DISTRIBUTION OF INCOME AND AGGREGATION OF DEMAND*

Francisco Marhuenda**

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** University of Alicante.

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ABSTRACT

We show that, under certain regularity conditions, if the distribution of income is price independent and satisfies a condition on the shape of its graph, then total market demand, $F(p)$, is monotone, i.e., given two positive prices p, and q, one has $(p - q) \cdot (F(p) - F(q)) < 0$. Similar assumptions on the distributions of endowments, yield a restricted monotonicity property on aggregate excess demand, where, now, wealth is determined by market prices. This is enough, however, to obtain uniqueness and stability of equilibrium for our Walrasian pure exchange model.

1 INTRODUCTION

A satisfactory theory linking the known microeconomic features of consumers and the assumed macroeconomic properties of aggregated systems is still missing. The major stumbling block to fill the gap between these two fields, is how to find a way around Sonnenschein's indeterminacy theorem. Sonnenschein's ([15, 16, 17]) result states that any continuous, mapping that satisfies Walras' law and is homogeneous of degree 0, coincides with the excess demand function of a certain economy.

This has some unpleasant consequences. One of these, is that it permits the theoretical possibility of many equilibria. Furthermore, it questions the regularity (in terms of smoothness, like differentiability) of market demand and the stability of the equilibrium prices.

The problem of smoothness of market demand was satisfactorily attacked in [4]. The authors there show that conveniently dispersed distributions of preferences and wealth lead to a continuous or even C^1 demand function. The interested reader is referred to [18] for a survey on market demand in large economies with non-convex preferences and for instances in which aggregation has a smoothing effect.

Two main lines of research have dealt with the remaining issues, i.e. uniqueness and stability of equilibrium prices. One of them was initiated by W. Hildenbrand ([9]) and developed by W. Härdle, W. Hildenbrand and M. Jerison in [8]. The gist of their method is to put restrictions on the *shape* of the distribution of *income*. The upshot of this approach is that one obtains that market demand is monotone (see Section below). In particular, this implies that market demand for a particular commodity is a decreasing function of its price, and that the weak axiom of revealed preference holds for the aggregate.

However, the restrictions imposed are rather stringent: Firstly, income is price

independent. Moreover, it is assumed that the distribution of income has a continuous non-increasing density with bounded support.

With respect to the first assumption, it has been extended to the case in which individual endowments are collinear ([9],[13]). In [1], the second assumption has been relaxed, to other types of densities for the case of identical consumers the Engel curves of whom have a specific functional form.

A second line of argument has been put forth by J. M. Grandmont ([5]). His approach is to impose restrictions on the *shape of the agents' characteristics* rather than on the distribution of income. In his work the use of a very particular linear structure on the space of demand functions, named the α-transform, is essential.

As a result the author proves very nice properties of the aggregate demand: market demand has a dominant diagonal Jacobian matrix and aggregate excess demand has the gross substituability property. It follows then that there is a unique equilibrium which is globally stable under the usual tâtonnement process.

There are still some unsatisfactory features in the aforementioned viewpoint: One of them is that the way the α-transform is used does not have a straightforward economics interpretation.

Further, the so-called hypothesis of *aggregate desirability* which also appears in a critical way, in the said work, is not satisfied by many of the working examples of demand models (e.g. CES, Diewert or Addilog). Even though one could argue that it may not be met individually and yet hold in the aggregate, it certainly puts some constraints on the type of demand functions allowed.

The present work goes back to the first formulation. The key idea is again to control the "pathologies" of the income term in Slutsky's equation

$$DF(p) = S(p) - A(p),$$

where DF is the Jacobian matrix of mean demand, S is the average Slutsky compensated matrix and A is the average income matrix. We do this by way of conditions in the shape of either income or initial endowments.

The novelty is twofold: First, in the case of price independent wealth, we allow for some increasing densities while the Law of Demand still holds for all prices. In particular, our result applies to some subfamily of lognormal distributions.

Secondly, for the case of pure market exchange economies, we obtain uniqueness of price equilibrium and local stability for the unique equilibrium, when initial endowments need not be necessarily collinear.

If one imposes a further restriction on individual demand functions, then it is possible to obtain global stability of the unique equilibrium.

One further characteristic of our present work, already present in [5], is that the matrix A does not need to be positive semidefinite, i.e. it may have some negative eigenvalues. We do make use however of the hypothesis that S is negative semidefinite.

The paper is organized as follows. In Section 2, we study the problem of aggregation demand for a model in which income is independent of prices. We introduce the notion of metonymy, which was already present in the work of W. Härdle, W. Hildenbrand and M. Jerison ([8]) and is also essential in the present one.

We prove that if the the shape of the density of the distribution of income satisfies a certain restriction, the Law of Demand holds for all prices.

Essentially the same proof shows that it is possible to obtain the monotonicity of market demand for all prices by imposing that the second eigenvalue of the Slutsky compensated term be strictly negative as prices go to the boundary of the positive orthant, for at least one agent in the economy. This is a condition on consumers'

preferences. It is satisfied, for example, if there is at least one consumer with a CES utility function.

In Section 3 we apply these ideas to the setting in which income is determined by the market price of initial endowments. The Law of Demand cannot hold in this case. However, a limited version of it can be shown, and this restricted variation is enough to yield uniqueness. By strengthening the assumptions, the weak axiom of revealed preference in the aggregate follows.

These properties of total market demand have a counterpart in the stability of the model under a "Walrasian auctioneer" mechanism.

In order to facilitate the reading all the proofs are relegated to the Appendix. The reader is referred there also for an explanation of the notation used through out this work.

2 MARKET DEMAND

We consider an economy with n goods and a continuum of agents. Consumers will be

distinguished from each other by their preferences and income, which for the moment will be assumed to be exogeneously given. Typically, the characteristics of each agent in the economy will be determined by his individual demand function and income.

Definition 2.1 *An individual demand function is a C^1 mapping $f : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$, satisfying the following properties:*

(i) Budget identity: $p \cdot f(p, w) = w$;

(ii) Weak Axiom of Revealed Preference: for every (p, w) and (p', w') in $\mathbb{R}^n_{++} \times \mathbb{R}_+$, $p' \cdot f(p, w) \leq w'$ implies $p \cdot f(p', w') \geq w$.

Here $p \in \mathbb{R}^n_{++}$ denotes the vector of prices and $w \in \mathbb{R}_+$ is the consumer's budget. It follows from this definition that individual demand functions are homogeneous of degree 0 in (p, w). We note that demand functions for consumers are defined only for prices in the interior of the positive orthant.

It is also a well known fact that the axiom of revealed preference as stated above implies that the Slutsky substitution matrix, $S(p, w)$, with entries given by

$$s_{ij} = \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial w} f_j \tag{2.1}$$

is negative semidefinite, i.e. for every $x \in \mathbb{R}^n$, $< S(p, w)x, x > \leq 0$. The rank of the Slutsky matrix can be at most $n - 1$, since $S(p, w) \cdot p = 0$.

Definition 2.2 *An individual demand function is said to be regular if it is continuously differentiable for $p \in \mathbb{R}^n_{++}$ and for every $(p, w) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, the Slutsky matrix $S(p, w)$ is symmetric.*

Apart from differentiability of the demand functions, $f(p, w)$, the other conditions can be derived if one assumes they arise from continuous, strictly convex and non-saturated preference relations.

We will assume that the behavior of each consumer is described by a demand function as in the above definition (which may be taken as the solution to the usual individual utility maximization problem) and an income level. Here we assume income to be exogenously given and independent of prices. The general case, with income depending on prices will be studied in the next Section.

