SHAPING LONG-RUN EXPECTATIONS IN PROBLEMS OF COORDINATION*

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A B S T R A C T

When far-sighted agents may adjust their behavior only gradually, the issue of equilibrium selection in games becomes one of tension between "history" and "expectations". This paper analyzes whether, in this context, a planner may intervene successfully through short-run policies which redirect expectations away from the inertia of undesired history. The possibilities and limitations of such approach to "expectation management" are studied in a game-theoretic framework where both the planner and the population are involved in a struggle to impose their (credible) commitment possibilities.

EN BLANCO

1.- INTRODUCTION

In intertemporal contexts with equilibrium multiplicity, expectations will generally enjoy a key role in determining eventual performance. History, on the other hand, will also play an important part if, as it is often reasonable to assume, adjustment can only be done gradually due to some sort of friction.

In those cases where history is the decisive factor, it is clear that the process may become mired in undesired inertias. On the other hand, if it is expectations that represent the key driving force of the process, its long-run behavior will be affected by inherent indeterminacy. In response to these two different sort of problems, there arises the question of whether some supplementary mechanism might be able to "close" the model, yielding both unambiguous and desired long-run performance.

A natural candidate of long economic tradition is the figure of a benevolent planner. In particular, one may ask the following questions.

(i) Can a planner liberate the economy from undesired inertias imposed by previous history ?

(ii) Can she redirect the population's expectations towards a desired outcome ?

Of course, the answer to the former questions could be made just trivial if the planner is allowed a quite unrestricted set of possibilities. The issues become interesting only when realistic constraints are imposed on the planner's scheme (e.g., anonymity, "simplicity", finite duration, individual rationality, etc.). Such requirements are considered below, formulated as restrictions on the set of strategies used by the planner in a certain commitment game played with the rest of the population. As we shall see, alternative restrictions in this respect will produce quite different conclusions.

When expectations are "right", the answer to question (i) in the affirmative will turn out to be a simple exercise in (short-run) subsidy policy. The answer to (ii), however, will generally be more subtle. At first glance, it is not even clear how a planner could have any effect on long-run expectations through policies that are required to be, in some sense, short run. In most of the literature, the role left to the "planner" (or government) in the process of shaping these expectations is often reduced to the intangible task of preaching "the Economics of Euphoria" (*sic,* Matsuyama (1991, p. 620)); that is, to promoting confidence and optimism in the population. As it vaguely stands, this does not seem a very fruitful idea. One of the basic objectives of this paper will be to attempt some precise formalization of the planner's "expectation-management activities", that may then allow a more productive discussion of these undoubtedly important issues.

The surging literature on the dichotomy of "history versus expectations" in dynamic models is the main source of motivation and inspiration for the present work. Through this literature (see, for example, Howit & McAfee (1988), Matsuyama (1991), Krugman (1991), or Matsui & Matsuyama (1991)),[(1)] we have much better understood the role of these two forces - history and expectations - in determining the long-run performance of dynamic models. History, on the one hand, will be a decisive force if adjustment is subject to large "friction" (for example, large adjustment costs). On the other hand, expectations will tend to be a crucial part of the story if phenomena such as externalities and complementarities have the potential of generating strong band-wagon effects. These considerations will naturally play a major role in our analysis.

The paper is organized as follows. In the next section, Section 2, I present the formal model. It builds upon that proposed by Matsui & Matsuyama (1991), whose analysis I shall also partly rely on. Section 3 contains the results and discussion. Finally, Section 4 closes the paper with a summary. For the sake of smooth exposition, formal proofs are relegated to the Appendix.

1 Related work with an emphasis on history rather than expectations in a context of international trade and dynamic specialization is Ethier (1982) or Panagariya (1986).

2.- THE MODEL

Consider the following setup. Time is measured continuously, $t \in [0,\infty)$. The economy's population remains unchanged throughout and is composed of a continuum of members with measure one. At every point in time, The members of the population are randomly matched pairwise in order to play a certain bilateral symmetric game. This game is simultaneous, involves only two actions, A and B, and two strict equilibria, (A,A) and (B,B). Such a game may be represented by the following payoff table:

	A	B
A	a,a	c,d
B	d,c	b,b

Table 1

where a > d, b > c. As explained below (Subsection 3.3), our formulation of the individual-rationality constraint shall derive from the interpretation of the above payoffs as deviations over a minimum level which any individual can guarantee for herself. Thus, if payoffs are so interpreted, all of them must be non-negative.

Players will be assumed to play an intertemporal sub-game perfect equilibrium of a dynamic game, whose description is postponed to Subsection 3.2 (cf. Definition 1). Prior to the start of such dynamic game, a "planner" will be allowed to intervene and commit to a certain policy to be applied in the course of it. In the present "reduced-form" context, it is natural to identify a planner's policy with the specification of a time profile of "taxes or subsidies". Let me first introduce verbally the four restrictions which shall be demanded from any admissible planner's policy.

(1) It has to be anonymous, i.e., a policy may not depend on the "name" of the individuals. This is a natural symmetry requirement which needs no further comment. It implies, on the one hand, that the tax or subsidy which an individual may receive at any point in time may only depend on her past history of actions. On the other hand, it requires that any funds needed to finance some possible "deficit" of the policy have to be collected uniformly. (Or, if a surplus results, its distribution should also be done uniformly.)

(2) Any admissible policy function will be required to vanish in the long run. Thus, in a sense, any admissible policy must prescribe intervention only in the "short-run".

(3) Its pattern of taxes and subsidies must respect, at all times, an appropriate individual-rationality constraint. Thus, in particular, the planner cannot impose a level of taxes which forces on some individuals a lower payoff than what they can ensure for themselves.

(4) The policy function cannot be too complex. Three different upper bounds on their complexity will be alternatively considered.

(4i) One scenario will require that the policy function be "open-loop", i.e., that its prescription at each point in time only depend on the corresponding "date", <u>not</u> on history.

(4ii) A second scenario will allow the planner to use (strictly) Markovian policy functions, i.e., functions whose argument is the current state of the system. Thus, in contrast with the open-loop policy functions, Markovian ones may depend on history. However, they may only depend on those features of it which are payoff-relevant (in particular, <u>not</u> on time). In our context, this will amount to restricting policy prescriptions at each point in time to be a function of the current action profile of the population.

(4iii) A third richer scenario will allow the planner to use <u>generalized</u> Markovian policies, i.e., policies which may depend on both time <u>and</u> the current state of the system. Obviously, this set of policy functions includes the union of the two former ones as a proper subset.

As it will be shown below, the possibilities enjoyed by the planner in each of the three different scenarios described are substantially different.

I now formalize matters, dividing the presentation of the model in three subsections: Policy functions (3.1); Equilibrium paths (3.2); The planning problem (3.3).

3.1. Policy functions.

By (1) and (4) above, a planner's policy may be represented by a pair of functions:[2]

$$\phi_A:[0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}, \tag{1a}$$
$$\phi_B:[0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}, \tag{1b}$$

which specify the subsidy $\phi_h(x,t)$ (tax, if negative) received by any individual who adopts action h = A, B at time t when the fraction of the population which plays A (or B) is x (or (1-x)).

If $\phi \equiv (\phi_A,\phi_B)$ is an *Open-Loop Policy* (OP), then it may only depend on its second argument, i.e., on time. The set of all open-loop policies will be denoted by Φ^o.

If ϕ is a (strictly) *Markovian Policy* (MP), it may only depend on its first argument. The set of all those (strictly) Markovian policies will be denoted by Φ^m.

If ϕ is a *Generalized Markovian Policy* (GMP), it may depend on both of its arguments. The set of all those generalized Markovian policies will be denoted by $\hat{\Phi}$. As mentioned, $\hat{\Phi} \supset (\Phi^m \cup \Phi^o)$.

