)$ is upper semi-continuous;
- (iii) there exists $t_\infty < +\infty$ for which $P(t_\infty) = 0$.

Assumption (iii) is by no means necessary: it is just the simplest way to guarantee that no firm (with $K_i = +\infty$) will ever wish to produce infinite output and that no best reply goes to infinity. On the other hand, it becomes quite natural if we admit that no market can be completely frictionless, so even if the seller's price is zero, the buyer cannot afford to buy infinite volume.

A cost function $C_i(\cdot)$ is called regular if it satisfies the following assumptions:

- (i) $C_i(\cdot)$ is non-decreasing;
- (ii) $C_i(\cdot)$ is lower semi-continuous;
- (iii) $C_i(0) = 0$.

In the following (except for a couple of counter-examples in Section 6) we shall restrict ourselves to models where any two cost functions coincide wherever both are finite, which we will call McManus models in recognition of his pioneering contribution to the domain, although he himself considered a much narrower class of models.

More technically, a McManus model is defined by a regular inverse demand function $P(\cdot)$, a regular cost function $C(\cdot)$, a (finite) set of firms N , and a production capacity $K \in \mathbb{R}_+ \cup \{+\infty\}$ for each firm $i \in N$. Each firm's effective cost function is assumed to be $C_i(x) = C(x)$ for $x \leq K_i$, $C_i(x) = +\infty$ for

$x > K_i$. We will additionally assume that $C(x) < +\infty$ for some $x > 0$ (otherwise, no firm would be able to produce anything; "dummy" firms with $K_i = 0$ are allowed nonetheless).

To analyze equilibria of the model, consider the best replies of one firm to a fixed total output of the others. For any $s \geq 0$, $K \in \mathbb{R}_+ \cup \{+\infty\}$, denote

$$R(s, K) = \text{Arg max}_{0 \leq x \leq K} P(s + x) \cdot x - C(x)$$

(the correspondence $R(\cdot, K)$ is upper semi-continuous for any K), $r(s, K) = \min R(s, K)$. Obviously, $r(\cdot, \cdot)$ is lower semi-continuous in both arguments and non-decreasing in the second one; $r(t_\infty, K) = 0$ for any K . It is worth noting that the correspondence $R(\cdot, \cdot)$ and the function $r(\cdot, \cdot)$ do not depend on the collection $\langle K_i \rangle_{i \in N}$ of a particular model.

Define $K^+ = \max_{i \in N} K_i$, $t^+ = \sum_{i \in N} K_i$, if $t^+ < +\infty$, then $T = [0, t^+]$, otherwise, $T = [0, +\infty[$; thus T is the set of feasible total outputs.

We will call an output vector $\langle x_i \rangle_{i \in N}$ relatively symmetric if $x_i < x_j$ implies $x_i = K_i$ for each $i, j \in N$ (i.e. only asymmetries attributable to inequalities in production capacities are allowed). If all K_i are the same, a relatively symmetric output vector is just symmetric.

For each feasible total output $t \in T$, there is a unique relatively symmetric output vector $\langle x_i^o(t) \rangle_{i \in N}$ satisfying $\sum_{i \in N} x_i^o(t) = t$. To define it formally, we denote, for each $x \in [0, K^+]$,

$$N(x) = \{i \in N \mid K_i \leq x\}, \quad n(x) = |N \setminus N(x)|,$$

$$\tau(x) = \sum_{i \in N(x)} K_i + n(x) \cdot x.$$

The function $\tau(x)$ is continuous and increasing on $[0, K^+]$, $\tau(K^+) = t^+$; therefore, there exists its inverse $\xi(t)$, continuous and increasing on $[0, t^+]$. It can easily be verified that the function $\sigma(t) = t - \xi(t)$ is at least non-decreasing on T . Moreover, all the functions $\tau(\cdot)$, $\xi(\cdot)$ and $\sigma(\cdot)$ are piecemeal linear, $\tau(\cdot)$ and $\sigma(\cdot)$ are concave, $\xi(\cdot)$ convex. Now we can define $x_i^o(t) = \min \{\xi(t), K_i\}$.

Remark. If there are n completely identical firms, then $\xi(t) = t/n$, $\sigma(t) = (n-1)t/n$, $x_i^o(t) = t/n$ for each $t \in T$, $i \in N$.

3. - MAIN RESULTS

First of all, let us prove a simple technical statement (more precisely, two related statements) based on the convexity of the cost function, which plays an important part in the following.

Lemma 1.- If $x^1 > x^2 > 0$, $0 < \Delta \leq x^2$, $0 < p_1 \leq p_2$ and

$$p_1 \cdot x^1 - C(x^1) \geq p_2 \cdot x^2 - C(x^2), \quad (1)$$

then

$$p_1 \cdot (x^1 - \Delta) - C(x^1 - \Delta) \geq p_2 \cdot (x^2 - \Delta) - C(x^2 - \Delta). \quad (2)$$

Moreover, if (1) is strict, then so is (2).

Indeed, the difference between the left-hand sides of (2) and (1) is $-p_1 \cdot \Delta + C(x^1) - C(x^1 - \Delta)$; for the right-hand sides, the difference is $-p_2 \cdot \Delta + C(x^2) - C(x^2 - \Delta)$. As $p_1 \leq p_2$, $x^1 > x^2$, and $C(\cdot)$ is convex, the first difference is bigger (not less, to be more precise), so (1) implies (2).

Lemma 1 helps to establish some important properties of the correspondence $R(\cdot, \cdot)$, in particular, connections between the functions $r(\cdot, K)$ for different K 's.

Lemma 2.—If $x=r(s,K)$, then, for any $s' \geq s$ and $K' \geq s+x-s'$,
 $r(s',K') \geq s+x-s'$.