A market economy, say E, is a triple, $((\mathcal{A}, \mu), \{f(\alpha, \cdot, \cdot)\}_{\alpha \in \mathcal{A}}, \omega)$, where $(\mathcal{A}, \mu)$ is a Borel space, $f(\alpha, \cdot, \cdot)$ is an individual demand function, and, for each $p \in \mathbb{R}^n_{++}$, the function $f(\cdot, p, \omega(\cdot))$, defined on $\mathcal{A}$, is continuous and integrable, i.e., $f(\cdot, p, \omega(\cdot)) \in L^1(\mathcal{A}) \cap C(\mathcal{A})$. The mapping $\omega : \mathcal{A} \to \mathbb{R}_+$, which is also integrable and continuous, represents the income level of each consumer. Thus, for each $\alpha \in \mathcal{A}$, the relevant economic features of consumer α are captured by $f(\alpha, \cdot, \cdot)$ and $\omega(\alpha)$. We shall also assume the measure μ to be atomless and regular, assigning strictly positive measure to open subsets of $\mathcal{A}$.

In this context, at a given price system p, total demand is defined by

$$F(p) = \int_{\mathcal{A}} f(\alpha, p, \omega(\alpha)) d\mu. \tag{2.2}$$

We assume that this integral is finite, $F(p)$, is differentiable and, furthermore, for p in the interior of the positive orthant we have,

$$\frac{\partial F}{\partial p_j}(p) = \int_{\mathcal{A}} \frac{\partial f}{\partial p_j}(\alpha, p, \omega(\alpha)) d\mu < \infty. \tag{2.3}$$

We shall enote by $DF(p)$ the Jacobian matrix of F at p.

Definition 2.3 *An economy E is said to be regular if 2.3 holds and for all $\alpha \in \mathcal{A}$, the mapping $f(\alpha, \cdot, \cdot)$ is a regular demand function.*

The *Law of Demand* is said to hold for the economy E, if total demand $F(p)$ is monotone, i.e. if for each $p, q \in \mathbb{R}^n_{++}$, with $p \neq q$,

$$(p - q) \cdot (F(p) - F(q)) < 0. \tag{2.4}$$

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The Law of Demand is easily obtained if it holds for each individual α at all budgets. This is the case, for example, if individual demand functions are derived from homothetic preferences. Another instance in which the Law of Demand has been obtained, [9], is for economies with identical consumers and a decreasing density of income. These results have been extended in [1] to an economy with identical consumers in which the Engel curves can be written as $\sum_{k=0}^{l} g_k(p)\phi_k(w)$ and the functions $\phi_k(w)$ satisfy certain restrictions. Economies with agents not necessarily identical have been studied in [10], again under the assumption of a decreasing density of income.

These may however be considered to be very restrictive cases. Furthermore, they do not seem to apply easily to the case of income dependent on prices, unless all endowments are collinear. In this Section we will be concerned with other situations in which the Law of Demand holds.

Recall now the

Lemma 2.4 *Let G be a cone of prices. The following conditions are equivalent*

(i) The Jacobian matrix $DF(p)$ is negative semidefinite for each $p \in G$.

(ii) For all prices $p, q \in G$, $(p - q) \cdot (F(p) - F(q)) \leq 0$.

The proof is similar to the usual case (see [12]), so we will omit it here. It is also a well known fact that the negative definiteness of $DF(p)$ implies the strict monotonicity of total demand, $F(p)$. We note, that the converse is not true.

For $w \in \mathbb{R}_+$, define $B(w) = \{\alpha \in \mathcal{A} : \omega(\alpha) = w\}$. That is, $B(w)$ is the set of consumers whose wealth is exactly w. The measure μ induces conditional distributions η_w on $B(w)$, for each w and a probability measure, v on $\mathbb{R}_+$ such that for any function $h \in L^1(\mathcal{A}, d\mu)$,

$$\int_{\mathcal{A}} h d\mu = \int_{\mathbb{R}_+} \left(\int_{B(w)} h d\eta_w \right) dv.$$

Definition 2.5 ([8]) *The measure μ is said to be metonymic if:*

(i) The measure v has an absolutely continuous density, ρ, with support contained in $\mathbb{R}_+$

(ii) For each $z \in \mathbb{R}^n$,

$$\int_{\mathbb{R}_+} \left(\int_{B(w)} \frac{\partial}{\partial w}(f \cdot z)^2 d\eta_w \right) \rho(w) dw = \int_{\mathbb{R}_+} \frac{\partial}{\partial w} \left(\int_{B(w)} (f \cdot z)^2 d\eta_w \right) \rho(w) dw$$

Condition (ii) in the above definition is fulfilled if the conditional distributions η_w do not depend on w. This is the case if $(\mathcal{A}, \mu) = (C \times D, \eta \otimes v)$ is a product space, in which (C, η) describes the distribution of types of consumers and (D, v) describes the distribution of wealth. Thus, the simplest case in which metonymy holds is if the distribution of types is the same at all income levels. This is not a necessary condition though, and it is also valid in many other instances. We refer the reader to [8] and [10] for a more detailed discussion of metonymy. Nevertheless, we remark that condition (ii) is equivalent to:

$$\text{for all } i, j, \qquad \int_{\mathbb{R}_+} \int_{B(w)} \frac{\partial}{\partial w}(f_i f_j) d\eta_w \rho(w) dw = \int_{\mathbb{R}_+} \frac{\partial}{\partial w} \left(\int_{B(w)} f_i f_j d\eta_w \right) \rho(w) dw.$$

Fix now $\alpha \in \mathcal{A}$, $p \in \mathbb{R}^n$, $p > 0$ and consider the quadratic form defined on S^{n-1},

$$S(\alpha, p, x) = \sum_{i,j=1}^n s_{ij}(\alpha, p) p_i p_j x_i x_j. \tag{2.5}$$

This function is continuous in all the arguments for $p \in \mathbb{R}^n_{++}$. From the properties of the compensated demand function, it follows that this form is negative semidefinite. In fact, for $x_0 = \frac{1}{\sqrt{n}}(1, \ldots, 1)$, we have $S(\alpha, p, x_0) = 0$. The eigenvalues of $S(\alpha, p, x)$ have the form,

$$\lambda_n(\alpha, p) \leq \lambda_{n-1}(\alpha, p) \leq \cdots \leq \lambda_2(\alpha, p) \leq \lambda_1(\alpha, p) = 0. \tag{2.6}$$

We define

$$\lambda(\alpha) = \sup_{p \in \mathbb{R}^n_{++}} \lambda_2(\alpha, p) \le 0. \tag{2.7}$$

and

$$\lambda = \int_{\mathcal{A}} \lambda(\alpha) d\mu.$$

Assume that $\lambda < 0$. Define also

$$\bar{\omega} = \int \omega(\alpha) d\mu \in \mathbb{R}^+.$$

Consider now the following quantity

$$\varepsilon = \min\{\frac{1}{2}, |\lambda|\delta\} > 0, \tag{2.8}$$

where, if π_0 denotes the orthogonal projection onto the plane $\{z : x_0 \cdot z = 0\}$, then δ is given by

$$\delta = \inf\{\|\pi_0(x)\|^2 : \|x\| = 1, \quad x \cdot x_0 \ge 0, \quad \|x - x_0\| \ge \frac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega} + 1)}\}.$$

By means of the quantity ε we can make precise the statement that by imposing certain restrictions on the shape of the density $\rho(w)$ the Law of Demand for a range of prices is obtained.

Theorem 2.6 *Let $\mathcal{E} = ((\mathcal{A}, \mu), \{f(\alpha, \cdot, \cdot)\}_{\alpha \in \mathcal{A}}, \omega)$ be a market economy and suppose the following hold:*

(i) The Economy $\mathcal{E}$ is regular.

(ii) The measure μ is metonymic.

(iii) $\lambda(\alpha) < 0$ on a set of positive measure.

Let ε be defined by 2.8. If

$$\int_{\{w:\rho'(w)>0\}} w^2 \rho'(w) dw < \varepsilon, \tag{2.9}$$

then the Law of Demand holds for all prices $p, q \in \mathbb{R}^n_{++}$, with $p \ne q$.