3.2. Equilibrium paths.

Let $x(t) \in [0,1]$ stand for the fraction of the population which plays A at time t. From Table 1, it follows that the current average payoff, gross of taxes and subsidies, associated to playing each action A and B, is respectively given by:

$$\varphi_A(x) = a\ x + c\ (1-x) \tag{2a}$$
$$\varphi_B(x) = d\ x + b\ (1-x) \tag{2b}$$

Along the game, players will be assumed subject to the following crucial restriction: At every point in time, they can only change their action at some

[2] For restrictions (2) and (3), see Subsection 3.3 below.

common probability rate $p > 0$, which is independent across players. This is viewed as the consequence of "friction", which may be provided with a variety of standard interpretations.[3] Thus, as viewed from any given t, the <u>first</u> future date d(t) at which such possibility will occur is a Poisson process with arrival rate p and distribution function:

$$\mathrm{Prob}_t \{d(t) \leq \tau\} = 1 - e^{-p(\tau - t)}. \tag{3}$$

Given the policy $\phi \equiv (\phi_A, \phi_B)$, and a profile path

$$\mathbf{x}: \mathbb{R}_+ \longrightarrow [0,1], \tag{4}$$

let:

$$S_A\big(x(t),t;\phi\big) \equiv \phi_A\big(x(t),t\big)\ x(t) \tag{5a}$$

$$S_B\big(x(t),t;\phi\big) \equiv \phi_B\big(x(t),t\big)\ \big(1-x(t)\big) \tag{5b}$$

denote the net subsidies of actions A and B, respectively, induced at time t by the policy ϕ and the profile path $\mathbf{x}$. Associated to them,

$$B\big(x(t),t;\phi\big) \equiv S_A\big(x(t),t;\phi\big) + S_B\big(x(t),t;\phi\big) \tag{6}$$

represents the planner's current "deficit" at each t.

Assuming, by the requirement of anonymity, that this deficit is uniformly distributed among all individuals of the population, the net instantaneous payoff of an individual who adopts action h = A,B, at some time t is equal to:

$$\varphi_h\big(x(t)\big) + \phi_h\big(x(t),t\big) - B\big(x(t),t;\phi\big) \tag{7}$$

Let θ denote the common discount rate of all population members. The net intertemporal payoff for any individual of the population who at time t chooses action h is then given by:

$$V_h(\phi,\mathbf{x},t) = \int_{s=t}^{\infty} \Big[\varphi_h\big(x(s)\big) + \phi_h\big(x(s),s\big) - B\big(x(s),s;\phi\big) + p\ V^*(s)\Big] e^{-(p+\theta)s}\, ds \tag{8}$$

3 Let me offer one of them in terms of a process of population renewal. It may be supposed, for example, that the population is not permanent (only its size is fixed), but that everyone of its members is subject to an independent "death probability" of p. The proposed formulation of friction then results if it is further assumed that only newcomers may change their actions.

where

$$V^*(s) = \max \left\{ V_A(\phi,x,s), V_B(\phi,x,s) \right\} \tag{9}$$

represents the maximum payoff attainable by an individual who is able to change her action at s.

Consider any given planner's policy ϕ, as described above. Once the planner has committed to it, the dynamic game confronted by the population is defined. The equilibrium concept which shall model their behavior is now introduced.

Definition 1: *Given policy ϕ and initial conditions $x_0 \in [0,1]$, the path* $\mathbf{x}(\cdot)$ *is a* **ϕ-equilibrium** [4] *if it is right differentiable,* $\mathbf{x}(0) = x_0$, *and, for all $t \in \mathbb{R}$:*

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt^+} &= p\,\big(1-\mathbf{x}(t)\big) \quad \text{if} \quad V_A(\phi,\mathbf{x},t) > V_B(\phi,\mathbf{x},t), \\ \frac{d\mathbf{x}(t)}{dt^+} &= -\,p\,\mathbf{x}(t) \quad \text{if} \quad V_A(\phi,\mathbf{x},t) < V_B(\phi,\mathbf{x},t), \\ -\,p\,\mathbf{x}(t) \le \frac{d\mathbf{x}(t)}{dt^+} &\le p\,\big(1-\mathbf{x}(t)\big) \quad \text{if} \quad V_A(\phi,\mathbf{x},t) = V_B(\phi,\mathbf{x},t). \end{aligned} \tag{10}$$

Verbally, given some initial conditions x_0, a ϕ-equilibrium is a path which represents an optimal intertemporal response by each member of the population to both policy ϕ and correct expectations on how the rest of the population will, in average, behave.

If the policy function ϕ is Markovian, it will be of interest to focus on *Markovian ϕ-equilibria.* As presently defined, they are those ϕ-equilibria whose law of motion is time-homogeneous (i.e., time does not enter into it as an explicit separate argument).

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4 For simplicity, I am incurring in the inaccuracy of labelling $\mathbf{X}(\cdot)$ as a ϕ-equilibrium rather than a ϕ-equilibrium path (it does not specify "out-of-equilibrium" behavior). However, such inaccuracy is inessential in our context since, given that each agent has infinitesimal weight in the population, only the "average" equilibrium path is needed in order to evaluate alternative responses at each point in time.

Definition 2: *Given policy* $\phi \in \Phi^m$, *and initial conditions* $x_0 \in [0,1]$, *the path* $\mathbf{x}(\cdot)$ *is a* **Markovian ϕ-equilibrium** *if it is right differentiable,* $\mathbf{x}(0) = x_0$ *and, for all* $t \in \mathbb{R}$, *(14) is satisfied with the requirement that, for every action* $h = A,B$, *there is a function* $\hat{V}_h:\Phi\times[0,1] \longrightarrow \mathbb{R}$ *such that their respective value functions admit the representation* $V_h(\phi,x,t) = \hat{V}_h(\phi,x(t))$.

3.3. The planner's decision problem

Let $\Phi \in \{\Phi^m, \Phi^o, \hat{\Phi}\}$ represent the set of admissible policies in the scenario under consideration. It may be conceived as the strategy set of the planner in the game she plays with the population.[5] In this game, the "strategy" of the population is a collection of policy-contingent action paths:

$$\mathbf{X} \equiv \{x(\cdot\,|\,\phi)\}_{\phi\in\Phi}, \tag{11}$$

which will be labelled a *pattern of (equilibrium) expectations.* Each $\mathbf{x}(\cdot\,|\,\phi)$ represents the ϕ-equilibrium which (given initial conditions x_0) the population plans on playing if the planner commits to policy ϕ. Such pattern of expectations, a datum for the planner, defines the essential component of her decision problem. In particular, it determines the set of admissible policies. For, as explained, any policy chosen by the planner will have to satisfy (in addition to the anonymity and simplicity - (1) and (4) above - which already incorporated into their formulation) the requirements of being "short-run" and respecting individual rationality.

The first requirement is formulated as follows. Given $\mathbf{X}$, any policy ϕ chosen by the planner is to satisfy:

$$\forall h = A,B,\ S_h(x(t\,|\,\phi),t;\phi) \longrightarrow 0 \text{ as } t \longrightarrow \infty. \tag{12}$$

5 The interaction between the population and the planner could be modelled as a two-stage game as follows. In the first stage, each individual of the population commits simultaneously (and independently) to a certain policy-contingent dynamic rule (Markovian or not) to be applied once the planner's policy is known. In the second stage, it is the planner who commits to a certain policy function. I abstract from such game-theoretic formalization since it would not add anything substantial to the analysis.