Indeed, for any $s' \geq s$ and $y \in [0, s+x-s']$, denote $\Delta = s' - s > 0$, $x^1 = x$, $x^2 = y + \Delta < x^1$, $p_1 = P(s+x^1)$, $p_2 = P(s'+y) = P(s+x^2) \leq p_1$. Now we have the strict version of (1); Lemma 1 implies the strict version of (2); therefore, $x^1 - \Delta$ as a reply to s' is better than y , so $y = r(s', K')$ is impossible and we have the inequality required.

Lemma 3.—The function $s + r(s, K)$ is non-decreasing in s on $[0, +\infty[$ for any $K \in [0, +\infty]$.

Follows directly from Lemma 2 ($K=K'$).

Lemma 4.—If $x^1 \in R(s^1, K)$ for some $K \geq 0$, then for any $s^2 > s^1$ and $K' \geq x^1 + s^1 - s^2$ either $x^1 + s^1 - s^2 \in R(s^2, K')$ or $x^1 + s^1 - s^2 < r(s^2, K')$.

Suppose $x^1 + s^1 - s^2 > r(s^2)$ and denote $\Delta = s^2 - s^1 > 0$, $x^2 = r(s^2) + \Delta$, $p_i = P(s^i + x^i)$ ($i=1,2$); then we have $x^1 > x^2$, $p_1 \leq p_2$, $p_1 = P(s^2 + x^1 - \Delta)$, $p_2 = P(s^2 + r(s^2))$. Now the optimality of x^1 for s^1 implies Inequality (1), hence, by Lemma 1, Inequality (2) is true, and this means exactly $x^1 - \Delta \in R(s^2)$.

The four lemmas do not give an exhaustive characterization of possible best reply correspondences in McManus models; they could be strengthened considerably, especially if we assumed positive costs. However, they are sufficient for all further results.

For any $t \in T$ denote $E(t) \subseteq \times_{i \in N} X_i$ the set of equilibria with the total output t ; each set $E(t)$ is closed.

Theorem 1.- For any $t \in T$, $E(t)$ is convex.

If $E(t) = \emptyset$, it is convex; suppose $E(t) \neq \emptyset$, and for each $i \in N$ denote x_i^- and x_i^+ , respectively, the lowest and the highest output x_i of firm i over all the equilibria from $E(t)$. So $x_i^- \in R(t - x_i^-, K_i)$, $x_i^+ \in R(t - x_i^+, K_i)$; let us show that, for any $x_0 \in [x_i^-, x_i^+]$, $x_0 \in R(t - x_0, K_i)$.

Suppose $x_i^+ > x_i^-$: there is nothing to prove otherwise. We may apply Lemma 4 with $x^1 = x_i^+$, $s^1 = t - x_i^+$, $s^2 = t - x_0$ (so $x^1 + s^1 - s^2 = x_0$), $K = K' = K_i$. If $x_0 \notin R(t - x_0, K_i)$, then $x_0 < r(t - x_0, K_i)$. Applying Lemma 3 for $K = K_i$, we obtain $x_i^- < r(t - x_i^-, K_i)$ - a contradiction.

Now if $\langle x_i \rangle_{i \in N}$ is a convex combination of two equilibria from $E(t)$, then each x_i satisfies $x_i^- \leq x_i \leq x_i^+$, so the equilibrium condition is satisfied.

Corollary.- $E(t) \neq \emptyset$ if and only if $\langle x_i^o(t) \rangle_{i \in N} \in E(t)$.

Easily follows from the relative symmetry of $E(t)$ itself.

The McManus correspondence $M(t)$ is defined on T as follows:

$$M(t) = \sigma(t) + R(\sigma(t), K^+).$$

Theorem 2.- For any $t \in T$, $E(t) \neq \emptyset$ if and only if $t \in M(t)$.

1. Suppose $t \in M(t)$ and show that $\langle x_i^o(t) \rangle_{i \in N}$ is an equilibrium. Indeed, we have $\xi(t) \in R(\sigma(t), K^+)$, so for any $i \in N$ satisfying $K_i \geq \xi(t)$ (i.e. $x_i^o(t) = \xi(t)$) the equilibrium condition is satisfied as $\sum_{j \in N \setminus \{i\}} x_j^o(t) = \sigma(t)$. On the other hand, for $K_i < \xi(t)$ the total output of the others is $t - K_i$; now we may apply Lemma 4 with $K' = K^+$, $K = K_i$, $x^1 = \xi(t)$, $s^1 = \sigma(t)$, $x = \xi(t)$, $s^2 = t - K_i$ (so $x^1 + s^1 - s^2 = K_i$), obtaining $K_i \in R(t - K_i, K_i)$ unless $K_i < r(t - K_i, K_i)$, which is impossible.

2. If $E(t) \neq \emptyset$, then, by Corollary to Theorem 1, $\langle x_i^o(t) \rangle_{i \in N} \in E(t)$, so $\xi(t) \in R(\sigma(t), K^+)$ and $t \in M(t)$.

Corollary.- The set $\{t \in T \mid E(t) \neq \emptyset\}$ is closed.

Easily follows from the previous corollary, continuity of $\xi(t)$ and $\sigma(t)$, and upper semi-continuity of $R(\cdot, \cdot)$.

Theorem 3.—If $t_0 \geq \sigma(t_0) + r(\sigma(t_0), K^+)$, then there exists $t \in [0, t_0]$ for which $t \in M(t)$.

Define $f(t) = \sigma(t) + r(\sigma(t), K^+)$; according to Lemma 3, $f(\cdot)$ is monotonic; by the assumption of the theorem, $f([0, t_0]) \subseteq [0, t_0]$. Now by Tarski's theorem (Tarski, 1955, Theorem 1), f has a fixed point $t = f(t)$; obviously, $t \in M(t)$.