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Remark 2.7 The number ε appearing in Expression 2.8 depends on the density ρ through integration. As stated, Theorem 2.6 gives only a sufficient condition (which has to be checked in each particular case) for the Law of Demand to hold.

With further hypotheses it is possible to obtain a positive number ε which depends only on the set of consumers, $\mathcal{A}$ and their demand functions but not on the possible densities ρ, for which the measure μ is metonymic. Namely, suppose

$$\lambda_0 = \sup\{\lambda(\alpha) : \alpha \in \mathcal{A}\} < 0,$$

and

$$\omega_0 = \inf\{\omega(\alpha) : \alpha \in \mathcal{A}\} > 0$$

hold. Then Theorem 2.6 is still valid for

$$\varepsilon = \min\{\frac{1}{2}, |\lambda_0|\delta\} > 0, \tag{2.10}$$

with

$$\delta = \inf\{\|\pi_0(x)\|^2 : \|x\| = 1, \quad x \cdot x_0 \geq 0, \quad \|x - x_0\| \geq \frac{\bar{\omega}_0}{n^{\frac{3}{2}}(2\bar{\omega}_0 + 1)}\}.$$

If this is the case, for a fixed set of consumers, $\mathcal{A}$, with regular demand functions $f(\alpha, \cdot, \cdot)$, in the sense of Theorem 2.6, it is possible to construct metonymic measures μ, which are *not always decreasing* and for which the Law of Demand holds.

Example 2.8 We present next an example of a function, $\rho(t)$, which gives an idea of what kind of distributions we may expect theorem 2.6 to apply to.

The density distribution is piecewise linear as given in Figure 1. It attains its peak at the point t_0. Suppose we require that the total population with income less than t_0 be $a < 1$, and assume that, for that sector, the distribution ρ is given by $\rho(t) = \alpha t$. A simple computation shows that given ε as in Theorem 2.6, we may take $t_0 = \frac{3\varepsilon}{2a}$, $\alpha = \frac{8a^3}{9\varepsilon^2}$ and this distribution satisfies $\int_{\{\rho' > 0\}} t^2 \rho'(t) dt \leq \varepsilon$.
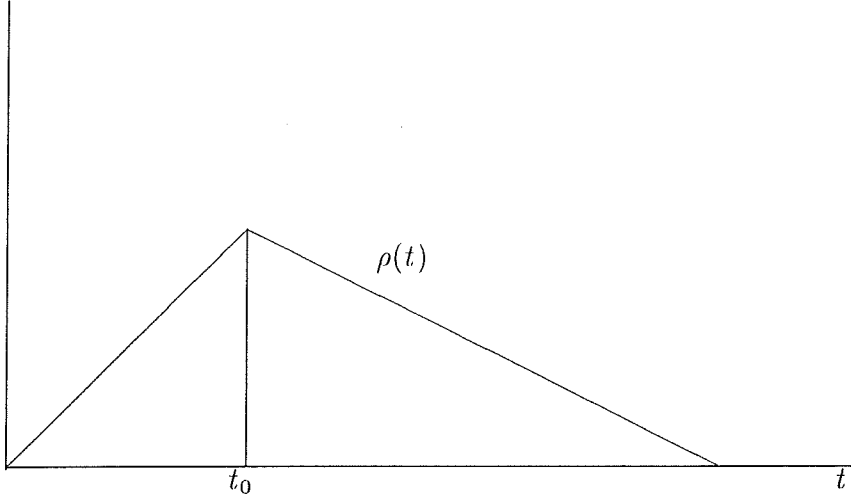
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Figure 1:

Example 2.9 It is possible to extend the above result to unimodal distributions. Namely, let $\rho(w)$ be a unimodal distribution, which attains its peak at the point $t_0 \in \mathbb{R}$. Let

$$M = \sup\{\rho'(w) : w \in [0, t_0]\}.$$

A simple calculation is therefore required to show that condition 2.9 in Theorem 2.6 is verified if $\frac{M t_0^3}{3} < \varepsilon$.

Example 2.10 Finally, we apply Example 2.9 to the case of the lognormal distribution:

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}\exp(-\frac{(\ln x - \mu)^2}{2\sigma^2}).$$

We find that this distribution attains its peak at the point $t_0 = e^{\mu - \sigma^2}$ and condition 2.9 in 2.6 is satisfied for

$$\frac{e^{-\sigma^2}}{\sigma\sqrt{2\pi}}|1 + \mu + \sigma^2 - \frac{\mu}{\sigma^2}| < \varepsilon$$

Remark 2.11 To compare the above result with the existing literature ([9],[10]) we note that we are assuming here additional regularity conditions, namely, that all demand functions be differentiable and preferences be strictly convex and non-saturated. In

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contrast, we obtain the Law of Demand for a certain type of not necessarily decreasing density distributions of income, without making strong assumptions on the Engel curves as in [1].

As noted by M. Jerison, it is not really necessary in Theorem 2.6 that $f(\alpha, p, w)$ be continuous in the first argument. It suffices to assume that $f(\alpha, p, w(\alpha))$ is integrable as a function of the variable α.

In contrast, the next result does make use of this hypothesis and it follows from Theorem 2.6 by taking into consideration the fact that $\lambda(\alpha)$ is a continuous function of α.

Theorem 2.12 *Suppose the following hold:*

  (i) *The Economy $\mathcal{E}$ is regular.*

  (ii) *The measure μ is metonymic.*

  (iii) *There is $\alpha_0 \in \mathcal{A}$, such that $\lambda(\alpha_0) < 0$.*

Then there is $\varepsilon > 0$ such that if

$$\int_{\{w: \rho'(w) > 0\}} w^2 \rho'(w) dw < \varepsilon,$$

the Law of Demand holds for all prices $p, q \in \mathbb{R}^n_{++}$.

We notice that Condition (iii), is a limitation on the consumer's individual preferences rather than one on dispersion. However, it is enough that it be satisfied for one consumer. Because, by continuity of $\lambda(\alpha)$, it is then verified for a set of consumers of positive measure. It is easily verified for CES utilitiy functions. Hence Condition (iii) is fulfilled as long as there is one agent in the economy with CES-type preferences. Remark 2.7 applies also here.

3 MARKET EXCHANGE ECONOMIES

We now apply the ideas in the preceding section to a competitive pure exchange economy. Thus, each trader's income is now given by the market value of his own endowment and is, therefore, price dependent.

In this new context, a market exchange economy will be a triple

$$((\mathcal{A}, \mu), \{f(\alpha, \cdot, \cdot)\}_{\alpha \in \mathcal{A}}, \omega).$$

The only difference with the previous Section is that $\omega \in L^1(\mathcal{A}, \mathbb{R}_+^n) \cap C(\mathcal{A}, \mathbb{R}_+^n)$, now represents initial endowments, rather than income. In other words, we suppose that each consumer $\alpha \in \mathcal{A}$ has an endowment $\omega(\alpha) \in \mathbb{R}_+^n$.

We let

$$\bar{\omega} = \int_{\mathcal{A}} \omega(\alpha) d\mu$$

denote the mean endowment. Total demand is defined by

$$F(p) = \int_{\mathcal{A}} f(\alpha, p, p \cdot \omega(\alpha)) d\mu, \tag{3.11}$$

and the market excess demand function is then

$$Z(p) = F(p) - \bar{\omega}. \tag{3.12}$$

As before, we assume $Z(p)$ to be a C^1 function and suppose that one may differentiate with respect to p under the integral sign.

It is immediately seen that $Z(p)$ is homogeneous of degree 0, bounded below by $\bar{\omega}$ and satisfies Walras' law: $p \cdot Z(p) = 0$. A positive price $p^* \in \mathbb{R}_{++}^n$ will be called an equilibrium price if $Z(p^*) = 0$.

