INDIVIDUALLY RATIONAL EQUAL LOSS PRINCIPLE FOR BARGAINING PROBLEMS*

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A B S T R A C T

The purpose of this paper is to provide a new solution concept for bargaining problems, which modifies Chun's Equal-Loss Solution in a way that ensures the individual rationality. We also consider its lexicographic extension which turns out to be both individually rational and Pareto Optimal. Characterizations of the proposed solutions are also provided.

I. INTRODUCTION.

A solution concept in axiomatic bargaining may be thought of as a compromise between incumbents. Among the (mutually consistent) principles characterizing a solution concept, there is usually a fairness notion, which makes it appealing as an arbitration scheme. Solution concepts include those based on equal changes in payoffs. By considering the "equal-gains" criterion from the status-quo, the *egalitarian solution* [Kalai (1977)] appears in a natural way, whereas by considering "equal-losses" from the ideal point two other bargaining solutions were found: the *YU*$^{\infty}$ *solution* [Yu (1973)], and the *equal-loss solution* [Chun (1988)].

In spite of their ethical appeal, the YU$^{\infty}$ and the equal-loss solutions exhibit some shortcomings worth taking into account:(1) in general, the YU$^{\infty}$ solution is multivalued; (2) as was pointed out by Thomson (1991), the equal-loss solution is not individually rational for more than two agents. Multivaluedness gives rise to indeterminacy of the proposal, whereas the lack of individual rationality puts the stability of the solution outcome into question.

In this paper we consider a new way of introducing the equal-loss principle for bargaining problems. A new solution, the *equal-loss* solution*, is proposed and axiomatically characterized. This solution can be viewed as a modification of the equal-loss solution, ensuring the individual rationality of the outcome. According to our solution a unique utility allocation is determined as follows: Starting from the ideal point, identical utility losses are applied to all agents with the condition that no one is below her status-quo level. If an agent reaches her status-quo, then she is kept at this level whereas the rest of the agents follow up by decreasing their utility levels by the same amount. A maximal element satisfying this property is then chosen.

Even though the equal-loss* solution is individually rational, it may fail to be fully Pareto Optimal (only Weak Pareto Optimality can be ensured). Pareto Optimality is obtained by considering the lexicographic extension of the equal-loss* solution. Recently, Chun & Peters (1991) proposed and axiomatically characterized the lexicographic extension of the equal-loss solution, which, in the same way as the equal-loss solution itself, may fail to be individually rational. Then, they suggested a modification of the lexicographic equal-loss solution, in order to guarantee individual rationality, and left its characterization as an open problem. Interestingly, the lexicographic equal-loss* solution coincides with the aforementioned modification of the lexicographic equal-loss solution. Thus, by characterizing the lexicographic equal-loss* solution, we also close the open problem posed by these authors.

Section 2 presents some preliminaries and definitions. Section 3 contains the characterization result for the equal-loss* solution. Section 4 is devoted to the definition and characterization of the lexicographic equal-loss* solution. Section 5, with some comments, closes the paper.

2. PRELIMINARIES.

Following Nash (1950), a n-person bargaining problem is a pair (S,d), where S is a subset of $\mathbb{R}^n$, and d is a point of S. $\mathbb{R}^n$ is the utility space, S is the feasible set and d is the disagreement point. The intended interpretation of (S,d) is as follows: the agents can achieve any point of S if they unanimously agree on it. Otherwise, they end up at d.

Given a class of n-person bargaining problems, a solution is a function F which associates to every problem in the class (S,d), a point F(S,d) in S, representing the agreement made by the agents.

Let $\sum^n$ be the class of bargaining problems (S,d) such that $S \subset \mathbb{R}^n$ is convex, closed and comprehensive (if $x \in S$, and $y \leq x$, then $y \in S$)[1], and such that there exists $x \in S$, with $x >> d$.

Whenever $(S,d) \in \sum^n$, we shall call IR(S,d) the set of individually rational points, i.e., $IR(S,d) = \{ x \in S \mid x \geq d\}$. PO(S) will denote the set of Pareto Optimal elements, and WPO(S) the set of weakly Pareto Optimal elements, i.e., $PO(S) = \{ x \in S \mid \text{if } y \geq x, \text{ then } y \notin S \}$, and $WPO(S) = \{x \in S \mid \text{if } y >> x, \text{ then } y \notin S\}$.

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1 Vector inequalities will be $\geq$, $>$ $>>$.

By considering $a_i(S,d) = \max \{x_i | x \in IR(S,d)\}$, $i = 1,\ldots,n$, we construct the *ideal point* $a(S,d)$, such that for every i, it gives the maximal obtainable utility levels of each agent, subject to the condition that all agents achieve at least the utility levels of the disagreement point.

A class of solutions for this kind of problems was proposed by Yu (1973), sharing the idea of looking for *the closest point in* IR(S,d) *to the ideal point.* Then, by means of considering a particular family of distances in $\mathbb{R}^n$, namely, $d_p(x,y) =[\sum_{i=1}^{n} |(x_i - y_i)|^p]^{1/p}$, $1 \leq p < \infty$, and $d_\infty(x,y) = \max_i |x_i - y_i|$, he obtained a family of solutions to the bargaining problem, namely those which minimize the adequate distance to the ideal point, and he called them YU^p, $1 \leq p \leq \infty$.

Strictly speaking, neither YU^1, nor YU^∞ are solutions to the bargaining problem, since the associated norms $\|.\|_1$, $\|.\|_\infty$ are not strictly convex, and therefore both YU^1, YU^∞ can be multi-valued [see Freimer & Yu (1976)].

It is worth mentioning that $YU^p(S,d) \in WPO(S) \cap IR(S,d)$, for $1 < p < \infty$, and $YU^q(S,d) \subset WPO(S) \cap IR(S,d)$, for $q = 1, \infty$.