Corollary.— Every McManus model has a relatively symmetric Cournot equilibrium.

We may choose $t_0 = \min \{t_\omega, t^+\}$, and apply Theorem 3 (followed by Theorem 2 and Corollary to Theorem 1).

Remark.— Vives (1990, p.317) suggested another way for deriving this statement (for identical firms) from Tarski's results. However, his approach heavily relies on the continuity of the strategy sets, while ours can easily be modified for a discrete version of the model. This will be accomplished in Section 5.

Theorem 4.—If $t_0 \leq \sigma(t_0) + x$ for some $t_0 \in T$, $x \in R(\sigma(t_0), K^+)$, then there exists $t \in T \cap [t_0, +\infty[$ for which $t \in M(t)$.

Define $R^*(s) = R(s, K^+) \cap [t_0 - s, +\infty[$; according to Lemma 4, $R^*(s) \neq \emptyset$ for $s \geq \sigma(t_0)$. Now we may define $r^*(s) = \min R^*(s)$ for $s \geq \sigma(t_0)$, and $f^*(t) = \sigma(t) + r^*(\sigma(t))$ for $t \geq t_0$. The function $f^*(\cdot)$ is non-decreasing (the analogue of Lemma 3 for $r^*(\cdot)$ is valid as well); choose $t^\infty = \max\{t_\infty, t_0\}$ if $t^+ = +\infty$ or $t^\infty = t^+$ otherwise, then $f^*([t_0, t^\infty]) \subseteq [t_0, t^\infty]$, so Tarski's theorem can be applied.

4.- SOME COMPARATIVE STATICS

The results of this section should be interpreted from the viewpoint of the consumers' welfare: we will describe several kinds of changes in the production sector which are always beneficial for the consumers. This study is by no means exhaustive and the producers' viewpoint is not represented at all.

In fact, all consumers are interested in a single characteristic: the equilibrium output (which determines the price). However, as we have no general uniqueness theorem, we have to be able to compare models with multiple equilibria. For two subsets A and B of the real line, we say that A dominates B if for each $a \in A$ there exists $b \in B$ such that $a \geq b$ and for each $b \in B$ there exists $a \in A$ such that $b \leq a$. Note that this does not imply $a \geq b$ for all $a \in A$ and $b \in B$. In fact, A dominates B if and only if $\inf A \geq \inf B$ and $\sup A \geq \sup B$.

Theorem 2 shows that all the information necessary to describe the possible equilibrium total outputs is given by the McManus correspondence. Having two McManus models, we say that the first model dominates the second one if

(1) the domain T of the McManus correspondence of the first model contains that of the second one, and

(2) the (set) value of the first correspondence dominates that of the second one at every point of the latter's domain.

Theorem 5. If one McManus model dominates another, then the set of equilibrium total outputs of the first model dominates that of the other.

If t is an equilibrium total output of the first model, then it satisfies the condition of Theorem 4 for the second model; if t is an equilibrium total output of the second model, then it satisfies the condition of Theorem 3 for the first one.

It is essential to have in mind that $\sigma(\cdot)$ only depends on the number and sizes of the firms involved, while $R(\cdot, \cdot)$ depends on the consumption sector and on the technology of production. If two McManus models have the same number of firms, the same list of K_i 's, and the same demand function, and, besides, the difference between their cost functions is non-negative and non-decreasing, then the model with lesser costs obviously dominates the other. (However, an arbitrary decrease in costs may well cause a decrease in the equilibrium total output.)

An important motive for McManus (1962, 1964) was the desire to describe how equilibrium total outputs react to changes in the composition of the production sector. Although he did not explicitly use the above concept of a set dominating another set, his results may most naturally be expressed in those terms. Theorem 5 reduces the problem to finding conditions for one model to dominate another with the same demand and cost functions, and allows us to strengthen McManus's results.

Lemma 5.—For any regular functions $P(\cdot)$ and $C(\cdot)$, any $s' \geq s \geq 0$, $K \geq 0$, and any $K' \geq K + s - s'$, the set $s' + R(s', K')$ dominates $s + R(s, K)$.

Immediately follows from Lemmas 2 and 4.

Theorem 6.—If there are two McManus models with the same set N , inverse demand function $P(\cdot)$ and cost function $C(\cdot)$, and if all K_j for $j \neq i$ are the same in both models while K_i in the first model is bigger, then the first model dominates the second one.

Let $K_i^{(1)} = K_i^{(2)} + \Delta$ with $\Delta > 0$; compare the functions $\tau^{(1)}$ and $\tau^{(2)}$. Obviously, $\tau^{(1)}(x) = \tau^{(2)}(x)$ for $x < K_i^{(2)}$; for $K_i^{(2)} \leq x < K_i^{(1)}$, $n^{(1)}(x) = n^{(2)}(x) + 1$, so $\tau^{(1)}(x) > \tau^{(2)}(x)$; for $x \geq K_i^{(1)}$, $N^{(1)}(x) = N^{(2)}(x)$ and $n^{(1)}(x) = n^{(2)}(x)$ but $\tau^{(1)}(x) = \tau^{(2)}(x) + \Delta$. Therefore, $\xi^{(1)}(t) \leq \xi^{(2)}(t)$, hence $\sigma^{(1)}(x) \geq \sigma^{(2)}(x)$. A reference to Lemma 5 accomplishes the proof.

Corollary.- If a new firm enters the market, then the new set of equilibrium total outputs dominates the old one.

We may assume that the new firm was present in the old model but with $K_i = 0$. (It is essential to stress that we do not consider the possibility of incumbent firms fighting the entrant.)