As for the requirement of individual rationality, it is *a priori* unclear what should be an appropriate reference point for this constraint. As suggested above, the benchmark which shall be proposed here derives from the interpretation of the payoffs included in Table 1 as deviations over the minimum payoff level which any individual can ensure for herself. Thus, given any pattern of expectations **X**, the individual rationality constraint required from any policy ϕ may be formulated as follows:

$$\forall t,\ \forall h = A,B,\ \varphi_h\big(x(t\,|\,\phi)\big) + \phi_h\big(x(t\,|\,\phi),t\big) - B\big(x(t\,|\,\phi),t;\phi\big) \geq 0. \tag{13}$$

Denote by $\mathscr{F}(\mathbf{X})$ the set of policy functions in $\boldsymbol{\Phi}$ which, given **X**, satisfy (12) and (13). It represents the choice set of the planner's decision problem. To complete the description of her decision problem, I now focus on what are postulated to be her preferences.

The planner will be assumed subject to the same discount factor θ as the population. Her intertemporal payoff will be identified with the discounted population-average payoffs, net of "implementation costs". Its first component (discounted population-average payoffs) is given by:

$$F(\phi,\mathbf{X}) = \int_{t=0}^{\infty} \Big[x(t\,|\,\phi)\ \varphi_A\big(x(t\,|\,\phi)\big) + \big(1-x(t\,|\,\phi)\big)\ \varphi_B\big(x(t\,|\,\phi)\big)\Big]\ e^{-\theta t}\ dt. \tag{14}$$

Note that, since the planner's budget must be balanced, action specific taxes and subsidies are exactly compensated by lump-sum transfers in the above expression (c.f. (8)).

The second term in the planner's payoff (implementation costs) is meant to reflect a variety of costs associated to implementing the policy (e.g. distortions induced by taxes and subsidies, monitoring activities, etc.). For simplicity, they will be assumed linked, at each time point in time, to the aggregate magnitude of current taxes and subsidies, as given by the increasing and continuous function

$$g:\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \tag{15}$$

which satisfies $g(0) = 0$. The discounted implementation costs over the complete time horizon are then given by:

$$G(\phi,X) = \int_{t=0}^{\infty} g\Big[\big|S_A\big(x(t|\phi),t;\phi\big)\big| + \big|S_B\big(x(t|\phi),t;\phi\big)\big|\Big]\, e^{-\theta t}\, dt, \tag{16}$$

where $|\cdot|$ denotes absolute value. Joining (14) and (16), the planner's total payoffs induced by a strategy profile (ϕ,X) become:

$$\pi(\phi,X) = F(\phi,X) - G(\phi,X).^{(6)} \tag{17}$$

The planner's problem can now be stated. Given $\mathbf{X} \equiv \{x(\cdot|\phi)\}_{\phi\in\Phi}$,

$$\textit{Choose } \phi^* \in \mathcal{F}(\mathbf{X}) \;\; s.t. \;\; \pi(\phi^*,\mathbf{X}) \geq \pi(\phi,\mathbf{X}),\; \forall\phi \in \mathcal{F}(\mathbf{X}). \tag{18}$$

Any policy function that is a solution to (18) will be called an **X**-*optimal policy* (or simply an "optimal policy", when there is no risk of ambiguity). To study the properties and long-run population behavior induced by such any such policy is the main objective of the analysis that follows.

6 Note that implementation costs are not incorporated into the population's payoffs. It could be assumed that they enter uniformly as lump-sum costs in a way analogous to the planner's deficit. Nothing essential would change in the analysis.

4.- ANALYSIS

I start with the following preliminary existence result.

Proposition 1: *Given any $x_0 \in [0,1]$ and $\phi \in \hat{\Phi}$, a ϕ-equilibrium* $\mathbf{x}(\cdot\,|\,\phi)$ *always exists. If $\phi \in \Phi^m$, the ϕ-equilibrium may be chosen Markovian.*

Proof: See the Appendix.

By virtue of the previous proposition, every policy $\phi \in \hat{\Phi}$ has at least one equilibrium path $\mathbf{x}^*(\cdot\,|\,\phi)$ - possibly Markovian, if $\phi \in \Phi^m$ - associated to it. Any collection of such equilibrium paths obtained as the function ϕ ranges over the set Φ under consideration is what has been labelled above a pattern of (equilibrium) expectations. Proposition 1 ensures that it is a well-defined object, i.e., it always exists.

A given pattern of expectations determines the range of population responses which the planner confronts when selecting the policy function. In general, one would like to ensure that the planner's decision problem is always well-defined, i.e., it has a solution. If Φ were finite, this would follow from a simple backwards-induction argument. However, since $\Phi \in \{\Phi^o\, ,\, \Phi^m\, ,\, \hat{\Phi}\}$ is not assumed finite, the existence of such a planner's optimal response may fail due to lack of continuity of $\mathbf{x}^*(\cdot\,|\,\phi)$ with respect to the policy function ϕ. (See Example 1 in the Appendix for an illustration of this problem.)

In order to circumvent this somewhat technical issue, the former concept of "optimal policy" is generalized as follows.

Definition 3: *Given a pattern of expectations* **X**, *and any $\varepsilon \geq 0$, an* ε-**optimal policy** *is a function ϕ^* that satisfies:*

$$\pi(\phi^*,\mathbf{X}) \geq (1-\varepsilon)\ \pi(\phi,\mathbf{X}),\ \forall\phi \in \mathscr{F}(\mathbf{X}).$$

As compared with the original definition of (X-)optimal policy, the concept formulated in the previous definition allows for the possibility that the planner may ignore profitable deviations from a certain policy which only entail arbitrarily "small" relative gains in payoff. [7] It is analogous to the concept of ε-equilibrium which is standard in Game Theory (see, for example, Radner (1980) or Fudenberg & Tirole (1983)). Since the planner's payoff function $\pi(\cdot)$ is bounded, it is clear that, for all $\varepsilon > 0$ and every given pattern of expectations, an ε-optimal policy always exists.

The first basic result of the paper is now stated and discussed.

Theorem 1: *Let $\Phi = \Phi^o$ and assume that $a > b$ and $a\text{-}d < b\text{-}c$ (i.e., equilibrium (A,A) Pareto-dominates equilibrium (B,B), but the latter is risk-dominant). There is some $\chi > 0$ such that if $(\theta/p) \le \chi$:*

(a) Given $x_0 \in [0,1]$, there is some $\bar{\varepsilon} > 0$ and a pattern of expectations $\tilde{X}$ s.t. if $\varepsilon \le \bar{\varepsilon}$, *every ε-optimal policy $\tilde{\phi}$ induces $\lim_{t\to\infty} \tilde{x}(t|\tilde{\phi}) = 1$.*

(b) Given $x_0 \in [0,1]$, there is a pattern of expectations $\hat{X}$ such that, for any $\varepsilon \ge 0$, every ε-optimal policy $\hat{\phi}$ induces $\lim_{t\to\infty} \hat{x}(t|\hat{\phi}) = 0$.

Proof: See the Appendix.

The interpretation of the previous result is as follows. Assume that "friction" θ/p is sufficiently small. (Here, of course, the speed of adjustment captured by p has to be assessed in terms of the magnitude of the time discount rate θ.) Then, Part (a) of Theorem 1 establishes that there is always some pattern of expectations and an associated open-loop ε-optimal policy which, for ε small, leads the economy towards the efficient profile $x = 1$ from any initial conditions x_0. This conclusion contrasts with the results of Matsui & Matsuyama (1991) who showed that, left to itself, the population will always remain deadlocked into the inefficient and risk-dominant profile $x = 0$ if it starts sufficiently close. In the present context, the planner will intervene in those cases by "internalizing", through feasible (open-loop) policies, the effects of a Pareto improving transition.

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7 Since the planner's payoff function is unbounded in the discount factor θ (which will itself be a crucial parameter in our discussion) it seems conceptually more appropriate to specify the value of ε as a proportion of the corresponding payoffs rather than in absolute terms.