Moreover, as was pointed out by Chun (1988), in the case n = 2, one of the elements in $YU^\infty(S,d)$ corresponds to that point x in $WPO(S) \cap IR(S,d)$ such that $|a_1(S,d) - x_1| = |a_2(S,d) - x_2|$. Taking this idea into account, Chun proposed a new solution concept, EL, *the equal-loss solution*, as a variant of the YU^{∞} solution, i.e. that point EL(S,d) = y in S such that $|a_i(S,d) - y_i| = |a_j(S,d) - y_j|$, $\forall$ i,j, and this common difference is minimum. Chun's defence of his proposed solution is made on the grounds of the *equal loss principle*, namely, equalization across agents of the losses from the ideal point, in a similar spirit to that of the egalitarian solution [Kalai (1977)], which equalizes the gains across agents, from the disagreement point. Nevertheless, a main shortcoming of this proposal is that, for more than two agents, it is not individually rational [see Thomson (1991)].

In order to avoid previous shortcomings, a modification of the equal-loss solution will be proposed. Some notation is first necessary.

For $A \subset \mathbb{R}^n$, δA represents the northeast frontier of set A; we shall denote by Co(A) the convex hull of set A, and by Com(A) the comprehensive hull of set A. CoCom(A) is simply the convex-comprehensive hull of set A.

Let $N = \{1,2,\ldots,n\}$ denote the set of indices in $\mathbb{R}^n$. For a given subset $Q \subseteq N$, N/Q will be the complement of Q on N. If z, $y \in \mathbb{R}^n$, by considering the partition {Q,N/Q} in the set of indices, we shall call $(z_Q,y_{N/Q})$ the vector $t \in \mathbb{R}^n$ such that $t_i=z_i$ if $i \in Q$; $t_i=y_i$ if $i \in N/Q$.

Let $(S,d) \in \sum^n$, $x \in S$, $Q \subset N$, and let q = card Q. We shall denote S^x_Q the projection of S on $\mathbb{R}^q$ given in the following way:

$$S^x_Q = \{ y_Q \mid (y_Q,x_{N/Q}) \in S \}.$$

Consider now the following definition:

Definition 1: *The equal-loss* solution* , $EL^* : \Sigma^n \longrightarrow \mathbb{R}^n$, associates to each problem $(S,d) \in \Sigma^n$ the unique point of S which satisfies:

(1) If $EL(S,d) \in IR(S,d)$, then $EL^*(S,d) = EL(S,d)$

(2) If $EL(S,d) \notin IR(S,d)$, let $Q=\{j \in N \ / \ EL_j(S,d) \geq d_j\}$. Then

$\forall \ i \in N/Q \quad EL^*_i(S,d) = d_i$

$\forall \ j \in Q$ apply the previous process to bargaining problem (S^d_Q, d_Q).

Since for the bipersonal case EL(S,d) is individually rational, this process finishes, and the equal-loss* solution is well defined.

It is straightforward to check that definition 1 is equivalent to the following definitions, where $\bar{S}$ is the comprehensive hull of set IR(S,d):

Definition 1': *The equal-loss* solution,* $EL^* : \Sigma^n \longrightarrow \mathbb{R}^n$, associates to each problem $(S,d) \in \Sigma^n$ the unique point of S which satisfies:

(1) If $EL(\bar{S},d) \in IR(S,d)$, then $EL^*(S,d) = EL(\bar{S},d)$

(2) If $EL(\bar{S},d) \notin IR(S,d)$, let $Q = \{ \ j \in N \ | \ EL_j(\bar{S},d) \geq d_j\}$. Then,

$EL^*_i \ (S,d) = d_i \quad \forall \ i \in N/Q$

$EL^*_j \ (S,d) = EL_j(\bar{S}^d_Q, d_Q) \quad \forall \ j \in Q.$

Definition 1'': The *equal-loss* solution*, $EL^* : \sum^n \longrightarrow \mathbb{R}^n$, associates to each problem $(S,d) \in \sum^n$ the alternative:

$$EL^*_i(S,d) = \begin{cases} d_i & \text{if } EL_i(\bar{S},d) < d_i \\ EL_i(\bar{S},d) & \text{if } EL_i(\bar{S},d) \geq d_i \end{cases} \qquad \forall i \in N$$

The intended interpretation of the equal-loss* solution is the following: it equalizes the losses from the ideal point *whenever it represents an acceptable agreement for all agents.* If it does not, it is because there are some agents that at the equal-loss solution are below their status-quo. In this case, we accept smaller losses for these agents, keeping them at their disagreement level, and only equalize losses from the ideal point for those agents who do not reach their disagreement utility level in the equal-decreasing procedure. In this way we find a compromise between the equal-loss principle and the possibility of agreement among all agents.

The relationship between EL(S,d) and EL*(S,d) has been given in Definition 1: They coincide whenever the (strict) application of the equal-loss principle is consistent with agreement among agents. The relationship between $YU^\infty(S,d)$ and EL*(S,d) is contained in Proposition 1.

In order to obtain this result, we first present a lemma:

Lemma 1.- Let $(S,d) \in \sum^n$, and $Q = \{i \in N \mid EL_i(S,d) \geq d_i\}$. Let a(S,d) be the ideal point. Consider now any $x \in IR(S,d)$.

Then, if $|a_k(S,d) - x_k| = \max \{|a_i(S,d) - x_i|\}$, $k \in Q$.

Proof:

The result is obvious if $EL(S,d) \in IR(S,d)$. Let us now look at the case in which $EL(S,d) \notin IR(S,d)$.

Consider $d = 0$. Since $x \in IR(S,d)$, then $x > 0$, and $a(S,d) - x \geq 0$. Let k be an index such that $a_k(S,d) - x_k \geq a_i(S,d) - x_i$, $\forall\, i \in N$.

As S is a comprehensive set, we can construct an element $y \in S$ such that $y_k = x_k$, $y_i \leq x_i$ if $i \neq k$, with $a_i(S,d) - y_i = a_j(S,d) - y_j$, $\forall\, i,j \in N$. Therefore, the equal-loss straight line ℓ_{EL} intersects S in an element z such that $z_k \geq 0$, and, in consequence, $EL_k(S,d) \geq 0$, and $k \in Q$. ∎

Proposition 1.- For every $(S,d) \in \sum^n$, $EL^*(S,d)$ belongs to $YU^{\infty}(S,d)$.