Remark.- Without the convexity of the costs, this statement cannot be universally true even if the demand function is linear (McManus, 1964, Fig. 3; Corchón, 1993, Example 2).

Theorem 7.- If there are two McManus models with the same set N , inverse demand function $P(\cdot)$ and cost function $C(\cdot)$, and if $K_j^{(1)} - K_i^{(1)} \geq 2\Delta > 0$ (for some $j, i \in N$) while $K_i^{(2)} = K_i^{(1)} + \Delta$, $K_j^{(2)} = K_j^{(1)} - \Delta$, and $K_k^{(2)} = K_k^{(1)}$ for all other $k \in N$, then the second model dominates the first one. (More equal redistribution of the existing production capacities among the firms is always beneficial to the consumers.)

As in the previous proof, compare the functions $\tau^{(1)}$ and $\tau^{(2)}$. Obviously, $\tau^{(1)}(x) = \tau^{(2)}(x)$ for $x < K_i^{(1)}$ and $x \geq K_j^{(1)}$; for $K_i^{(1)} \leq x < K_i^{(2)}$, $n^{(1)}(x) = n^{(2)}(x) - 1$, so $\tau^{(1)}(x) < \tau^{(2)}(x)$; for $K_i^{(2)} \leq x < K_j^{(2)}$, $N^{(1)}(x) = N^{(2)}(x)$

and $n^{(1)}(x)=n^{(2)}(x)$ but $\tau^{(1)}(x)=\tau^{(2)}(x)-\Delta$. Now for $K_j^{(2)} \leq x < K_j^{(1)}$, $\tau^{(1)}(x)$ increases faster than $\tau^{(2)}(x)$ (which may not even be defined everywhere if $K_j^{(1)}$ was the unique maximal capacity); nonetheless, $\tau^{(1)}(K^{+(1)})=t^+=\tau^{(2)}(K^{+(2)})$. Therefore, $\xi^{(1)}(t) \geq \xi^{(2)}(t)$, hence $\sigma^{(1)}(x) \leq \sigma^{(2)}(x)$. Again a reference to Lemma 5 accomplishes the proof.

To conclude this section, we will consider a widely discussed question of the limit of Cournot equilibrium when the number of firms tends to infinity. In contrast to usual approaches, but in accordance with our previous results, we will impose no restriction (beyond regularity) on the demand function.

Suppose a regular demand and cost function to be fixed, and consider the set of natural numbers as the set N of firms. Now a sequence of coherent McManus models is defined by a set of functions $K_i(n)$ ($i \in N$, $n \in N$) satisfying the following conditions:

- 1) $K_i(n+1) \geq K_i(n) \geq 0$ for each i, n ;
- 2) $N(n) = \{i \in N \mid K_i(n) > 0\}$ is finite for each $n \in N$.

Given a coherent sequence of McManus models, we have an increasing sequence of functions $\tau^{(n)}(x)$; we will call the sequence of models crowding

if $\lim_{n \rightarrow \infty} \tau^{(n)}(x) = +\infty$ for some $x > 0$ (this implies $\tau^{(n)}(x) \rightarrow +\infty$ for all $x > 0$ where it is defined, and $\sum_{i \in N} K_i(n) \rightarrow +\infty$). In fact, it is possible to check whether a given sequence is crowding without referring to the mapping τ : Choose arbitrary $K > 0$ and denote $M(n) = \{i \in N \mid K_i(n) < K\}$, $m(n) = |N \setminus M(n)|$; now a sequence is crowding if and only if either $m(n) \rightarrow +\infty$ or $\sum_{i \in M(n)} K_i(n) \rightarrow +\infty$ (a particular choice of K is unessential).

Define $\sigma_0 = \lim_{x \rightarrow 0} C(x)/x$; as $C(\cdot)$ is regular, σ_0 exists and $0 \leq \sigma_0 < +\infty$. At the competitive equilibrium the price should be σ_0 , and if $C(\cdot)$ is strictly convex at zero, then each firm's output should be infinitesimal.

Theorem 8. - If $P(t^0) > \sigma_0$, then for each $t^* < t^0$ and each crowding sequence of models, there exists natural n^* such that $E^{(n)}(t) = \emptyset$ for all $t \leq t^*$, $n \geq n^*$. (If $P(\cdot)$ is continuous, this means that the equilibrium price tends to σ_0 .)

First of all, there must be $K_i(n) > 0$ for some i, n ; without any loss in generality, assume $K = \max_{i \in N} K_i(1) > 0$ (by the monotonicity, $K^+(n) \geq K$ for all n). For each $s \leq t^*$, $r(s, K) > 0$ - a positive profit is feasible as $P(t^0) > \sigma_0$; define

$$x = \min_{0 \leq s \leq t^*} r(s, K) > 0$$

and choose n^* so that $\tau^{(n)}(x) > t^*$ for $n \geq n^*$. Now we have $\xi^{(n)}(t) < x$ for $t \leq t^*$, $n \geq n^*$, hence $\sigma^{(n)}(t) > t - x$ and $\sigma^{(n)}(t) + r(\sigma^{(n)}(t), K^+(n)) \geq \sigma^{(n)}(t) +$

$r(\sigma^{(n)}(t), K) > t$. Thus $t \notin M^{(n)}(t)$ for all $t \leq t^*$, $n \geq n^*$; the statement of the theorem now follows from Theorem 2.

Interestingly, if the sequence $\tau^{(n)}(x)$ has a finite limit $\tau^\infty(x)$, we may define the limit McManus correspondence $M^\infty(t)$, which determines the limits of equilibria. We may thus speak about a Cournot model with infinite number of firms, a finite number of which play an oligopoly game against the background of infinitely many finitely small competitors, each producing at its maximal capacity (the number of "real" oligopolists may be different at different equilibria).