However, the above theorem also indicates that the planner alone cannot ensure that the previous transition will take place if policies are restricted to being open-loop. In particular, building upon a result by Matsui & Matsuyama (1991; Lemma 1), its Part (b) establishes that if the population's pattern of expectations (as captured by $\hat{X}$ in the above statement) are too "committed" to the risk dominant (inefficient) equilibrium, no ε-optimal policy may prevent the economy from settling down irreversibly in the inefficient profile.

Thus, combining the two parts of Theorem 1, we conclude that, if the planner's policies are restricted to being open-loop, there remains in the model an essential degree of long-run indeterminacy which the planner alone is unable to remedy. The population has the option of inducing any of the two polar configurations ($x = 0$ or $x = 1$) by "adopting" a suitable pattern of expectations. As presently explained, this sharply contrasts with the ensuing analysis.

In contrast with the OP scenario studied so far, the next result assumes that the planner can use (only) Markovian policy functions, i.e., it is assumed that $\Phi = \Phi^m$. Under these circumstances, Proposition 1 (see above) indicates that the pattern of expectations under consideration may be chosen to include only Markovian ϕ-equilibria. In this case, they will be labelled a *Markovian pattern of expectations.*

Theorem 2: *Let* $\Phi = \Phi^m$, *and* $a > b$. $\exists \chi > 0$, $\exists \bar{\varepsilon} > 0$, *such that if* $\theta/p \leq \chi$ *and* $\varepsilon \leq \bar{\varepsilon}$, *given any Markovian pattern of expectations* $\hat{X}$, *every* ε*-optimal policy* $\hat{\phi}$ *induces* $\lim_{t \to \infty} x(t|\hat{\phi}) = 1$.

Proof: See the Appendix.

The previous result establishes that, provided the population restricts to only Markovian patterns of expectations, the planner's ε-optimal policy will always close the model towards the efficient population profile $x = 1$, provided both friction and ε are sufficiently small.

In connection with this result, there arises the question as to how crucial it is for the previous result the restriction to Markovian patterns of expectations. It is easy to construct examples which show that, indeed, this requirement cannot be dispensed with. The intuition here is simple to explain. If expectations are non-Markovian, the population may link its current behavior to the prevailing "date". But then, for some such date sufficiently far into the future, the population behavior after it need not concern an "ε-optimizing planner". In particular, where (and whether) this behavior converges in the very long run need not affect her policy choice, as long as desirable behavior materializes in the earlier part of the process.

Despite the previous considerations, if the planning problem is viewed from a somewhat "looser" perspective the desired long-run efficient performance can be ensured, even for non-Markovian patterns of expectations. This is established by the following result.

Theorem 3: *Let $\Phi \geq \Phi^m$ and assume that $a > b$. There exists some $\phi^* \in \Phi$ for which $\forall \varepsilon > 0$, $\exists \chi > 0$ such that if $(\theta/p) \leq \chi$, ϕ^* is an ε-optimal policy for any pattern of expectations* $\mathbf{X}$. *Furthermore,* $\lim_{t \to \infty} \mathbf{x}(t|\phi^*) = 1$.

Proof: See the Appendix.

The previous result asserts that, if Markovian policies are available, there is some given policy ϕ^* which, once determined any "tolerance bound" ε, yields, if friction is sufficiently small, an ε-policy equilibrium that induces long-run efficient performance for any pattern of expectations. It may be helpful to emphasize the three distinctive features of this result:

(a) The policy function considered is independent of the pattern of expectations.

(b) The conclusion applies to any type of expectations, Markovian or not.

(c) The tolerance bound ε is dependent on the extent of friction. [8]

Thus, if only approximate standards of optimality are required, points (a) and (b) indicate that the planner has quite strong expectation-management possibilities with Markovian policies. However, point (c) reflects what, in contrast with Theorem 2, is the weak aspect of this result: although the optimality standards captured by ε can become arbitrarily stringent, they are linked to the prevailing extent of friction.

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8 Indeed, the proof of Theorem 3 essentially relies on the following simple insight (see the Appendix for the formal argument). If the planner commits to a Markovian policy ϕ^* which guarantees that action A obtains a "slightly larger" instantaneous payoff than B for all population profiles, a large fraction of the population will turn to doing action A after some time along any ϕ^*-equilibrium. When this happens, no actual intervention is necessary any further in order to ensure convergence to the efficient profile. Thus, when friction is sufficiently small, this policy must be approximately optimal to any given degree (once its individual rationality is confirmed) since, clearly, any alternative policy can only yield a higher instantaneous payoff to the planner during a predetermined (and finite) time span.

5.- SUMMARY

This paper has addressed a theoretical question apparently by-passed by the recent literature on equilibrium selection. Namely, the issue of how can expectations be formed or managed when they are the key force behind the selection of a particular equilibrium.

Rather than resorting to stress the importance of "promoting confidence and optimism" in the population, the model has focused on the impact which tangible policy measures may enjoy in shaping the public's expectations. The nature of these policies has been restricted in different respects. Specifically, they have been required to be "short-run", "simple", "anonymous" and "individually rational", in appropriate versions of these concepts adapted to the present context.

The analysis has focused on the extent to which the optimal planner's policy will unambiguously select among an otherwise non-unique range of long-run expectations of the model. Not surprisingly, the conclusion in this respect has been seen to depend crucially on the particular set of admissible planner's policies under consideration.

First, if these are restricted to being open-loop, then suitable patterns of population's expectations may lead, even in the presence of an optimal policy choice by the planner, to any of the two polar population profiles of the model. Thus, in this case, the planner is left with essentially no power to close the model unambiguously.

The situation is very different if the planner relies on Markovian policies and the population responds to all of them with a Markovian equilibrium path. Under these circumstances, every ε-optimal policy (for ε sufficiently small) will lead the process to the efficient population profile in the long run. In any case, and even if the population does not behave in such a Markovian fashion, there is always a certain Markovian policy which is approximately optimal and induces long-run convergence to the efficient profile if friction is sufficiently small.

A P P E N D I X

<u>Example 1</u>: *A pattern of expectations for which no optimal policy exists.*

Assume initial conditions $x_0 = \frac{b-c}{(a-d) + (b-c)}$. That is, $\varphi_A(x_0) = \varphi_B(x_0)$.

Denote by $\bar{\phi}(\cdot)$ the "non-interventionist" policy, i.e., the policy with $\bar{\phi}_A(\cdot) \equiv \bar{\phi}_B(\cdot) \equiv 0$. Clearly, the two paths $x'(\cdot\,|\,\bar{\phi})$ and $x''(\cdot\,|\,\bar{\phi})$ given by:

$$x'(t\,|\,\bar{\phi}) = x_0\, e^{-pt}$$

$$x''(t\,|\,\bar{\phi}) = 1 - (1-x_0)\, e^{-pt}$$

define $\bar{\phi}$-equilibria. The first one has $x'(t\,|\,\bar{\phi}) \longrightarrow 0$ as $t \longrightarrow \infty$, whereas the second yields $x''(t\,|\,\bar{\phi}) \longrightarrow 1$. Obviously, if friction θ/p is small enough, the planner's payoff for the latter $\bar{\phi}$-equilibrium is higher than for the former.

In order to illustrate matters in the simplest possible fashion, let us ignore the constraints (12) and (13) contemplated by the concept of ε-optimality and have the set of admissible policies be restricted to the subset $\tilde{\Phi} \subset \Phi$ of constant functions. These functions have:

$$\forall x,\ \forall t,\ \forall h = A,B,\ \phi_h(x,t) = \rho_h,\ \rho_h \in \mathbb{R}. \tag{19}$$

Thus, in effect, we make $\mathscr{F}(\mathbf{X}) = \tilde{\Phi}$ for all pattern of expectations $\mathbf{X}$. Building upon the idea used in this case, it should be clear how to adapt it to the context covered in the text.