We now introduce the concept of metonymy in this setting. For $s \in \mathbb{R}_+^n$ we let $G(s)$ be the set of agents in the economy whose initial endowment is s, i.e.

$$G(s) = \{\alpha \in \mathcal{A} : \omega(\alpha) = s\}.$$

The measure μ induces a conditional distribution, say η_s on each $G(s)$ along with a measure ν on $\mathbb{R}^n_+$, such that if $h \in L^1(\mathcal{A}, \mathbb{R}^n_+)$, then

$$\int_{\mathcal{A}} h\, d\mu = \int_{\mathbb{R}^n_+} \left(\int_{G(s)} h\, d\eta_s \right) d\nu.$$

Recall that the Slutsky equation now reads,

$$\frac{\partial f_i}{\partial p_j}(\alpha, p, p \cdot \omega(\alpha)) = S_{ij}|_{(\alpha, p, p \cdot \omega(\alpha))} + (\omega_j(\alpha) - f_j) \frac{\partial f_i}{\partial w}|_{(\alpha, p, p \cdot \omega(\alpha))}.$$

Definition 3.1 *We say that the measure μ is metonymic if the following conditions hold:*

(i) *The measure ν has an absolutely continuous density, denoted by $g(s)$, supported in $\mathbb{R}^n_+$.*

(ii) *For $k = 1, \ldots, n$, for all $y \in \mathbb{R}^n$ and for all prices p,*

$$\int_{\mathbb{R}^n_+} \int_{G(s)} \left(\frac{\partial}{\partial s_k} < s - f, y >^2 \right) d\eta_s g(s)\, ds = \int_{\mathbb{R}^n_+} \frac{\partial}{\partial s_k} \left(\int_{G(s)} < s - f, y >^2 d\eta_s \right) g(s)\, ds$$

Note that if we assume that the measure μ is a product-type measure of the form $\mu = \eta \times \nu$, as in the preceding Section, then part (ii) follows from part (i). Namely, $(\mathcal{A}, \mu) = (C \times D, \eta \otimes \upsilon)$, where, now (C, η) describes the "types" of consumers and (D, υ) the allocation of initial endowments among each type. As in the price-independent case, it is easy to verify that condition (ii) is equivalent to the following one

for all i, j, k $\quad \int_{\mathbb{R}^n_+} \int_{G(s)} \frac{\partial}{\partial s_k}((s_i - f_i)(s_j - f_j))\, d\eta_s g(s)\, ds = \int_{\mathbb{R}^n_+} \frac{\partial}{\partial s_k} \left(\int_{G(s)} (s_i - f_i)(s_j - f_j)\, d\eta_s \right) g(s)\, ds.$

Our conditions on the shape of the distribution of initial endowments will be expressed by requirements on the derivatives of the density $g(s)$. Consider,

$$B_k^+ = \{ s \in \mathbb{R}^n_+ : \frac{\partial g(s)}{\partial s_k} > 0 \}.$$

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We will make use of the following result,

Given a vector $e \in \mathbb{R}^n_+$ denote by $H(e) = \{x \in \mathbb{R}^n : x \cdot e = 0\}$. The following is a variant of lemma **6.1** in [13],

Lemma 3.2 ([13]) *Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a C^1 function, $\mathcal{H}$ a cone contained in $\mathbb{R}^n_{++}$ and e any vector in $\mathbb{R}^n_+$. Then the following are equivalent:*

(i) For every $p \in \mathcal{H}$ the Jacobian matrix $DF(p)$ is negative semidefinite on $H(e)$.

(ii) $(p - q) \cdot (F(p) - F(q)) \leq 0$ for every $p, q \in \mathcal{H}$ with $p \neq q$ and $p \cdot e = q \cdot e$.

We can now state a restricted version of the Law of Demand. This will be enough, though, to guarantee uniqueness and will also provide stability of equilibrium in a closed cone of prices.

In order to exhibit the flexibility of our methods, we will consider now a slightly different variant of hypothesis (iii) in Theorem 2.6. Namely, we suppose that, for all consumers, the rank of the Slutsky matrix is the maximum it may have, i.e. we assume its rank is exactly $n - 1$. In this case, the eigenvalues of $S(\alpha, p, x)$ given by 2.5, now have the form,

$$\lambda_n(\alpha, p) \leq \lambda_{n-1}(\alpha, p) \leq \cdots \leq \lambda_2(\alpha, p) < \lambda_1(\alpha, p) = 0. \tag{3.13}$$

Consider a closed cone of prices $\mathcal{H} \subset \mathbb{R}^n_{++}$. We let now

$$\lambda(\alpha) = \sup\{\lambda_2(\alpha, p) : p \in \mathcal{H} \cap S^{n-1}_+\} < 0, \tag{3.14}$$

since $\mathcal{H} \cap S^{n-1}_+$ is compact. Likewise, we define

$$\lambda = \int_A \lambda(\alpha) d\mu$$

and

$$v = \inf_{p \in S_+^{n-1}} p \cdot \bar{\omega}.$$

Consider ε defined by

$$\varepsilon = \frac{|\lambda| v}{n \|\bar{\omega}\|_1} > 0, \tag{3.15}$$

where for $z \in \mathbb{R}^n$, $\|z\|_1 = \sum_{k=1}^n |z|_k$. We have,

Proposition 3.3 *If the following hold:*

(i) The Economy is regular.

(ii) The measure μ is metonymic.

(iii) For every consumer, the Slutsky matrix has rank $n - 1$.

Let $\mathcal{H} \subset \mathbb{R}_{++}^n$ be a closed cone of prices. Let $\varepsilon > 0$ be defined by Equation 3.15. If for all $k = 1, \dots, n$

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon, \tag{3.16}$$

then the excess demand function satisfies,

$$(p - q) \cdot (Z(p) - Z(q)) < 0 \tag{3.17}$$

for all prices $p, q \in \mathcal{H}$ with $p \neq q$ and $p \cdot \bar{\omega} = q \cdot \bar{\omega}$.

Remark 3.4 The proof of Proposition 3.3 shows that condition (iii) in it can be replaced by the following one:

(iii) $\lambda(\alpha) < 0$ *on a set of positive measure.*

Remark 3.5 The number ε appearing in Proposition 3.3 is computed using data obtained from the Slutsky matrix of all consumers. Hence, equation 3.16 restricts the

shape of the graph of the density distribution of income according to the consumers' taste.

As observed in Remark 2.7, the quantity ε depends, in principle on the density ρ. It is also easy to verify from the proof of Proposition 3.3 that if

$$\lambda_0 = \sup\{\lambda(\alpha) : \alpha \in \mathcal{A}\} < 0,$$

and

$$\omega_0 = \inf\{\omega(\alpha) : \alpha \in \mathcal{A}\} > 0,$$

then the Proposition holds for

$$\varepsilon = \frac{|\lambda_0|v}{n\|\bar{\omega}_0\|_1}$$

which does not depend on ρ.

Remark 3.6 We observe ([13]), that 3.17 cannot hold for all prices. Indeed, let $p \in \mathbb{R}^n_{++}$ such that $Z(p) \notin \mathbb{R}^n_+$. There is $q \in \mathbb{R}^n_{++}$ such that $q \cdot Z(p) < 0$. For $\lambda \in \mathbb{R}$ consider

$$
\begin{aligned}
(\lambda p - q)(Z(\lambda p) - Z(q)) &= \lambda p(Z(\lambda p) - Z(q)) - q \cdot (Z(\lambda p) - Z(q)) \\
&= \lambda p(Z(p) - Z(q)) - q \cdot Z(p).
\end{aligned}
$$

For small enough λ the last term can be made positive.

However, we can still use this result to show that if the distribution of initial endowments is consistent with 3.16, then there is a unique equilibrium. Let $\partial \mathbb{R}^n_{++}$ denote the boundary of $\mathbb{R}^n_{++}$, i.e. $\partial \mathbb{R}^n_{++} = \{p \geq 0 : \text{ for some } 0 \leq i \leq n, \quad p_i = 0\}$.

Theorem 3.7 *Assume the following hold:*

(i) The Economy is regular.

(ii) The measure μ is metonymic.