5.- DISCRETE COURNOT MODEL

Let us now consider a discrete version of the Cournot model: suppose that the commodity in question is not completely divisible but produced and sold in integer numbers of minimal units (bottles, boxes, carloads, etc.). In other words, suppose that each K_i is an integer and each firm's strategy set is the set of integers contained in $[0, K_i]$ (or in $[0, +\infty[$). The set T of feasible total outputs is now the set of integers contained in the above-defined T . The inverse demand and cost functions may still be considered as defined everywhere, although only their values at integers are now essential.

We start with the observation that Lemmas 1 - 4 obviously remain valid for integer x and s if $R(\cdot, \cdot)$ and $r(\cdot, \cdot)$ are restricted to integers. The only obstacle to repeating the same proofs is the fact that $\xi(t)$ and $\sigma(t)$ are not necessarily integers for an integer t ($\tau(\cdot)$ maps integers into integers). To overcome, or rather bypass, the difficulty, we define $\sigma^-(t)$ and $\sigma^+(t)$ as, respectively, the biggest integer not exceeding, and the smallest integer not less than $\sigma(t)$; obviously, $\sigma^+(t) = \sigma^-(t)$ if and only if $\sigma(t)$ is an integer, otherwise $\sigma^+(t) = \sigma^-(t) + 1$, and, in either case, $\sigma^+(t) = \sigma^-(t+1)$. Now we may define $\xi^+(t)$ and $\xi^-(t)$ in a similar way, starting with $\xi(t)$, or directly as $\xi^-(t) = t - \sigma^+(t)$, $\xi^+(t) = t - \sigma^-(t)$: the result will be the same.

Define the upper and lower McManus correspondences on T as follows:

$$M^+(t) = \sigma^+(t) + R(\sigma^+(t), K^+), \quad M^-(t) = \sigma^-(t) + R(\sigma^-(t), K^+).$$

Theorem 9.- For any $t \in T$, $E(t) \neq \emptyset$ if and only if $t \in M^-(t)$ and $t \in M^+(t)$.

1. Let t be a common fixed point for both McManus correspondences. By definition,

$$t = \sum_{i \in N(\xi(t))} K_i + n(\xi(t)) \cdot \xi(t),$$

so

$$\sum_{i \in N(\xi(t))} K_i + n(\xi(t)) \cdot \xi^-(t) \leq t, \quad \sum_{i \in N(\xi(t))} K_i + n(\xi(t)) \xi^+(t) \geq t.$$

Therefore, there exist integers $n^+, n^- \geq 0$ such that $n^+ + n^- = n(\xi(t))$ and $n^+ \xi^+(t) + n^- \xi^-(t) + \sum_{i \in N(\xi(t))} K_i = t$. Now define the equilibrium $\langle x_i^o \rangle_{i \in N}$ needed: if $K_i \leq \xi^-(t)$, then $x_i^o = K_i$; among $n(\xi(t)) = n(\xi^-(t))$ firms with $K_i > \xi^-(t)$ (hence $K_i \geq \xi^+(t)$), n^+ produce $\xi^+(t)$ and n^- others produce $\xi^-(t)$ (which firm produces which output is unessential). The equilibrium conditions are satisfied for the same reasons as in Theorem 2 above.

2. Suppose that $E(t) \neq \emptyset$. There is some similarity between the following and the proof of Theorem 1, but the necessity to restrict oneself to integers causes certain complications. Denote $x^+ = \max_{E(t)} \max_{i \in N} x_i$ (the biggest output of a single firm over all the equilibria from $E(t)$); we have

$x^+ \in R(t-x^+, K_i)$ for some $K_i \leq K^+$. We also have $x^+ \geq \xi^+(t)$ because the total output would be less than t otherwise. Now we are to make the three following steps.

1) $x^+ \in R(t-x^+, K^+)$. Otherwise, $x^+ < r(t-x^+, K^+)$, so $(t-x^+) + r(t-x^+, K^+) > t$, hence $(t-x) + r(t-x, K^+) > t$ for any $x \leq x^+$, and this contradicts the assumption $E(t) \neq \emptyset$ (take as x any x_i for $K_i = K^+$ at any equilibrium from $E(t)$).

2) $\xi^-(t) \in R(\sigma^+(t), K^+)$. Otherwise, by Lemma 4, $\sigma^+(t) + r(\sigma^+(t), K^+) > t$, hence, by Lemma 2, $(t-x) + r(t-x, K) > t$ for any $x \leq \xi^-(t)$, $K \geq x$. This means that for any $\langle x_i \rangle_{i \in N} \in E(t)$ and any $i \in N$ either $x_i > \xi^-(t)$ or $x_i = K_i$; however, the conditions imply $\sum_{i \in N} x_i > t$.

3) $\xi^+(t) \in R(\sigma^-(t), K^+)$. Otherwise, by Lemma 4 applied to x^+ and $\sigma^-(t)$, we would have $\sigma^-(t) + r(\sigma^-(t), K^+) > t$, hence, by Lemma 3, $\sigma^+(t) + r(\sigma^+(t), K^+) > t$, contradicting the previous step.

Remark.- In a discrete model, it is natural to call an output vector $\langle x_i \rangle_{i \in N}$ relatively symmetric if $x_i < x_j - 1$ implies $x_i = K_i$ for each $i, j \in N$ (a difference of just one unit may be attributed to the impossibility to divide the total output equally). The above proof shows that each non-empty $E(t)$ contains a relatively symmetric equilibrium.