Proof:

Assume $d = 0$. Let $x^* = EL^*(S,d)$. By construction, $x^* \in IR(S,d)$, and therefore $x^* > 0$, and $a(S,d) - x^* \geq 0$. Suppose that x^* does not minimize $d_{\infty}[a(S,d),S]$. Then there exists $y \in IR(S,d)$ such that

$$\max\, [a_i(S,d) - x_i^*] > \max\, [a_i(S,d) - y_i] = a_k(S,d) - y_k.$$

Then we can distinguish the following cases:

(i) If $EL(\bar{S},d) \in IR(S,d)$, then $EL(\bar{S},d) = EL^*(S,d)$, and therefore $a_i(S,d) - x_i^* = a_j(S,d) - x_j^*$, $\forall\, i,j \in N$. In consequence, $a_k(S,d) - x_k^* > a_k(S,d) - y_k$, and $x_k^* < y_k$. But both x^*, $y \in \delta IR(S,d) = WPO[IR(S,d)]$, and therefore there exists $h \in N$ such that $y_h \leq x_h^*$. In consequence, $a_k(S,d) - y_k < a_k(S,d) - x_k^* = a_h(S,d) - x_h^* \leq a_h(S,d) - y_h$. Contradiction.

(ii) If $EL(\bar{S},d) \notin IR(S,d)$, by Lemma 1 over $(\bar{S},d)$, we know that k is in $Q=\{i \in N \ / \ EL_i(\bar{S},d) \geq d_i\}$ .Moreover, let $a_t(S,d) - x_t^* = \max\ [a_i(S,d) - x_i^*]$. Again by Lemma 1, $t \in Q$, and therefore, $a_t(S,d) - x_t^* = a_k(S,d) - x_k^* > a_k(S,d) - y_k$, and $x_k^* < y_k$. Reasoning as before, we conclude that there exists $h \in N$ such that $y_h \leq x_h^*$.

Let us now notice that there must be $h \in Q$ such that $y_h \leq x_h^*$. Suppose this is not true. Then, define z in the following way: $z_i = 0$ if $i \in N/Q$, $z_i = y_i$, in the case where $i \in Q$. Let now consider z_Q, $x_Q^* \in \mathbb{R}^q$. We have $z_Q >> x_Q^*$. Furthermore, $x_Q^* = EL[\bar{S}_Q^d, d_Q] = EL(S_Q^d, d_Q)$, since $EL[\bar{S}_Q^d, d_Q]$ is in the set $IR(\bar{S}_Q^d, d_Q)$, and therefore $x_Q^* \in WPO(S_Q^d)$. In consequence, $z_Q \notin S_Q^d$, and thus $y \notin S$. Contradiction.

Thus, there exists $h \in Q$ such that $y_h \leq x_h^*$. In this case, $a_k(S,d) - y_k < a_k(S,d) - x_k^* = a_h(S,d) - x_h^* \leq a_h(S,d) - y_h$. Contradiction.$\blacksquare$

Chun (1988) noticed that for 2-person bargaining problems $EL(S,d) \in YU^{\infty}(S,d)$. Since for $n = 2$, $EL(S,d)$ is individually rational, Chun's remark turns out to be a particular case of Proposition 1.

3. CHARACTERIZATION OF THE EQUAL-LOSS* SOLUTION.

In order to characterize the equal-loss* solution, we consider the following axioms:

(WPO) *Weak Pareto Optimality.* For all $(S,d) \in \Sigma^n$, $F(S,d) \in WPO(S)$.

(AN) *Anonimity.* For all $(S,d) \in \sum^n$ and for all permutations $\pi: N \longrightarrow N$, $F[\pi(S),\pi(d)] = \pi[F(S,d)]$.

(T.INV) *Translation Invariance.* For all $(S,d) \in \sum^n$ and for all $t \in \mathbb{R}^n$, $F(S+\{t\}, d+t) = F(S,d) + t$.

(CONT) *Continuity.* For each sequence $\{(S^k,d^k)\} \subset \sum^n$, and every $(S,d) \in \sum^n$, if S^k converges onto S in the Hausdorff Topology, and $d^k=d \ \forall \ k$, then $F(S^k,d^k)$ converges onto $F(S,d)$.

(IIIA) *Independence of Individually Irrational Alternatives.* For all (S,d), $(S',d') \in \sum^n$, if $IR(S,d) = IR(S',d')$, then $F(S,d) = F(S',d')$.

(W.MON) *Weak Monotonicity.* For all (S,d), $(S',d') \in \sum^n$, if $S \subset S'$, and $d=d'$, and $a(S,d) = a(S',d')$, then $F(S,d) \le F(S',d')$.

(R.ID.MON) *Rational Ideal Point Monotonicity.* $\forall$ (S,d), $(S,d') \in \sum^n$, $d \le d'$ $\forall$ i, if $a_j(S,d) = a_j(S,d')$ for $j \neq i$, $a_i(S,d) \ge a_i(S,d')$ and $F_i(S,d) \ge d'_i$, then $F_i(S,d') \le F_i(S,d)$.

WPO requires for there to be no feasible alternative in which all agents are better off than they are at the solution outcome; AN says that the names of the agents do not affect the solution outcome. T.INV requires the choice of origin for the utility functions to be irrelevant. CONT implies that small variations in the opportunity set without changes in the disagreement point cause small variations in the solution.

IIIA asks the solution not to take into account those alternatives which are not individually rational.

W.MON was introduced by Kalai & Smorodinski (1975) for two- person bargaining problems, and was extended to n-person bargaining problems by Roth (1979). This property says that, if the feasible set expands in such a way that neither the disagreement point nor the ideal point change, then no agent may be worse-off.

R.ID.MON, requires that an increase of disagreement point in such a way that the ideal point for agent i does not increase, but this change does not affect the other agents's ideal point, would not benefit her, unless she is better off at the new disagreement level. This axiom can be viewed as a weakening of the *Ideal Point Monotonicity* (ID.MON.), introduced by Chun (1988), which requires that a decrease of an agent's ideal point, while the feasible set remains fixed, would not benefit her. For $n = 2$, this axiom is essentially equivalent to that of *Disagreement Point Monotonicity* [Thomson (1987)], which requires that an increase of an agent's utility level at the disagreement point, *ceteris paribus*, will not hurt her [see Chun (1988)].