(iii) If $\{p_n\}_{n=1}^{\infty} \subset \mathbb{R}^n_{++}$ converges to $p \in \partial\mathbb{R}^n_{++}$, then

$$\lim_{n \to \infty} \|F(p_n)\| = +\infty.$$

(iv) For every consumer, the Slutsky matrix has rank $n - 1$.

Then there is $\varepsilon > 0$ such that if for all $k = 1, \ldots, n$

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon,$$

there is a unique equilibrium price, p^*.

Furthermore, there is closed cone of prices $\mathcal{H} \subset \mathbb{R}^n_{++}$, containing p^*, such that for all $q \in \mathcal{H}$, which is not collinear with p^*, we have that $p^* \cdot Z(q) > 0$ (the weak axiom of revealed preference for the aggregate holds in $\mathcal{H}$).

We are restricted to a closed cone to guarantee to ourselves that the substitution effect is bounded away from 0 for at least one consumer. Thus, assuming this, we can dispose of the cone $\mathcal{H}$. Recall the definition of $\lambda(\alpha)$ given in 3.13.

Theorem 3.8 *Let the following hold:*

(i) The Economy is regular.

(ii) The measure μ is metonymic.

(iii) If $\{p_n\}_n \subset \mathbb{R}^n_{++}$ converges to $p \in \partial\mathbb{R}^n_{++}$, then

$$\lim_{n \to \infty} \|F(p_n)\| = +\infty.$$

(iv) There is $\alpha_0 \in \mathcal{A}$, such that $\lambda(\alpha_0) < 0$.

Then there is $\varepsilon > 0$ such that if for all $k = 1, \ldots, n$

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon,$$

there is a unique equilibrium price, p^*. Furthermore, for all $q \in \mathbb{R}_{++}^n$, not collinear with p^*, we have that $p^* \cdot Z(q) > 0$ (the weak axiom of revealed preference for the aggregate).

4 STABILITY AND A TATONNEMENT PROCESS

In this Section we suggest a price adjustment process which, for the economies considered, converges to the unique equilibrium. Consider the standard tâtonnement process

$$\dot{p}(t) = Z(p(t)), \tag{4.18}$$

where $\dot{p} = \frac{\partial p}{\partial t}$ i.e. prices move in the direction of excess demand.

It is a standard result that if the weak axiom of revealed preference as stated in Theorem 3.8 holds the the aggregate, then the unique equilibrium price is a globally stable equilibrium point of the system of differential equations 4.18. Thus we can state,

Proposition 4.1 *If the following hold:*

(i) The Economy is regular.

(ii) The measure μ is metonymic.

(iii) There is $\alpha_0 \in \mathcal{A}$, such that $\lambda(\alpha_0) < 0$.

(iv) If $\{p_n\}_n \subset \mathbb{R}_{++}^n$ converges to $p \in \partial \mathbb{R}_{++}^n$, then $\lim_{n \to \infty} \|F(p_n)\| = +\infty$.

Then there is $\varepsilon > 0$ such that if for all $k = 1, \ldots, n$

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon,$$

the unique equilibrium price given in Theorem 3.8 is asymptotically stable under the tâtonnement 4.18.

5 FINAL REMARKS

We have presented a model of a pure exchange economic system in which total market demand is monotone in prices. This allows one to obtain results on uniqueness and stability of the equilibrium price.

The aim of our work has been to do this through as realistic hypotheses as possible on the distribution of total expenditure and, at the same time, without imposing too restrictive conditions on consumers' preferences. We will now compare our results with the existing literature.

With respect to the line followed in [9], [10] and [1], the novelty here is that we allow for some increasing densities of income without making assumptions on the Engel curves.Our methods can be further extended to the more general setting in wich wealth is determined by the market price of initial endowments.

The alternative approach to the problem of aggregation of demand, followed in [5], makes rather strong assumptions on both, the consumers preferences (aggregate desirability) and the distribution of 'characteristics' (the conditional densities in each class of α-transform are all the same, and all the agents in that class have equal wealth). In return, the author obtains that one does not need to make any reference to individual rationality other than homogeneity and Walras' law.

In the near future, it is hoped that empirical estimates will be obtained in order to test whether the observable distributions satisfy the hypotheses proposed here. It is also hoped that these findings will be extended so as to incorporate production, and a temporal framework.

APPENDIX

Let S_+^{n-1} denote the positive orthant of the unit sphere, $S_+^{n-1} = \{x \in R_{++}^n : \|x\| = 1\}$, where n is the number of commodities.

We will use the following conventional notation: $\mathbb{R}_+^l = \{z \in \mathbb{R}^l : z \geq 0, z \neq 0\}$ $\mathbb{R}_{++}^l = \{z \in \mathbb{R}^l : z_i > 0 \text{ for all } i = 1, \ldots, l\}$. Similarly, we denote the usual inner product in $\mathbb{R}^n$ by $< x, y >= x \cdot y$. We will need the following norms in $\mathbb{R}^n$: $\|z\| = \sqrt{\sum_i z_i^2}$ and $\|z\|_1 = \sum_i |z_i|$. The inequality $\|z\|_1 \leq \sqrt{n}\|z\|$ is a standard result.

We will also make use of the following,

Lemma 1 *Let $g \in C^1(\mathbb{R}^n, \mathbb{R})$. Let G be a convex subset of $\mathbb{R}^n$ and let $a, b \in G$ with $g(a) = 0$. Suppose for all $x \in G$, and $i = 1, \ldots, n$, $\|\frac{\partial g}{\partial x_i}\| < \alpha$. Then*

$$\|g(b)\| < \alpha \sqrt{n}\|b - a\|.$$

Proof of lemma 1:

Let $h(t) = g(bt + (1 - t)a)$. Clearly, $h(0) = g(a) = 0$, $h(1) = g(b)$, so

$$
\begin{aligned}
\|g(b)\| &= \|h(1) - h(0)\| = \|\int_0^1 h'(t)dt\| \\
&= \|\sum_{i=1}^n (b_i - a_i) \int_0^1 \frac{\partial g}{\partial x_i}\|_{(bt+(1-t)a)}dt\| \\
&\leq \sum_{i=1}^n \|b_i - a_i\| \int_0^1 \|\frac{\partial g}{\partial x_i}\|dt \\
&< \alpha \sum_{i=1}^n \|b_i - a_i\| \\
&\leq \alpha \sqrt{n}\|b - a\|.
\end{aligned}
$$

$\square$

Proof of Theorem 2.6:

From the remark following lemma 2.4 we only have to show that the Jacobian matrix $DF(p)$ is negative definite. By the Slutsky equation, for each $\alpha \in \mathcal{A}$,

$$\frac{\partial f_i}{\partial p_j}(\alpha, p, t) = s_{ij}(\alpha, p, t) - f_j \frac{\partial f_i}{\partial t}|_{(\alpha, p, t)}.$$

Let $y \in \mathbb{R}^n$ and define

$$\bar{S}(p, y) = \sum_{i,j=1}^{n} \int_{\mathcal{A}} s_{ij}(\alpha, p, \omega(\alpha)) y_i y_j d\mu \qquad (A.2)$$

$$\bar{A}(p, y) = \sum_{i,j=1}^{n} \int_{\mathcal{A}} f_j \frac{\partial f_i}{\partial t}|_{(\alpha, p, \omega(\alpha))} y_i y_j d\mu. \qquad (A.3)$$

Thus, $F(p)$ is monotone if and only if for all prices $p \in \mathbb{R}_{++}^n$ and $y \in \mathbb{R}^n$, $\bar{S}(p, y) < \bar{A}(p, y)$. We will show this indirectly.