Partition now the set $\tilde{\Phi}$ into the following three sub sets:

$$\tilde{\Phi}_1 = \{\ \phi \in \tilde{\Phi}\colon \phi_A(\cdot) - \phi_B(\cdot) > 0\ \},$$
$$\tilde{\Phi}_2 = \{\ \phi \in \tilde{\Phi}\colon \phi_A(\cdot) - \phi_B(\cdot) < 0\ \},$$
$$\tilde{\Phi}_3 = \{\ \phi \in \tilde{\Phi}\colon \phi_A(\cdot) - \phi_B(\cdot) = 0\ \}.$$

Consider now the pattern of equilibrium expectations $\mathbf{X}^* = \left\{ x^*(\cdot\,|\,\phi) \right\}_{\phi \in \tilde{\Phi}}$ defined as follows:

(a) For $\phi \in \tilde{\Phi}_1$, $x^*(\cdot|\phi) = x''(\cdot|\bar{\phi})$, as described above.
(b) For $\phi \in \tilde{\Phi}_2$, $x^*(\cdot|\phi) = x'(\cdot|\bar{\phi})$, as described above.
(c) For $\phi \in \tilde{\Phi}_3$, $x^*(t|\phi) = x_0$, for all $t \geq 0$.

Clearly, since the function $g(\cdot)$ (cf. (15)) has been assumed continuous with $g(0) = 0$, no payoff-maximal policy exists in response to the previous pattern of expectations when friction θ/p is small (cf. (17)). However, for any arbitrary $\varepsilon > 0$, a policy $\phi^* \in \tilde{\Phi}_1$, $\phi^*(\cdot) = (\rho_A^*, \rho_B^*)$, with $\rho_B^* = 0$ and $\rho_A^* = \eta$ for sufficiently small $\eta > 0$ (cf. (17)) induces an ε-optimal policy.

Proof of Proposition 1: The proof involves the following two steps:
(i) The continuous-time framework can be approximated by one where time intervals are discrete, and for which an intertemporal equilibrium can be ensured from available existence results.
(ii) As the length of the time interval converges to zero, any discrete-time equilibrium converges to one in continuous time.

Consider a framework as described in the text, except that agents can only change their actions at points in time $\tau = 0, \delta, 2\delta, 3\delta, \ldots$, for some $\delta > 0$. In analogy with the continuous-time context, the fraction of the population which is allowed to change their action at each point in time is given by $p\delta$, where $p > 0$ and δ is chosen so that $p\delta < 1$.

In this context, we may invoke the existence result of Jovanovic & Rosenthal (1988) to assert that, given any policy function ϕ and some initial conditions $x_0 \in [0,1]$, a subgame perfect equilibrium exists. Along this equilibrium, denote by $W_A(\tau;\delta)$ and $W_B(\tau;\delta)$ the discounted expected value of adopting action A or B at $\tau = 0,\delta,2\delta,\ldots$, and by $\hat{x}(\tau;\delta)$ the corresponding fraction of the population which plays action A. In analogy with (10) in Definition 1, we must have:

$$\begin{aligned}
\hat{x}(\tau+\delta;\delta)-\hat{x}(\tau;\delta) &= p\delta\left(1-\hat{x}(\tau;\delta)\right) \quad \text{if} \quad W_A(\tau;\delta) > W_B(\tau;\delta),\\
\hat{x}(\tau+\delta;\delta)-\hat{x}(\tau;\delta) &= -\,p\delta\,\hat{x}(\tau;\delta) \quad \text{if} \quad W_A(\tau;\delta) < W_B(\tau;\delta), \qquad (20)\\
-\,p\delta\,\hat{x}(\tau;\delta) \leq \hat{x}(\tau+\delta;\delta)-\hat{x}(\tau;\delta) &\leq p\delta\left(1-\hat{x}(\tau;\delta)\right) \quad \text{if} \quad W_A(\tau;\delta) = W_B(\tau;\delta).
\end{aligned}$$

Extend now the functions $W_A(\cdot;\delta)$, $W_B(\cdot;\delta)$, and $\hat{x}(\cdot;\delta)$ to all $t \geq 0$, making:

$$\forall t \in [\tau,\tau+\delta),\ W_h(t;\delta) = W_h(\tau;\delta),\ h = A,B,\ \hat{x}(t) = \hat{x}(\tau;\delta). \tag{21}$$

Given δ, the functions $W_A(\cdot;\delta)$, $W_B(\cdot;\delta)$, and $\hat{x}(\cdot;\delta)$ are absolutely bounded functions defined on the real line. This set is compact with respect to the topology of pointwise convergence. Thus, consider any sequence $\{\delta(k)\}$, $\delta(k) \longrightarrow 0$, and the associated functional sequences $\{W_A(\cdot;\delta(k))\}$, $\{W_B(\cdot;\delta(k))\}$, and $\{\hat{x}(\cdot;\delta(k))\}$. There is some subsequence $\{\delta(k_j)\} \subseteq \{\delta(k)\}$ such that:

$$\begin{aligned} &\{W_A(\cdot;\delta(k_j))\} \longrightarrow W_A^*(\cdot),\\ &\{W_B(\cdot;\delta(k_j))\} \longrightarrow W_B^*(\cdot),\\ &\{\hat{x}(\cdot;\delta(k_j))\} \longrightarrow x^*(\cdot). \end{aligned} \tag{22}$$

These functions obviously define a continuous-time ϕ-equilibrium, as described in Definition 1. This completes the proof of the first part of the Proposition.

The proof of its second part requires showing that if the policy function ϕ belongs to Φ^m, the corresponding ϕ-equilibrium can be chosen Markovian, i.e., with its strategies only depending on the current state. (9) As before, the proof relies on showing that, for the above discretization of the time framework, a Markovian equilibrium always exists. This involves a standard fixed-point argument which is now sketched.

Consider, for each action $h = A,B$, a function U_h: $[0,1] \longrightarrow \mathbb{R}$, with the following interpretation: $U_h(x)$ represents the discounted payoff of playing action h when the current profile is $x \in [0,1]$. With this interpretation, any individual which has to choose an action when the current profile is x will chose action A with any probability $\alpha(x)$ that solves the following problem:

$$\max_{\alpha(x)\in[0,1]} \quad \alpha(x)\, U_A(x) + \big(1-\alpha(x)\big)\, U_B(x). \tag{23}$$

9 Note that the existence of a Markovian equilibrium does not follow from the work of Jovanovic & Rosenthal (1988). Indeed, they state in one of their remarks that, although such existence is conjectured, they have been unable to show it for their very general context. In the present particular context, it is easy to confirm it, as shown in the text.

Denote by Γ the set of all functions α: $[0,1] \longrightarrow [0,1]$ and by Ξ the set of all pairs of functions $U \equiv \big(U_A(\cdot),U_B(\cdot)\big)$, which are uniformly bounded by some M. Further endow the sets Γ and Ξ with the topology of pointwise convergence. Define then the correspondence

$$\psi: \Xi \longrightarrow\!\!\!\!\rightarrow \Gamma, \tag{24}$$

where for all $U \in \Xi$, $\psi(U)$ consists of all those functions α which, for every $x \in [0,1]$, have $\alpha(x)$ be a solution to (23). This correspondence is obviously closed, convex, and upper-hemicontinuous.