Theorem 10.- If $t_0 \leq \sigma^-(t_0) + r(\sigma^-(t_0), K^+)$ for some $t_0 \in T$, then $T \cap [t_0, +\infty[$ contains a common fixed point of both M^- and M^+ .

Let us define $f^-(t) = \sigma^-(t) + r(\sigma^-(t), K^+)$, $f^+(t) = \sigma^+(t) + r(\sigma^+(t), K^+)$ for $0 \leq t \leq t^\infty = \min\{t_\infty, t^+\}$; the functions are monotonic (Lemma 3) selections of M^- and M^+ , respectively. Note that $f^-([t_0, t^\infty]) \subseteq [t_0, t^\infty]$; therefore, for each $t \in [t_0, t^\infty]$ satisfying $f^-(t) \geq t$, Tarski's theorem may be applied, showing the existence of a fixed point $t^* = f^-(t^*) \geq t$. Let t^* be the biggest fixed point of f^- on the closed interval $[t_0, t^\infty]$. If $\sigma^-(t^*) = \sigma^+(t^*)$ (which is inevitably the case if $t^* = t^+$), then M^- and M^+ coincide at t^* and the theorem is proved; suppose $\sigma^-(t^*) < \sigma^+(t^*)$, i.e. $\sigma^+(t^*) = \sigma^-(t^*) + 1$. Now, if it were that $t^* + 1 \leq f^-(t^* + 1)$, there would exist a bigger fixed point, hence $f^-(t^* + 1) < t^* + 1$ and, by the monotonicity, $f^-(t^* + 1) = t^*$, which means $t^* \in M^+(t^*)$, as $f^-(t+1) = f^+(t)$ for all t .

Remark.- Tarski (1955) has proved a theorem about the existence of common fixed points for a family of functions. However, our functions f^- , f^+ do not satisfy the assumptions ($f^-(f^+(t))$ need not coincide with $f^+(f^-(t))$). By the way, it would be wrong to suppose that every fixed point of M^- is necessarily a fixed point of M^+ .

Corollary.- Every discrete McManus model has a relatively symmetric equilibrium.

As $f^-(0) \geq 0$, we may apply Tarski's theorem (to the interval $[0, t^\infty]$) and obtain the existence of a fixed point t^0 of M^- ; Theorems 10 and 9 then prove the statement.

Exact analogues of Theorems 3 and 4 for discrete models could be proved as well as those of the results of Section 4, but I can see no point in actually doing that. In fact, a modification of Lemma 2 making it completely symmetric to Lemma 4 would be required.

6.- COUNTER-EXAMPLES: CONTINUOUS MODELS

One of the most interesting features of Novshek's result is its necessity part: if an inverse demand function violates his conditions, there exists a list of cost functions for which no equilibrium exists. It seems possible that the present existence theorem might be supplemented with the "dual" necessity statement; unfortunately, I have been unable to prove it so far. The two results of this section, being generalizations of Novshek's examples, produce a step in this direction.

Theorem 11.- For any $\sigma \geq 0$, $F, K_1, K_2, U > 0$, there exists a regular inverse demand function $P(\cdot)$ such that $P(t) = 0$ for $t > U$ and the Cournot duopoly model with identical cost functions

$$C(x) = \begin{cases} \sigma x + F, & x > 0, \\ 0, & x = 0, \end{cases}$$

(constant marginal cost and positive fixed cost) for both firms and with upper restrictions on the outputs K_1, K_2 has no equilibrium.

Denote $\Delta = \min \{K_1/10, K_2/10, U/11\}$, $p_1 = ((5F/3) + 2\sigma\Delta)/2\Delta$,
 $p_2 = (F + 6\sigma\Delta)/6\Delta$ (so $p_1 > p_2 > \sigma$),

$$P(t) = \begin{cases} p_1, 0 \leq t \leq 3\Delta, \\ p_2, 3\Delta < t \leq 11\Delta, \\ 0, 11\Delta < t. \end{cases}$$

Let us describe the best replies for the model, using the notation $u(s, x) = P(s+x) \cdot x - C(x)$.

For $0 \leq x \leq 3\Delta - s$ (if $s \leq 3\Delta$) we have $u(s, x) = p_1 x - C(x)$; for $3\Delta - s < x \leq 11\Delta - s$, $u(s, x) = p_2 x - C(x)$; for $x > 11\Delta - s$, $u(s, x) = -C(x)$. As $u(s, x)$ is increasing for $x > 0$ in the first two cases and negative in the third case (if $x > 0$), there are (no more than) three potential optimal replies for any $s \geq 0$: $x=0$, $x=3\Delta-s$, $x=11\Delta-s$.

The first option is only optimal when a positive profit is unreachable; denote $v_1(s) = u(s, 3\Delta-s) = p_1 \cdot (3\Delta-s) - C(3\Delta-s)$, $v_2(s) = u(s, 11\Delta-s) = p_2 \cdot (11\Delta-s) - C(11\Delta-s)$. Both functions v_i are decreasing and v_1 decreases faster; $v_1(s) = v_2(s) = 2F/3$ when $s = \Delta$; $v_2(s) = 0$ for $s = 5\Delta$. Therefore, $R(s) = \{3\Delta-s\}$ for $s < \Delta$; $R(s) = \{11\Delta-s\}$ for $\Delta < s < 5\Delta$; $R(s) = \{0\}$ for $s > 5\Delta$. At each "switching point", $R(s)$ contains two elements: $R(\Delta) = \{2\Delta, 10\Delta\}$, $R(5\Delta) = \{0, 6\Delta\}$.