Let $p \in \mathbb{R}_{++}^n$ and $\alpha \in \mathcal{A}$. Define

$$s_{ij}^p(\alpha, p) = p_i p_j s_{ij}(\alpha, p, \omega(\alpha))$$

$$a_{ij}^p(\alpha, p) = p_i p_j f_j \frac{\partial f_i}{\partial t}|_{(\alpha, p, \omega(\alpha))}$$

and the matrices,

$$S(\alpha, p) = \{s_{ij}^p(\alpha, p)\}_{ij}$$

$$A(\alpha, p) = \{a_{ij}^p(\alpha, p)\}_{ij}.$$

We consider now the quadratic forms,

$$S(p, x) = \int_{\mathcal{A}} < S(\alpha, p)x, x > d\mu \qquad (A.4)$$

$$= \sum_{i,j=1}^{n} \int_{\mathcal{A}} s_{ij}(\alpha, p, \omega(\alpha)) x_i x_j p_i p_j d\mu$$

$$A(p, x) = \int_{\mathcal{A}} < A(\alpha, p)x, x > d\mu \qquad (A.5)$$

$$= \sum_{i,j=1}^{n} \int_{\mathcal{A}} f_j(\alpha, p, \omega(\alpha)) \frac{\partial f_i}{\partial t}(\alpha, p, \omega(\alpha)) x_i x_j p_i p_j d\mu$$

$$S(\alpha, p, x) = < S(\alpha, p)x, x > \qquad (A.6)$$

$$= \sum_{i,j=1}^{n} s_{ij}(\alpha, p, \omega(\alpha)) p_i p_j x_i x_j.$$

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We first note that the theorem follows if we can show that for all $p \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n$, $S(p,x) - A(p,x) < 0$, where S and A are given by equations A.4 and A.5. Because, given $y \in \mathbb{R}^n$, let $x_i = \frac{1}{p_i} y_i$, then

$$\bar{S}(p,y) - \bar{A}(p,y) = S(p,x) - A(p,x) < 0.$$

It is also enough to prove that $S(p,x) - A(p,x) < 0$ for $x \in S^{n-1} = \{x \in R^n : \|x\| = 1\}$. For then, if $z \in \mathbb{R}^n \setminus \{0\}$, there is $r = \|z\| \neq 0$ such that $x = \frac{1}{r}z \in S^{n-1}$, and $S(p,z) - A(p,z) = r^2(S(p,x) - A(p,x)) < 0$.

Since the eigenvalues of $S(\alpha, p, x)$ in A.6 are given in Equation 2.6, we see that there is an orthonormal transformation $M \in O(n)$, taking $x_0 = \frac{1}{\sqrt{n}}(1,\dots,1)$ to $(1,0,\dots,0)$ and such that $MS(\alpha,p)M^{-1}$ is the diagonal matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Let $\lambda(\alpha) = \sup\{\lambda_2(\alpha,p) : p \in \mathbb{R}^n_{++}\} \leq 0$. Let $y = Mx$ and denote by π and π_0, the orthogonal projections onto the planes $\{z : z_1 = 0\}$, and $\{z : x_0 \cdot z = 0\}$, respectively. Note that $M \circ \pi_0 = \pi \circ M$. For each $(\alpha,p) \in \mathcal{A} \times \mathbb{R}^n_{++}$, we have,

$$
\begin{aligned}
\sum_{i,j} s_{i,j}(\alpha,p,\omega(\alpha))p_i p_j x_i x_j
&= \;\; <S(\alpha,p)x, x> \\
&= \;\; <SM^{-1}y, M^{-1}y> \\
&= \;\; <MSM^{-1}y, y> \\
&= \;\; \lambda_2(\alpha,p)y_2^2 + \cdots + \lambda_n(\alpha,p)y_n^2 \\
&\leq \;\; \lambda_2(\alpha,p)(y_2^2 + \dots + y_n^2) \\
&\leq \;\; \lambda(\alpha)(y_2^2 + \dots + y_n^2) \\
&= \;\; \lambda(\alpha) <\pi(y), \pi(y)> \\
&= \;\; \lambda(\alpha) <\pi(Mx), \pi(Mx)> \\
&= \;\; \lambda(\alpha) <M\pi_0(x), M\pi_0(x)> \\
&= \;\; \lambda(\alpha) <\pi_0(x), \pi_0(x)> .
\end{aligned}
$$

(A.7)

Let

$$\lambda = \int_{\mathcal{A}} \lambda(\alpha) d\mu.$$

Since $\lambda(\alpha) \leq 0$ and $\lambda(\alpha) < 0$ on a set of positive measure, we have that $\lambda < 0$ By integrating, we have that,

$$S(p, x) \leq \lambda < \pi_0(x), \pi_0(x) >= \lambda \|\pi_0(x)\|^2. \tag{A.8}$$

We let

$$\delta = \inf\{\|\pi_0(x)\|^2 : \|x\| = 1, \quad x \cdot x_0 \geq 0, \quad \|x - x_0\| \geq \frac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega} + 1)}\}$$

and define $\varepsilon > 0$ by

$$\varepsilon = \min\{\frac{1}{2}, |\lambda|\delta\} > 0, \tag{A.9}$$

where

$$\bar{\omega} = \int \omega(\alpha) d\mu.$$

Thus, for all $p \in \mathbb{R}^n_{++}$ if $\|x - x_0\| \geq \frac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega}+1)}$, $\|x\| = 1$, and $x \cdot x_0 \geq 0$ then $S(p, x) < -\varepsilon$.

Suppose now that ρ satisfies $\int_{\rho'>0} t^2 \rho' < \varepsilon$.

We study first the term $A(p, x)$. We have that

$$
\begin{aligned}
A(p, x) &= \sum_{i,j=1}^{n} \int_{\mathcal{A}} f_j \frac{\partial f_i}{\partial t}|_{(\alpha, p, \omega(\alpha))} x_i x_j p_i p_j d\mu \\
&= \int_{\mathcal{A}} (\sum_j f_j x_j p_j)(\sum_i \frac{\partial f_i}{\partial t}|_{(\alpha, p, \omega(\alpha))} x_i p_i) d\mu \\
&= \frac{1}{2} \int_{\mathcal{A}} \frac{\partial}{\partial t} (\sum_j f_j x_j p_j)^2 d\mu \\
&= \frac{1}{2} \int_{\mathbb{R}_+} (\int_{G(t)} \frac{\partial}{\partial t} (\sum_j f_j x_j p_j)^2 d\eta_t) \rho(t) dt. \tag{A.10}
\end{aligned}
$$

In particular, we have that $A(p, x_0) = \frac{\bar{\omega}}{n}$.

Since the measure μ is metonymic, twice the last term in A.10 equals

$$\int_{\mathbb{R}_+} \frac{\partial}{\partial t} (\int_{G(t)} (\sum_j f_j x_j p_j)^2 d\eta_t) \rho(t) dt.$$

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By integrating by parts this is the same as,

$$-\int_{\mathrm{R}_+}\int_{G(t)}(\sum_j f_j x_j p_j)^2 d\eta_t \rho'(t)dt,$$

which can be decomposed into,

$$-\int_{\rho'>0}\int_{G(t)}(\sum_j f_j x_j p_j)^2 d\eta_t \rho'dt$$

$$-\int_{\rho'\leq 0}\int_{G(t)}(\sum_j f_j x_j p_j)^2 d\eta_t \rho'dt. \tag{A.11}$$