Consider now the function

$$\varphi: \Gamma \longrightarrow \Xi, \tag{25}$$

where, for all $\alpha \in \Gamma$, $U(\alpha) = \big(U_h(\alpha)(\cdot)\big)_{h=A,B}$ defines, given the parameters of the model, the value of playing each action h action when the population is adopting the Markovian strategy $\alpha(\cdot)$ when performing any change of action. This function is clearly continuous. Thus, consider the product correspondence:

$$\psi{\times}\varphi: \Xi{\times}\Gamma \longrightarrow\!\!\!\!\rightarrow \Xi{\times}\Gamma. \tag{26}$$

This correspondence is well defined if M above is chosen large enough. Moreover, it satisfies the hypotheses of the standard Fixed-Point theorems of Fan (1952) or Glicksberg (1952). Therefore, it has one fixed point, which defines a Markovian equilibrium. The proof of the Proposition is thus complete. ∎

<u>Proof of Theorem 1</u>: I start with some preliminary notation. Let $\zeta \in (0,1)$ be defined by:

$$\zeta \equiv \frac{b-c}{(a-d) + (b-c)}. \tag{27}$$

The population profile ζ represents the threshold such that $\varphi_A(x) \geq \varphi_B(x)$ if, and only if, $x \geq \zeta$. Let $\gamma(x) \equiv x\ \varphi_A(x) + (1-x)\ \varphi_B(x)$ denote the average payoff induced by population profile $x \in [0,1]$. Define:

$$\tilde{\zeta} \equiv \max\ \{\ 0,\ \frac{(b-c) + (b-d)}{(a-d) + (b-c)}\ \}. \tag{28}$$

If $\tilde{\zeta} > 0$, it represents the unique interior population profile such that $\gamma(\tilde{\zeta}) = \gamma(0) = b$. Moreover, it can be easily verified that:

$$\forall x \geq \tilde{\zeta},\ \gamma'(x) > 0. \tag{29}$$

Finally, we shall denote

$$\xi \equiv \max\ \{\ \zeta\ ,\ \tilde{\zeta}\ \}. \tag{30}$$

For convenience in the exposition, I first address the proof of Part (b). Choose $\delta > 0$ sufficiently small, and let $\hat{X} = \{\hat{x}(\cdot\,|\,\phi)\}_{\phi\in\Phi^o}$ be a pattern of expectations which satisfies, for any $\phi \in \Phi^o$ and $\underline{t} \geq 0$, the following condition:

$$\Big[\forall t \geq \underline{t},\ |\phi_h(\cdot,t)| \leq \delta,\ h = A,B\Big] \Longrightarrow \Big[\forall t \geq \underline{t},\ \frac{d\hat{x}(t\,|\,\phi)}{dt^+} = -\ p\ \hat{x}(t\,|\,\phi)\Big]. \tag{31}$$

First, the existence of a pattern of expectations satisfying (31) needs to be confirmed. It follows directly from a straightforward adaptation of Lemma 1 in Matsui & Matsuyama (1991), whose argument is now sketched for the sake of completeness.

Consider any $\phi \in \Phi^o$ and $\underline{t} \geq 0$ which meet the hypothesis of (31). The essential point to show is that, if the path $\hat{x}(\cdot\,|\,\phi)$ is governed after $\underline{t}$ by the differential equation in (31), i.e., it is given by:

$$\hat{x}(t\,|\,\phi) = \hat{x}(\underline{t}\,|\,\phi)\ e^{-p(t-\underline{t})},\ \forall t \geq \underline{t}, \tag{32}$$

then:

$$V_B(\phi,\hat{x},t) \geq V_A(\phi,\hat{x},t),\ \forall t \geq \underline{t}. \tag{33}$$

By the hypothesis of (31), $|\phi_h(\cdot,t)| \leq \delta$ for each $h = A,B$ and all $t \geq \underline{t}$. Thus, (33) obtains if:

$$\forall t \geq \underline{t},\quad \int_{s=t}^{\infty} \Big[\varphi_B\big(\hat{x}(s\,|\,\phi)\big) - \varphi_A\big(\hat{x}(s\,|\,\phi)\big) - \delta\Big]\ e^{-(p+\theta)s}\ ds \geq 0, \tag{34}$$

which, from (32), can be rewritten as follows:

$$\forall t \geq \underline{t},\ e^{-(p+\theta)t} \int_{s=0}^{\infty} \left[\left((a-d)+(b-c)\right)\left(\zeta - \hat{x}(t|\phi)e^{-ps}\right) - \delta\right] e^{-(p+\theta)s}\, ds \geq 0, \tag{35}$$

where recall the definition of ζ in (27). Computation of the above integral yields the following equivalent expression:

$$\forall t \geq \underline{t},\ [(a-d)+(b-c)] \left[\frac{\zeta}{p+\theta} - \frac{\hat{x}(t|\phi)}{2p+\theta} \right] - \frac{\delta}{p+\theta} \geq 0, \tag{36}$$

or, multiplying by p and rearranging terms:

$$\forall t \geq \underline{t},\ [(a-d)+(b-c)] \left[\frac{\zeta}{1+\theta/p} - \frac{\hat{x}(t|\phi)}{2+\theta/p} \right] - \frac{\delta}{1+\theta/p} \geq 0. \tag{37}$$

Under the assumption that a-d < b-c, we have $\zeta > 1/2$. Thus, there is some $\chi > 0$ such that if $\theta/p \leq \chi$, (37) holds for sufficiently small δ, as desired.

The previous argument shows that, for all $\phi \in \Phi^o$ and $\underline{t} \geq 0$ which meet the hypothesis of (31), there is a continuation equilibrium path after $\underline{t}$ which satisfies (32) and (33) for <u>any</u> profile $x(\underline{t}|\phi)$ prevailing at $\underline{t}$. Thus, taking such population behavior after $\underline{t}$ as given, an adaptation of the existence proof of Proposition 1 to the time interval $[0,\underline{t}]$ implies (using a "backwards induction" argument) that a ϕ-equilibrium which satisfies condition (31) always exists. But then, since the hypothesis of this condition must hold for every ε-optimal policy $\hat{\phi}$ at some finite $\underline{t}$ (recall (12)), it follows that $\hat{x}(t|\hat{\phi}) \longrightarrow 0$ as $t \longrightarrow \infty$. This completes the proof of Part (b) of Theorem 1.

I address next the proof of Part (a) of the theorem. It is decomposed into the following two steps.

(i) There is a policy $\tilde{\phi}$ such that an associated $\tilde{\phi}$-equilibrium $\hat{x}(\cdot|\tilde{\phi})$ exists such that $\hat{x}(t|\tilde{\phi}) \longrightarrow 1$ and $\tilde{\phi}$ is "feasible", i.e., satisfies (12) and (13) above.

(ii) A pattern of expectations $\tilde{X}$ may be constructed which includes the preceding $\tilde{\phi}$-equilibrium and is such that every open-loop policy $\phi \neq \tilde{\phi}$ yields a lower planner's payoff if friction is sufficiently small.

Step (i). Define the path:

$$\frac{dx^*(t)}{dt^+} = p\big(1 - x^*(t)\big);\ x^*(0) = x_0, \tag{38}$$

and consider the open-loop policy function $\tilde{\phi} \equiv (\tilde{\phi}_A, \tilde{\phi}_B)$: $\mathbb{R}_+ \longrightarrow \mathbb{R}^2$ defined as follows:[10]

$$\begin{aligned} \tilde{\phi}_A(t) &= \max\ \Big\{\ 0,\ dx^*(t) + b[1-x^*(t)] - \big(ax^*(t) + c[1-x^*(t)]\big)\ \Big\}, \\ \tilde{\phi}_B(t) &= 0. \end{aligned} \tag{39}$$

If the population predicts the path (38) and the policy function is given by (39), this path obviously defines a $\tilde{\phi}$-equilibrium since then action A becomes weakly dominant for all t.