A Cournot equilibrium of the model should consist of x_1 and x_2 such that $x_2 \in R(x_1)$, $x_1 \in R(x_2)$; let us show that this is impossible. Indeed, if $0 \leq x_1 < \Delta$, then $x_1 + x_2 = 3\Delta$, hence $x_2 > 2\Delta$ and there should be $x_1 + x_2 = 11\Delta$, which is

impossible. If $\Delta < x_1 < 5\Delta$, then $x_1 + x_2 = 11\Delta$, hence $x_2 > 6\Delta$ and x_1 should be 0. If $x_1 = \Delta$, then x_2 may fit in either of the cases considered, but in both cases $x_1 \in R(x_2)$ is impossible. If $x_1 \geq 5\Delta$, then $x_2 = 0$ or $x_2 = 6\Delta$; in the first case, $x_1 = 3\Delta$, in the second one, $x_1 = 0$.

The theorem is proved.

Remark.- It can easily be seen that if any number of firms with the same costs and $K \geq \min\{K_1, K_2\}$ is added to the model, the set of equilibria will remain empty.

Theorem 12.- For any $\sigma_2 > \sigma_1 \geq 0$ and $K_1, K_2, U > 0$, there exists a regular inverse demand function $P(\cdot)$ such that $P(t) = 0$ for $t > U$ and the Cournot duopoly model with linear cost functions defined by marginal costs σ_1, σ_2 and with upper restrictions on the outputs K_1, K_2 has no equilibrium.

Denote $\delta = (\sigma_2 - \sigma_1)/2$, $p_1 = \sigma_1 + 10\delta = \sigma_2 + 8\delta$, $p_2 = \sigma_1 + 4\delta = \sigma_2 + 2\delta$, $p_3 = \sigma_1 + \delta = \sigma_2 - \delta$, $\Delta = \min\{K_1, K_2, U\}/15$. Define

$$P(t) = \begin{cases} p_1, & 0 \leq t \leq 2\Delta, \\ p_2, & 2\Delta < t \leq 4\Delta, \\ p_3, & 4\Delta < t \leq 15\Delta, \\ 0, & 15\Delta < t. \end{cases}$$

Without repeating the considerations of the previous proof, I note that the best replies of firm 1 are computed by comparing $v_1^1(s)=(p_1-\sigma_1)\cdot(2\Delta-s)=20\delta\Delta-10\delta s$, $v_1^2(s)=(p_2-\sigma_1)\cdot(4\Delta-s)=16\delta\Delta-4\delta s$, $v_1^3(s)=(p_3-\sigma_1)\cdot(15\Delta-s)=15\delta\Delta-\delta s$, and 0, while the best replies of firm 2 are found by comparing $v_2^1(s)=(p_1-\sigma_2)\cdot(2\Delta-s)=16\delta\Delta-8\delta s$, $v_2^2(s)=(p_2-\sigma_2)\cdot(4\Delta-s)=8\delta\Delta-2\delta s$, and 0. Every function v_i^k is decreasing, and the smaller is k , the faster it decreases (for each $i=1,2$); so the "switching points" for each firm i are found by solving the equations $v_i^k(s)=v_i^m(s)$ and $v_i^k(s)=0$.

Omitting the simple calculations, I state that $R_1(s)=\{2\Delta-s\}$ for $0\leq s<5\Delta/9$, $R_1(s)=\{15\Delta-s\}$ for $5\Delta/9\leq s\leq 15\Delta$, $R_1(s)=\{0\}$ for $15\Delta\leq s$, and $R_1(5\Delta/9)=\{13\Delta/9, 130\Delta/9\}$; $R_2(s)=\{2\Delta-s\}$ for $0\leq s<4\Delta/3$, $R_2(s)=\{4\Delta-s\}$ for $4\Delta/3\leq s\leq 4\Delta$, $R_2(s)=\{0\}$ for $4\Delta\leq s$, and $R_2(4\Delta/3)=\{2\Delta/3, 8\Delta/3\}$.

Now supposing the existence of an equilibrium x_1, x_2 , we see that if $0\leq x_1<4\Delta/3$, then $x_1+x_2=2\Delta$, hence $x_2>2\Delta/3>5\Delta/9$, hence $x_1+x_2=15\Delta$: a contradiction; if $4\Delta/3<x_1\leq 4\Delta$, then $x_1+x_2=4\Delta$, so $x_1\in R_1(x_2)$ is impossible; if $x_1=4\Delta/3$, then one of the previous cases should take place; if, at last, $x_1>4\Delta$, then $x_2=0$ and $x_1=2\Delta$: a contradiction again.

Remark 1.- As U may be assumed arbitrarily small, the statement can be extended to two firms with different marginal costs at zero. The extension to a greater number of firms apparently requires some more complicated constructions.

Remark 2.- To a reader unhappy with the discontinuity of demand in the proofs of Theorems 11 and 12, I would point out that the absence of a Nash equilibrium is a robust property of a normal form game, so the discontinuous functions $P(\cdot)$ may be approximated with continuous or even smooth ones (the proofs would have become much more cumbersome if we started with the approximations). Besides, there is no problem with interpretation: Assuming the presence behind the scene of at least one more commodity with a fixed price, such a demand function is generated e.g. by a concave piece-meal linear utility (a step in the demand for an angle in the utility) provided the income is independent of the price on the market under consideration.

7. - COUNTER-EXAMPLES: DISCRETE MODELS

Now we are to consider two counter-examples concerning discrete versions of the Cournot model. They show a difference between linear costs and general convex ones.

For discrete McManus models with linear costs, there is no problem with the existence of an equilibrium: If profitable production is at all possible, any maximum-point of the function $(P(\sum_{i \in N} x_i) - \sigma) \cdot \sum_{i \in N} x_i$, where σ is the constant marginal cost, is obviously an equilibrium (Kukushkin, 1993). This statement remains valid whatever restrictions might be imposed on the outputs if the set of feasible outputs of each firm is compact (finite or not).