Now note that in case $\|x - x_0\| < \frac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega}+1)}$, we must have that $A(p,x) > 0$, because, from Equation A.11, for $i = 1,\ldots,n$,

$$
\begin{aligned}
|\frac{\partial A}{\partial x_i}| &\leq \int_{\rho'>0}\int_{G(t)}|\sum_j f_j x_j p_j|f_i p_i d\eta_t \rho'dt - \int_{\rho'\leq 0}\int_{G(t)}|\sum_j f_j x_j p_j|f_i p_i d\eta_t \rho'dt\\
&\leq \int_{\rho'>0}\int_{G(t)}(\sum_j f_j p_j)f_i p_i d\eta_t \rho'dt - \int_{\rho'\leq 0}\int_{G(t)}(\sum_j f_j p_j)f_i p_i d\eta_t \rho'dt\\
&\leq \int_{\rho'>0}\int_{G(t)}t^2 d\eta_t \rho'dt - \int_{\rho'\leq 0}\int_{G(t)}t^2 d\eta_t \rho'dt\\
&= 2\int_{\rho'>0}\int_{G(t)}t^2 d\eta_t \rho'dt - \int_{\mathrm{R}_+}\int_{G(t)}t^2 d\eta_t \rho'dt\\
&\leq 2\varepsilon - \int_{\mathrm{R}_+}\int_{G(t)}t^2 d\eta_t \rho'dt\\
&= 2\bar{\omega} + 2\varepsilon\\
&\leq 2\bar{\omega} + 1, \tag{A.12}
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
\bar{\omega} &= \int_{\mathcal{A}}\omega(\alpha)d\mu\\
&= \int_{\mathrm{R}_+}\int_{G(t)}t d\eta_t \rho dt\\
&= \int_{\mathrm{R}_+}t\rho(t)dt\\
&= -\frac{1}{2}\int_{\mathrm{R}_+}t^2\rho'(t)dt > 0.
\end{aligned}
$$

Equation A.12 holds also for x in the open unit ball (we need this in order to apply Lemma 1). Thus, if $A(p,x) = 0$, then, by Lemma 1,

$$A(p,x_0) = \frac{\bar{\omega}}{n} < (2\bar{\omega}+1)\sqrt{n}\|x_0 - x\|.$$

So, $\|x_0 - x\| > \dfrac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega}+1)}$.

Hence, we may assume that $\|x - x_0\| \geq \dfrac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega}+1)}$. The integral over the region $\{t : \rho'(t) \leq 0\}$ appearing in A.11 is positive and will cause no problems. Thus we will concentrate on the region $\{t : \rho'(t) > 0\}$.

Fix a price $p \in \mathbb{R}^n_{++}$ and define

$$A^+(p, x) = \frac{1}{2}\int_{\{t:\rho'(t)>0\}}\int_{G(t)}(\sum_i f_i x_i p_i)^2 d\eta_t \rho'(t)dt$$

$$A^-(p, x) = \frac{1}{2}\int_{\{t:\rho'(t)\leq 0\}}\int_{G(t)}(\sum_i f_i x_i p_i)^2 d\eta_t \rho'(t)dt.$$

Note, that $A^-(p, x) \geq 0$. The theorem will follow if we can show that $|A^+(p, x)| < |S(p, x)|$, for appropriate x. Since the quadratic forms S and A^+ satisfy $S(p, -x) = S(p, x)$ and $A^+(p, -x) = A^+(p, x)$ it is enough to show that $S(p, x) + A^+(p, x) < 0$ for x in the half sphere $S_0^{n-1} = \{z \in S^{n-1} : x_0 \cdot z \geq 0\}$. Thus, we may restrict ourselves to x lying on S_0^{n-1} and $\|x - x_0\| \geq \dfrac{\bar{\omega}}{n^{\frac{3}{2}}(2\bar{\omega}+1)}$. But then,

$$|A^+(p, x)| = \frac{1}{2}\int_{\rho'>0}\int_{G(t)}(\sum_i f_i(\alpha, p, t)x_i p_i)^2 d\eta_t \rho' dt \tag{A.13}$$

$$\leq \int_{\rho'>0} t^2 \rho' < \varepsilon. \tag{A.14}$$

Hence, from the way ε was chosen, we have that $|S(p, x)| > A^+(p, x)$. Therefore,

$$S(p, x) - A(p, x) = S(p, x) - A^+(p, x) - A^-(p, x)$$

$$\leq S(p, x) - A^+(p, x)$$

$$\leq 0,$$

and the proof is finished. $\qquad\square$

We start, now the preliminaries to prove Proposition 3.3. Let

$$H_p = \{z \in \mathbb{R}^n : \sum_{i=1}^n p_i x_i \bar{\omega}_i = 0, z_1^2 + \ldots + z_n^2 = 1\}. \tag{A.15}$$

Recall that π_0 denotes the orthogonal projection from $\mathbb{R}^n$ onto the plane $\{z : z \cdot x_0 = 0\}$. We have,

Lemma 2 *If $z \in H_p$, then $\|\pi_0(z)\| \geq \frac{1}{\sqrt{n}}$.*

Proof of lemma 2:

First, $\pi_0(z) = z - (z \cdot x_0)x_0$, so

$$\|\pi_0(z)\|^2 = 1 - (z \cdot x_0)^2.$$

On the other hand, $z \cdot x_0 = \cos(z, x_0)$. Thus, the maximum of $(z \cdot x_0)^2$ is attained when the angle between z and x_0 is minimum, i.e. on the edges of the positive orthant. By symmetry we may assume that $z = (0, z_2, \ldots, z_{n-1}, \sqrt{1 - \sum_{i=2}^{n-1} z_i^2})$. A simple computation shows that $z \cdot x_0$ attains its maximum, for $z_2 = \ldots = z_n = \frac{1}{\sqrt{n-1}}$. So,

$$(z \cdot x_0)^2 \leq 1 - \frac{1}{n}$$

and the Lemma follows. $\qquad\square$

Proof of Proposition 3.3:

By Lemma 3.2 it is enough to show that for each $p \in \mathcal{H}$, $DF(p)$ is negative semidefinite on $H(\bar{\omega})$. Since $DF(p)$ is homogeneous of degree -1 in p, it is enough to show this for $p \in \mathcal{H} \cap S_+^{n-1}$.

From now on, we fix $p \in \mathcal{H} \cap S_+^{n-1}$, consider,

$$S_{ij}(p) = \int_{\mathcal{A}} s_{ij}(\alpha, p, p \cdot \omega(\alpha)) d\mu \qquad (A.16)$$

$$A_{ij}(p) = \int_{\mathcal{A}} (\omega_j(\alpha) - f_j) \frac{\partial f_i}{\partial t}\Big|_{(\alpha, p, p \cdot \omega(\alpha))} d\mu \qquad (A.17)$$

$$S(p, x) = \sum_{i,j=1}^{n} S_{ij}(p) p_i p_j x_i x_j$$

$$A(p, x) = \sum_{i,j=1}^{n} A_{ij}(p) p_i p_j x_i x_j,$$

where $s_{ij}(\alpha, p, p \cdot \omega(\alpha))$ are the entries of the Slutsky's matrix.

As before we let

$$\lambda(\alpha) = \sup\{\lambda_2(\alpha, p) : p \in \mathcal{H} \cap S_+^{n-1}\} < 0,$$

since $\mathcal{H} \cap S_+^{n-1}$ is compact. Likewise, we define

$$\lambda = \int_\mathcal{A} \lambda(\alpha)d\mu.$$

Then by Lemma 2 and Formula A.8 in the proof of Theorem 2.6, it follows that, for $x \in H_p$,

$$S(p, x) \le \frac{\lambda}{n}.$$

Note also, that

$$v = \inf_{p \in S_+^{n-1}} p \cdot \bar{\omega} = \inf_{p \in S_+^{n-1}} \|\bar{\omega}\| \cos(p, \bar{\omega}) > 0,$$

because it follows from metonymy that $\bar{\omega}$ is in the interior of $\mathbb{R}_+^n$.

Let

$$\varepsilon = \frac{|\lambda|v}{n\|\bar{\omega}\|_1} > 0,$$

where for $z \in \mathbb{R}^n$, $\|z\|_1 = \sum_{k=1}^n |z|_k$. The Proposition is a consequence of Claims 1, 2 and 3 below.

Recall the definition of H_p in A.15,

Claim 1 *For each $x \in H_p$, $S(p, x) + A(p, x) < 0$.*

Let $x \in H_p$, $y_i = x_i p_i$ and define

$$a(p) = p \cdot \bar{\omega} \ge v > 0.$$

From the remark before Claim 1 we see that $a(p)S(p, x) < \frac{\lambda v}{n}$.