It is now verified that, along the path $x^*(\cdot)$, policy $\tilde{\phi}$ is indeed "short-run" (i.e., satisfies (12)) and is individually rational (satisfies (13)). The first of this requirements is immediate since, beyond the time t for which $x^*(t) = \zeta$, (39) prescribes no further intervention. In order to verify individual rationality, it is enough to confirm that:

$$\begin{aligned} &\varphi_B\big(x^*(t)\big) + \phi_B\big(x^*(t),t\big) - B\big(x^*(t),t;\tilde{\phi}\big) = \\ &dx^*(t) + b[1-x^*(t)] - x^*(t)\ \max\ \Big\{0,\big((d-a)x^*(t) + (b-c)[1-x^*(t)]\big)\Big\} \geq 0 \end{aligned} \tag{40}$$

Define:

$$\rho(x) \equiv dx + b(1-x) - x\ \big((d-a)x + (b-c)(1-x)\big) \tag{41}$$

The inequality in (40) will obtain if

$$\hat{\rho} \equiv \min\ \{\rho(x):\ 0 \leq x \leq 1\} \geq 0. \tag{42}$$

Differentiating in (41), it is verified that the function $\rho(\cdot)$ is convex, and its (global) minimum is attained at $x = \tilde{\zeta}/2$ (cf. (28)). If $\tilde{\zeta} = 0$, $\hat{\rho} = \rho(0) = b > 0$. If instead $\tilde{\zeta} > 0$, straightforward computation shows that:

$$\rho(\tilde{\zeta}/2) = \frac{4ab - (c+d)^2}{4[(a-d)+(b-c)]} \tag{43}$$

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10 For notational simplicity, the first (redundant) argument of the open-loop policy functions is not explicitly included throughout this proof.

Consider first the case where $d \leq b$.[11] Then, the facts that $a \geq b > c$ and $a > d$ directly imply that $\rho(\tilde{\zeta}/2) > 0$. Thus, suppose alternatively that $d > b$. Two subcases need to be considered. First, when $d-b > b-c$. This possibility may be ignored since it implies that $\tilde{\zeta} = 0$. Therefore, assume that $d-b \leq b-c$. In this case, it may be computed:

$$\begin{aligned} 4ab - (c+d)^2 &= 4ab - (d-b)d - bd - c^2 - 2cd \\ &> 4ab - (b-c)d - bd - c^2 - 2cd \\ &= 4ab - 2bd - cd - c^2 > 0. \end{aligned} \tag{44}$$

Thus, from (43), $\rho(\tilde{\zeta}/2) > 0$, as desired. This completes step (i) of the proof.

Step (ii). Choose a collection of population profile paths $\tilde{X} = \{\tilde{x}(\cdot\,|\,\phi)\}_{\phi\in\Phi^o}$ as follows. For policy $\tilde{\phi}$ (cf. (39)), $\tilde{x}(\cdot\,|\,\tilde{\phi}) = x^*(\cdot)$, as defined in (38). For any other $\phi \neq \tilde{\phi}$, construct $\tilde{x}(\cdot\,|\,\phi)$ to satisfy, for some $\hat{G} > 0$ and $T > 0$, the following condition:

$$\frac{1}{T}\int_{t=0}^{T} G(\phi,\tilde{X}) \leq \hat{G} \Rightarrow \frac{1}{T}\int_{t=0}^{T} \tilde{x}(\cdot\,|\,\phi) \leq \xi/2. \tag{45}$$

If $\hat{G}$ is sufficiently small and (given $\hat{G}$) T is large enough, the line of proof used above for Part (b) can be easily adapted to establish that, under low friction θ/p, every $\tilde{x}(\cdot\,|\,\phi)$ can be chosen to be a ϕ-equilibrium.[12]

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11 In fact, this possibility is incompatible with the assumed risk-dominance of the inefficient equilibrium (B,B). However, we consider explicitly this possibility since the argument used here will be applied below to prove som results below where such risk-dominance is not assumed.

12 To avoid repetitiveness, I dispense with a detailed argument. Informally, if the average subsidy in [0,T] is low, there are appropriate expectations (cf.(37)) which shall lead the population to spend a large fraction of this time span in a neighborhood of the profile x=0. In order to have such expectations be part of an equilibrium, one simply needs to construct "continuation paths" (see above) in a suitable fashion. At this point, one must rely on the fact (which follows again from Lemma 1 of Matsui & Matsuyama (1991)) that all equilibrium paths which start close to x=0 must remain close to it unless the planner's subsidies yield otherwise.

I now argue that, when confronting a pattern of expectations $\tilde{X} = \{\tilde{x}(\cdot\,|\,\phi)\}_{\phi\in\Phi^o}$ as described, policy $\tilde{\phi}$ is optimal for the planner if θ/p is small and T is large enough. Let

$$\hat{t} = -\frac{1}{p}\,ln\,(1-\xi). \tag{46}$$

Then,

$$\forall t \geq \hat{t},\ \tilde{x}(t\,|\,\tilde{\phi}) \geq \xi. \tag{47}$$

Because of (30), we have that, for all $\phi \in \Phi^o$:

$$\int_{t=\hat{t}}^{\infty} \left[\tilde{x}(t|\tilde{\phi})\ \varphi_A\big(\tilde{x}(t|\tilde{\phi})\big) + \big(1-\tilde{x}(t|\tilde{\phi})\big)\ \varphi_B\big(\tilde{x}(t|\tilde{\phi})\big)\right] e^{-\theta t}\ dt \ \geq$$

$$\int_{t=\hat{t}}^{\infty} \left[\tilde{x}(t|\phi)\ \varphi_A\big(\tilde{x}(t|\phi)\big) + \big(1-\tilde{x}(t|\phi)\big)\ \varphi_B\big(\tilde{x}(t|\phi)\big)\right] e^{-\theta t}\ dt. \tag{48}$$

For simplicity in the argument, fix p (and $\hat{t}$). Then, there exists some H, independent of θ, such that:

$$\forall\phi \in \Phi^o,\ F(\tilde{\phi},\tilde{X}) \geq F(\phi,\tilde{X}) - H. \tag{49}$$

Combining (45) and (49), it follows that (given $\hat{G}$) θ can be chosen sufficiently small and T large enough such that $\pi(\tilde{\phi},\tilde{X}) > \pi(\phi,\tilde{X})$ for all $\phi \neq \tilde{\phi}$. This completes the proof of the Theorem. ▮

<u>Proof of Theorem 2</u>: Consider any given pattern of expectations $\hat{X} = \{\hat{x}(\cdot\,|\,\phi)\}_{\phi\in\Phi^m}$ and let $\hat{\phi}$ be a policy function such that:

$$\forall t \geq 0,\ \hat{x}(t\,|\,\hat{\phi}) \leq \xi. \tag{50}$$

(For simplicity in the argument, I assume implicitly that initial conditions $x_0 \leq \xi$. Otherwise, the ε-optimality of the type of policy function described in (51) below follows immediately, for any $\varepsilon > 0$, from a direct argument.)

It will be shown first that, if friction θ/p is sufficiently small, any such $\hat{\phi}$ can <u>not</u> be ε-optimal for sufficiently small $\varepsilon > 0$. Given some $\eta > 0$, sufficiently small, consider the Markovian policy function $\phi^* \equiv (\phi_A^*, \phi_B^*)$: $[0,1] \longrightarrow \mathbb{R}$, defined as follows:(13)

$$\begin{aligned} \phi_A^*(x) &= \max \left\{ 0, \varphi_B(x) - \varphi_A(x) + \eta \right\}, \\ \phi_B^*(x) &= 0. \end{aligned} \tag{51}$$

Since such policy function renders the instantaneous payoff of action A strictly larger than that of B at every point in time, it must be that:

$$\frac{d\hat{x}(t|\phi^*)}{dt^+} = p\left(1-\hat{x}(t|\phi^*)\right). \tag{52}$$

This obviously implies that ϕ^* satisfies requirement (12) - no subsidy at all is required after some date if η is chosen small enough. On the other hand, by pursuing the argument used in the proof of Theorem 1, it can also be shown to satisfy (13). Thus, ϕ^* is a feasible policy belonging to $\mathcal{F}(\hat{X})$.