Our aim is to present a characterization result for the equal-loss* solution. In order to do so, let us start with the following lemmas:

Lemma 2.- The equal-loss* solution satisfies weak monotonicity.

<u>Proof:</u>

Let (S,d), (S',d') $\in \sum^n$ such that S$\subset$S', d=d', and a(S,d) = a(S',d'). Let $\bar{S}$ = Com[IR(S,d)], $\bar{S}$'= Com[IR(S',d')]. Then $\bar{S}\subset\bar{S}$', and a($\bar{S}$,d) = a($\bar{S}$',d'). Since the equal-loss solution, EL, satisfies W.MON [see Chun (1988)], we get EL($\bar{S}$,d) $\leq$ EL($\bar{S}$',d'). Now, by considering definition 1'', we obtain EL*(S,d) $\leq$ EL*(S',d'). ■

Lemma 3.- The equal-loss* solution satisfies rational ideal point monotonicity.

<u>Proof:</u>

Let (S,d), (S,d') $\in \sum^n$ such that a_j(S,d) = a_j(S,d') $\forall j\neq i$, a_i(S,d) $\geq$ a_i(S,d'), d $\leq$ d'(1), and EL^*_i(S,d) $\geq$ d'_i (2). Let $\bar{S}$ = Com [IR(S,d)], $\bar{S}$'=Com [IR(S',d')], then $\bar{S}\subset\bar{S}$'. Taking into account that the equal-loss solution satisfies W.MON and ID.MON [confront Chun (1988)], we can apply ID. MON. to ($\bar{S}$,d'), ($\bar{S}$,d) and W.MON to ($\bar{S}$,d'),($\bar{S}$',d') concluding that EL_i($\bar{S}$',d') $\leq$ EL_i($\bar{S}$,d) (3).

We shall analyze two possible cases:

(i) if $EL_i(\bar{S},d) \geq d_i$, then $EL_i^*(S,d) = EL_i(\bar{S},d)$, moreover we have that:

$$EL_i^*(S,d') = \begin{cases} EL_i(\bar{S},d') \overset{(3)}{\leq} EL_i(\bar{S},d) = EL_i^*(S,d) \\ d'_i \overset{(2)}{\leq} EL_i^*(S,d) \end{cases}$$

(ii) if $EL_i(\bar{S},d) < d_i$, then $EL_i^*(S,d) = d_i$. Now, by considering (1) and (2), we get $d'_i = d_i$, and taking into account (3), $EL_i(\bar{S}',d') < d_i = d'_i$. Therefore, $EL_i^*(S,d') = d'_i = d_i$. ■

Lemma 4.- If F is a solution to the bargaining problem such that F is WPO, CONT and IIIA, then $F(S,d) \in IR(S,d)$, $\forall\ (S,d) \in \sum^n$.

<u>Proof:</u>

Let $(S,d) \in \sum^n$ such that $WPO(S) = PO(S)$, and let us denote $S^1 = Com\{IR(S,d)\}$. Suppose $F(S,d) \notin IR(S,d)$, and consider (S,d) and (S^1,d). Then, by IIIA, $F(S,d) = F(S^1,d)$. Now, taking into account that F is WPO, if $F(S,d) \notin IR(S,d)$ it follows that $F(S,d) \notin S^1$. Contradiction.

Finally, for an arbitrary element in $\sum^n$, we apply CONT. ■

Theorem 1.- The equal- loss* solution is the only solution satisfying WPO, AN, T.INV, IIIA, W. MON, R.ID.MON, and CONT.

Proof:

Obviously, EL* satisfies WPO, AN, T.INV. IIIA and CONT. Moreover, EL* verifies W.MON and R.ID.MON [lemmas 2,3].

In order to prove uniqueness, let F be a solution for which the axioms hold, and consider a problem $(S,d) \in \sum^n$, such that $WPO[IR(S,d)] = PO[IR(S,d)]$. We shall analyze two possible cases:

(i) If $EL(\bar{S},d) \in IR(S,d)$, by T.INV. we can assume $a(S,d) = (1,\dots,1)$.

Let $EL^*(S,d) = x$, $S^1 = Com\ [IR(S,d)]$, $p \in \Delta^{(n-1)}$ a normal vector to the supporting hyperplane of S at x, $H_-(x) = \{y \in \mathbb{R}^n |\ py \le px\}$, and finally, $S^2 = \{\ y \in H_-(x)\ |\ y \le a(S,d)\}$.

Let us now choose y^i as the maximal weakly Pareto Optimal point in S^2 such that $y^i_i = a_i(S,d) = 1$, $y^i_j = y^i_k = \alpha^i$, $\forall$ j,k $\neq$ i. Let $\alpha^* = \min \alpha^i$, and define $z^i \in \mathbb{R}^n$, such that $z^i_i = a_i(S,d) = 1$, $z^i_j = \alpha^*$, if $j \neq i$. Finally, let $S^3 = CoCom\ \{x,z^1,\dots,z^n\}$.

By taking $d^* = (\alpha^*,\dots,\alpha^*)$, $d^* \le d$, we get that $a(S^3,d^*) = (1,\dots,1)$, and by applying WPO and AN, $F(S^3,d^*) = x$. By applying W.MON. to (S^3,d^*) and (S^2,d^*), we conclude that $F(S^2,d^*) = x$. Now, by considering R.ID.MON. for (S^2,d^*) and (S^2,d), we get that $F(S^2,d) \le x$, and taking into account that the solution satisfies WPO, and $F(S^2,d) \in IR(S^2,d)$ [lemma 4], we conclude $F(S^2,d) = x$. By W.MON. on (S^2,d) and (S^1,d), we obtain $F(S^1,d) = x$, and taking into account that $F(S^1,d) \in IR(S^1,d)$, we can apply W.MON. to (S^1,d) and (S,d), concluding that $F(S,d) = x$.