For general convex costs, we only have the previous results, whose proofs do not survive imposing arbitrary restrictions on the outputs. The following shows that this is due not to any deficiency of the proofs but to a real fact: If the commodity in question is discrete in its nature, and if e.g. one of two firms can only produce it in duos and the other in trios, then an equilibrium always exists if the firms have identical linear costs but may not exist at all for general convex (even though identical) costs. I cannot assert any economic importance of the fact: my only goal is to

clarify the reasons for the existence of an equilibrium in a discrete model from a purely mathematical viewpoint.

Theorem 13.—For any regular convex but non-linear cost function $C(\cdot)$, there exist a regular demand function $P(\cdot)$ and $a, b > 0$ such that the discrete McManus duopoly where the set of feasible outputs for one firm is $\{0, a, 2a\}$ and for the other $\{0, b\}$ has no equilibrium.

As the cost function is non-linear, there must be a point where the (lower) marginal cost exceeds the average one; take it as b . Now find the minimal point of the interval $[0, b]$ where the (upper) marginal cost is not less than the average cost at b , and take it as $2a$. Denote $\beta = C(b)/b$, $\alpha = (C(2a) - C(a))/a$, $\delta = (\beta - \alpha)/4$; by definition, $\alpha < \beta$, so $\delta > 0$. Now define the demand function:

$$P(2a+b) = \beta - \delta = \alpha + 3\delta;$$

$$P(2a) = P(b) = P(a+b) = \beta + \delta = \alpha + 5\delta;$$

$$P(a) = \alpha + 11\delta.$$

If $x_1 = 0$ or $x_1 = a$, the optimal reply of firm 2 is $x_2 = b$ as $P(a+b) > \beta$; if $x_1 = 2a$, firm 2 cannot obtain a positive profit ($P(2a+b) < \beta$), so the optimal reply is $x_2 = 0$. On the other hand, the optimal reply for firm 1 to $x_2 = 0$ is

$x_1=a$ ($2P(2a)-P(a)<\alpha$), and its optimal reply to $x_2=b$ is $x_1=2a$ ($2P(2a+b)-P(a+b)>\alpha$). Thus no outcome bundle is an equilibrium.

Theorem 13 shows that a difference in the discretization of the strategy sets means the possibility of non-existence (for non-linear costs). If all strategy sets are discretized in the same way, but not evenly, an equilibrium may again not exist. This time I am not ready to produce a theorem, so we have to restrict ourselves to an example.

Consider a McManus duopoly with the following cost and inverse demand functions:

$$C(x) = \begin{cases} 0, & 0 \leq x \leq 4, \\ 90(x-4), & 4 \leq x \leq 5, \\ 136(x-5)+90, & 5 \leq x; \end{cases}$$

$$P(t) = \begin{cases} 260, & 0 \leq t \leq 9, \\ 225, & 9 < t \leq 11, \\ 200, & 11 < t \leq 12, \\ 0, & 12 < t. \end{cases}$$

Suppose that the feasible outputs of each of the two firms form the set $\{0,1,2,3,4,5,7\}$ (for some mysterious reasons which we will not try to explain here). If the output 6 were feasible, an equilibrium would exist

because of the results of the previous section (and 11 would be the only equilibrium total output). As it is, there is no equilibrium in the model.

Since the firms are identical, we have to consider just one best reply function. Omitting the straightforward calculations, I state that $R(s)=\{7\}$ as long as $s \leq 4$ (for $s=4$, $x=7$ brings the profit of 1213, while $x=5$ brings 1210); for $s=5$, $x=4$ brings the profit of 1040, while $x=5$ brings 1035 and $x=7$, 1038, hence $R(5)=\{4\}$ (here lies the key point of the example: Lemma 3 does not hold!); for $s=7$, $x=5$ brings the profit of 910, while $x=4$ brings 900, hence $R(7)=\{5\}$. Therefore, only output pairs $\langle 4,7 \rangle$, $\langle 5,4 \rangle$, and $\langle 7,5 \rangle$ could be equilibria, but neither of them is one.

8.- CONCLUDING REMARKS

Now let me try to summarize the message of this paper as well as of other recent research in the area, cited in Introduction.

First of all, Kakutani's theorem, and convexity in general, does not play a central role in the existence problem for Cournot equilibrium as it apparently plays e.g. in the competitive equilibrium context. No less important are other causes for the existence of equilibria such as the separability of utilities ("almost" identical linear costs) and Tarski's theorem (decreasing incremental revenues in the duopoly case; "almost" identical convex costs); besides, the general case of Novshek's theorem remains unexplained (I would guess that some connection with Tarski's theorem might be found eventually) but it certainly has nothing to do with convexity. (One should not be misled by the convexity of costs assumed in the main result of this paper: a more rigorous formulation would speak of non-decreasing incremental costs instead as, in fact, McManus (1964) did).

Second, "non-convex" causes for the existence of an equilibrium act for discrete versions of the Cournot model too. Admittedly, such exotic models as that of Theorem 13 or, even more so, of the example following it cannot claim much significance for economic theory. However, the basic

discrete model with integer outputs may be regarded as even more natural than the (continuous) Cournot model itself, which, after all, is just the limit case of the integer model when the unit of measurement becomes even smaller and smaller. The fact that it is not always necessary to assume the unlimited divisibility to guarantee the existence of an equilibrium (however convenient the assumption might be for computation, geometric interpretation, etc.) bears on the fundamental properties of widely used mathematical tools for economic analysis. In this light, it seems extremely important to find out to what extent Novshek's general result survives the discretization of the strategy sets.

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