Consider now some given $\xi' > \xi$ and define:

$$\tilde{t} \equiv \frac{1}{p}\,[ln(1-x_0) - ln(1-\xi')]. \tag{53}$$

We have that:

$$\forall t \geq \tilde{t},\ \hat{x}(t|\phi^*) \geq \xi'. \tag{54}$$

Thus, denoting by $\bar{g}$ some appropriate (finite) upper bound on instantaneous implementation costs (dependent on η, but independent of θ or p), we can write:

$$\begin{aligned} \pi(\phi^*,\hat{X}) &> -\int_{t=0}^{\tilde{t}} \bar{g}e^{-\theta t}\,dt + \int_{t=\tilde{t}}^{\infty} \gamma(\xi')e^{-\theta t}\,dt \\ &= -\frac{\bar{g}}{\theta}\left(1 - e^{-\theta\tilde{t}}\right) + \frac{\gamma(\xi')}{\theta}\,e^{-\theta\tilde{t}} \end{aligned} \tag{55}$$

On the other hand, we have:

$$\pi(\hat{\phi},\hat{X}) < \int_{t=0}^{\infty} \gamma(\xi)\,e^{-\theta t}\,dt = \frac{\gamma(\xi)}{\theta}\,. \tag{56}$$

13 For notational simplicity, the second (redundant) argument of the open-loop policy functions is not explicitly included throughout this proof.

Therefore, given $\varepsilon > 0$, the inequality

$$(1-\varepsilon)\ \pi(\phi^*,\hat{X}) > \pi(\hat{\phi},\hat{X}), \tag{57}$$

will hold if:

$$(1-\varepsilon)\ [\frac{\gamma(\xi')}{\theta} - \frac{\bar{g}}{\theta}\left(1 - e^{-\theta\tilde{t}}\right)] > \frac{\gamma(\xi)}{\theta}\ . \tag{58}$$

Multiplying through by $\theta > 0$, (29) implies that (58) obtains for sufficiently small ε as long as

$$\theta\tilde{t} = \frac{\theta}{p}\ [ln(1-x_0) - ln(1-\xi')] \tag{59}$$

is small enough. This, in turn, depends on friction being sufficiently small, which establishes the desired preliminary conclusion.

In order to complete the proof of the theorem, we proceed as follows. By the previous considerations, given any pattern of expectations $\hat{X}$, there exists some $\bar{\varepsilon} > 0$ such that if policy ϕ is ε-optimal for $\varepsilon \leq \bar{\varepsilon}$, then $\hat{x}(t|\phi) \geq \xi$ at some time t. But then, if $\hat{X}$ is Markovian, it must be the case (since $x_0 < \xi$) that $\hat{x}(t'|\phi) \geq \xi$ for all $t' > t$. (Essentially, the argument here is that since the autonomous one-dimensional dynamical system points rightwards at $x = \xi$, the process cannot leave the subinterval $[\xi,1]$ once it has entered it.) On the other hand, the requirement that ϕ be "short-run" (cf. (12)) requires that:

$$\forall\delta > 0,\ \exists T\!:\ \forall t \geq T,\ \left|S_h\left(\hat{x}(t|\phi),t;\phi\right)\right| \leq \delta. \tag{60}$$

This implies that:

$$\begin{array}{lll} \text{(i)}\ \ \phi_A\left(\hat{x}(t|\phi),t\right) \longrightarrow 0, & \text{and} & \\ \text{(ii)}\ \ \hat{x}(t|\phi) \longrightarrow 1, & \text{or} & \phi_B\left(\hat{x}(t|\phi),t\right) \longrightarrow 0. \end{array} \tag{61}$$

By (ii) above, either the desired convergence obtains or the subsidy to action B vanishes in the limit. If the former cannot be ensured, then the latter and (i) imply that action A yields a strictly higher instantaneous payoff than B beyond some point in time. This also guarantees the convergence $\hat{x}(t|\phi) \longrightarrow 1$, thus completing the proof of the Theorem. ∎

Proof of Theorem 3: Consider the policy ϕ^* defined in (51) for some sufficiently small η. It needs to be shown that, given any $\varepsilon > 0$, there exists some $\chi > 0$ such that if $\theta/p \le \chi$, then:

$$\pi(\phi^*,\mathbf{X}) \ge (1-\varepsilon)\ \pi(\phi,\mathbf{X}) \tag{62}$$

for all $\phi \in \Phi$ and any pattern of expectations $\mathbf{X}$. Define:

$$M(\theta) \equiv \int_0^\infty a\ e^{-\theta t}\ dt = \frac{a}{\theta}, \tag{63}$$

i.e., the maximum discounted payoff which the planner may obtain over all initial conditions and patterns of expectations. To show the desired conclusion, it is enough to prove that, for any pattern of expectations $\mathbf{X}$,

$$\pi(\phi^*,\mathbf{X}) \ge (1-\varepsilon)\ M(\theta). \tag{64}$$

Define ξ as in (30) and let $\hat{\xi} \ge \xi$ be such that, for all $x \ge \hat{\xi}$,

$$\begin{aligned} x\ \varphi_A(x) + (1-x)\ \varphi_B(x) &\ge (1-\frac{\varepsilon}{2})\ a, \\ \varphi_A(x) - \varphi_B(x) &\ge \eta. \end{aligned} \tag{65}$$

Furthermore, denote

$$\hat{t} \equiv \frac{1}{p}\ [ln(1-x_0) - ln(1-\hat{\xi})], \tag{66}$$

and, as above, let $\bar{g}$ be some appropriate (finite) upper bound on instantaneous implementation costs along the path $\mathbf{x}(\cdot\,|\phi^*)$. (Recall that such upper bound can be chosen independently of θ and p.) We can then write:

$$\begin{aligned} \pi(\phi^*,\mathbf{X}) &\ge \int_{t=\hat{t}}^{\infty} (1-\frac{\varepsilon}{2})a\ e^{-\theta t}\ dt - \int_{t=0}^{\hat{t}} \bar{g}\ e^{-\theta t}\ dt \\ &= \Big((1-\varepsilon)a + \frac{\varepsilon}{2}\ a\Big)\int_{t=\hat{t}}^{\infty} e^{-\theta t}\ dt - \int_{t=0}^{\hat{t}} \bar{g}\ e^{-\theta t}\ dt \\ &= (1-\varepsilon)a \int_{t=0}^{\infty} e^{-\theta t}\ dt \\ &\quad + \frac{\varepsilon}{2}\ a\int_{t=\hat{t}}^{\infty} e^{-\theta t}\ dt - \Big(\bar{g} + (1-\varepsilon)a\Big) \int_{t=0}^{\hat{t}} e^{-\theta t}\ dt \end{aligned} \tag{67}$$

Thus,

$$\pi(\phi^*,X) \geq (1-\varepsilon)\ M(\theta) = (1-\varepsilon)a \int_{t=0}^{\infty} e^{-\theta t}\, dt \tag{68}$$

if, and only if:

$$\frac{\varepsilon}{2}\, a \int_{\hat{t}}^{\infty} e^{-\theta t}\, dt \geq \left(\bar{g} + (1-\varepsilon)a\right) \int_{t=0}^{\hat{t}} e^{-\theta t}\, dt, \tag{69}$$

which, from (66), is obviously satisfied if θ/p is sufficiently small. This completes the proof of the theorem. ∎

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