(ii) If $EL(\bar{S},d) \notin IR(S,d)$ we will show that $F(S,d) = EL^*(S,d)$ by means of mathematical induction. Let us denote $Q = \{ i \in N \mid EL_i(\bar{S},d) \geq d_i \}$.

(ii)-(a). Let $P = N/Q = \{j\}$. By AN we can assume $j = n$. Now, by T.INV we can take $a_i(S,d) = 1 \ \forall\ i \neq n$. Thus $EL^*(S,d) = (x_1,..,x_{n-1},d_n)$ with $x_1 = x_2 = \dots = x_{n-1} = y$. Again by T.INV, assume $d_n = y$ concluding that $EL^*(S,d) = (x_1,..,x_n) = x$.

Define now the sets S^1, T and C in the following way:

$S^1 = Com\{IR(S,d)\}$

T is a closed, comprehensive and convex subset of $\mathbb{R}^n$ such that:

$IR(T,d) = IR(S,d)$

$(T^d_Q,d_Q) = Com\ IR(S^d_Q,d_Q)$

$\{x \in T \mid x \notin IR(T,d)\}$ is as great as possible.

Notice that we can find $e \in \mathbb{R}^n$, $e \leq d$, such that $e_n = d_n = x_n$, and $a(T,e) = (1,..,1)$. Moreover $Com\{IR(T,d)\} = S^1$ by construction.

$C = CoCom\ \{\ x,\ (z^1_Q,d_n),\ (z^2_Q,d_n),..,(z^{n-1}_Q,d_n),\ (d^{**}_Q,a_n(S,d))\}$, where $z^i_Q, d^*_Q \in \mathbb{R}^{n-1}$ are elements defined for (S^d_Q,d_Q) in the same way as z^i, d^* for (S,d) in (i).

Then, by (i), $F(T,e) = x$. Take now (T,e) and (T,d). Then, by R.ID.MON., and since $F(T,d) \in IR(T,d)$ [Lemma 4], we conclude $F_n(T,d) = x_n = d_n$. By IIIA on (T,d) and (S^1,d), $F(S^1,d) = F(T.d)$. Since $(d^*_Q,d_n) \leq d$ we can apply R.ID.MON to (S^1,d), $(S^1,(d^*_Q,d_n))$ and get $F(S^1,(d^*_Q,d_n)) \leq F(S^1,d)$, then, by Lemma 4, $F_n(S^1,(d^*_Q,d_n)) = x_n$. Considering W.MON for $(S^1,(d^*_Q,d_n))$, $(C,(d^*_Q,d_n))$ and again taking into account Lemma 4, $F_n(C,(d^*_Q,d_n)) = x_n$. By applying WPO and AN to $(C,(d^*_Q,d_n))$, we obtain $F_i(C,(d^*_Q,d_n)) = x_i\ \forall i \in Q$, and in consequence, $F(C,(d^*_Q,d_n)) = x$. By W.MON we get that $F(S^1,(d^*_Q,d_n)) = x$. By applying R.ID.MON to $(S^1,(d^*_Q,d_n))$ and (S^1,d) we get that $F(S^1,d) \leq x$ and taking into account that the solution satisfies WPO and $F(S^1,d) \in IR(S^1,d)$ [lemma 4], we conclude $F(S^1,d) = x$. Finally, by IIIA, $F(S,d) = x$.

(ii)(b) In the case whereby $p = \text{card } P = k$, assume that $F(S,d) = EL^*(S,d) = x$.

(ii)(c) Let now $p = k+1$. We assume $j \in P$, $j=1,2,..,k+1$ by AN. Now, by T.INV take $a_i(S,d) = 1$ $\forall i \in Q$ and then $EL^*(S,d) = (d_1,..,d_{k+1},x_{k+2},..,x_n)$ with $x_{k+2} = \ldots\ldots = x_n = y$. Again by T.INV assume $d_j = y$, $\forall$ $j \in P$, therefore $EL^*(S,d) = (x_1, x_2,.., x_n)$.

Define now the sets S^1, ${}^jT$ and C in the following way:

$S^1 = \text{Com}\{IR(S,d)\}$

For $j=1,.,k+1$, ${}^jT \subset \mathbb{R}^n$ is a closed, comprehensive and convex set such that:

$IR({}^jT,d) = IR(S,d)$

$({}^jT^d_{H_j},d_{H_j}) = \text{Com } IR(S^d_{H_j},d_{H_j})$ where $H_j = N/\{j\}$ for $j = 1,..,k+1$.

$\{x \in {}^jT|\ x \notin IR({}^jT,d)\}$ is as great as possible.

Notice that $\forall j \in P$ we can find, a vector ${}^je \in \mathbb{R}^n$, ${}^je \leq d$ $\forall j$, such that ${}^je_j = {}_jd = {}_1x$ and $a_j({}^jT,{}^je) = a_i({}^jT,d) = 1$ $\forall i \in Q$, $a_k({}^jT,{}^je) = a_k(S,d)$ $k \neq j$ $k \in P$. Moreover $\text{Com}\{IR({}^jT,d)\} = S^1$ $\forall j \in P$ by construction.

$C = \text{CoCom}$ $\{x,$ (d_P,z^1_Q), $(d_P,z^2_Q),..,(d_P,z^{n-k-1}_Q),(a_1(S,d),d_2,..,d_{k+1},d^*_Q),\ldots$ $..,(d_1,d_2,..,d_k,a_{k+1}(S,d),d^*_Q)\}$, where $z^i_Q, d^*_Q \in \mathbb{R}^{n-k-1}$ are elements defined for (S^d_Q,d_Q) in the same way as z^i, d^* for (S,d) in (i).