On the other hand, we have that,

$$
\begin{aligned}
a(p)A(p, x) &= a(p) \sum_{i,j} \int_\mathcal{A} (\omega_j - f_j) \frac{\partial f_i}{\partial t} y_i y_j d\mu \\
&= a(p) \int_\mathcal{A} <\omega - f, y><\frac{\partial f}{\partial t}, y> d\mu
\end{aligned}
$$

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$$= \sum_k \bar{\omega}_k \int_{\mathcal{A}} < \omega - f, y > < p_k \frac{\partial f}{\partial t}, y > d\mu$$

$$= \sum_k \bar{\omega}_k \int_{\mathbf{R}_+^n} \int_{G(s)} < s - f, y > < p_k \frac{\partial f}{\partial t}, y > d\eta_s g(s) ds$$

Note that

$$\frac{\partial f(\alpha, p, p \cdot s)}{\partial s_k} = p_k \frac{\partial f(\alpha, p, p \cdot s)}{\partial t},$$

so we can write the last term as

$$\sum_k \bar{\omega}_k \int_{\mathbf{R}_+^n} \int_{G(s)} < s - f, y > < \frac{\partial f}{\partial s_k}, y > d\eta_s g(s) ds,$$

which is the same as

$$\sum_k \bar{\omega}_k \left(\int_{\mathbf{R}_+^n} \int_{G(s)} y_k < s - f, y > d\eta_s g(s) ds - \frac{1}{2} \int_{\mathbf{R}_+^n} \int_{G(s)} \frac{\partial}{\partial s_k} < s - f, y >^2 d\eta_s g(s) ds \right).$$

But the first term vanishes because

$$\sum_k \bar{\omega}_k y_k = \bar{\omega} \cdot y = 0,$$

since $x \in H_p$. Using now metonymy and integration by parts, we get

$$\int_{\mathbf{R}_+^n} \int_{G(s)} \frac{\partial}{\partial s_k} < (s - f), y >^2 d\eta_s g(s) ds = - \int_{\mathbf{R}_+^n} \int_{G(s)} < (s - f), y >^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds,$$

and, as before, we have to worry only about the regions $B_k^+ = \{s \in \mathbb{R}_+^n : \frac{\partial g(s)}{\partial s_k} > 0\}$.

Define,

$$A^+(p, x) = \frac{1}{2} \sum_k \bar{\omega}_k \int_{B_k^+} \int_{G(s)} \left(\sum_j (s_j - f_j) p_j x_j \right)^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds.$$

We first bound the inner integrand. Note that

$$
\begin{aligned}
\left(\sum_j (s_j - f_j(\alpha, p, p \cdot s)) p_j x_j \right)^2 &= \left(\sum_j s_j p_j x_j - \sum_j f_j(\alpha, p, p \cdot s) p_j x_j \right)^2 \\
&\leq \left(\sum_j s_j p_j x_j \right)^2 + \left(\sum_j f_j(\alpha, p, p \cdot s) p_j x_j \right)^2 \\
&\leq \left(\sum_j s_j p_j \right)^2 + \left(\sum_j f_j(\alpha, p, p \cdot s) p_j \right)^2 \\
&= 2(p \cdot s)^2,
\end{aligned}
\tag{A.18}
$$

because $0 \leq x_i^2 \leq 1$ and $p \cdot f(\alpha, p, p \cdot s) = p \cdot s$. Since we are assuming, $p \in S_+^{n-1}$, we have that $p \cdot s \leq \|s\|$. Thus,

$$
\begin{aligned}
A^+(p, x) &\leq \sum_k \bar{\omega}_k \int_{B_k^+} \int_{G(s)} \|s\|^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds \\
&= \sum_k \bar{\omega}_k \int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds
\end{aligned}
\tag{A.19}
$$

Hence, if $\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon$, then

$$
\begin{aligned}
A^+(p, x) &\leq \varepsilon \sum_k \bar{\omega}_k \\
&= \varepsilon \|\bar{\omega}\|_1 \\
&\leq \frac{|\lambda| v}{n} \\
&\leq |a(p) S(p, x)|
\end{aligned}
\tag{A.20}
$$

It follows that $a(p)(S(p, x) + A(p, x)) < 0$. Hence $S(p, x) + A(p, x) < 0$ as advertised.

Claim 2 *For all $x \in \mathbb{R}^n$ such that $\sum_i p_i x_i \bar{\omega}_i = 0$, we have that $S(p, x) + A(p, x) < 0$.*

Let $x \in \mathbb{R}^n$ such that $\sum_i p_i x_i \bar{\omega}_i = 0$. Let $z = \frac{1}{\|x\|} x \in H_p$. Then from Claim 1, we have that $S(p, x) + A(p, x) = \|x\|^2 (S(p, z) + A(p, z)) < 0$.

Let us define now the following quantities,

$$
\bar{S}(p, y) = \sum_{i,j=1}^n S_{ij}(p) y_i y_j
$$

$$
\tag{A.21}
$$

and

$$
\bar{A}(p, y) = \sum_{i,j=1}^n A_{ij}(p) y_i y_j,
$$

where $S_{ij}(p)$ and $A_{ij}(p)$ are defined, respectively, by equations A.16 and A.17. We have the following

Claim 3 *For all $y \in \mathbb{R}^n$ such that $\sum_i y_i \bar{\omega}_i = 0$, we have that $\bar{S}(p, y) + \bar{A}(p, y) < 0$.*

Let $y \in \mathbb{R}^n$ such that $\sum_i y_i \bar{\omega}_i = 0$. Let $x_i = y_i / p_i$. Then

$$\sum_i p_i x_i \bar{\omega}_i = \sum_i y_i \bar{\omega}_i = 0,$$

and, by Claim 2,

$$0 > S(p, x) + A(p, x) = \bar{S}(y) + \bar{A}(y)$$

which finishes the proof of the theorem. $\qquad \square$

Proof of Theorem 3.7:

It is a standard result (see [3]), that if $Z(p)$ is continuous, bounded below, and satisfies Walras' law and property (iv), then there is $p^* \in \mathbb{R}^n_{++}$ such that $Z(p^*) = 0$.

Next, we note that, by (iv), there is $\delta > 0$ such that if $d(p, \partial \mathbb{R}^n_{++}) \leq \delta$, then $Z(p) \neq 0$. Let $\mathcal{H}$ be a closed cone, such that if $p \in \mathbb{R}^n_{++} \setminus \mathcal{H}$, then $d(p, \partial \mathbb{R}^n_{++}) \leq \delta$. In particular, the set of equilibrium prices is contained in $\mathcal{H}$.

It follows from Proposition 3.3, that there is ε, such that if for $k = 1, \ldots, n$

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \varepsilon,$$

then the excess demand function satisfies,

$$(p - q) \cdot (Z(p) - Z(q)) < 0$$

for all prices $p \neq q \in \mathcal{H}$ with $p \cdot \bar{\omega} = q \cdot \bar{\omega}$.

Fix an equilibrium price $p^* \in \mathcal{H}$, and let $q \in \mathcal{H}$, not collinear with p^*. Let $\lambda = \frac{q \cdot \bar{\omega}}{p^* \cdot \bar{\omega}} > 0$. Then $\lambda p^* \neq q$ and

$$-\lambda p^* \cdot Z(q) = (\lambda p^* - q) \cdot (Z(\lambda p^*) - Z(q)) < 0,$$

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since $Z(\lambda p^*) = Z(p^*) = 0$, $q \cdot Z(q) = 0$ and the restricted monotonicity property of Proposition 3.3.

This implies that $p^* \cdot Z(q) > 0$ for any other $q \in \mathcal{H}$, not collinear with p^*. In particular, $Z(q) \neq 0$ for any other $q \in \mathcal{H}$, not collinear with p^*. Therefore, p^* is the only possible equilibrium price in $\mathcal{H}$ and hence in $\mathbb{R}^n_{++}$. $\qquad \square$

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