Thus, by the induction hypothesis [(ii)-(b)], we conclude $F({}^{j}T, {}^{j}e) = x$ $\forall j=1,..,k+1$. Take now $({}^{j}T, {}^{j}e)$ and $({}^{j}T,d)$. Then, by R.ID.MON., and since $F({}^{j}T,d) \in IR({}^{j}T,d)$ [Lemma 4], we get $F_j({}^{j}T,d) = d_j$ $\forall j \in P$. By IIIA on $({}^{j}T,d)$ and (S^1,d), for $j=1,..,k+1$, we can conclude that $F(S^1,d) = F({}^{j}T,d)$ $\forall j \in P$. Since $(d_P,d_Q^*) \le d$ we can apply R.ID.MON to (S^1,d), $(S^1,(d_P,d_Q^*))$ getting $F(S^1,(d_P,d_Q^*)) \le F(S^1,d)$, then, by Lemma 4, $F_j(S^1,(d_P,d_Q^*)) = F_j(S^1,d)$ $\forall j \in P$. Considering now W.MON for $(S^1,(d_P,d_Q^*))$, $(C,(d_P,d_Q^*))$ and again taking into account Lemma 4, $F_j(C,(d_P,d_Q^{**})) = x_j$ $\forall j \in P$. By applying WPO and AN to $(C,((d_P,d_Q^*))$, $F_i(C,(d_P,d_Q^*)) = x_i$ $\forall i \in Q$, and in consequence $F(C,(d_P,d_Q^*)) = x$. By W.MON we get that $F(S^1,(d_P,d_Q^*)) = x$. By applying R.ID.MON to $(S^1,(d_P,d_Q^*))$ and (S^1,d), we obtain that $F(S^1,d) \le x$, and taking into account that the solution satisfies WPO and $F(S^1,d) \in IR(S^1,,d)$ [lemma 4] we conclude $F(S^1,d) = x$. Now, by IIIA, $F(S,d) = x$.

Finally, for an arbitrary element in $\sum^n$, we apply CONT. $\blacksquare$

4. THE LEXICOGRAPHIC EQUAL-LOSS* SOLUTION.

By introducing the Equal-Loss* solution for bargaining problems we have solved the main shortcoming of the Equal-Loss Solution, namely, the lack of individual rationality . Nevertheless, another problem remains, as was the case in the Equal-Loss Solution, the lack of full Pareto Optimality.

Recently Chun & Peters (1991) presented the Lexicographic Equal-Loss Solution, as a way of ensuring Pareto Optimality when starting from the Equal-Loss Solution. Unfortunately, the lexicographic equal-loss solution fails to be individually rational. Thus, Chun & Peters proposed a modification of the lexicographic equal-loss solution in order to get individual rationality, in the following way: Starting with $(S,d) \in \sum^n$, consider $\overline{S} = \text{Com}[\text{IR}(S,d)]$, and then take $\overline{\text{LEL}}(S,d) = \text{LEL}(\overline{S},d)$. It can be proved that $\overline{\text{LEL}}(S,d)$ turns out to be individually rational and Pareto Optimal. The characterization of this new solution was left as an open problem.

The aim of this Section is to introduce and axiomatically characterize the Lexicographic Equal-Loss* Solution, as a way of ensuring Pareto Optimality from the Equal-Loss* Solution. In a similar spirit, we start from the Equal-Loss* Solution. If it is not Pareto Optimal, then we use a lexicographic procedure in order to achieve a Pareto Optimal element. Interestingly, by means of the aforementioned procedure, we will end up at

$\overline{LEL}(S,d)$. Thus, by characterizing the lexicographic equal-loss* solution, we close the open problem proposed by Chun and Peters (1991).

Let $>^{\ell}$ be the lexicographical ordering on $\mathbb{R}^n$, i.e. $x >^{\ell} y$ $(x,y \in \mathbb{R}^n)$ if there is $i \in N$ with $x_i > y_i$, and $x_j = y_j$, for all $j < i$. Let $\alpha:\mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that for each $x \in \mathbb{R}^n$ there is a permutation π of N with $\alpha(x) = \pi(x)$, and $\alpha_1(x) \leq \alpha_2(x) \leq \ldots \leq \alpha_n(x)$. Then, the *lexicographic maximin ordering* $>^{\ell m}$ on $\mathbb{R}^n$ is defined by $x >^{\ell m} y$ $(x,y \in \mathbb{R}^n)$ if $\alpha(x) >^{\ell} \alpha(y)$.

Then we have the following definition:

Definition 2.- For a given problem $(S,d) \in \Sigma^n$, the *lexicographic equal loss* solution* $LEL^*:\Sigma^n \longrightarrow \mathbb{R}^n$, assigns to (S,d) the unique point of S defined in the following way:

(i) Choose $t \in \mathbb{R}^n$, such that $EL^*(S^*,d^*) = k$, $\forall\ i \in N$, $a_i(S^*,d^*) = 0$ $\forall\ i \in N$ such that $EL_i(S,d) \geq d_i$, $S^* = \{S+t\}$, $d^* = d+t$.

(ii) find a maximal element in S^* with respect to $>^{\ell m}$, x^*

(iii) $LEL^*(S,d) = x^* - t$.

LEL* is well-defined, and can be viewed as a modification of the equal-loss* solution, in order to ensure Pareto Optimality. Thus, by starting from EL*(S,d), and by exhausting all possible gains of the agents (without damaging any one), we reach the lexicographic equal-loss* solution. Notice that $LEL^*(S,d) = \overline{LEL}(S,d)$.

There exists a simple procedure to find LEL*(S,d). First, take S^1 = Com[IR(S,d)]. Now, decrease the utility of the n agents in $N^1=N$ equally from $a(S^1,d)$ along $a(S^1,d)-\zeta^1 e_{N^1}$ ($\zeta^1 \geq 0$), until a boundary point of S^1 is reached, say z^1. If $z^1 \in PO(S^1)$, then $z = z^1$. Otherwise, let $N^2 \subset N$ be the largest possible subset of agents whose utilities can be equally increased in a nonnegative direction starting from z^1, i.e., follow the direction $z^1 + \zeta^2 e_{N^2}$ ($\zeta^2 > 0$). Let z^2 be the maximal point in this direction and still in S^1. If $z^2 \in PO(S^1)$, then $z = z^2$. Otherwise we continue along the direction $z^2 + \zeta^3 e_{N^3}$ ($\zeta^3 > 0$), where $N^3 \subset N^2$ is the largest possible subset of agents for which an increase along $z^2 + \zeta^3 e_{N^3}$ is still possible. Etc. In this way we end up, after a finite number of steps, at a point $z \in PO(S^1)$. It is not hard to show that z = LEL*(S,d), by adapting Lemma 3 in Imai (1983) to this context.

In order to characterize the lexicographic equal-loss* solution, we introduce some additional axioms:

(PO) *Pareto Optimality* $\forall$ $(S,d) \in \sum^n$, $F(S,d) \in PO(S)$.

(R.W.MON) *Restricted Weak Monotonicity* $\forall$ (S,d), $(S',d') \in \sum^n$, if $S \subseteq S'$, $d = d'$ and $S_{d,-i} = S'_{d',-i}$ for all i, then $F(S',d') \geq F(S,d)$, where x_{-i} is the (n-1) dimensional vector obtained after deleting the ith component of x, and $S_{d,-i} \equiv$ the closure of $\{x_{-i} \mid x \in S, x \leq a(S,d)\}$.

(RIAIP) *Rational Independence of Alternatives other than Ideal Point.*

$\forall$ (S,d), (S',d') $\in \sum^n$, if S' $\subseteq$ S, a(S,d) = a(S',d') and d $\leq$ d' F(S,d) $\in$ IR(S',d'), then F(S',d') = F(S,d).

(P.O) requires that the solution outcome should exhaust all gains from cooperation.

(R.W.MON), introduced for 2-person problems by Kalai and Smorodinski (1975) states that an expansion of the feasible set which does not affect the ideal point should not hurt any agent. It was noted by Roth (1979) that a straightforward extension of this axiom may be incompatible with Pareto Optimality for more than 2-person bargaining problems. The version of the axiom introduced here, compatible with Pareto Optimality, was presented by Imai (1983), and also used by Chun & Peters (1991).

Finally, (RIAIP) is a modification of the axiom (IAIP), Independence of Alternatives other than Ideal Point, introduced by Roth (1977), in which it was required that, if the feasible set should shrink and the disagreement point increases without affecting the ideal point, and the solution outcome for the original point is still feasible *and individually rational* for the smaller problem, then the solution outcome for the smaller problem should be the same as for the original one.

In order to prove the characterization result for the lexicographic equal-loss* solution, let us start by presenting some lemmas:

Lemma 5.- The lexicographic equal-loss* solution satisfies restricted weak monotonicity.

Proof:

Let (S,d), (S',d') $\in \sum^n$ such that S $\subseteq$ S', d = d', and $S_{d,-i} = S'_{d',-i}$, for all i. Let T = Com[IR(S,d)], T' = Com[IR(S',d')]. Then T $\subseteq$ T', and for all i, $T_{d,-i} = T'_{d',-i}$. Since the lexicographic equal-loss solution, LEL, satisfies R.W.MON. [see Chun & Peters (1991)], we get LEL(T',d') $\geq$ LEL(T,d). Now, by the definition of $\overline{LEL}$, we get $\overline{LEL}$(S',d') $\geq$ $\overline{LEL}$(S,d), and therefore, LEL*(S',d') $\geq$ LEL*(S,d).■

Lemma 6.- The lexicographic equal-loss* solution satisfies rational independence of alternatives other than ideal point.

Proof:

Let (S,d), (S',d') $\in \sum^n$ such that S' $\subseteq$ S, a(S,d) = a(S',d'), d$\leq$d' and LEL*(S,d) $\in$ IR(S',d'). Let T = Com[IR(S,d)], T' = Com[IR(S',d')]. Thus, since d$\leq$d', T' $\subseteq$ T and a(T',d) = a(T,d), given that LEL*(S,d) $\in$ IR(S,d). Then, LEL*(S,d) = LEL*(T,d), and as LEL*(S,d) $\in$ IR(S',d'), LEL*(T,d) $\in$ T'. Now, since the lexicographic equal-loss solution LEL satisfies IAIP [confront Chun & Peters (1991)], we conclude that LEL(T',d') = LEL(T,d). Then, by definition of $\overline{LEL}$, we get $\overline{LEL}$(S',d') = $\overline{LEL}$(S,d) and therefore LEL*(S',d') = LEL*(S,d).■

Theorem 2.- The Lexicographic Equal-Loss* Solution is the only solution on $\sum^n$ satisfying IIIA, PO, AN, T.INV., R.W.MON. and RIAIP.

Proof:

For the proof of the Theorem, some additional notation is needed. Given $(S,d) \in \sum^n$, Int(S) is the *interior* of S. Moreover, given $z,p \in \mathbb{R}^n$, we shall denote $H(p,pz) \equiv \{x \in \mathbb{R}^n |\ px \le pz\}$.

LEL* satisfies R.W.MON and RIAIP (Lemmas 5,6). It is straightforward to check that LEL* satisfies IIIA, PO, AN and T.INV.

Note that $\forall(S,d)\in\sum^n$, if F(S,d) is a solution satisfying PO and IIIA, then we can show in a similar way as in Lemma 4, that $F(S,d)\in IR(S,d)$.

We try to sketch the idea of the proof. The proof uses the procedure for finding LEL*(S,d) as described above. We then we have to find $z^1,\dots,z^T$, to obtain z^T = LEL*(S,d). First, by T.Inv. we may assume that the ideal point has all coordinates equal to one. The main step lies in the construction of a sequence of problems, whose solution outcome is $z^1,\dots,z^T$.

Consider a solution F for which all six axioms hold. Given $(\bar{S},\bar{d}) \in \sum^n$, let $S = Com[IR(\bar{S},\bar{d})]$, and by T.INV. suppose that $a(\bar{S},\bar{d}) = e_N$. Let $d' \le \bar{d}$, $d'\in Int(S)$ such that $d'_i = d'_j = 1 - \delta$, $\forall\ i,j \in N$, and $a(S,d') = e_N$. Equivalently, we may well take, by T.Inv, $d = 0$, and $a(S,d) = \delta e_N$, $\delta > 0$. Now let $\{z^t\}_{t=1}^T$ be the sequence as defined in the process of finding LEL*(S,d). We will show that $F(S,d) = z^T$.Then, by PO we know that $F(S,d) \in IR(S,\bar{d})$, and by RIAIP we obtain $F(S,\bar{d}) = z^T$. Then, by IIIA, $F(\bar{S},\bar{d}) = z^T$.

In order to prove that $z^T = F(S,d)$ we construct auxiliary problems. Let $M^t = N/N^t$, $p^t = e_{M^t}$, $\forall\ t = 1,2,\ldots,T$, where $M^1 = \varnothing$ and $p^1 = 0$. We define:

$$S^{1t} \equiv H(e_N, \Sigma z^1_t) \cap \left[\bigcap_{k=1}^{t} H(p^k, p^k z^k) \right] \cap (\delta e_N - \mathbb{R}^n_+) \qquad \forall\ t = 1,\ldots,T$$

$$S^{2t} \equiv S^{1t} \cap H(p^{t+1}, p^{t+1} z^{t+1}) \qquad \forall\ t = 1,\ldots,T-1$$

$$S^{3t} \equiv H(e_N, \Sigma z^t_1) \cap S \qquad \forall\ t = 1,\ldots,T$$

$$S^{4t} \equiv S^{1t} \cap S \qquad \forall\ t = 1,\ldots,T$$

Then, by reasoning in an identical way to Chun & Peters' (1991), main theorem, with the only substitution of IAIP by RIAIP, we get $F(S,d) = z^T$. ∎

The main idea is that all the auxiliary problems will have z^t as a solution. The first problem, S^{11} is symmetric, and its solution is z^1, by PO and AN. Thus, by applying RIAIP, we get $F(S^{21},d) = F(S^{31},d) = F(S^{41},d) = z^1$. Now, by induction, and by using R.W.MON. and RIAIP, we obtain that the solution for any problem t (t>2) must be greater or equal than the solution outcome for (t-1), z^{t-1}. Now, by PO we conclude that it is equal to z^t.

5. FINAL REMARKS.

The equal-loss principle is an attractive one when dealing with problems of bankruptcy, property rights and taxation [see Auman & Maschler (1985), Young (1987) (1988)], and is one which has been traditionally used in distributive justice problems. Yet, its application to bargaining problems may fail to be individually rational. This suggests the convenience of an adequate modification in the form of applying it to axiomatic bargaining. The object of this paper has been to do this, by proposing the equal-loss* solution.

In order to ensure Pareto Optimality (and not only Weak Pareto Optimality), Chun & Peters (1991) recently presented a variation of the equal-loss solution, namely, the *lexicographic equal-loss solution*, LEL(S,d), in which, by starting with EL(S,d), they construct a lexicographic extension of EL(S,d), by means of increasing the utility levels of some agents, without damaging the rest, and looking for a maximal element in this way. It is worth pointing out that by means of this modification, the lack of individual rationality problem remains unsolved. Thus, they propose a modification of the lexicographic equal-loss solution, in order to ensure individual rationality, in the following way: For a given $(S,d) \in \sum^n$, consider $\bar{S} = \text{Com}[IR(S,d)]$, and define $\overline{LEL}(S,d) = LEL(\bar{S},d)$. It is worth noticing that this solution turns out to be individually rational and Pareto Optimal, and they propose its characterization as an open problem.

In this paper we solve the proposed problem, but we come to the solution in a different way. First, we solve the lack of individual rationality *for the equal-loss solution*, and then, we face the problem of full Pareto Optimality by properly considering a lexicographic extension of our solution.

The two proposed solutions, namely the *equal-loss* solution and its lexicographic extension* are axiomatically characterized.

It is worth noticing the relationship between the results provided in this paper and the characterizations of both the equal-loss and the lexicographic equal-loss solutions:

Chun (1988) characterizes the equal-loss solution by means of six axioms: WPO, AN, T.INV, W.MON, CONT and ID.MON. In order to characterize the equal-loss* solution apart from WPO, AN, T.INV, W.MON and CONT, we also consider *Rational Ideal Point Monotonicity* (R.ID.MON), which can be viewed as a weakening of *Ideal Point Monotonicity* (ID.MON), without taking into account those alternatives which are not individually rational (IIIA).

Chun & Peters (1991) characterize the lexicographic equal-loss solution by means of five axioms: PO, AN, T.INV, R.W.MON and IAIP. Our characterization of the lexicographic equal-loss* solution is made by introducing *Independence of Individually irrational Alternatives* (IIIA), and *Rational Independence of Alternatives other than the Ideal Point*

(RIAIP) [a weaker requirement than *Independence of Alternatives other than Ideal Point* (IAIP)], and considering exactly the same axioms as in Chun & Peters: PO, AN, T.INV and R.W.MON.

Table 1 summarizes axiomatic properties of the solutions we introduce in this paper together with the Nash (N), the Kalay-Smorodinsky (K), the egalitarian (E), the lexicographic egalitarian (LE), the equal-loss (EL) and the lexicographic equal-loss (LEL) solutions.

TABLE 1

	N	K	E	LE	EL	LEL	EL*	LEL*
Weak Pareto Optimality	yes	yes	yes	yes	yes	yes	yes	yes
Pareto Optimality	yes	no	no	yes	no	yes	no	yes
Anonimity	yes	yes	yes	yes	yes	yes	yes	yes
Translation Invariance	yes	yes	yes	yes	yes	yes	yes	yes
Continuity	yes	yes	yes	no	yes	no	yes	no
Individual Rationality	yes	yes	yes	yes	no	no	yes	yes
Independence of individually irrational alternatives	yes	yes	yes	yes	no	no	yes	yes
Weak Monotonicity	no	yes	yes	no	yes	no	yes	no
Restricted Weak Monotonicity	no	yes	yes	yes	yes	yes	yes	yes
Ideal Point Monotonicity	no	no	no	no	yes	yes	no	no
Rational Ideal Point Monotonicity	no	no	no	no	yes	yes	yes	yes
Independence of Alternatives other than Ideal Point	no	no	no	no	yes	yes	no	no
Rational Independence of Alternatives other than Ideal Point	no	no	no	no	yes	yes	yes	yes

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